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# Approximate Nash equilibria in large nonconvex aggregative games

Kang Liu\*, Nadia Oudjane†, Cheng Wan‡

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## Abstract

This paper shows the existence of  $\mathcal{O}(\frac{1}{n^\gamma})$ -Nash equilibria in  $n$ -player noncooperative sum-aggregative games where the players' cost functions depend only on their own action and the average of all players' actions, is lower semicontinuous in the former, while  $\gamma$ -Hölder continuous in the latter. Neither the action sets nor the cost functions need to be convex. For an important class of sum-aggregative games which includes congestion games with  $\gamma$  being 1, a gradient-proximal algorithm is used to construct an  $\mathcal{O}(\frac{1}{n})$ -Nash equilibria with at most  $\mathcal{O}(n^3)$  iterations. These results are applied to a numerical example of demand-side management of the electricity system. The asymptotic performance of the algorithm is illustrated when  $n$  tends to infinity.

**Keywords.** Shapley-Folkman lemma, sum-aggregative games, nonconvex game, large finite game,  $\epsilon$ -Nash equilibrium, gradient-proximal algorithm, congestion game

**MSC Class** Primary: 91A06; secondary: 90C26

## 1 Introduction

This paper studies approximate pure-Nash equilibria (PNE for short) for  $n$ -player noncooperative games involving non-convexities in players' costs or action sets. The goal is to show the existence of such approximate equilibria under certain conditions, and to propose an algorithm allowing their effective calculation, in some specific cases. In particular, this paper focuses on a specific class of noncooperative games (which includes congestion games) referred to as sum-aggregative games (Selten [42], Corchón [9], Jensen [22]). The cost of each player depends on the weighted sum of the other players' decisions. These games find practical applications in various fields in political science, economics, social biology, and engineering, such as voting [32, 36], market competition [31], public goods provision [2, 15], rent seeking [11], population dynamics [17], traffic analysis [10, 27], communications network control [26, 33] and electrical system management [18, 21]. However, in these real-life situations, the players' action sets and their cost functions are often *nonconvex*.

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This work is motivated by concrete applications for which it is unreasonable to neglect the *nonconvexities* inherent to the problem. In particular, we are interested in demand-side management in electrical systems [20], where each flexible consumer is considered as a player trying to minimize her electricity bill by modulating her consumption (e.g. electric vehicle charging). We design a game where the players' bills (costs) depend on both their own consumption and the total consumption of all players in order to ensure that the Nash equilibria or  $\epsilon$ -Nash equilibria attain the goal of shaving the peak demand and smoothing the load curve of the power grid. Mathematically, we are interested in the existence and computation of Nash or  $\epsilon$ -Nash equilibria of the game. In this context, the technical constraints of flexible electrical devices (such as the battery of an electric vehicle) limit the number of feasible electrical consumption profiles. They also generally imply nonconvex action sets, for example, only allowing for discrete power consumption profiles. However, most theoretical or algorithmic results concerning games are limited to the convex framework, in which the players' action sets as well as their cost functions are convex. Our motivation is precisely to propose theoretical and algorithmic tools considering large *nonconvex* sum-aggregative games, in order to address demand-side management applications in a relevant way.

In the convex framework, a PNE is known to exist under mild regularity conditions (see, for example, Rosen [38]). Outside the convex framework, it is generally difficult to provide *existence* results for PNE and *approximation algorithms* with performance guarantees.

