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# REDUCIBILITY OF $n$ -ARY SEMIGROUPS: FROM QUASITRIVIALITY TOWARDS IDEMPOTENCY

MIGUEL COUCEIRO, JIMMY DEVILLET, JEAN-LUC MARICHAL, AND PIERRE MATHONET

ABSTRACT. Let  $X$  be a nonempty set. Denote by  $\mathcal{F}_k^n$  the class of associative operations  $F: X^n \rightarrow X$  satisfying the condition  $F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$  whenever at least  $k$  of the elements  $x_1, \dots, x_n$  are equal to each other. The elements of  $\mathcal{F}_1^n$  are said to be quasitrivial and those of  $\mathcal{F}_n^n$  are said to be idempotent. We show that  $\mathcal{F}_1^n = \dots = \mathcal{F}_{n-2}^n \subseteq \mathcal{F}_{n-1}^n \subseteq \mathcal{F}_n^n$  and we give conditions on the set  $X$  for the last inclusions to be strict. The class  $\mathcal{F}_1^n$  was recently characterized by Couceiro and Devillet [2], who showed that its elements are reducible to binary associative operations. However, some elements of  $\mathcal{F}_n^n$  are not reducible. In this paper, we characterize the class  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$  and show that its elements are reducible. We give a full description of the corresponding reductions and show how each of them is built from a quasitrivial semigroup and an Abelian group whose exponent divides  $n - 1$ .

## 1. INTRODUCTION

Let  $X$  be a nonempty set, let  $|X|$  be its cardinality, and let  $n \geq 2$  be an integer. An  $n$ -ary operation  $F: X^n \rightarrow X$  is said to be *associative* if

$$\begin{aligned} F(x_1, \dots, x_{i-1}, F(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ = F(x_1, \dots, x_i, F(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}), \end{aligned}$$

for all  $x_1, \dots, x_{2n-1} \in X$  and all  $1 \leq i \leq n - 1$ . The pair  $(X, F)$  is then called an  *$n$ -ary semigroup*. This notion is due to Dörnte [8] and has led to the concept of  $n$ -ary group, which was first studied by Post [13].

In [7] the authors investigated associative  $n$ -ary operations that are determined by binary associative operations. An  $n$ -ary operation  $F: X^n \rightarrow X$  is said to be *reducible* to an associative binary operation  $G: X^2 \rightarrow X$  if there are  $G^m: X^{m+1} \rightarrow X$  ( $m = 1, \dots, n - 1$ ) such that  $G^{n-1} = F$ ,  $G^1 = G$ , and

$$G^m(x_1, \dots, x_{m+1}) = G^{m-1}(x_1, \dots, x_{m-1}, G(x_m, x_{m+1})), \quad m \geq 2.$$

The pair  $(X, F)$  is then said to be the  *$n$ -ary extension* of  $(X, G)$ . In that case, we also say that  $F$  is the  *$n$ -ary extension* of  $G$ .

Also, an  $n$ -ary operation  $F: X^n \rightarrow X$  is said to be

- *idempotent* if  $F(x, \dots, x) = x$  for all  $x \in X$ ,
- *quasitrivial* [1, 11] (or *conservative* [14]) if  $F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$  for all  $x_1, \dots, x_n \in X$ .

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Clearly, any quasitrivial  $n$ -ary operation is idempotent. As we will illustrate below, the converse is not true, even for associative operations.

The quest for conditions under which an associative  $n$ -ary operation is reducible to an associative binary operation gained an increasing interest since the pioneering work of Post [13] (see, e.g., [1, 2, 4, 7, 9, 10]). A necessary and sufficient condition for reducibility was given by Dudek and Mukhin [7] using the concept of *neutral element*. Recall that an element  $e \in X$  is said to be *neutral* for  $F: X^n \rightarrow X$  if

$$(1) \quad F((k-1) \cdot e, x, (n-k) \cdot e) = x, \quad x \in X, k \in \{1, \dots, n\}.$$

Here and throughout, for any  $k \in \{0, \dots, n\}$  and any  $x \in X$ , the notation  $k \cdot x$  stands for the  $k$ -tuple  $x, \dots, x$ . For instance, we have

$$F(3 \cdot x, 0 \cdot y, 2 \cdot z) = F(x, x, x, z, z).$$

Throughout this paper we also denote the set of neutral elements for an operation  $F: X^n \rightarrow X$  by  $E_F$ . Recall that for any binary operation  $G: X^2 \rightarrow X$  we have  $|E_G| \leq 1$ .

Dudek and Mukhin [7, Lemma 1] proved that if an associative operation  $F: X^n \rightarrow X$  has a neutral element  $e$ , then it is reducible to the associative operation  $G_e: X^2 \rightarrow X$  defined by

$$(2) \quad G_e(x, y) = F(x, (n-2) \cdot e, y), \quad x, y \in X.$$

Furthermore, it was recently observed [2, Corollary 2.3] that all the quasitrivial associative  $n$ -ary operations are reducible to associative binary operations. However, there are associative operations that are neither quasitrivial nor reducible to any binary operation; for instance, the associative and idempotent ternary operation  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $F(x, y, z) = x - y + z$  (see, e.g., [16] or more recently [12]).

The observations above show that it is natural to seek conditions under which an idempotent  $n$ -ary semigroup is reducible to a semigroup. To this extent, we will investigate certain subclasses of idempotent  $n$ -ary semigroups that contain the quasitrivial ones. In this direction, we will consider classes where the condition

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$$

holds on at least some subsets of  $X^n$ . More precisely, for a set  $S \subseteq \{1, \dots, n\}$ , let

$$D_S^n = \{(x_1, \dots, x_n) \in X^n : \forall i, j \in S, x_i = x_j\},$$

and, for every  $k \in \{1, \dots, n\}$ , let

$$D_k^n = \bigcup_{\substack{S \subseteq \{1, \dots, n\} \\ |S| \geq k}} D_S^n = \bigcup_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=k}} D_S^n.$$

Thus, the set  $D_k^n$  consists of those tuples of  $X^n$  for which at least  $k$  components are equal to each other. In particular,  $D_1^n = X^n$  and  $D_n^n = \{(x, \dots, x) : x \in X\}$ .

For every  $k \in \{1, \dots, n\}$ , denote by  $\mathcal{F}_k^n$  the class of those associative  $n$ -ary operations  $F: X^n \rightarrow X$  that satisfy

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}, \quad \text{whenever } (x_1, \dots, x_n) \in D_k^n.$$

We say that these operations are *quasitrivial on  $D_k^n$* .

Thus defined,  $\mathcal{F}_1^n$  is exactly the class of quasitrivial associative  $n$ -ary operations and  $\mathcal{F}_n^n$  is exactly the class of idempotent associative  $n$ -ary operations. It follows directly from the definition of the classes  $\mathcal{F}_k^n$  that  $\mathcal{F}_1^n = \mathcal{F}_2^n = \dots = \mathcal{F}_n^n$  if  $|X| \leq 2$ . Therefore, throughout

the rest of this paper we assume that  $|X| \geq 3$ . Since the sets  $D_k^n$  are nested in the sense that  $D_{k+1}^n \subseteq D_k^n$  for  $1 \leq k \leq n-1$ , the classes  $\mathcal{F}_k^n$  clearly form a filtration, that is,

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \dots \subseteq \mathcal{F}_n^n.$$

Quite surprisingly, we have the following result, which shows that this filtration actually reduces to three nested classes only.

**Proposition 1.1.** *For every  $n \geq 3$ , we have  $\mathcal{F}_1^n = \mathcal{F}_{n-2}^n$ .*

The proof is deferred until Section 2, and so are the proofs of the other results in this introduction.

