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# ON THE SYMMETRIC VERSION OF SEAKI THEOREM AND FLAT DENSITIES

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ABSTRACT. It is shown that for any  $\alpha \in ]\frac{1}{2}, 1[$  there exists a symmetric probability measure  $\sigma$  on the torus such that the Hausdorff dimension of its support is  $\alpha$  and  $\sigma * \sigma$  is absolutely continuous with flat continuous Radon-Nikodym derivative. Namely, we obtain a symmetric version of Seaki Theorem but the flat Radon-Nikodym derivative of  $\sigma * \sigma$  can not be a Lipschitz function.

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*Key words and phrases:* symmetric measure, singular measure, convolution, Hausdorff dimension, distance of Hausdorff, Seaki theorem, flat densities.

## 1. INTRODUCTION

Seaki's theorem assert that there exists a singular measure  $\sigma$  with support of Lebesgue measure zero such that  $\sigma * \sigma$  is absolutely continuous and in addition the Radon-Nikodym derivative of  $\sigma * \sigma$  has a uniformly convergent Fourier series. It is any easy exercise to show that the measure  $\sigma$  can not be symmetric. Nevertheless, using the ideas of Körner's proof of Seaki Theorem [2], we shall obtain a symmetric version of Saeki theorem. In fact, we shall prove that there exist a symmetric singular measure for which the convolution is absolutely continuous with continuous Radon-Nikodym derivative. Our principal motivation is connected to the question raised in [1] on the existence of singular symmetric measure  $\sigma$  with absolutely continuous such that the Radon-Nikodym derivative is flat. We recall that the function  $f$  is flat if  $\|f - 1\|_\infty < \varepsilon$ , for some  $\varepsilon \in [0, 1)$ . The subject of this note is to establish that such measure exist. For that, we will essentially follows Körner's proof of Seaki Theorem [2].

This note is organized as follows. In section 2, we state our main result and the fundamental theorem need it for its proof. In section 3, we present the ingredients of need it in the proof of the fundamental theorem, and we conclude by presenting its proof.

## 2. THE MAIN THEOREM AND ITS PROOF

We start by stating our main theorem.

**Theorem 2.1.** *Given  $\varepsilon > 0$  we can find a symmetric measure  $\sigma$  with support of Hausdorff dimension  $\frac{1}{2}$  and a continuous function  $f : \mathbb{T} \rightarrow \mathbb{R}$  such that  $\|f - 1\|_\infty < \varepsilon$ , and  $\sigma * \sigma = f d\lambda$ , where  $\lambda$  is a Lebesgue measure.*

The main ingredients of Theorem 2.1 is contained in the Körner's proof of Seaki Theorem [2]. Following Körner, we denote by  $\mathcal{F}_s$  the space of non-empty closed symmetric subsets of  $\mathbb{T}$  equipped with the Hausdorff distance  $d_H$  define by

$$\forall (E, F) \in \mathcal{F}_s, d_H(E, F) = \sup_{e \in E} d(e, F) + \sup_{f \in F} d(E, f).$$

It is an easy exercise to verify that  $(\mathcal{F}_s, d_H)$  is a complete metric space, in fact, we have more  $(\mathcal{F}_s, d_H)$  is compact (see [3, Ch. IV.]). As in [2], we consider the metric space  $(\mathcal{E}_s, d_{\mathcal{E}_s})$ , where  $\mathcal{E}_s$  is consisting of ordered pairs  $(E, \mu)$  where  $E \in \mathcal{F}_s$  and  $\mu$  is a symmetric probability measure

with  $\text{supp}(\mu) \subset E$  and  $\widehat{\mu}(r) \rightarrow 0$  as  $|r| \rightarrow +\infty$  and  $d_{\mathcal{E}_s}$  is defined by  $\forall((E, \mu), (F, \sigma)) \in \mathcal{E}_s$ ,  $d_{\mathcal{E}_s}((E, \mu), (F, \sigma)) = d_H(E, F) + \sup_{r \in \mathbb{Z}} |\widehat{\mu}(r) - \widehat{\sigma}(r)|$ .