When players have a finite number of actions, Mondrer and Shapley [30] show that potential games, where a so-called potential function exists, admit PNEs. As a matter of fact, every finite potential game is isomorphic to a finite congestion game introduced by Rosenthal [39], where players have equal weights and non-player-specific resource cost functions. Recall that, in a congestion game, resources are shared among players, with each resource having a cost function depending on the aggregate load applied to it. However, when players have player-specific weights and/or resource cost functions, a potential function no longer necessarily exists. The existence of PNEs is then not guaranteed, except in some particular cases (cf. Milchtaich [29]). In integer-splittable congestion games where the unequally integer-weighted players can split their weight into unit-weight blocks, the existence of PNE is shown by Tran-Thanh et al. [44] for the case where a pure strategy consists in a single resource and the non-player-specific resource cost functions are convex and monotone, and by Meyers [28] for the case where the non-player-specific resource cost functions are linear. For games with discrete (but not necessarily finitely many) strategies, Sagratella [40] proposes a branching method to compute the whole solution set of PNEs. He proves the existence of PNEs for a particular class of such games and proposes an algorithm leading to one of the equilibria. However, when the players' cost functions are nonconvex and/or their action sets are nonconvex but not necessarily finite, there is no general result for the existence of PNEs.

Even in the convex setting, convergent algorithms for the computation of ( $\epsilon$ -)PNE in games are known only for some special cases. For example, for potential games or (strongly) monotone games. A common approach is to solve the variational inequality characterizing the PNEs in such games (cf. Facchinei and Pang [13] and the references therein). Scutari et al. 2014 [41] consider generic  $n$ -player games which need not be large nor aggregative, but with strongly monotone inequality characterizing the PNE. They use proximal best-reply algorithms to solve this variational inequality. Paccagnan, Kamgarpour and Lygeros 2016 [35] consider a specific convex aggregative game and use a decentralized gradient projection

algorithm to solve the strongly monotone variational inequality characterizing the PNE. Paccagnan et al. 2019 [34] studies  $\epsilon$ -PNE in convex large aggregative games with coupling constraints. Their methodology is close to ours in the sense that they only look for an  $\epsilon$ -PNE (which they call Wardrop equilibrium) instead of an exact one. They use respectively a decentralized gradient projection algorithm and a decentralized best-reply (to the aggregate term) algorithm to solve the variational inequality characterizing this Wardrop equilibrium. Compared with their work, our novelty consists in studying *nonconvex* games. We proceed by first finding an  $\epsilon$ -PNE of an auxiliary convex game. We then develop a specific algorithm to ensure that the resulting  $\epsilon$ -PNE verifies a particular condition called *the stability condition*, which is necessary for recovering an  $\epsilon$ -PNE of the original nonconvex game.

The originality of this paper is to circumvent the nonconvexity by exploiting the fact that large sum-aggregative games approach a convex framework when the number of players is large. The counterpart of this approach is to search for an *additively  $\epsilon$ -PNE* (cf. Definition 2.1) instead of an exact PNE. The main inspirations of the present work are [43] in economics and [48] in optimization. Starr 1969 [43] was interested in computing general equilibria for nonconvex competitive economy in terms of price and quantity, while Wang 2017 [48] considered large scale nonconvex separable optimization problems coupled by sum-aggregative terms. In both cases, the authors proposed to convexify the problem, taking advantage of the large number of agents or subproblems to bound the error induced by the convexification, thanks to the Shapley-Folkman Lemma (cf. Lemma 5.4). Roughly speaking, the Shapley-Folkman Lemma states that the Minkowski sum of a finite number of sets in Euclidean spaces is close to convex, when the number of sets is very large compared with their dimensions. It has been applied in nonconvex optimization for its convexification effect. Aubin and Ekeland [1] used the lemma to derive an upper bound on the duality gap in an additive, separable nonconvex optimization problem. Since, quite a few papers have extended or sharpened this result (cf. Ekeland and Temam [12], Bertsekas and coauthors [4, 7], Pappalardo [37], Kerdreux et al. [24], Bi and Tang [8]). These theoretical results have found applications in engineering, such as the large-scale unit commitment problem [3, 25] and optimization of Plug-in Electric Vehicle charging [47], optimization of multi-carrier communication systems [50], supply-chain management [46], and spatial graphical model estimation [14].

**Main contributions.** The main contribution of the present paper is threefold.

(C1) Theoretically, Proposition 2.4 and Theorem 2.7 give the existence of  $\mathcal{O}(\frac{1}{n^\gamma})$ -PNEs for  $n$ -player nonconvex sum-aggregative games where the players' cost functions depend only on their own action and the average of all the players' actions, is lower semicontinuous in the former, and  $\gamma$ -Hölder continuous in the latter. Neither the action sets nor the cost functions need to be convex.