We observe that the class  $\mathcal{F}_1^n = \mathcal{F}_2^n = \dots = \mathcal{F}_{n-2}^n$  was characterized by Couceiro and Devillet [2] who showed that all its elements are reducible. More precisely, the following result summarizes [2, Corollary 3.8] and [2, Corollary 3.11].

**Proposition 1.2.** *If  $F: X^n \rightarrow X$  is an associative quasitrivial operation, then  $|E_F| \leq 2$  and  $F$  has either one or two binary reductions. Furthermore, the binary reductions depend on  $E_F$  as follows.*

- (a) *If  $E_F = \emptyset$ , then the operation  $G: X^2 \rightarrow X$  defined for every  $x, y \in X$  by*

$$G(x, y) = F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$$

*is the only binary reduction of  $F$ , and  $G$  is quasitrivial.*

- (b) *If  $E_F = \{e\}$ , then the operation  $G_e: X^2 \rightarrow X$  defined for every  $x, y \in X$  by*

$$G_e(x, y) = F(x, (n-2) \cdot e, y) = F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$$

*is the only binary reduction of  $F$ , and  $G_e$  is quasitrivial.*

- (c) *If  $E_F = \{e_1, e_2\}$  (with  $e_1 \neq e_2$ ), then the operations  $G_{e_1}, G_{e_2}: X^2 \rightarrow X$  defined for every  $x, y \in X$  by*

$$G_{e_1}(x, y) = F(x, (n-2) \cdot e_1, y) \quad \text{and} \quad G_{e_2}(x, y) = F(x, (n-2) \cdot e_2, y)$$

*are the only binary reductions of  $F$  (and  $G_{e_1} \neq G_{e_2}$ ). Neither of  $G_{e_1}$  and  $G_{e_2}$  is quasitrivial and, in this case, the identity  $F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$  does not hold.*

Proposition 1.2 is of particular interest since the class of associative and quasitrivial binary operations was characterized by Langer in [11, Theorem 1].

In this paper, we provide a characterization of the class  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ . We show that all of its elements are also reducible to binary associative operations. We give a full description of the possible reductions of the operations in this class.

Let us begin with the particular case when all the elements in  $X$  are neutral. Recall that a group  $(X, G)$  with neutral element  $e$  has *bounded exponent* if there exists an integer  $m \geq 1$  such that  $G^{m-1}(m \cdot x) = e$  for any  $x \in X$  (with the usual convention that  $G^0(x) = x$  for every  $x \in X$ ). In that case, the *exponent* of the group is the smallest integer having this property. The following result provides a description of the class of  $n$ -ary semigroups containing only neutral elements. It was stated without proof in [6, p. 2] in the framework of  $n$ -ary groups, but it can be easily extended to  $n$ -ary semigroups by using [6, Corollary 4]. For the sake of completeness, we provide a direct proof that basically uses [7, Lemma 1].

**Theorem 1.3.** *Let  $F: X^n \rightarrow X$  ( $n \geq 3$ ) be an associative operation. Then  $E_F = X$  if and only if  $(X, F)$  is the  $n$ -ary extension of an Abelian group whose exponent divides  $n-1$ .*

Abelian groups having bounded exponent play a central role in this first result, but also in the next theorems. We recall that Prüfer and Baer (see, e.g., [15, Corollary 10.37]) showed that if an Abelian group has bounded exponent, then it is isomorphic to a direct sum of cyclic groups. Hence, the exponent of an Abelian group divides  $n - 1$  if and only if the Abelian group is isomorphic to a direct sum of cyclic groups whose orders divide  $n - 1$ .

Theorem 1.3 also highlights the fact that an  $n$ -ary associative operation may have several reductions, associated with distinct neutral elements. For instance, the ternary sum on  $\mathbb{Z}_2$  has two neutral elements, namely 0 and 1. It is reducible to the operations  $G_0, G_1: \mathbb{Z}_2^2 \rightarrow \mathbb{Z}_2$  defined by  $G_0(x, y) = x + y \pmod{2}$  and  $G_1(x, y) = x + y + 1 \pmod{2}$ , respectively. We can also easily see that the semigroups  $(\mathbb{Z}_2, G_0)$  and  $(\mathbb{Z}_2, G_1)$  are isomorphic. In fact, this result can be generalized to other underlying sets: all reductions obtained in this way are isomorphic, as stated in the following result.

**Proposition 1.4.** *Let  $F: X^n \rightarrow X$  ( $n \geq 3$ ) be an associative operation such that  $E_F \neq \emptyset$ . Then every reduction of  $F$  is of the form  $G_e$  for some  $e \in E_F$ . Moreover, if  $e_1, e_2 \in E_F$ , then  $(X, G_{e_1})$  and  $(X, G_{e_2})$  are isomorphic.*

In order to state one of the main results of this paper, we shall make use of the following classes of operations. Recall that an element  $a \in X$  is said to be an *annihilator* for  $F: X^n \rightarrow X$  if  $F(x_1, \dots, x_n) = a$  whenever  $a \in \{x_1, \dots, x_n\}$ .

**Definition 1.5.** For every integer  $m \geq 1$ , let  $\mathcal{H}_m$  be the class of binary operations  $G: X^2 \rightarrow X$  such that there exists a subset  $Y \subseteq X$  with  $|Y| \geq 3$  for which the following assertions hold.

- (a)  $(Y, G|_{Y^2})$  is an Abelian group whose exponent divides  $m$ .
- (b)  $G|_{(X \setminus Y)^2}$  is associative and quasitrivial.
- (c) Any  $x \in X \setminus Y$  is an annihilator for  $G|_{(\{x\} \cup Y)^2}$ .

Note that  $\mathcal{H}_1 = \emptyset$ . As we will see, all operations in  $\mathcal{H}_m$  are associative, and the set  $Y$  is unique. In fact, the family of classes  $\mathcal{H}_m$  is the key for the characterization of the classes  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ .

**Theorem 1.6.** *Every  $G \in \mathcal{H}_m$  is associative ( $m \geq 1$ ). If  $G \in \mathcal{H}_{n-1}$ , then its  $n$ -ary extension  $F = G^{n-1}$  is in  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ . Conversely, for every  $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$  we have that  $|E_F| \geq 3$ , and the reductions of  $F$  are exactly the operations  $G_e$  for  $e \in E_F$  and they lie in  $\mathcal{H}_{n-1}$ .*

As an immediate corollary we solve the reducibility problem for operations in  $\mathcal{F}_{n-1}^n$ .

**Corollary 1.7.** *Every operation in  $\mathcal{F}_{n-1}^n$  is reducible to a binary associative operation.*

Theorem 1.6 is of particular interest as it enables us to easily construct  $n$ -ary operations in  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ . For instance, for any integers  $n \geq 3$  and  $p \geq 1$ , the operation of the cyclic group  $(\mathbb{Z}_n, +)$  is in  $\mathcal{H}_{np}$ , and thus the operation associated with its  $(np + 1)$ -ary extension is in  $\mathcal{F}_{np}^{np+1} \setminus \mathcal{F}_1^{np+1}$ .

To give another example, consider the chain  $(X, \leq) = (\{1, 2, 3, 4, 5\}, \leq)$  together with the operation  $G: X^2 \rightarrow X$  defined by the following conditions:

- $(\{1, 2, 3\}, G|_{\{1,2,3\}^2})$  is isomorphic to  $(\mathbb{Z}_3, +)$ ,
- $G|_{\{4,5\}^2} = \vee|_{\{4,5\}^2}$ , where  $\vee: X^2 \rightarrow X$  is the maximum operation for  $\leq$ ,
- for any  $x \in \{1, 2, 3\}$ ,  $G(x, 4) = G(4, x) = 4$  and  $G(x, 5) = G(5, x) = 5$ .

