



DE LA RECHERCHE À L'INDUSTRIE

A Discontinuous Galerkin-based surface domain decomposition method for the electromagnetic wave scattering problem

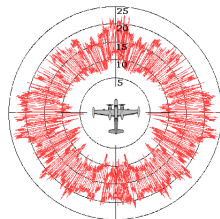
Justine Labat Agnès Pujols Muriel Sesques | CEA-CESTA

Journes Jeunes Chercheuses Jeunes Chercheurs sur le thème de la propagation d'ondes en régime harmonique
November 23-24, 2020

Physical context : Radar detection

Scattering of **electromagnetic waves** on a target
to determine its **radar cross-section**

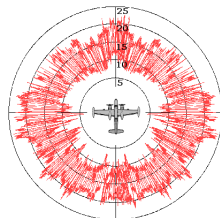
Make an object **stealthy** = Reduce its radar signature



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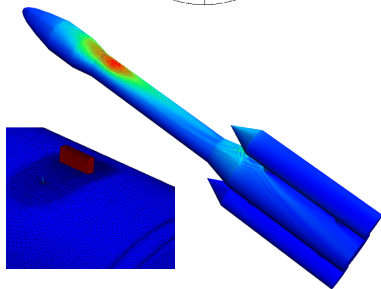
Make an object **stealthy** = Reduce its radar signature



Mathematical context : Numerical simulation

- Solve the **time-harmonic Maxwell equations**
- Simulate **infinite** domains
- Take into account **complex** geometries (large-scale objects, multi-scale effects, ...)
- Guarantee **accuracy, efficiency and robustness**

→ **Limitations due to high numerical costs ...**



Monopole antenna in the presence of a
dielectric object on a launcher
(ISAE Workshop, 2016)

The **Boundary Element Method** is very well-adapted for solving scattering problems . . .

. . . but can also be limited by computational resources !

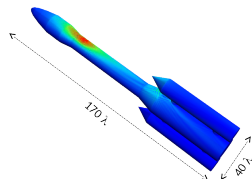
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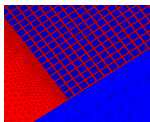
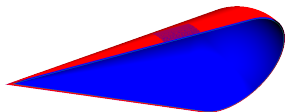
Combine it with a **Surface Domain Decomposition** in order to tackle the following issues :

- Robust and accurate simulations for a **wide range** of frequencies

Domain length	Mesh points
10λ	$> 20\,000$
100λ	$\approx 2\,000\,000$
$1\,000 \lambda$	$\approx 200\,000\,000$
$10\,000 \lambda$	$\approx 20\,000\,000\,000$



- **Geometry-adaptive** strategies to handle multi-scale structures



100×100 metallic patch
on the half cone-sphere
(ISAE Workshop, 2016)

- Simulation-based engineering using **High Performance Computing**

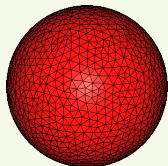
- Development of a *Discontinuous Galerkin*-based Surface Domain Decomposition Method
 - ▶ Acoustics : N.-A. Messai and S. Pernet (2019)
 - ▶ Electromagnetism : Z. Peng, K.-H. Lim, J.-F. Lee (2013), Z. Peng, R. Hiptmair, Y. Shao (2016)

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Model problem

$$Au = f \quad \text{on } \Gamma$$

Initial surface Γ

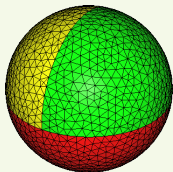


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Decomposition $\bigcup \Gamma_n$



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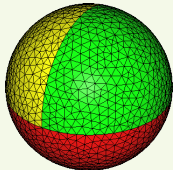
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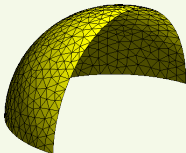
Sub-problems

$$Au_n^0 = f \quad \text{on } \Gamma_n$$

Decomposition $\cup \Gamma_n$



Partial surface Γ_n



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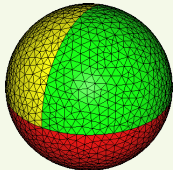
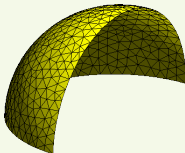
$$Au = f \quad \text{on } \Gamma$$

Sub-problems

$$Au_n^0 = f \quad \text{on } \Gamma_n$$

Iterative coupling

$$Au_n^{p+1} = f_n^p \quad \text{on } \Gamma_n$$

Decomposition $\cup \Gamma_n$ Partial surface Γ_n 

Interactions between

- elements coming from two **different** sub-domains
- elements on the **interface** with **all** the other elements

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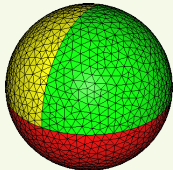
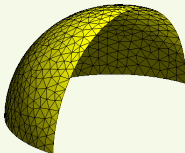
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Decomposition $\cup \Gamma_n$ Partial surface Γ_n 

Interactions between

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- Discretization using the **Boundary Element Method**

- ▶ Physics : **Conducting** or **dielectric** material
- ▶ Geometry : **Conformal** or **non-conformal** meshes
- ▶ Solvers : Direct solvers or compression method + **iterative** coupling
- ▶ Features : Massively parallel architecture on TERA-1000



- 1 Discontinuous Galerkin-based surface domain decomposition method
 - Model problem
 - Continuous formulation
 - Discrete formulation

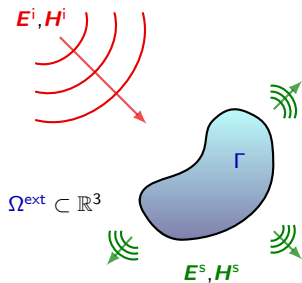
- 2 Preliminary numerical results
 - Test-cases
 - Qualitative comparison
 - Numerical convergence
 - Eigenspectrum of preconditioned system

- 3 Conclusion and perspectives

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- 3 Conclusion and perspectives



- Ω^{ext} : exterior domain
- Γ : boundary of Ω^{ext}
- $\mathbf{E}^i, \mathbf{H}^i$: incident fields
- $\mathbf{E}^s, \mathbf{H}^s$: scattered fields
- \mathbf{E}, \mathbf{H} : total fields

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \quad \mathbf{H} = \mathbf{H}^i + \mathbf{H}^s$$