Finally, we consider the metric space  $(\mathcal{G}_s, d_{\mathcal{G}_s})$ , where  $\mathcal{G}_s$  is consisting of those  $(E, \mu) \in \mathcal{E}_s$  such that  $\mu * \mu = f_\mu d\lambda$  with  $f_\mu$  is continuous,  $d_{\mathcal{G}_s}$  is given by, for all  $((E, \mu), (F, \sigma)) \in \mathcal{G}_s$

$$d_{\mathcal{G}_s}((E, \mu), (F, \sigma)) = d_{\mathcal{E}_s}((E, \mu), (F, \sigma)) + \|f_\mu - f_\sigma\|_\infty.$$

For the proof of our main result, we need the following Lemma.

**Lemma 1.** *The metric spaces  $(\mathcal{E}_s, d_{\mathcal{E}_s})$  and  $(\mathcal{G}_s, d_{\mathcal{G}_s})$  are complete.*

The proof of the lemma is leaved to reader. We need also the following crucial lemma

**Lemma 2.** *Let  $\alpha \in [\frac{1}{2}, 1)$  and  $\mathcal{H}_n$  be the subset of consisting of those  $(E, \mu) \in \mathcal{G}_s$  such that we can find a finite collection of intervals  $\mathcal{I}$  symmetric (which means if  $I \in \mathcal{I}$  the  $-I$  is in  $\mathcal{I}$ ) with*

$$E \subseteq \bigcup_{I \in \mathcal{I}} I \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha + \frac{1}{n}} < \frac{1}{n}.$$

*Then  $\mathcal{H}_n$  is open dense set in  $(\mathcal{G}_s, d_{\mathcal{G}_s})$ .*

The proof of Lemma 2 is similar to that in [2], we need only to point out that the metric  $d_\psi$  defined in [2] verify  $d_\psi \geq d_{\mathcal{G}_s}$ .

At this, we stat the fundamental result of this note.

**Theorem 2.2.** *Let  $\alpha \in [\frac{1}{2}, 1)$ . The complement of the set*

$$\mathcal{H}_\alpha = \{(E, \mu) \in \mathcal{G}_s : E \text{ has Hausdorff dimension } \alpha\}$$

*is of first category in  $(\mathcal{G}_s, d_{\mathcal{G}_s})$ .*

Obviously, Theorem 2.1 follows from Theorem 2.2.

### 3. THE PROOF OF THEOREM 2.2.

The fondamental ingredient of the proof of Theorem 2.2 is based on the following lemmas. The first one is standard in Probability and Martingale theory. and its the proof can be found in [2].

**Lemma 3.** *Suppose that  $0 < Np < 1$  and  $m \geq 2$ . Then, if  $Y_1, Y_2, \dots, Y_N$  are independent random variables with*

$$\mathbb{P}(Y_j = 1) = p, \quad \mathbb{P}(Y_j = 0) = 1 - p,$$

*it follows that*

$$\mathbb{P}\left(\sum_{j=1}^N Y_j \geq m\right) \leq \frac{2(Np)^m}{m!}.$$

Before stating the second one. We recall the following definition.

**Definition 1.** A sequence  $W_r$  is said to be a martingale with respect to a sequence  $X_r$  of random variables if

- (i)  $\mathbb{E}(|W_j|) < \infty$ .
- (ii)  $\mathbb{E}(W_{r+1}|X_0, \dots, X_r) = W_r$ .