Then we have  $G \in \mathcal{H}_{3p}$  for any integer  $p \geq 1$  and so  $G^{3p}$  is in  $\mathcal{F}_{3p}^{3p+1} \setminus \mathcal{F}_1^{3p+1}$ .

Now we give a reformulation of Theorem 1.6 that is not based on binary reductions.

**Theorem 1.8.** *If  $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ , then, setting  $Y = E_F$ , we have that  $|Y| \geq 3$  and the following assertions hold.*

- (a)  $(Y, F|_{Y^n})$  is the  $n$ -ary extension of an Abelian group whose exponent divides  $n-1$ .
- (b)  $F|_{(X \setminus Y)^n}$  is associative, quasitrivial, and has at most one neutral element.
- (c) For all  $x_1, \dots, x_n \in X$  and  $i \in \{1, \dots, n-1\}$  such that  $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$  we have

$$F(x_1, \dots, x_n) = F(x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n).$$

Conversely, if an operation  $F$  satisfies these conditions for some  $Y \subseteq X$  with  $|Y| \geq 3$ , then  $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$  and  $E_F = Y$ .

Proposition 1.1 shows that all operations in  $\mathcal{F}_{n-2}^n$  are quasitrivial. The examples we just presented show that there are operations in  $\mathcal{F}_{n-1}^n$  that are not quasitrivial, for some  $n \geq 3$  and some sets  $X$ . Theorem 1.6 enables us to provide necessary and sufficient conditions on the set  $X$  for such operations to exist.

**Definition 1.9.** For any integer  $m \geq 2$ , let  $c_m$  denote the cardinality of the smallest Abelian group with at least three elements whose exponent divides  $m$ .

**Proposition 1.10.** *For every  $n \geq 3$ , we have  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \emptyset$  if and only if  $|X| \geq c_{n-1}$ .*

**Corollary 1.11.** *For any integer  $n \geq 3$ , let  $p$  be the least odd prime divisor of  $n-1$  if  $n-1$  is not a power of 2; otherwise, set  $p = 4$ . The following assertions hold.*

- (a) *If  $n$  is even, then  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \emptyset$  if and only if  $|X| \geq p$ .*
- (b) *If  $n$  is odd, then  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \emptyset$  if and only if  $|X| \geq \min(4, p)$ .*

Finally, we observe that if  $(X, \leq)$  is a semilattice that is not a chain, then the  $n$ -ary operation  $F: X^n \rightarrow X$  defined by  $F(x_1, \dots, x_n) = x_1 \vee \dots \vee x_n$  is in  $\mathcal{F}_n^n$ . However, it is not in  $\mathcal{F}_{n-1}^n$  since  $F((n-1) \cdot x, y) \notin \{x, y\}$  whenever  $x$  and  $y$  are not comparable, i.e.,  $x \vee y \notin \{x, y\}$ . Since such a semilattice structure exists on every set  $X$  such that  $|X| \geq 3$ , we obtain the following result.

**Proposition 1.12.** *For every  $n \geq 2$ , we have  $\mathcal{F}_n^n \setminus \mathcal{F}_{n-1}^n \neq \emptyset$  if and only if  $|X| \geq 3$ .*

In Section 2 we give the proofs of the results above, using some more technical statements that may have interest on their own. In Section 3 we introduce and investigate an alternative hierarchy of subclasses of idempotent operations. We end the paper by some concluding remarks in Section 4.

## 2. TECHNICALITIES AND PROOFS OF THE MAIN RESULTS

Let us begin with Proposition 1.1, which essentially follows from the very definition of the classes  $\mathcal{F}_k^n$ .

*Proof of Proposition 1.1.* We only need to prove that  $\mathcal{F}_{n-2}^n \subseteq \mathcal{F}_1^n$ , and so we can assume that  $n \geq 4$ . Let  $F \in \mathcal{F}_{n-2}^n$  and let us show by induction that for every  $k \in \{1, \dots, n\}$  we have

$$(3) \quad F(k \cdot x_1, x_{k+1}, \dots, x_n) \in \{x_1, x_{k+1}, \dots, x_n\}, \quad x_1, x_{k+1}, \dots, x_n \in X.$$

By the definition of  $\mathcal{F}_{n-2}^n$ , condition (3) holds for any  $k \in \{n-2, n-1, n\}$ . Let us now assume that it holds for some  $k \in \{2, \dots, n\}$  and let us show that it still holds for  $k-1$ . Using associativity and idempotency, we have

$$\begin{aligned} F((k-1) \cdot x_1, x_k, \dots, x_n) &= F(F(n \cdot x_1), (k-2) \cdot x_1, x_k, \dots, x_n) \\ &= F(k \cdot x_1, F((n-2) \cdot x_1, x_k, x_{k+1}), \dots, x_n). \end{aligned}$$

By the induction hypothesis, the latter expression lies in  $\{x_1, x_k, \dots, x_n\}$ .

Thus, Equation (3) holds for every  $k \in \{1, \dots, n\}$ . Using it for  $k = 1$ , we obtain that  $F$  is quasitrivial.  $\square$

In Theorem 1.3 and Proposition 1.4 we deal with neutral elements. We first state and prove some intermediate results concerning such elements. The following two lemmas were stated and proved in [5, Theorem 3] for  $n$ -ary groups. We provide a proof of the first one that does not use the  $n$ -ary group structure but basically uses [7, Lemma 1], and we give a slightly different proof for the second one in the framework of  $n$ -ary semigroups.

**Lemma 2.1.** *Let  $F: X^n \rightarrow X$  be an associative operation and let  $e \in E_F$ . Then, for any  $x_1, \dots, x_{n-1} \in X$  we have*

$$F(x_1, \dots, x_{n-1}, e) = F(x_1, \dots, e, x_{n-1}) = \dots = F(e, x_1, \dots, x_{n-1}).$$

Moreover, for any  $x \in X$  the restriction  $F|_{(\{x\} \cup E_F)^n}$  is symmetric.

*Proof.* Let  $x_1, \dots, x_{n-1} \in X$  and let  $G_e$  be the reduction of  $F$  defined by (2). For  $i \in \{1, \dots, n-1\}$  we have  $G_e(x_i, e) = x_i = G_e(e, x_i)$ , which proves the first part of the statement for  $n = 2$ . For  $n \geq 3$  we have

$$F(x_1, \dots, x_i, e, x_{i+1}, \dots, x_{n-1}) = G_e^{n-2}(x_1, \dots, x_{i-1}, G_e(x_i, e), x_{i+1}, \dots, x_{n-1}),$$

and the first part of the statement follows from the fact that each  $x_i$  commutes with  $e$  in  $G_e$ . The second part is a direct consequence of the first part.  $\square$

**Lemma 2.2.** *Let  $F: X^n \rightarrow X$  be an associative operation such that  $E_F \neq \emptyset$ . Then  $F$  preserves  $E_F$ , i.e.,  $F(E_F^n) \subseteq E_F$ .*

*Proof.* Let  $e_1, \dots, e_n \in E_F$  and let us show that  $F(e_1, \dots, e_n) \in E_F$ . By Lemma 2.1 and associativity of  $F$ , for any  $x \in X$  we have

$$\begin{aligned} & F((n-1) \cdot F(e_1, \dots, e_n), x) \\ &= F(F(e_1, (n-1) \cdot e_2), F(e_1, (n-1) \cdot e_3), \dots, F(e_1, (n-1) \cdot e_n), x) \\ &= F((n-1) \cdot e_1, x) = x. \end{aligned}$$

Similarly, for any  $x \in X$  we can show that

$$F(i \cdot F(e_1, \dots, e_n), x, (n-i-1) \cdot F(e_1, \dots, e_n)) = x, \quad i \in \{0, \dots, n-2\}.$$

Thus  $F(e_1, \dots, e_n)$  satisfies (1), i.e.,  $F(e_1, \dots, e_n) \in E_F$ .  $\square$

Combining Lemmas 2.1 and 2.2, we immediately derive the following result.