- Time-harmonic dependence $\exp(+i\omega t)$

Time-harmonic Maxwell equations (in vacuum)

$$\begin{cases} \nabla \times \mathbf{E}^s + i\kappa Z_0^{-1} \mathbf{H}^s = 0 & \text{in } \Omega^{\text{ext}} \\ \nabla \times \mathbf{H}^s - i\kappa Z_0 \mathbf{E}^s = 0 & \text{in } \Omega^{\text{ext}} \end{cases}$$

- κ : wave-number
- Z_0 : impedance coefficient

Perfect conductor condition

$$\mathbf{n} \times \mathbf{E}^s = -\mathbf{n} \times \mathbf{E}^i \quad \text{on } \Gamma$$

- \mathbf{n} : exterior unit normal vector

Silver-Müller radiation condition

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| (Z_0 \mathbf{H}^s \times \hat{\mathbf{x}} - \mathbf{E}^s) = 0 \quad \text{unif. in } \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

- The scattered electromagnetic fields can be represented by the **Stratton-Chu formulas**

$$\mathbf{E}^s(\mathbf{x}) = -i\kappa\mathcal{T}\mathbf{J}(\mathbf{x}) \quad \mathbf{H}^s(\mathbf{x}) = \frac{1}{Z_0}\mathcal{K}\mathbf{J}(\mathbf{x}) \quad \mathbf{x} \in \Omega^{\text{ext}}$$

where $\mathbf{J} = Z_0(\mathbf{n} \times \mathbf{H})$ is the **total surface electric current** and

$$\mathcal{T}\mathbf{J}(\mathbf{x}) = \frac{1}{\kappa^2}\nabla(\mathcal{S}\text{div}_\Gamma\mathbf{J})(\mathbf{x}) + \mathcal{S}\mathbf{J}(\mathbf{x}) \quad \mathcal{K}\mathbf{J}(\mathbf{x}) = \nabla \times \mathcal{S}\mathbf{J}(\mathbf{x}) \quad \mathcal{S}\lambda(\mathbf{x}) = \int_\Gamma G(\mathbf{x}, \mathbf{y})\lambda(\mathbf{y}) \, d\mathbf{s}_y$$

where $G(\mathbf{x}, \mathbf{y}) = \frac{\exp(-i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$ is the outgoing Green function of the Helmholtz equation

- The scattered electromagnetic fields can be represented by the **Stratton-Chu formulas**

$$\mathbf{E}^s(\mathbf{x}) = -i\kappa \mathcal{T} \mathbf{J}(\mathbf{x}) \quad \mathbf{H}^s(\mathbf{x}) = \frac{1}{Z_0} \mathcal{K} \mathbf{J}(\mathbf{x}) \quad \mathbf{x} \in \Omega^{\text{ext}}$$

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- From **jump relations** through Γ , we deduce two kinds of **Boundary Integral Equations**¹

Electric Field Integral Equation

$$i\kappa \mathcal{T} \mathbf{J} = \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}) \quad \text{on } \Gamma$$

where $\mathcal{T} : \mathbf{H}_t^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\operatorname{rot}_\Gamma, \Gamma)$

$$\mathcal{T} \mathbf{J} = \mathbf{n} \times (\mathcal{T} \mathbf{J} \times \mathbf{n})$$

Magnetic Field Integral Equation

$$\frac{1}{2} \mathbf{J} - \mathcal{K} \mathbf{J} = Z_0(\mathbf{n} \times \mathbf{H}^i) \quad \text{on } \Gamma$$

where $\mathcal{K} : \mathbf{H}_t^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$

$$\mathcal{K} \mathbf{J} = \text{p.v.}(\mathbf{n} \times \mathcal{K} \mathbf{J})$$

where $\mathbf{H}_t^s(D_\Gamma, \Gamma) = \{\mathbf{J} \in \mathbf{H}^s(\Gamma) \mid D_\Gamma \mathbf{J} \in H^s(\Gamma) \text{ and } \mathbf{n} \cdot \mathbf{J} = 0 \text{ on } \Gamma\}$

- The variational formulations associated with the **EFIE** and **MFIE** formulations read as :

Find $\mathbf{J} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ such that

$$\text{(EFIE)} \quad a(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$$

$$\text{(MFIE)} \quad b(\mathbf{J}, \mathbf{v}) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{rot}_\Gamma, \Gamma)$$

where

$$a(\mathbf{J}, \mathbf{v}) = i\kappa \langle \mathbf{T}\mathbf{J}, \mathbf{v} \rangle_\Gamma = \frac{1}{i\kappa} \langle \mathbf{S} \text{div}_\Gamma \mathbf{J}, \text{div}_\Gamma \mathbf{v} \rangle_\Gamma + i\kappa \langle \mathbf{S}\mathbf{J}, \mathbf{v} \rangle_\Gamma$$

$$b(\mathbf{J}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{J}, \mathbf{v} \rangle_\Gamma - \langle \mathbf{K}\mathbf{J}, \mathbf{v} \rangle_\Gamma$$

with $\mathbf{S} : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ is the **Dirichlet trace** of single-layer potential operator \mathbf{S}

In **(EFIE)**, we have used the **Stokes formula** on the **closed** surface Γ

$$\int_\Gamma \nabla_\Gamma u \cdot \mathbf{v} \, ds = - \int_\Gamma u (\text{div}_\Gamma \mathbf{v}) \, ds$$

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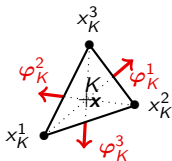
$$\text{(EFIE)} \quad a(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$$

$$\text{(MFIE)} \quad b(\mathbf{J}, \mathbf{v}) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{rot}_\Gamma, \Gamma)$$

- Discretization space $\mathcal{V}^h \subset \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$: The **Raviart-Thomas div-conforming boundary elements** of smallest degree

$$\mathbf{J} \approx \mathbf{J}^h(\mathbf{x}) = \sum_{K \in \Gamma^h} J_{K,1}^h \boldsymbol{\varphi}_K^1(\mathbf{x}) + J_{K,2}^h \boldsymbol{\varphi}_K^2(\mathbf{x}) + J_{K,3}^h \boldsymbol{\varphi}_K^3(\mathbf{x}) \quad \mathbf{x} \in \Gamma^h \text{ (triangulation of } \Gamma)$$