**Lemma 4.** Let  $\delta > 0$  and let  $W_r$  be a martingale with respect to a sequence  $X_r$  of random variables. Write  $Y_{r+1} = W_{r+1} - W_r$ . Suppose that

$$\mathbb{E}(e^{\lambda Y_{r+1}}|X_0, X_1, \dots, X_r) \leq e^{a_{r+1} \frac{\lambda^2}{2}}.$$

for all  $\lambda < \delta$  and some  $a_{r+1} > 0$ . Suppose further that  $A \geq \sum_{r=1}^N a_r$ . Then, provided that  $0 \leq x < A\delta$ , we have

$$\mathbb{P}(|W_N - W_0|) \leq \exp\left(\frac{-x^2}{2A}\right).$$

Lemma 4 is known as Hoeffding-Azuma's inequality. By applying Lemma 3, we get the following lemma. For its complete proof, we refer to [2].

**Lemma 5.** Let  $\gamma \in ]0, 1[$  and  $\varepsilon > 0$ , we can find an integer  $M = M(\gamma, \varepsilon) \geq 1$  such that the following property holds. Suppose  $n \geq 2$ ,  $n^\gamma \geq N$  and  $X_1, X_2, \dots, X_N$  are independent symmetric random variables each uniformly distributed on

$$\Gamma_n \stackrel{\text{def}}{=} \left\{ \frac{r}{n} : r \in \left\{ -1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1 \right\} \right\}.$$

Then, with probability at least  $1 - \frac{\varepsilon}{n}$ ,

$$\sum_{j=1}^N \left( \delta_{X_j} \left( \left\{ \frac{r}{n} \right\} \right) + \delta_{-X_j} \left( \left\{ \frac{r}{n} \right\} \right) \right) < M.$$

for all  $r \in \left\{ -1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1 \right\}$ .

The key lemma is the following lemma.

**Lemma 6.** Suppose  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence with  $\varphi(n) \xrightarrow[n \rightarrow \infty]{} +\infty$ . If  $\gamma \in ]0, 1[$  and  $\varepsilon > 0$ , there exists a integer  $M(\gamma) \stackrel{\text{def}}{=} M$  and  $n_0(\varphi, \gamma, \varepsilon) \stackrel{\text{def}}{=} n_0$  with the following property. Suppose that  $n > n_0$ ,  $n$  is odd,  $n^\gamma \geq N$  and  $X_1, X_2, \dots, X_N$  are independent symmetric random variables each uniformly distributed on

$$\Gamma_n \stackrel{\text{def}}{=} \left\{ \frac{r}{n} : r \in \left\{ -1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1 \right\} \right\}.$$

Then, if we write  $\sigma \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^N (\delta_{X_j} + \delta_{-X_j})$ , we have

$$\left| \sigma * \sigma \left( \frac{k}{n} \right) - \frac{1}{2n} \right| \leq \varepsilon \frac{\varphi(n) \sqrt{\ln(n)}}{N \sqrt{n}},$$

and

$$\sigma \left( \frac{k}{n} \right) \leq \frac{M}{N}$$

for all  $k \in \{-1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1\}$ , with probability at least  $\frac{1}{2}$ .

As a corollary of the lemme 6 we have the following lemma

**Lemma 7.** Suppose  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence with  $\varphi(n) \xrightarrow[n \rightarrow \infty]{} +\infty$ . If  $\gamma \in ]0, 1[$  and  $\varepsilon > 0$ , there exists a integer  $M(\gamma) \stackrel{\text{def}}{=} M$  and  $n_0(\varphi, \gamma, \varepsilon) \stackrel{\text{def}}{=} n_0$  with the following property. Suppose that  $n > n_0$ ,  $n$  is odd,  $n^\gamma \geq N$ , we can find  $N$  points

$$x_j \in \Gamma_n \stackrel{\text{def}}{=} \left\{ \frac{r}{n} : r \in \left\{ -1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1 \right\} \right\},$$

such that writing  $\sigma \stackrel{\text{def}}{=} \frac{1}{2N} \sum_{j=1}^N (\delta_{x_j} + \delta_{-x_j})$ , we have

$$\left| \sigma * \sigma \left( \frac{k}{n} \right) - \frac{1}{2n} \right| \leq \varepsilon \frac{\varphi(n) \sqrt{\ln(n)}}{N \sqrt{n}},$$

and

$$\sigma \left( \frac{k}{n} \right) \leq \frac{M}{N}$$

for all  $k \in \{-1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1\}$ .