This contribution makes use of similar approaches to Starr [43] and Wang [48], who both used the same technique. Starr (resp. Wang) first found an economy equilibrium of the convexified economy (resp. an optimum of the convexified optimization problem), then used the Shapley-Folkman lemma to find a profile of choices by the agents (resp. subproblems) that are almost all feasible except for very few agents (resp. subproblems), such that the aggregate term remains unchanged. Then, they showed that such a profile is

an approximation of the exact equilibrium (resp. optimum). Our first contribution consists of two novelties. Firstly, in Proposition 2.4 we extend this approach to nonconvex sum-aggregative games to show the existence of  $\epsilon$ -PNE, by the “disaggregation” of an exact PNE of an auxiliary convexified game. Secondly, in Theorem 2.7 we show that one can also construct an  $\epsilon$ -PNE of the nonconvex game by the “disaggregation” of an  $\epsilon$ -PNE (instead of an exact PNE) of the auxiliary convexified game provided that some *stability condition* is satisfied. This second novelty is more significant, and it is crucial for our next contribution.

(C2) Algorithmically, for an important class of  $n$ -player nonconvex sum-aggregative games including congestion games with  $\gamma$  being 1 and where action sets are compact subsets of Euclidean spaces, we present an iterative gradient-proximal algorithm to compute an  $\epsilon$ -PNE of the convexified game which satisfies the above mentioned stability condition (Proposition 3.1). Then, Theorem 3.2 ensures the performance of this algorithm which allows to recover an  $\mathcal{O}(\frac{1}{n})$ -PNE of the original nonconvex game with at most  $\mathcal{O}(n^3)$  iterations. We also provide an extremely fast, easy and distributed method to obtain an  $\mathcal{O}(\frac{1}{\sqrt{n}})$ -mixed-strategy Nash equilibrium after the same number of iterations (Corollary 3.3).

(C3) Practically, the interest of this approach is demonstrated in Section 4, where a numerical simulation with the gradient-proximal algorithm is done for a demand-side management problem involving flexible electric vehicle charging.

**Notations.** In a Euclidean space,  $\|\cdot\|$  denotes the  $l^2$ -norm. For a point  $x \in \mathbb{R}^d$  and a subset  $\mathcal{X}$  of  $\mathbb{R}^d$ ,  $d(x, \mathcal{X}) := \inf_{y \in \mathcal{X}} \{\|x - y\|\}$  is the distance from the point to the subset. For two subsets  $\mathcal{X}$  and  $\mathcal{Y}$  of  $\mathbb{R}^d$ , their Minkowski sum is the set  $\{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$ . For  $x \in \mathbb{R}^d$  and  $r \in \mathbb{R}^+$ ,  $B(x, r) := \{y \in \mathbb{R}^d \mid \|y - x\| \leq r\}$ , the  $r$ -radial ball centered on  $x$ .

For a matrix  $A \in \mathbb{R}^d \times \mathbb{R}^q$ ,  $\|\cdot\|_2$  is the 2-norm of matrices:  $\|A\|_2 := \sqrt{\lambda_{\max}(A^T A)}$  where  $A^T$  is the transpose of  $A$  and  $\lambda_{\max}(A^T A)$  stands for the largest eigenvalue of the matrix  $A^T A$ .

The proof of Proposition 2.3 and the lemmata used for the proof is given in Appendix A. All the other proofs and intermediate results are gathered in Appendix B.