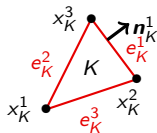
with



$$\boldsymbol{\varphi}_K^i(\mathbf{x}) = \frac{1}{2|K|} (\mathbf{x} - \mathbf{x}_K^i)$$

and

$$J_{K,i}^h = \int_{e_K^i} \mathbf{J}^h \cdot \mathbf{n}_K^i \, d\sigma$$



- The variational formulations associated with the **EFIE** and **MFIE** formulations read as :

Find $\mathbf{J} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ such that

$$\text{(EFIE)} \quad a(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$$

$$\text{(MFIE)} \quad b(\mathbf{J}, \mathbf{v}) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{rot}_\Gamma, \Gamma)$$

- Discretization space $\mathcal{V}^h \subset \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$: The **Raviart-Thomas div-conforming boundary elements** of smallest degree
- The discrete formulations associated with the **EFIE** , **MFIE** and **CFIE** formulations read as :

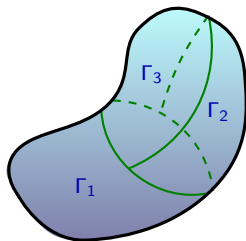
Find $\mathbf{J}^h \in \mathcal{V}^h$ such that for any $\mathbf{v}^h \in \mathcal{V}^h$

$$\text{(EFIE)} \quad a^h(\mathbf{J}^h, \mathbf{v}^h) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v}^h \right\rangle_\Gamma$$

$$\text{(MFIE)} \quad b^h(\mathbf{J}^h, \mathbf{v}^h) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v}^h \right\rangle_\Gamma$$

$$\text{(CFIE)} \quad \alpha a^h(\mathbf{J}^h, \mathbf{v}^h) + (1 - \alpha) b^h(\mathbf{J}^h, \mathbf{v}^h) = \alpha \left\langle \mathbf{E}_t^i, \mathbf{v}^h \right\rangle_\Gamma + (1 - \alpha) Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v}^h \right\rangle_\Gamma$$

(with $0 < \alpha < 1$)



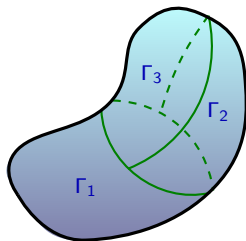
- $\Gamma = \bigcup_{n=1}^N \Gamma_n$
- $\gamma_{nm} = \Gamma_n \cap \Gamma_m = \gamma_{mn}$
- $\gamma_n = \partial\Gamma_n = \bigcup_{m \in \mathcal{I}_n} \gamma_{nm}$
- τ_{nm} : ext. normal vector to γ_{nm}
- τ_n : exterior normal vector to γ_n

- Write the EFIE on the partial surface Γ_n

Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_m}, \Gamma_m)$ such that

$$i\kappa \langle T\mathbf{J}, \mathbf{v} \rangle_{\Gamma_n} = \langle \mathbf{E}_t^i, \mathbf{v} \rangle_{\Gamma_n} \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_n}, \Gamma_n)$$

$$\text{with } T\mathbf{J} = \frac{1}{\kappa^2} \nabla_{\Gamma} (S \text{div}_{\Gamma} \mathbf{J}) + S\mathbf{J}$$



$$\bullet \Gamma = \bigcup_{n=1}^N \Gamma_n$$

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$\bullet \boldsymbol{\tau}_{nm}$: ext. normal vector to γ_{nm}

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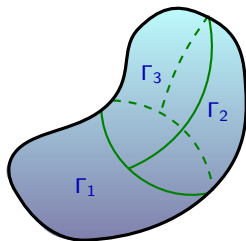
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$$\text{with } T\mathbf{J} = \frac{1}{\kappa^2} \nabla_{\Gamma} (S \text{div}_{\Gamma} \mathbf{J}) + S\mathbf{J}$$

- Use local integration by parts on Γ_n

$$\int_{\Gamma_n} \nabla_{\Gamma} u \cdot \mathbf{v} \, ds = - \int_{\Gamma_n} u (\text{div}_{\Gamma} \mathbf{v}) \, ds + \int_{\gamma_n} u (\boldsymbol{\tau}_n \cdot \mathbf{v}) \, d\sigma$$



- $\Gamma = \bigcup_{n=1}^N \Gamma_n$
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- Write the EFIE on the partial surface Γ_n

Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_m}, \Gamma_m)$ such that

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$$\text{with } T\mathbf{J} = \frac{1}{\kappa^2} \nabla_{\Gamma}(\text{Sdiv}_{\Gamma} \mathbf{J}) + S\mathbf{J}$$

- Use local integration by parts on Γ_n

$$\int_{\Gamma_n} \nabla_{\Gamma} u \cdot \mathbf{v} \, ds = - \int_{\Gamma_n} u(\text{div}_{\Gamma} \mathbf{v}) \, ds + \int_{\gamma_n} u(\boldsymbol{\tau}_n \cdot \mathbf{v}) \, d\sigma$$

- Sum over all the sub-domains

$$\sum_{n=1}^N \langle \text{Sdiv}_{\Gamma} \mathbf{J}, \boldsymbol{\tau}_n \cdot \mathbf{v} \rangle_{\gamma_n} = \sum_{\gamma_{nm}} \langle \text{Sdiv}_{\Gamma} \mathbf{J}, [\mathbf{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}$$

where the jump operator : $[\mathbf{v}]_{\gamma_{nm}} = \boldsymbol{\tau}_{nm} \cdot \mathbf{v}_n + \boldsymbol{\tau}_{mn} \cdot \mathbf{v}_m$

- While the classical EFIE variational formulation for **BEM** reads as

Find $\mathbf{J} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ such that

$$a(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$$

the EFIE variational formulation for **DG-SDDM** can be written as

Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_m}, \Gamma_m)$ such that

$$a(\mathbf{J}, \mathbf{v}) + c(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \bigoplus_{n=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_n}, \Gamma_n)$$

$$c(\mathbf{J}, \mathbf{v}) = -\frac{1}{i\kappa} \sum_{\gamma_{nm}} \langle S \text{div}_\Gamma \mathbf{J}, [\mathbf{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}$$