Now, let us emphasize that we need only to give a sketch of a the proof of the lemma 6.

*Proof.* Let  $M = M(\gamma, \frac{1}{4})$  be as in Lemma 5. Fix  $\frac{r}{n} \in \Gamma_n$  and define  $Y_1, Y_2, \dots, Y_N$  as follows. If  $\sum_{v=1}^{j-1} \left( \delta_{X_v} \left( \frac{u}{n} \right) + \delta_{-X_v} \left( \frac{u}{n} \right) \right) < M$ , for all  $u$  with  $1 \leq |u| \leq n$ , set

$$\begin{aligned}
Y_j = & -\frac{2j-1}{2n} + \frac{1}{4} \left( \delta_{2X_j} \left( \left\{ \frac{r}{n} \right\} \right) + \delta_{-2X_j} \left( \left\{ \frac{r}{n} \right\} \right) \right) + \\
& \frac{1}{2} \sum_{v=1}^{j-1} \left\{ \delta_{X_v+X_j} \left( \left\{ \frac{r}{n} \right\} \right) + \delta_{-(X_v+X_j)} \left( \left\{ \frac{r}{n} \right\} \right) \right\} + \\
& \frac{1}{2} \sum_{v=1}^{j-1} \left\{ \delta_{X_v-X_j} \left( \left\{ \frac{r}{n} \right\} \right) + \delta_{X_j-X_v} \left( \left\{ \frac{r}{n} \right\} \right) \right\}.
\end{aligned}$$

Otherwise  $Y_j = 0$ . Put  $W_0 = 0$  and  $W_j = \sum_{v=1}^{j-1} Y_v$ . it is a nice exercise to show that  $\mathbb{E}(Y_j | X_1, \dots, X_{j-1}) = 0$ . Thus the sequence  $W_j$  is a martingale with respect to  $X_1, \dots, X_N$ . Following Körner proof, we get that

$$\mathbb{E}(e^{\lambda Y_j} | X_1, \dots, X_N) \leq \exp \left\{ \frac{N}{n} 4(1+M^2)\lambda^2 \right\}.$$

We can thus apply Lemma 4 with

$$A = 8 \frac{N^2}{n} (M^2 + 1) \text{ and } x = \varepsilon \frac{N\phi(n)\sqrt{\ln(n)}}{\sqrt{n}},$$

since  $\phi(n) \xrightarrow{n \rightarrow +\infty} +\infty$  we can choose  $n_0(\phi, \gamma, \varepsilon) = n_0$ . Therefore

$$\mathbb{P} \left\{ |W_n| \geq \varepsilon \frac{N\phi(n)\sqrt{\ln(n)}}{\sqrt{n}} \right\} \leq \frac{1}{4n},$$

for all  $n \geq n_0$ . To finish the proof, observe that

$$|W_N| = \left| \sum_{j=1}^N Y_j \right| = \left| (N\sigma * N\sigma) \left( \left\{ \frac{r}{n} \right\} \right) - \frac{N^2}{2n} \right|.$$

It follows with probability at least  $1 - \frac{1}{2n}$ , that we have

$$\sum_{v=1}^N (\delta_{X_j} + \delta_{-X_j}) \left( \left\{ \frac{r}{n} \right\} \right) < M,$$

for all  $r$  with  $1 \leq |r| \leq n$  and

$$\left| (N\sigma * N\sigma) \left( \left\{ \frac{r}{n} \right\} \right) - \frac{1}{2n} \right| < \varepsilon \frac{N\phi(n)\sqrt{\ln(n)}}{\sqrt{n}}.$$

Hence

$$\left| (\sigma * \sigma) \left( \left\{ \frac{r}{n} \right\} \right) - \frac{1}{2n} \right| < \varepsilon \frac{\phi(n)\sqrt{\ln(n)}}{N\sqrt{n}}.$$

and the proof of the lemma is complete.  $\square$

Now we rewrite the lemma 7 in more usable form. More precisely, we will exhibit a function  $g$  in order to prove the following lemma.