**Corollary 2.3.** *If  $(X, F)$  is an  $n$ -ary monoid, then  $(E_F, F|_{E_F^n})$  is a symmetric  $n$ -ary monoid.*

*Proof of Theorem 1.3.* (Sufficiency) Obvious.

(Necessity) Suppose that  $X = E_F$ . Let  $e \in E_F$  and let  $G_e: X^2 \rightarrow X$  be the corresponding reduction of  $F$  defined by (2). Recall that  $e$  is the (unique) neutral element of  $G_e$  by (2). By Corollary 2.3, we have that  $F$  is symmetric. Thus, we have that  $G_e$  also is symmetric. Moreover, since  $G_e$  is a binary reduction of  $F$  and  $E_F = X$ , it follows that

$$G_e(G_e^{n-2}((n-1) \cdot x), y) = y = G_e(y, G_e^{n-2}((n-1) \cdot x)), \quad x, y \in X,$$

which shows that  $G_e^{n-2}((n-1) \cdot x) \in E_{G_e}$  for any  $x \in X$ . However, since  $E_{G_e} = \{e\}$ , we have that  $G_e^{n-2}((n-1) \cdot x) = e$  for any  $x \in X$ . Thus,  $(X, G_e)$  is an Abelian group whose exponent divides  $n-1$ .  $\square$

The following result follows immediately from Theorem 1.3.

**Corollary 2.4.** *If  $(X, F)$  is an  $n$ -ary monoid, then  $(E_F, F|_{E_F^n})$  is the  $n$ -ary extension of an Abelian group whose exponent divides  $n - 1$ .*

*Proof of Proposition 1.4.* The first part of the statement follows from [2, Proposition 3.3]. Moreover, the associativity of  $F$  and the definition of neutral elements ensure that the map  $\psi: X \rightarrow X$  defined by

$$\psi(x) = F(e_2, x, (n-2) \cdot e_1)$$

is a bijection and that  $\psi^{-1}(x) = F((n-2) \cdot e_2, x, e_1)$ . We then have

$$\begin{aligned} G_{e_2}(\psi(x), \psi(y)) &= F(F(e_2, x, (n-2) \cdot e_1), (n-2) \cdot e_2, F(e_2, y, (n-2) \cdot e_1)) \\ &= F(F(e_2, x, (n-2) \cdot e_1), F((n-1) \cdot e_2, y), (n-2) \cdot e_1) \\ &= F(F(e_2, x, (n-2) \cdot e_1), y, (n-2) \cdot e_1) \\ &= F(e_2, F(x, (n-2) \cdot e_1, y), (n-2) \cdot e_1) \\ &= \psi(G_{e_1}(x, y)), \end{aligned}$$

which completes the proof.  $\square$

Let us now prove Theorem 1.6. To this extent, we first state and prove some intermediate results. We have the following remarkable lemma, which characterizes the existence of a pair of neutral elements for  $F \in \mathcal{F}_{n-1}^n$  by means of two identities.

**Lemma 2.5.** *Let  $F \in \mathcal{F}_{n-1}^n$  and let  $a, b \in X$  such that  $a \neq b$ . Then  $a, b \in E_F$  if and only if  $F((n-1) \cdot a, b) = b$  and  $F(a, (n-1) \cdot b) = a$ .*

*Proof.* (Necessity) Obvious.

(Sufficiency) For any  $x \in X$ , we have

$$\begin{aligned} F((n-1) \cdot a, x) &= F((n-2) \cdot a, F(a, (n-1) \cdot b), x) \\ &= F(F((n-1) \cdot a, b), (n-2) \cdot b, x) = F((n-1) \cdot b, x), \end{aligned}$$

which implies that  $F((n-1) \cdot a, x) = F((n-1) \cdot b, x) = x$  for any  $x \in X$ . Indeed, for  $x \in \{a, b\}$  this relation follows from idempotency, and for  $x \notin \{a, b\}$  we have

$$F((n-1) \cdot a, x) = F((n-1) \cdot b, x) \in \{a, x\} \cap \{b, x\} = \{x\},$$

due to the definition of  $\mathcal{F}_{n-1}^n$ . Similarly, we get  $F(x, (n-1) \cdot a) = x = F(x, (n-1) \cdot b)$  for any  $x \in X$ . It follows from these relations, together with associativity of  $F$ , that for any  $k \in \{1, \dots, n-2\}$ , the maps  $\psi_k, \xi_k: X \rightarrow X$  defined by

$$\begin{aligned} \psi_k(x) &= F(k \cdot a, x, (n-k-1) \cdot a) \\ \xi_k(x) &= F(k \cdot b, x, (n-k-1) \cdot b) \end{aligned}$$

are bijections with inverse maps  $\psi_{n-k-1}$  and  $\xi_{n-k-1}$ , respectively. It then follows that, for any  $k \in \{1, \dots, n-2\}$ , we have  $F(k \cdot a, x, (n-k-1) \cdot a) = \psi_k(x) = x$  for every  $x \in X$ . Indeed, for  $x = a$ , this relation follows from idempotency, and for  $x \neq a$ , we have  $\psi_k(x) \in \{a, x\}$  and  $\psi_k(x) \neq a$ . Similarly, we can show that  $F(k \cdot b, x, (n-k-1) \cdot b) = \xi_k(x) = x$  for every  $x \in X$ , which shows that  $a, b \in E_F$ .  $\square$

Given an associative operation  $F: X^n \rightarrow X$ , we can define the sequence  $(F^q)_{q \geq 1}$  of  $(qn - q + 1)$ -ary associative operations inductively by the rules  $F^1 = F$  and

$$F^q(x_1, \dots, x_{qn-q+1}) = F^{q-1}(x_1, \dots, x_{(q-1)n-q+1}, F(x_{(q-1)n-q+2}, \dots, x_{qn-q+1})),$$



for any integer  $q \geq 2$  and any  $x_1, \dots, x_{qn-q+1} \in X$ .

The following proposition shows that every  $n$ -tuple that violates the quasitriviality condition for  $F \in \mathcal{F}_{n-1}^n$  belongs to  $E_F^n$ .

**Proposition 2.6.** *Let  $F \in \mathcal{F}_{n-1}^n$ . For any  $a_1, \dots, a_n \in X$  such that  $F(a_1, \dots, a_n) \notin \{a_1, \dots, a_n\}$ , we have that  $a_1, \dots, a_n, F(a_1, \dots, a_n) \in E_F$ . Moreover,  $F|_{(X \setminus E_F)^n}$  is quasitrivial.*

*Proof.* The case  $n = 2$  is trivial. So assume that  $n \geq 3$ . Let us prove by induction on  $k \in \{1, \dots, n-1\}$  that for every  $a_1, a_2, \dots, a_{k+1} \in X$  the condition

$$F((n-k) \cdot a_1, a_2, \dots, a_{k+1}) \notin \{a_1, \dots, a_{k+1}\}$$

implies  $a_1, \dots, a_{k+1} \in E_F$ . For  $k = 1$ , there is nothing to prove. We thus assume that the result holds true for a given  $k \in \{1, \dots, n-2\}$  and we show that it still holds for  $k+1$ . Now, consider elements  $a_1, \dots, a_{k+2}$  such that

$$(4) \quad F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) \notin \{a_1, \dots, a_{k+2}\}.$$

We first prove that  $a_1, a_2 \in E_F$ .

If  $a_1 = a_2$ , then  $a_1, \dots, a_{k+2} \in E_F$  by the induction hypothesis.