- While the classical EFIE variational formulation for **BEM** reads as

Find $\mathbf{J} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ such that

$$a(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$$

the EFIE variational formulation for **DG-SDDM** can be written as

Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_m}, \Gamma_m)$ such that

$$a(\mathbf{J}, \mathbf{v}) + c(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v} \right\rangle_\Gamma \quad \forall \mathbf{v} \in \bigoplus_{n=1}^N \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_n}, \Gamma_n)$$

(Anti-)Symmetrization¹ :
$$c(\mathbf{J}, \mathbf{v}) = -\frac{1}{i\kappa} \sum_{\gamma_{nm}} \langle S \text{div}_\Gamma \mathbf{J}, [\mathbf{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}} \pm \frac{1}{i\kappa} \sum_{\gamma_{nm}} \underbrace{\langle S \text{div}_\Gamma \mathbf{v}, [\mathbf{J}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}}_{\text{consistent term}}$$

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Penalization¹ : $p(\mathbf{J}, \mathbf{v}) = \frac{\beta}{\kappa} \sum_{\gamma_{nm}} \underbrace{\langle [\mathbf{J}]_{\gamma_{nm}}, [\mathbf{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}}_{\text{stabilizing term}}$

1. Reed and Hill (1973), Babuška et al. (1973), Cockburn et al. (2004), Cohen et al. (2006), Grote et al. (2007), . . .

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Drawbacks of the discontinuous formulation

- **No equivalence** between the classical formulation and the discontinuous one
- Flux terms : The vector fields \mathbf{v} and \mathbf{J} require more regularity (to make sense)
- Penalty terms : Use of a **<<weaker>>** inner product than $L^2(\gamma_{nm})$

- The variational formulations associated with the **EFIE** and **MFIE** formulations read as :

Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{H}_t^0(\text{div}_{\Gamma_m}, \Gamma_m)$ such that

$$\text{(EFIE)} \quad \tilde{a}(\mathbf{J}, \mathbf{v}) = a(\mathbf{J}, \mathbf{v}) + c(\mathbf{J}, \mathbf{v}) + p(\mathbf{J}, \mathbf{v}) = \left\langle \mathbf{E}_t^i, \mathbf{v} \right\rangle_{\Gamma} \quad \forall \mathbf{v} \in \bigoplus_{n=1}^N \mathbf{H}_t^0(\text{div}_{\Gamma_n}, \Gamma_n)$$

$$\text{(MFIE)} \quad b(\mathbf{J}, \mathbf{v}) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v} \right\rangle_{\Gamma} \quad \forall \mathbf{v} \in \bigoplus_{n=1}^N \mathbf{H}_t^0(\text{rot}_{\Gamma_n}, \Gamma_n)$$

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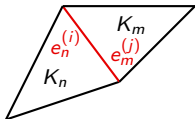
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- Discretization space $\mathcal{V}^h = \bigoplus \mathcal{V}_m^h$ with $\mathcal{V}_m^h \subset \mathbf{H}_t^0(\text{div}_{\Gamma_m}, \Gamma_m)$: The **Raviart-Thomas div-conforming boundary elements** of smallest degree

$$\mathbf{J}|_{\Gamma_m} \approx \mathbf{J}_m^h(\mathbf{x}) = \sum_{K \in \Gamma_m^h} J_{K,1}^{h,m} \varphi_K^1(\mathbf{x}) + J_{K,2}^{h,m} \varphi_K^2(\mathbf{x}) + J_{K,3}^{h,m} \varphi_K^3(\mathbf{x}) \quad \mathbf{x} \in \Gamma_m^h \text{ (triangulation of } \Gamma_m)$$



such that
$$[\mathbf{J}^h]_{e_n^{(i)}} = [\mathbf{J}^h]_{e_m^{(j)}} = \frac{1}{|e_n^{(i)}|} J_{K_n,i}^{h,n} + \frac{1}{|e_m^{(j)}|} J_{K_m,j}^{h,m}$$

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- The discrete formulations associated with the **EFIE** , **MFIE** and **CFIE** formulations read as :

Find $\mathbf{J}^h \in \mathcal{V}^h$ such that for any $\mathbf{v}^h \in \mathcal{V}^h$

$$\text{(EFIE)} \quad \tilde{a}^h(\mathbf{J}^h, \mathbf{v}^h) = \left\langle \mathbf{n} \times (\mathbf{E}^i \times \mathbf{n}), \mathbf{v}^h \right\rangle_{\Gamma}$$

$$\text{(MFIE)} \quad b^h(\mathbf{J}^h, \mathbf{v}^h) = Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v}^h \right\rangle_{\Gamma}$$

$$\text{(CFIE)} \quad \alpha \tilde{a}^h(\mathbf{J}^h, \mathbf{v}^h) + (1 - \alpha) b^h(\mathbf{J}^h, \mathbf{v}^h) = \alpha \left\langle \mathbf{E}_t^i, \mathbf{v}^h \right\rangle_{\Gamma} + (1 - \alpha) Z_0 \left\langle \mathbf{n} \times \mathbf{H}^i, \mathbf{v}^h \right\rangle_{\Gamma}$$

(with $0 < \alpha < 1$)

- Interior Penalty Discontinuous Galerkin formulation for surface domain decomposition
 - ▶ Less degrees of freedom than a classic IPDG formulation
 - ▶ Discrete ellipticity \implies well-posedness of the discrete weak formulations

$$\|\mathbf{J}^h\|_{IPDG}^2 = \|\mathbf{J}^h\|_{\mathbf{L}^2(\Gamma)}^2 + \sum_{n=1}^N \|\operatorname{div}_{\Gamma_n} \mathbf{J}_n^h\|_{\mathbf{L}^2(\Gamma_n)}^2 + \beta \sum_{n=1}^N \|[\mathbf{J}^h]\|_{\mathbf{L}^2(\gamma_n)}^2$$

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- What about the penalization term

$$p^h(\mathbf{J}^h, \mathbf{v}^h) = \beta \sum_{\gamma_{nm}} \langle [\mathbf{J}^h]_{\gamma_{nm}}, [\mathbf{v}^h]_{\gamma_{nm}} \rangle_{\gamma_{nm}} \quad ?$$

- ▶ How to choose the penalty parameter β ?

examples : $\beta = \frac{c}{h}$ $\beta = c |\log(h)|$ where $c > 0$ constant

- ▶ Which impact on the accuracy? on the condition number?