## 2 Existence of $\epsilon$ -PNE in large nonconvex sum-aggregative games

### 2.1 A nonconvex sum-aggregative game and its convexification

Consider an  $n$ -player noncooperative game  $\Gamma$ . The players are indexed over  $N = \{1, 2, \dots, n\}$ . Each player  $i \in N$  has an action set  $\mathcal{X}_i \subset \mathbb{R}^d$ , which is closed and bounded but not necessarily convex. Let  $\tilde{\mathcal{X}}_i := \text{conv}(\mathcal{X}_i)$  be the convex hull of  $\mathcal{X}_i$  (which is also closed and bounded) and denote  $\mathcal{X} := \prod_{i \in N} \mathcal{X}_i$ ,  $\tilde{\mathcal{X}} := \prod_{i \in N} \tilde{\mathcal{X}}_i$ ,  $\tilde{\mathcal{X}}_{-i} := \prod_{j \in N_{-i}} \tilde{\mathcal{X}}_j$  where  $N_{-i} := N \setminus \{i\}$ . Let constant  $\Delta > 0$  be such that, for all  $i \in N$ , the compact set  $\mathcal{X}_i$  has diameter  $|\mathcal{X}_i| := \max_{x_i, y_i \in \mathcal{X}_i} \|x_i - y_i\|$  that is not greater than  $\Delta$ .

As usual, let  $x_{-i}$  denote the profile of actions of all the players except that of player  $i$ . Each player  $i$  has a real-valued cost function  $f_i$  defined on  $\mathcal{X}_i \times \tilde{\mathcal{X}}_{-i}$ , which has the following

specific form:

$$f_i(x_i, x_{-i}) := \theta_i \left( x_i, \frac{1}{n} \sum_{j \in N} A_j x_j \right), \quad \text{for any } x_i \in \mathcal{X}_i, x_{-i} \in \tilde{\mathcal{X}}_{-i}, \quad (2.1)$$

where each  $A_j$  is a  $q \times d$  matrix for all  $j \in N$ , and  $\theta_i$  is a real-valued function defined on  $\mathcal{X}_i \times \Omega$ , with  $\Omega \subset \mathbb{R}^q$  a neighborhood of  $\{\frac{1}{n} \sum_{j \in N} A_j y_j \mid y_j \in \tilde{\mathcal{X}}_j, \forall j \in N\}$ .

Let constant  $M > 0$  be such that  $\|A_i\|_2 \leq M$  for each  $i \in N$ .

Remark that  $\Gamma$  is a *weighted sum-aggregative game*. It is a generalization of *sum-aggregative games*, which correspond to the specific case where, for each  $j \in N$ ,  $A_j$  reduces to the identity matrix. For the sake of simplicity, we still call  $\Gamma$  a sum-aggregative game.

**Definition 2.1** (Additively  $\epsilon$ -pure Nash equilibrium). For a constant  $\epsilon \geq 0$ , an *additively  $\epsilon$ -pure Nash equilibrium (additively  $\epsilon$ -PNE)*  $x^\epsilon \in \mathcal{X}$  in game  $\Gamma$  is a profile of actions of the  $n$  players such that, for each player  $i \in N$ ,

$$f_i(x_i^\epsilon, x_{-i}^\epsilon) \leq f_i(x_i, x_{-i}^\epsilon) + \epsilon, \quad \text{for any } x_i \in \mathcal{X}_i.$$

If  $\epsilon = 0$ , then  $x^\epsilon$  is a *pure Nash equilibrium (PNE)*.

For the sake of simplicity, we omit “additively” whenever it causes no confusion.

For nonconvex games (where either action sets or cost functions are not convex), the existence of a PNE is not clear. This paper uses an auxiliary convexified version of the nonconvex game, which is helpful both in the proof of the existence of an  $\epsilon$ -PNE of the nonconvex game and in the construction of such an approximate PNE.