If  $a_1 \neq a_2$ , then we prove that  $F((n-1) \cdot a_1, a_2) = a_2$  and  $F(a_1, (n-1) \cdot a_2) = a_1$ , which show that  $a_1, a_2 \in E_F$  by Lemma 2.5.

- For the sake of a contradiction, assume first that  $F((n-1) \cdot a_1, a_2) = a_1$ . Then, for  $\ell \geq 1$  we have

$$(5) \quad \begin{aligned} & F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) \\ &= F^{\ell+1}(((n-k-1) + \ell(n-2)) \cdot a_1, (\ell+1) \cdot a_2, \dots, a_{k+2}). \end{aligned}$$

Choosing  $\ell = n-k-1$  and using idempotency of  $F$ , we obtain

$$F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) = F^2((n-1) \cdot a_1, (n-k) \cdot a_2, a_3, \dots, a_{k+2}).$$

Since the left-hand side of this equation does not lie in  $\{a_1, \dots, a_{k+2}\}$  by (4), we obtain

$$F((n-k) \cdot a_2, a_3, \dots, a_{k+2}) \notin \{a_1, \dots, a_{k+2}\}.$$

By the induction hypothesis, we have  $a_2, \dots, a_{k+2} \in E_F$ . Then choosing  $\ell = n-2$  in (5) and using idempotency and the fact that  $a_2 \in E_F$ , we obtain

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) \\ &= F^{n-1}(((n-k-1) + (n-2)^2) \cdot a_1, (n-1) \cdot a_2, \dots, a_{k+2}) \\ &= F^2((n-k) \cdot a_1, (n-1) \cdot a_2, a_3, \dots, a_{k+2}) \\ &= F((n-k) \cdot a_1, a_3, \dots, a_{k+2}). \end{aligned}$$

By the induction hypothesis, we have  $a_1 \in E_F$ . We then have  $F((n-1) \cdot a_1, a_2) = a_2 \neq a_1$ , a contradiction.

- Assume now that  $F(a_1, (n-1) \cdot a_2) = a_2$ . Then, for  $\ell \geq 1$  we have

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) \\ &= F^{\ell+1}((n-k-1+\ell) \cdot a_1, (\ell(n-2)+1) \cdot a_2, \dots, a_{k+2}). \end{aligned}$$

For  $\ell = k$ , using idempotency and the fact that  $k(n-2)+1 = n-k+(k-1)(n-1)$ , we obtain

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \dots, a_{k+2}) \\ &= F^2((n-1) \cdot a_1, (n-k) \cdot a_2, a_3, \dots, a_{k+2}). \end{aligned}$$

Thus,  $F((n-k) \cdot a_2, a_3, \dots, a_{k+2}) \notin \{a_1, \dots, a_{k+2}\}$ . By the induction hypothesis, we have  $a_2, \dots, a_{k+2} \in E_F$ . It follows that  $F(a_1, (n-1) \cdot a_2) = a_1 \neq a_2$ , a contradiction.

Now, since  $a_2 \in E_F$ , it commutes with all other arguments of  $F$  by Lemma 2.1. Also, by (4) we have

$$F((n-k-1) \cdot a_1, a_3, \dots, a_{k+2}, a_2) \notin \{a_1, \dots, a_{k+2}\},$$

and thus  $a_3 \in E_F$ . Repeating this argument, we have that  $a_1, \dots, a_{k+2} \in E_F$ .

It follows from the induction that if  $F(a_1, \dots, a_n) \notin \{a_1, \dots, a_n\}$ , then  $a_1, \dots, a_n \in E_F$ . Finally we have  $F(a_1, \dots, a_n) \in E_F$  by Lemma 2.2. The second part is straightforward.  $\square$

Proposition 1.2 shows that a quasitrivial  $n$ -ary semigroup cannot have more than two neutral elements. The next result shows that an operation in  $\mathcal{F}_{n-1}^n$  is quasitrivial whenever it has at most two neutral elements.

**Corollary 2.7.** *An operation  $F \in \mathcal{F}_{n-1}^n$  is quasitrivial if and only if  $|E_F| \leq 2$ .*

*Proof.* (Necessity) This follows from Proposition 1.2.

(Sufficiency) Suppose that  $F$  is not quasitrivial, i.e., there exist  $a_1, \dots, a_n \in X$  such that  $F(a_1, \dots, a_n) \notin \{a_1, \dots, a_n\}$ . Since  $F$  is idempotent, we must have  $|\{a_1, \dots, a_n\}| \geq 2$  and so  $|\{a_1, \dots, a_n, F(a_1, \dots, a_n)\}| \geq 3$ . We also have  $\{a_1, \dots, a_n, F(a_1, \dots, a_n)\} \subseteq E_F$  by Proposition 2.6. Therefore we have  $|E_F| \geq 3$ .  $\square$

**Proposition 2.8.** *Let  $F \in \mathcal{F}_{n-1}^n$  and suppose that  $|E_F| \geq 3$ . Then, any element  $x \in X \setminus E_F$  is an annihilator of  $F|_{(\{x\} \cup E_F)^n}$ . Moreover,  $F|_{(X \setminus E_F)^n}$  is quasitrivial and has at most one neutral element.*

*Proof.* Let  $x \in X \setminus E_F$  and  $e \in E_F$  and let us show that  $F(k \cdot x, (n-k) \cdot e) = x$  for any  $k \in \{1, \dots, n-1\}$ . If  $k = 1$ , then this equality follows from the definition of a neutral element. Now, suppose that there exists  $k \in \{2, \dots, n-1\}$  such that  $F(k \cdot x, (n-k) \cdot e) \neq x$ . Since  $x \in X \setminus E_F$ , by Proposition 2.6 we must have  $F(k \cdot x, (n-k) \cdot e) = e$ . But then, using the associativity of  $F$ , we get

$$\begin{aligned} F((n-1) \cdot x, e) &= F((n-1) \cdot x, F(k \cdot x, (n-k) \cdot e)) \\ &= F(k \cdot x, (n-k) \cdot e) = e, \end{aligned}$$

and we conclude by Lemma 2.5 that  $x \in E_F$ , which contradicts our assumption. Thus, we have

$$(6) \quad F(k \cdot x, (n-k) \cdot e) = x, \quad k \in \{1, \dots, n-1\}.$$

Now, let us show that  $F(k \cdot x, e_{k+1}, \dots, e_n) = x$  for any  $k \in \{1, \dots, n-1\}$  and any  $e_{k+1}, \dots, e_n \in E_F$ . To this extent, we only need to show that

$$F(k \cdot x, e_{k+1}, \dots, e_n) = F((k+1) \cdot x, e_{k+2}, \dots, e_n),$$

for any  $k \in \{1, \dots, n-1\}$  and any  $e_{k+1}, \dots, e_n \in E_F$ . So, let  $k \in \{1, \dots, n-1\}$  and  $e_{k+1}, \dots, e_n \in E_F$ . Using (6) and the associativity of  $F$  we get

$$\begin{aligned} F(k \cdot x, e_{k+1}, \dots, e_n) &= F((k-1) \cdot x, F(2 \cdot x, (n-2) \cdot e_{k+1}), e_{k+1}, \dots, e_n) \\ &= F(k \cdot x, F(x, (n-1) \cdot e_{k+1}), e_{k+2}, \dots, e_n) \\ &= F((k+1) \cdot x, e_{k+2}, \dots, e_n), \end{aligned}$$

which completes the proof by idempotency of  $F$  and Lemma 2.1. For the second part of the proposition, we observe that  $F|_{(X \setminus E_F)^n}$  is quasitrivial by Proposition 2.6. Also, using (6) and the associativity of  $F$ , for any  $x, y \in X \setminus E_F$  and any  $e \in E_F$  we obtain

$$\begin{aligned} F((n-1) \cdot x, y) &= F((n-1) \cdot x, F(e, (n-1) \cdot y)) \\ &= F(F((n-1) \cdot x, e), (n-1) \cdot y) = F(x, (n-1) \cdot y), \end{aligned}$$

which shows that  $F|_{(X \setminus E_F)^n}$  cannot have more than one neutral element.  $\square$

*Proof of Theorem 1.6.* In order to show that every  $G \in \mathcal{H}_m$  is associative, we have to compare the expressions  $G(G(x_1, x_2), x_3)$  and  $G(x_1, G(x_2, x_3))$  for all  $x_1, x_2, x_3$  in  $X$ . Clearly these expressions are equal if all their arguments are either in  $Y$  or in  $X \setminus Y$  since the restriction of  $G$  to these subsets is associative. If one argument, say  $x_i$ , is in  $X \setminus Y$  and the others are in  $Y$ , then both expressions are equal to  $x_i$  by Property (c) of Definition 1.5. For the same reason, if the arguments  $x_i, x_j$  are in  $X \setminus Y$  and the third one in  $Y$ , then both expressions are equal to  $G(x_i, x_j)$ .