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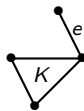
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- ▶ Which impact on the accuracy? on the condition number?

- Accurate evaluation of new weak singular integrals under the form

$$\frac{1}{4\pi} \int_K \int_e \frac{\exp(-i\kappa|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} d\sigma_x d\sigma_y$$



- ▶ Singularity subtraction : Haninen, Taskinen and Sarvas (2006), Lenoir and Salles (2012)
- ▶ Contribution assembling : Salles (2013, PhD thesis), Collino (2020, technical report)
- ▶ Singularity cancellation : Xu, Song, Pan and Sheng (2018)

- The discrete EFIE formulation can be put under the form of a linear system to solve

$$\mathbf{M} \mathbf{x} = \mathbf{f}$$

where \mathbf{x} is the **unknown**, \mathbf{f} is the right hand-side,

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$$\mathbf{M} = \begin{array}{|c|c|c|} \hline \mathbf{M}_1 & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \hline \mathbf{M}_{21} & \mathbf{M}_2 & \mathbf{M}_{23} \\ \hline \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_3 \\ \hline \end{array}$$

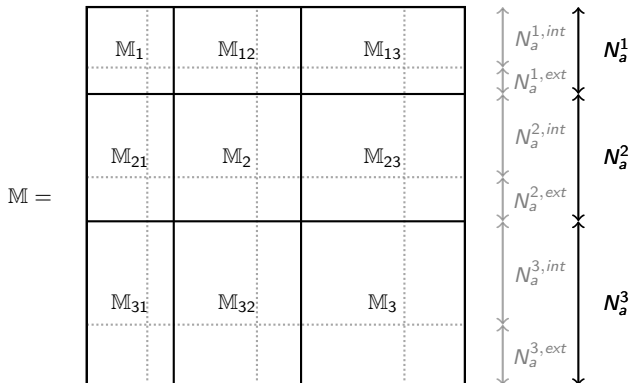
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$$\mathbf{M} = \underbrace{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}}_{\mathbf{A} \text{ (EFIE matrix)}} + \underbrace{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}}_{\mathbf{C} \text{ (Flux matrix)}} + \underbrace{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}}_{\mathbf{P} \text{ (Penalty matrix)}}$$

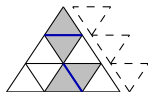
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- Classical EFIE

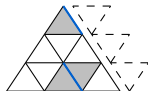
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- Classical EFIE
- "Half" EFIE

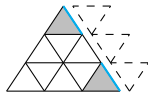
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- Classical EFIE
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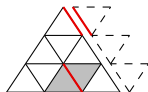
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- Classical EFIE
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- Double flux

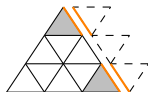
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- Classical EFIE
- "Half" EFIE
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- Double flux
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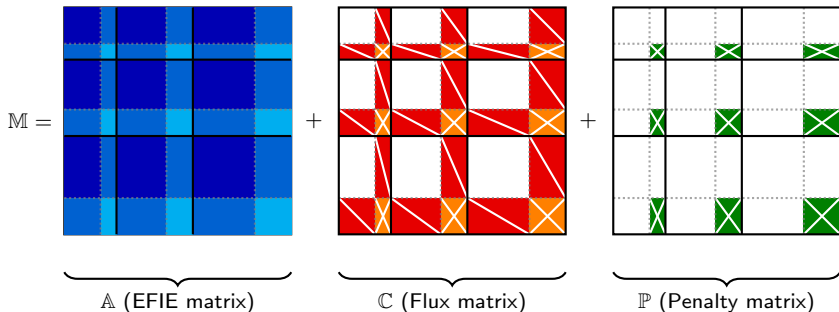
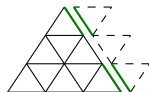
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- Classical EFIE
- "Half" EFIE
- "Single" EFIE
- Double flux
- Single flux
- Penalisation

- The discrete CFIE formulation can be put under the form of a linear system to solve

$$\mathbb{M} \mathbf{x} = \mathbf{f}$$

- Instead of **assembling** the matrix \mathbb{M} then **inverting** its, we implement an **iterative solution**

M_1	M_{12}	M_{13}	x_1	f_1
M_{21}	M_2	M_{23}	x_2	f_2
M_{31}	M_{32}	M_3	x_3	f_3

=

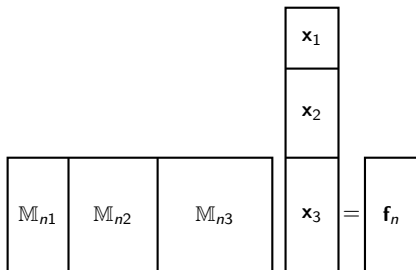
- The discrete CFIE formulation can be put under the form of a linear system to solve

$$\mathbb{M} \mathbf{x} = \mathbf{f}$$

- Instead of assembling the matrix \mathbb{M} then inverting its, we implement an **iterative solution**
 - We restrict the linear system to a subdomain $n = 1, \dots, N$

that is equivalent to solve

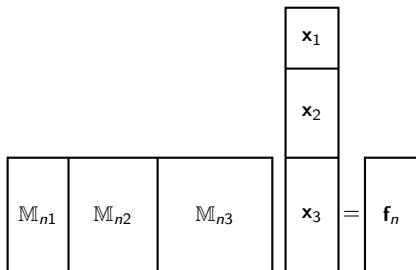
$$\mathbb{M}_n \mathbf{x}_n = \mathbf{f}_n - \sum_{m \neq n} \mathbb{M}_{nm} \mathbf{x}_m$$



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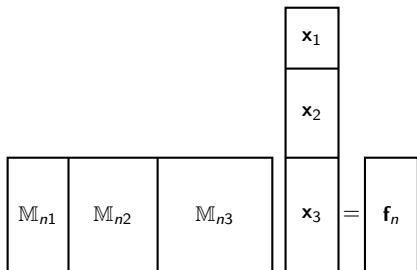
$$\mathbb{M}_n \mathbf{x}_n = \mathbf{f}_n - \sum_{m \neq n} \mathbb{M}_{nm} \mathbf{x}_m$$

- The iterative procedure is then defined by
 - $\mathbb{M}_n \mathbf{x}_n^{(0)} = \mathbf{f}_n$
 - $\mathbb{M}_n \mathbf{x}_n^{(p+1)} = \mathbf{f}_n - \sum_{m \neq n} \mathbb{M}_{nm} \mathbf{x}_m^{(p)}$
 - while it does not converge ...