**Lemma 8.** *Suppose  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence with  $\varphi(n) \xrightarrow[n \rightarrow \infty]{} +\infty$ . If  $\gamma \in ]0, 1[$  and  $\varepsilon > 0$ , there exists a integer  $M(\gamma) \stackrel{\text{def}}{=} M$  and  $n_0(\varphi, \gamma, \varepsilon) \stackrel{\text{def}}{=} n_0$  with the following property. Suppose that  $n > n_0$ ,  $n$  is odd,  $n^\gamma \geq N$ , we can find  $N$  points*

$$x_j \in \Gamma_n \stackrel{\text{def}}{=} \left\{ \frac{r}{n} : r \in \left\{ -1, \dots, \frac{-1}{n}, \frac{1}{n}, \dots, 1 \right\} \right\},$$

such that, writing

$$g = \frac{n}{N} \sum_{1 \leq |j| \leq N} \mathbb{I}_{[x_j - \frac{1}{4n}, x_j + \frac{1}{4n}]},$$

with  $x_{(-j)} = -x_j$ , we have  $g * g$  continuous and

- (1)  $\|g * g - 1\|_\infty \leq 2\varepsilon \frac{\phi(n) \sqrt{n \ln(n)}}{N}$ .
- (2)  $|g(t)| \leq \frac{2nM}{N}$  for all  $t \in \mathbb{T}$ .

*Proof.* Observe that we have

$$g = \sigma * 2n \mathbb{I}_{[-\frac{1}{2n}, \frac{1}{2n}]}$$

It follows that

$$g * g = \sigma * \sigma * 2n \mathbb{I}_{[-\frac{1}{4n}, \frac{1}{4n}]} * 2n \mathbb{I}_{[-\frac{1}{4n}, \frac{1}{4n}]} = \sigma * \sigma * 2n \Delta_n,$$

where  $\Delta_n(x) = \max\{0, 1 - 2n|x|\}$ . By the way, we get

$$g * g\left(\frac{r}{n}\right) = 2n (\sigma * \sigma) \left( \left\{ \frac{r}{n} \right\} \right).$$

$\square$

From now the rest of the proof follows the path of Körner's proof and this finish the proof of Theorem 2.2 and the proof of the main result of this note is done.

**Remark.** *Lemma 25 from [2] tell us that we can approximate uniformly a continuous function by a sequence of infinitely differentiable functions but not in any space  $\Lambda_\psi$  where  $\psi$  is a positive strictly increasing continuous function which satisfy*

$$\psi(0) = 0 \quad \text{and} \quad \frac{\psi(t)}{t^\beta} \xrightarrow[t \rightarrow 0^+]{} 0, \quad \beta < \alpha - \frac{1}{2},$$

for some given (in advance)  $\alpha \in ]\frac{1}{2}, 1[$ .



## REFERENCES

- [1] E. H. E. Abdalaoui, M. Lemańczyk, *Approximate transitivity property and Lebesgue spectrum*, Monatsh. Math. 161 (2010), no. 2, 121-144. (p. 2)
- [2] T. Kröner, *On a theorem of Saeki concerning convolution squares of singular measures*, Bull. Soc. Math. France 136 (2008), no. 3, 439-464. (pp. 2, 3, 4, and 7)
- [3] C. Kuratowski, *Casimir Topologie. I et II. (French) [Topology. I and II] Part I with an appendix by A. Mostowski and R. Sikorski. Reprint of the fourth (Part I) and third (Part II) editions. Éditions Jacques Gabay, Sceaux, 1992.* (p. 2)

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