Now, we consider  $G \in \mathcal{H}_{n-1}$  and define  $F = G^{n-1}$ . Then we have  $E_F = Y$ . Indeed, conditions (a) and (c) of Definition 1.5 imply directly that  $Y \subseteq E_F$ . Moreover if  $x \notin Y$ , then still by condition (c) we have  $F((n-1) \cdot x, y) = x \neq y$  for  $y \in Y$ , so  $x \notin E_F$ .

Next, we show that  $F(k \cdot x, y, (n-k-1) \cdot x) \in \{x, y\}$  for every  $x, y \in X$ . If  $x \in Y$ ,  $x$  is a neutral element, so this expression is equal to  $y$ . If  $x \in X \setminus Y$ , then either  $y \in Y$  and this expression is equal to  $x$  (by condition (c)), or  $y \in X \setminus Y$ , and this expression is in  $\{x, y\}$  (by condition (b)). Finally,  $F \notin \mathcal{F}_1^n$  by Corollary 2.7, since  $|E_F| = |Y| \geq 3$ .

Now we prove the converse statement and consider  $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ . Setting  $Y = E_F$  we have  $|Y| \geq 3$  by Corollary 2.7. By Proposition 1.4, every reduction of  $F$  is of the form  $G_e$  for some  $e \in E_F$ .

Finally, we show that  $G_e \in \mathcal{H}_{n-1}$ . We have that  $(Y, G_e|_{Y^2})$  is an Abelian group whose exponent divides  $n-1$  by Corollary 2.4 and Proposition 1.4. Also, we have that  $G_e|_{(X \setminus Y)^2}$  is quasitrivial by Propositions 1.2 and 2.8. Finally, we have that any  $x \in X \setminus Y$  is an annihilator for  $G_e|_{(\{x\} \cup Y)^2}$  by Proposition 2.8.  $\square$

*Proof of Corollary 1.7.* This follows from Proposition 1.2 and Theorem 1.6.  $\square$

*Remark 1.* In the proof of Corollary 1.7 we used [2, Corollary 3.11] which is based on results obtained by Ackerman [1]. In the appendix we provide an alternative proof of Corollary 1.7 that does not make use of [2, Corollary 3.11].

*Proof of Theorem 1.8.* If  $F \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ , then by Theorem 1.6 we have  $|E_F| \geq 3$  and for every  $e \in E_F$ ,  $G_e$  is in  $\mathcal{H}_{n-1}$ . Then  $G_e|_{Y^2}$  is a reduction of  $F|_{Y^n}$  and (a) holds true. Also  $G_e|_{(X \setminus Y)^2}$  is a quasitrivial reduction of  $F|_{(X \setminus Y)^n}$ , so (b) holds true by Proposition 1.2. Finally, if  $x_i, x_{i+1}$  satisfy the conditions of (c), we have  $G_e(x_i, x_{i+1}) = x = G_e(x, x)$ , so that (c) holds true.

Let us now assume that an operation  $F$  satisfies conditions (a), (b), and (c). By (a), there exists an Abelian group  $(Y, G_Y)$  whose exponent divides  $n-1$  such that  $(Y, F|_{Y^n})$  is the  $n$ -ary extension of  $(Y, G_Y)$ . We denote by  $e$  the neutral element of  $G_Y$ . We also define the operation  $G: X^2 \rightarrow X$  by  $G(x, y) = F(x, (n-2) \cdot e, y)$  for every  $x, y \in X$ . We now show that  $G$  is in  $\mathcal{H}_{n-1}$ . It is easy to see that  $G|_{Y^2} = G_Y$ . Then, by condition (c),  $G|_{(X \setminus Y)^2}(x, y) = F((n-1) \cdot x, y)$ , so  $G|_{(X \setminus Y)^2}$  is the unique quasitrivial reduction of  $F|_{(X \setminus Y)^n}$  (see Proposition 1.2). Finally, condition (c) also implies that any  $x \in X \setminus Y$  is an annihilator for  $G|_{(\{x\} \cup Y)^2}$ . Then by Theorem 1.6,  $G$  is associative and we have  $G^{n-1} \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ . We conclude the proof by showing that  $G^{n-1} = F$ . To this aim we compare

$G^{n-1}(x_1, \dots, x_n)$  and  $F(x_1, \dots, x_n)$  for every  $(x_1, \dots, x_n) \in X^n$ . We already showed that both expressions coincide if  $(x_1, \dots, x_n)$  belongs to  $Y^n$  or  $(X \setminus Y)^n$ . Otherwise, let us denote by  $\sigma_1, \dots, \sigma_r$  the integers such that  $\{i : x_i \in X \setminus Y\} = \{\sigma_1, \dots, \sigma_r\}$  and  $\sigma_1 < \dots < \sigma_r$ . By condition (c) there exist integers  $a_1, \dots, a_r$  such that

$$F(x_1, \dots, x_n) = F(a_1 \cdot x_{\sigma_1}, \dots, a_r \cdot x_{\sigma_r}).$$

This expression is equal to  $G^{r-1}(x_{\sigma_1}, \dots, x_{\sigma_r})$  because  $G|_{(X \setminus Y)^2}$  is a quasitrivial reduction of  $F|_{(X \setminus Y)^n}$ . Using condition (c) in Definition 1.5 for  $G \in \mathcal{H}_{n-1}$  we get that this expression is equal to  $G^{n-1}(x_1, \dots, x_n)$ .  $\square$

*Proof of Proposition 1.10.* If  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \emptyset$ , then Theorem 1.6 implies that there is a subset  $Y \subseteq X$  and an Abelian group  $(Y, G)$  whose exponent divides  $n-1$  and  $|Y| \geq 3$ . This shows that  $|X| \geq |Y| \geq c_{n-1}$ . Conversely, assume that  $|X| \geq c_{n-1}$ . Then we choose a subset  $Y \subseteq X$  such that  $|Y| = c_{n-1} \geq 3$  and we endow  $Y$  with an operation  $G_Y$  such that  $(Y, G_Y)$  is an Abelian group whose exponent divides  $n-1$ .

Let us consider the operation  $G: X^2 \rightarrow X$  defined by the conditions that any  $x \in X \setminus Y$  is an annihilator for  $G|_{(\{x\} \cup Y)^2}$ , that  $G|_{Y^2} = G_Y$ , and that  $G(x, y) = y$  for any  $x, y \in X \setminus Y$ . Then we have  $G \in \mathcal{H}_{n-1}$  and so  $G^{n-1} \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$  by Theorem 1.6, which concludes the proof.  $\square$

*Proof of Corollary 1.11.* By Proposition 1.10 it is sufficient to compute  $c_{n-1}$  in the two cases.