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that is equivalent to solve

$$\mathbb{M}_n \mathbf{x}_n = \mathbf{f}_n - \sum_{m \neq n} \mathbb{M}_{nm} \mathbf{x}_m$$

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 - while it does not converge ...

- Convergence depends on the **condition number** of the **(block-diagonal) preconditioned** matrix

$$\tilde{\mathbb{P}}^{-1} \mathbb{M} \mathbf{x} = \tilde{\mathbb{P}}^{-1} \mathbf{f}$$

where

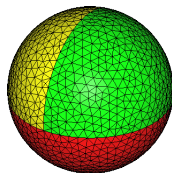
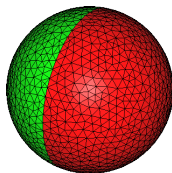
$$\tilde{\mathbb{P}}_{nm} = \begin{cases} \mathbb{M}_n & \text{if } m = n \\ \mathbf{0} & \text{if } m \neq n \end{cases}$$

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- 3 Conclusion and perspectives

- Validation of the **EFIE** and **CFIE** formulations

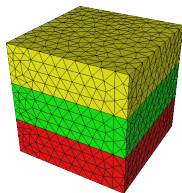
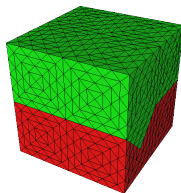
- Comparison of **electric currents** on Γ

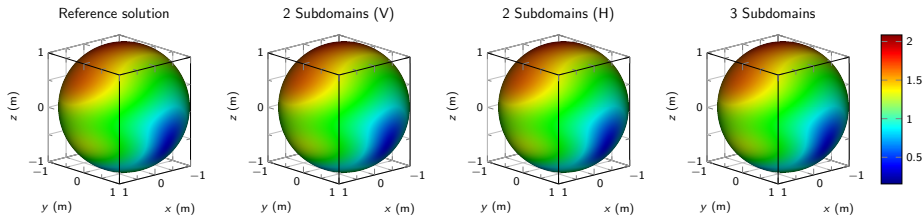
- ▶ Qualitative comparison (module)
- ▶ **Jump error** with respect to the mesh size h
- ▶ **L^2 -error** with respect to the **mesh size** h
- ▶ **L^2 -error** with respect the **penalty coefficient** β



- Comparison of **radar cross-sections**

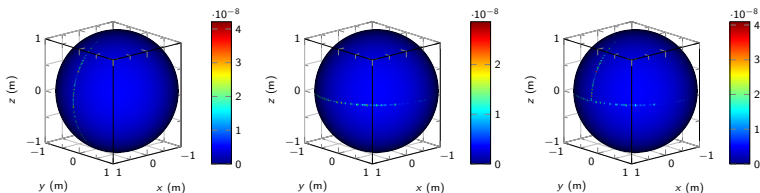
- ▶ Qualitative comparison (module)
- ▶ **L^2 -error** with respect to the **mesh size** h
- ▶ Impact of the penalization term ?

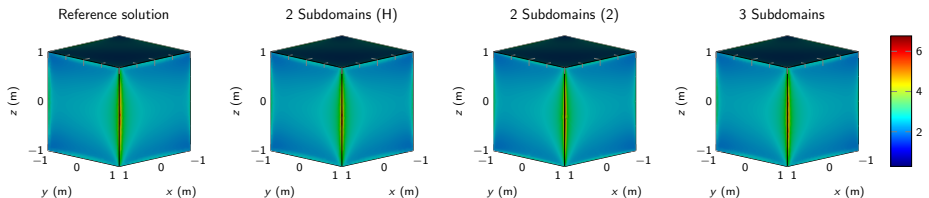




Test-case

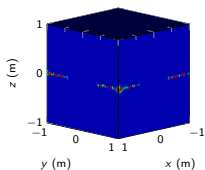
Sphere
EFIE
150 MHz



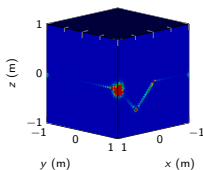


Test-case

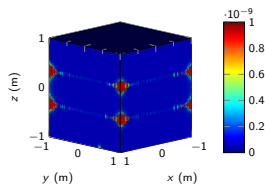
Cube
CFIE
150 MHz



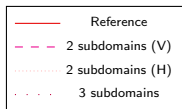
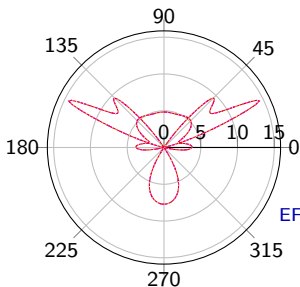
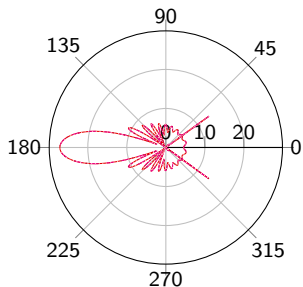
Max. error : $4.588 \cdot 10^{-8}$



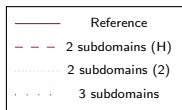
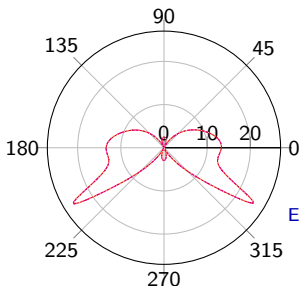
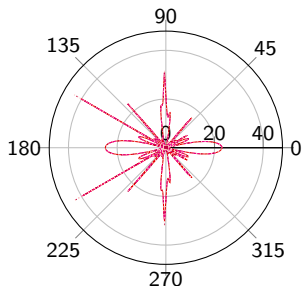
Max. error : $7.0238 \cdot 10^{-6}$



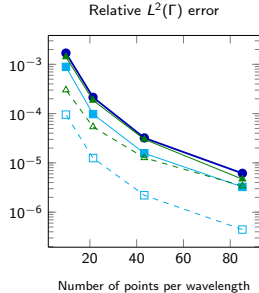
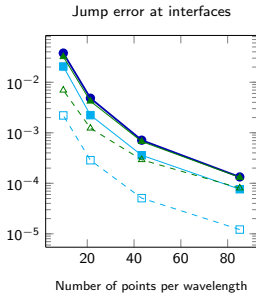
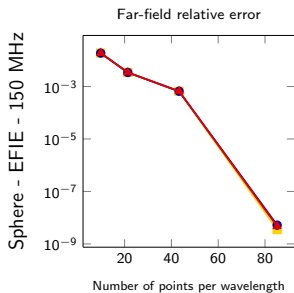
Max. error : $1.363 \cdot 10^{-5}$



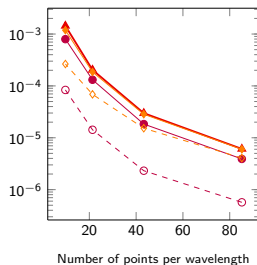
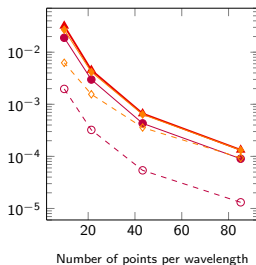
EFIE - 600 MHz / CFIE - 300 MHz



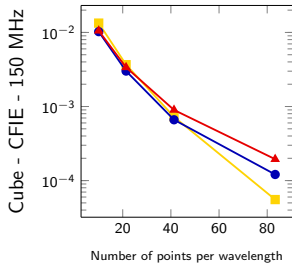
EFIE - 1.2 GHz / CFIE - 300 MHz



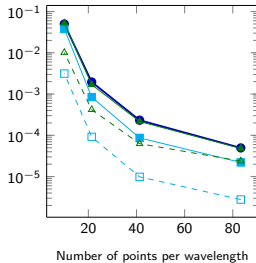
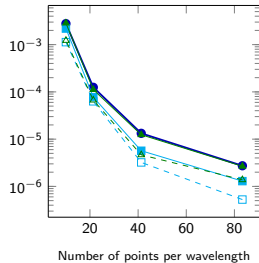
- 2 subdomains (H) with $\beta = 0$
- 2 subdomains (V) with $\beta = 0$
- ▲ 3 subdomains with $\beta = 0$
- 2 subdomains with $\beta = \frac{1}{h}$
- ▲ 2 subdomains with $\beta = |\log(h)|$
- 2 subdomains with $\beta = \frac{10}{h}$
- △ 2 subdomains with $\beta = 10|\log(h)|$
- 3 subdomains with $\beta = \frac{1}{h}$
- ◆ 3 subdomains with $\beta = |\log(h)|$
- 3 subdomains with $\beta = \frac{10}{h}$
- ◇ 3 subdomains with $\beta = 10|\log(h)|$



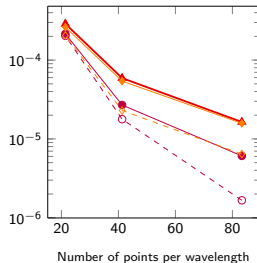
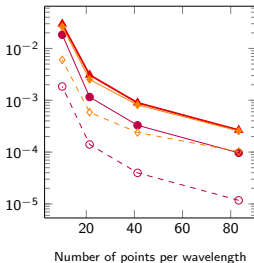
Far-field relative error



Jump error at interfaces

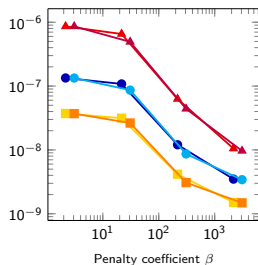
Relative $L^2(\Gamma)$ error

- 3 subdomains with $\beta = 0$
- 2 subdomains (V) with $\beta = 0$
- ▲ 2 subdomains (2) with $\beta = 0$
- 2 subdomains (V) with $\beta = \frac{1}{h}$
- ▲ 2 subdomains (V) with $\beta = |\log(h)|$
- 2 subdomains (V) with $\beta = \frac{10}{h}$
- △ 2 subdomains (V) with $\beta = 10|\log(h)|$
- 2 subdomains (2) with $\beta = \frac{1}{h}$
- ◆ 2 subdomains (2) with $\beta = |\log(h)|$
- 2 subdomains (2) with $\beta = \frac{10}{h}$
- ◇ 2 subdomains (2) with $\beta = 10|\log(h)|$

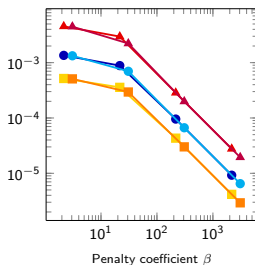
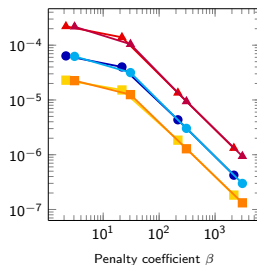


Test-case : Sphere - EFIE

Far-field relative error



Jump error at interfaces

Relative $L^2(\Gamma)$ error

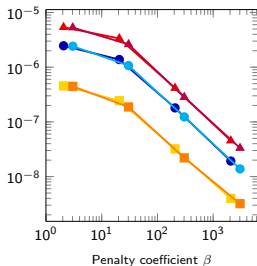
■ 150 MHz with $\beta = \frac{\xi}{h}$
■ 150 MHz with $\beta = c|\log(h)|$

● 300 MHz with $\beta = \frac{\xi}{h}$
● 300 MHz with $\beta = c|\log(h)|$

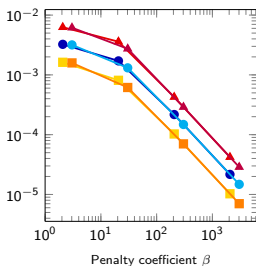
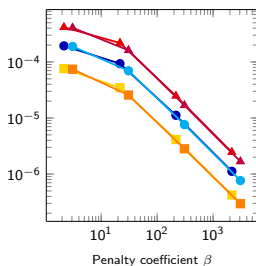
▲ 600 MHz with $\beta = \frac{\xi}{h}$
▲ 600 MHz with $\beta = c|\log(h)|$