- (a) The cyclic group of order  $p$  is an Abelian group with at least three elements whose exponent divides  $n-1$ , hence  $c_{n-1} \leq p$ . On the other hand, let  $(Y, G)$  be any Abelian group with at least three elements whose exponent  $m$  divides  $n-1$ . Let  $q$  be a prime divisor of  $m$ ; then  $q$  divides  $n-1$ , hence  $q$  is odd. From the definition of the exponent it follows that  $Y$  contains an element of order  $q$ , thus  $|Y| \geq q$ . Since  $q$  divides  $n-1$ , we have  $q \geq p$  by the minimality of  $p$ . Therefore,  $|Y| \geq q \geq p$ , which shows that  $c_{n-1} \geq p$ .
- (b) If  $p = 3$ , then we can take the group  $\mathbb{Z}_p$  as in the previous case; if  $p \geq 5$ , then we can take the group  $\mathbb{Z}_2^2$  (with exponent 2 dividing  $n-1$ ) in order to see that  $c_{n-1} \leq \min(4, p)$ . Conversely, let  $(Y, G)$  be any Abelian group with at least three elements and with exponent  $m$  such that  $m$  divides  $n-1$ . If  $m$  has an odd prime divisor  $q$ , then we can conclude that  $|Y| \geq q \geq p \geq \min(4, p)$  just as in (a). If  $m$  has no odd prime divisors, then  $m$  is a power of 2, and then  $|Y|$  is even, which together with  $|Y| \geq 3$  implies that  $|Y| \geq 4 \geq \min(4, p)$ . Thus, we conclude that  $c_{n-1} \geq \min(4, p)$ .  $\square$

### 3. AN ALTERNATIVE HIERARCHY

For any integer  $k \geq 1$ , let  $S_k^n$  be the set of  $n$ -tuples  $(x_1, \dots, x_n) \in X^n$  such that  $|\{x_1, \dots, x_n\}| \leq k$ . Of course, we have  $D_k^n \subseteq S_{n-k+1}^n$  for  $k \in \{1, \dots, n\}$ . Also, we have  $S_k^n \subseteq S_{k+1}^n$  for  $k \in \{1, \dots, n-1\}$ . Now, denote by  $\mathcal{G}_k^n$  the class of those associative  $n$ -ary operations  $F: X^n \rightarrow X$  satisfying

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}, \quad (x_1, \dots, x_n) \in S_k^n.$$

We say that these operations are *quasitrivial on  $S_k^n$* .

It is not difficult to see that if  $F \in \mathcal{G}_k^n$ , then  $F \in \mathcal{F}_{n-k+1}^n$ . Actually, we have  $\mathcal{G}_1^n = \mathcal{F}_n^n$  and  $\mathcal{G}_n^n = \mathcal{F}_1^n$ . These are the only classes when  $n = 2$ , and thus we assume throughout this section that  $n \geq 3$ . Due to Proposition 1.1, we have that  $\mathcal{G}_n^n = \dots = \mathcal{G}_3^n$  is exactly the class

of quasitrivial associative  $n$ -ary operations, and hence we only need to consider operations in  $\mathcal{G}_2^n$ . The counterpart of Theorem 1.6 can then be stated as follows.

**Theorem 3.1.** *If  $n$  is odd and  $G \in \mathcal{H}_2$ , then its  $n$ -ary extension  $F = G^{n-1}$  is in  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$ . Conversely, for every  $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$  we have  $|E_F| \geq 3$ ,  $n$  is odd, the reductions of  $F$  are exactly the operations  $G_e$  for  $e \in E_F$ , and they lie in  $\mathcal{H}_2$ .*

*Proof.* If  $n$  is odd and  $G \in \mathcal{H}_2$ , then  $n - 1$  is even, and so  $G \in \mathcal{H}_{n-1}$ . Therefore by Theorem 1.6,  $F = G^{n-1}$  is in  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n = \mathcal{F}_{n-1}^n \setminus \mathcal{G}_n^n$ . We have shown in the proof of Theorem 1.6 that  $E_F = Y$ . In order to show  $F \in \mathcal{G}_2^n$ , we need to show that if  $x_1, \dots, x_n \in \{x, y\}$ , then  $F(x_1, \dots, x_n) \in \{x, y\}$ . If  $x$  or  $y$  is in  $X \setminus Y$ , this follows from Proposition 2.8. If  $\{x, y\} \subseteq Y$ , then if  $k$  arguments are equal to  $x$  and  $n - k$  are equal to  $y$ ,  $F(x_1, \dots, x_n) = F(k \cdot x, (n - k) \cdot y)$  because  $(Y, G|_{Y^2})$  is an Abelian group. Since  $n$  is odd, the parity of  $k$  and of  $n - k$  are different. Since  $(Y, G|_{Y^2})$  has exponent 2, this expression is equal to  $x$  (resp.  $y$ ) when  $k$  is odd (resp. even).

Conversely, if  $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n \subseteq \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ , then by Theorem 1.6, we have  $|E_F| \geq 3$ , all the reductions of  $F$  are exactly the operations  $G_e$  for  $e \in E_F$  and they lie in  $\mathcal{H}_{n-1}$ . In particular, for any  $e \in E_F$ , we have that  $(E_F, G_e)$  is an Abelian group whose exponent divides  $n - 1$ . However, since the neutral element is the only idempotent element of a group and since  $G_e(e', e') \in \{e, e'\}$  for any  $e, e' \in E_F$ , it follows that  $G_e(e', e') = e$  for any  $e, e' \in E_F$ , i.e., for any  $e \in E_F$  we have that  $(E_F, G_e)$  is a group of exponent 2. Therefore, we conclude that  $(E_F, F|_{E_F^n})$  is the  $n$ -ary extension of an Abelian group of exponent 2. Also, since 2 divides  $n - 1$  we conclude that  $n$  is odd.  $\square$

Theorem 3.1 is particularly interesting as it enables us to construct easily  $n$ -ary operations in  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$ . For instance, consider the set  $X = \{1, 2, 3, 4, 5, 6\}$  together with the operation  $G: X^2 \rightarrow X$  defined by the following conditions:

- $(\{1, 2, 3, 4\}, G|_{\{1,2,3,4\}^2})$  is isomorphic to  $(\mathbb{Z}_2^2, +)$ ,
- $G|_{\{5,6\}^2} = \pi_1|_{\{5,6\}^2}$ , where  $\pi_1: X^2 \rightarrow X$  is defined by  $\pi_1(x, y) = x$  for any  $x, y \in X$ ,
- for any  $x \in \{1, 2, 3, 4\}$ ,  $G(x, 5) = G(5, x) = 5$  and  $G(x, 6) = G(6, x) = 6$ .

Then for any integer  $p \geq 1$ , we have that the operation associated with any  $(2p + 1)$ -ary extension of  $(\{1, 2, 3, 4, 5, 6\}, G)$  is in  $\mathcal{G}_2^{2p+1} \setminus \mathcal{G}_{2p+1}^{2p+1}$  by Theorem 3.1.

We now state a reformulation of Theorem 3.1 that does not make use of binary reductions. We omit the proof of this result as it is a straightforward adaptation of the proof of Theorem 1.8.

**Theorem 3.2.** *If an operation  $F$  is in  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$ , then  $n$  is odd and setting  $Y = E_F$  we have  $|Y| \geq 3$  and the following assertions hold.*

- (a)  $(Y, F|_{Y^n})$  is the  $n$ -ary extension of an Abelian group of exponent 2.
- (b)  $F|_{(X \setminus Y)^n}$  is associative, quasitrivial, and has at most one neutral element.
- (c) For all  $x_1, \dots, x_n \in X$  and  $i \in \{1, \dots, n-1\}$  such that  $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$  we have

$$F(x_1, \dots, x_n) = F(x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n).$$

Conversely, if  $n$  is odd and  $F$  is an operation that satisfies these conditions for some  $Y \subseteq X$  such that  $|Y| \geq 3$ , then  $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$  and  $E_F = Y$ .