Test-case : Cube - CFIE

Far-field relative error



Jump error at interfaces

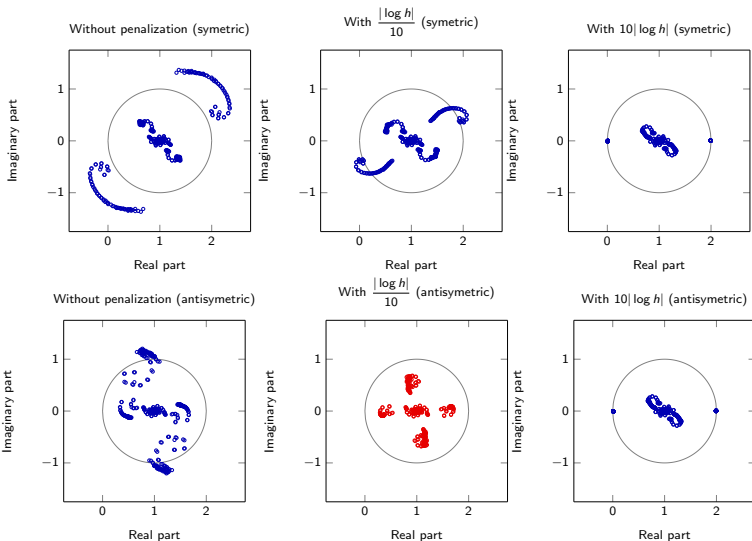
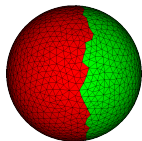
Relative $L^2(\Gamma)$ error

■ 150 MHz with $\beta = \frac{c}{h}$
■ 150 MHz with $\beta = c|\log(h)|$

● 300 MHz with $\beta = \frac{c}{h}$
● 300 MHz with $\beta = c|\log(h)|$

▲ 600 MHz with $\beta = \frac{c}{h}$
▲ 600 MHz with $\beta = c|\log(h)|$

Test-
case
Sphere
CFIE
300 MHz

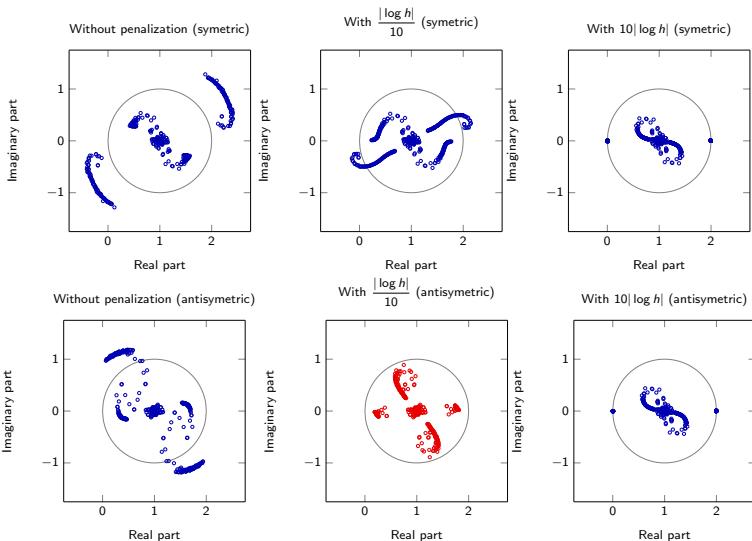
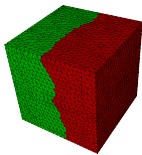


Test-
case

Cube

CFIE

300 MHz

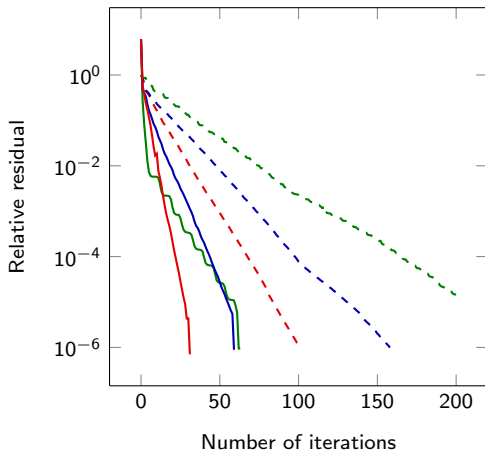
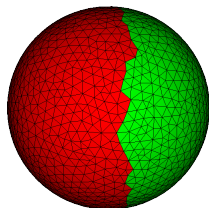


Test-case

Sphere

CFIE

300 MHz



— Antisymmetric formulation with $\beta = \frac{|\log h|}{10}$

— Symetric formulation with $\beta = \frac{|\log h|}{10}$

— Antisymmetric formulation with $\beta = 10|\log h|$

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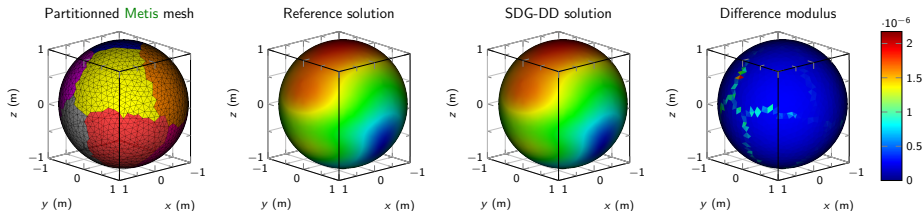
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Conclusion

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- Numerical validation using direct solver : EFIE/CFIE and symmetric/antisymmetric formulations
- First results on numerical solution using GMRES solver

Work in progress

- Iterative solution : comparison of pre-conditioners and study of eigenvalue distribution
- Scalability with respect to the number of subdomains : convergence and accuracy



Perspectives

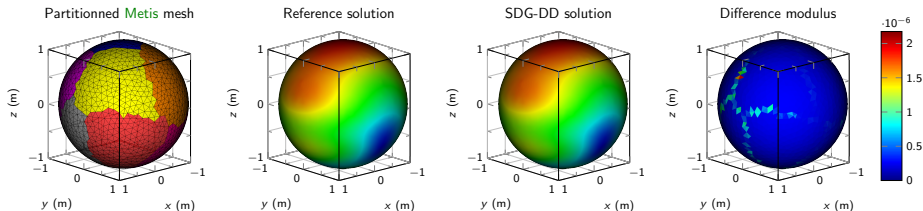
- Parallel performance and efficiency tests
- Extension to non-conformal meshes and dielectric material

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Thank you for your attention !