We end this section with the counterpart of Proposition 1.10 and Corollary 1.11 for operations in  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$ .

**Corollary 3.3.** *We have  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n \neq \emptyset$  if and only if  $n$  is odd and  $|X| \geq 4$ .*

*Proof.* (Necessity) By Theorem 3.1, we have that  $n$  is odd and there exists a subset  $Y \subseteq X$  and an Abelian group  $(Y, G)$  of exponent 2 such that  $|Y| \geq 3$ . Since  $(Y, G)$  is of exponent 2 we have  $|X| \geq |Y| \geq 4$ .

(Sufficiency) Let  $Y \subseteq X$  such that  $|Y| = 4$ . We can endow  $Y$  with an operation  $G_Y$  such that  $(Y, G_Y)$  is an Abelian group of exponent 2 that is isomorphic to  $(\mathbb{Z}_2^2, +)$ . Let us consider the operation  $G: X^2 \rightarrow X$  defined by the following conditions:

- $G|_{Y^2} = G_Y$ .
- $G(x, y) = y$  for any  $x, y \in X \setminus Y$ .
- Any  $x \in X \setminus Y$  is an annihilator for  $G|_{(\{x\} \cup Y)^2}$ .

It is not difficult to see that  $G \in \mathcal{H}_2$  (see Definition 1.5). Thus, we have  $G^{n-1} \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$  by Theorem 3.1, which concludes the proof.  $\square$

#### 4. CONCLUDING REMARKS

In this paper we characterized the class  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ , i.e., the class of those associative operations  $F: X^n \rightarrow X$  that are not quasitrivial but satisfy the condition  $F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$  whenever at least  $n-1$  of the elements  $x_1, \dots, x_n$  are equal to each other (Theorems 1.6 and 1.8). These characterizations enabled us to obtain necessary and sufficient conditions on the cardinality of  $X$  so that  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \emptyset$ . Moreover, we proved that any operation in  $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$  is reducible to a binary associative operation (Corollary 1.7). Finally, we characterized the class  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$ , i.e., the class of those associative operations  $F: X^n \rightarrow X$  that are not quasitrivial but satisfy the condition  $F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$  whenever  $|\{x_1, \dots, x_n\}| \leq 2$  (Theorems 3.1 and 3.2). As a byproduct of these characterizations, we obtained necessary and sufficient conditions on the cardinality of  $X$  for which  $\mathcal{G}_2^n \setminus \mathcal{G}_n^n \neq \emptyset$ .

The main results of this paper thus characterize several relevant subclasses of associative and idempotent  $n$ -ary operations. However, the characterization of the class  $\mathcal{F}_n^n$  of associative and idempotent  $n$ -ary operations still eludes us. This and related enumeration results [2, 3] constitute a topic of current research.

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#### APPENDIX A. ALTERNATIVE PROOF OF COROLLARY 1.7

We provide an alternative proof of Corollary 1.7 that does not use [2, Corollary 2.3].

To this extent, we first prove the following general result.

**Proposition A.1.** *Let  $F \in \mathcal{F}_n^n$ . The following assertions are equivalent.*

- (i)  $F$  is reducible to an associative and idempotent operation  $G: X^2 \rightarrow X$ .
- (ii)  $F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$  for any  $x, y \in X$ .

*Proof.* The implication (i)  $\Rightarrow$  (ii) is straightforward. Now, let us show that (ii) implies (i). So, suppose that

$$(7) \quad F((n-1) \cdot x, y) = F(x, (n-1) \cdot y) \quad x, y \in X,$$

and consider the operation  $G: X^2 \rightarrow X$  defined by  $G(x, y) = F((n-1) \cdot x, y)$  for any  $x, y \in X$ . It is not difficult to see that  $G$  is associative and idempotent. Now, let  $x_1, \dots, x_n \in X$  and let us show that  $G^{n-1}(x_1, \dots, x_n) = F(x_1, \dots, x_n)$ . Using repeatedly (7) and the idempotency of  $F$  we obtain

$$\begin{aligned} G^{n-1}(x_1, \dots, x_n) &= F^{n-1}((n-1) \cdot x_1, (n-1) \cdot x_2, \dots, (n-1) \cdot x_{n-1}, x_n) \\ &= F^{n-1}((2n-3) \cdot x_1, x_2, (n-1) \cdot x_3, \dots, (n-1) \cdot x_{n-1}, x_n) \\ &= \dots \\ &= F^{n-1}(((n-2)(n-1)+1) \cdot x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ &= F(x_1, \dots, x_n), \end{aligned}$$

which shows that  $F$  is reducible to  $G$ .  $\square$

*Remark 2.* Let  $\leq$  be a total ordering on  $X$ . An operation  $F: X^n \rightarrow X$  is said to be  $\leq$ -preserving if  $F(x_1, \dots, x_n) \leq F(x'_1, \dots, x'_n)$  whenever  $x_i \leq x'_i$  for any  $i \in \{1, \dots, n\}$ . One of the main results of Kiss and Somlai [9, Theorem 4.8] is that every  $\leq$ -preserving operation  $F \in \mathcal{F}_n^n$  is reducible to an associative, idempotent, and  $\leq$ -preserving binary operation. To this extent, they first show [9, Lemma 4.1] that any  $\leq$ -preserving operation  $F \in \mathcal{F}_n^n$  satisfies

$$F((n-1) \cdot x, y) = F(x, (n-1) \cdot y) \quad x, y \in X.$$

Thus, we conclude that [9, Theorem 4.8] is an immediate consequence of [9, Lemma 4.1] and Proposition A.1 above.

The following result is the key for the alternative proof of Corollary 1.7.

**Proposition A.2.** *Let  $F \in \mathcal{F}_{n-1}^n$ . The following assertions are equivalent.*

- (i)  $F$  is reducible to an associative and quasitrivial operation  $G: X^2 \rightarrow X$ .
- (ii)  $F$  is reducible to an associative and idempotent operation  $G: X^2 \rightarrow X$ .
- (iii)  $F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$  for any  $x, y \in X$ .
- (iv)  $|E_F| \leq 1$ .

*Proof.* The equivalence (i)  $\Leftrightarrow$  (ii) and the implication (iii)  $\Rightarrow$  (iv) are straightforward. Also, the equivalence (ii)  $\Leftrightarrow$  (iii) follows from Proposition A.1. Now, let us show that (iv) implies (iii). So, suppose that  $|E_F| \leq 1$  and suppose to the contrary that there exist  $x, y \in X$  with  $x \neq y$  such that  $F((n-1) \cdot x, y) \neq F(x, (n-1) \cdot y)$ . We have two cases to consider. If  $F((n-1) \cdot x, y) = y$  and  $F(x, (n-1) \cdot y) = x$ , then by Lemma 2.5 we have that  $x, y \in E_F$ , which contradicts our assumption on  $E_F$ . Otherwise, if  $F((n-1) \cdot x, y) = x$  and  $F(x, (n-1) \cdot y) = y$ , then we have

$$\begin{aligned} x &= F((n-1) \cdot x, y) = F((n-1) \cdot x, F(n \cdot y)) \\ &= F(F((n-1) \cdot x, y), (n-1) \cdot y) = F(x, (n-1) \cdot y) = y, \end{aligned}$$

which contradicts the fact that  $x \neq y$ .  $\square$

*Proof of Corollary 1.7.* This follows from Proposition A.2 and [7, Lemma 1].  $\square$

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