**Definition 2.2** (Convexified game and Generators). The *convexified game*  $\tilde{\Gamma}$  associated with  $\Gamma$  is a noncooperative game played by  $n$  players. Each player  $i \in N$  has an action set  $\tilde{\mathcal{X}}_i$  and a real-valued cost function  $\tilde{f}_i$  defined on  $\tilde{\mathcal{X}}$  as follows: for all  $x \in \tilde{\mathcal{X}}$ ,

$$\tilde{f}_i(x_i, x_{-i}) = \inf_{(\alpha^k)_{k=1}^{d+1} \in \mathcal{S}_d; z^k \in \mathcal{X}_i, \forall k} \left\{ \sum_{k=1}^{d+1} \alpha^k f_i(z^k, x_{-i}) \mid x_i = \sum_{k=1}^{d+1} \alpha^k z^k \right\}, \quad (2.2)$$

where  $\mathcal{S}_d := \{\alpha = (\alpha^k)_{k=1}^{d+1} \in \mathbb{R}^{d+1} \mid \forall k, \alpha^k \geq 0, \sum_{k=1}^{d+1} \alpha^k = 1\}$  denotes the probability simplex of dimension  $d$ .

A set of  $d+1$  points  $(z^k, k = 1, \dots, d+1)$  in  $\mathcal{X}_i$  that attains the minimum in (2.2) is called a *generator for  $(i, x)$* .

In the sequel, notation  $Z(i, x)$  refers to a certain generator for  $(i, x)$ , and  $\alpha(i, x)$  refers to the corresponding  $(d+1)$ -dimensional vector of coefficients.

For a lower semicontinuous (l.s.c.) function, equation (2.2) just defines its convex hull (cf. Lemma 5.5). This particular form of definition is proposed in [5].

PNE and additively  $\epsilon$ -PNE are similarly defined for the convexified game  $\tilde{\Gamma}$ .

The remainder of this subsection is dedicated to a preliminary analysis of the convexified game.

First let us introduce an assumption that will hold in this section. It ensures the existence of generators for all  $(i, x) \in N \times \mathcal{X}$  (cf. Lemma 5.5 for a proof).

**Assumption 1.**

- (1) For any player  $i \in N$ , for any  $y \in \Omega$ , function  $x_i \mapsto \theta_i(x_i, y)$  is l.s.c. on  $\mathcal{X}_i$ .  
(2) There exists constants  $H > 0, \gamma > 0$  such that, for all  $i \in N$ , for all  $x_i \in \mathcal{X}_i$ , function  $y \mapsto \theta_i(x_i, y)$  is  $(H, \gamma)$ -Hölder on  $y \in \Omega$ , i.e.

$$|\theta_i(x_i, y') - \theta_i(x_i, y)| \leq H \|y' - y\|^\gamma. \quad (2.3)$$

**Remark 2.1.** It is straightforward from Assumption 1 that  $f_i(\cdot, x_{-i})$  is l.s.c. in  $x_i \in \mathcal{X}_i$  for any fixed  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ .

According to Lemma 5.5,  $x_i \mapsto \tilde{f}_i(x_i, x_{-i})$  is convex and l.s.c on  $\tilde{\mathcal{X}}_i$ , its subdifferential exists and let it be denoted by  $\partial_i \tilde{f}_i(\cdot, x_{-i})$ . Then, for each  $x_i \in \tilde{\mathcal{X}}_i$ ,  $\partial_i \tilde{f}_i(x_i, x_{-i})$  is a nonempty convex subset of  $\mathbb{R}^d$ .

**Proposition 2.3** (Existence of PNE in  $\tilde{\Gamma}$ ). *Under Assumption 1, the convexified game  $\tilde{\Gamma}$  admits a PNE.*

*Proof.* It results from Theorem 5.3 in Appendix A. □

**Remark 2.2.** Theorem 5.3 extends Rosen's theorem on the existence of PNE in games with convex continuous cost functions [38] to the case where the cost functions are only l.s.c. instead of being continuous with respect to the players' own actions. This extension is not trivial. Our proof follows the same lines as Rosen, and relies on the use of the Kakutani's fixed point theorem [23]. In particular, we show in Lemma 5.1 that the conditions required to apply the Kakutani's theorem are fulfilled in our case of l.s.c. convex games.

The following example shows that even the continuity of  $f_i$  on  $\mathcal{X}_i$  cannot guarantee the continuity of  $\tilde{f}_i$  on  $\tilde{\mathcal{X}}_i$ , so that Rosen's theorem is not sufficient here.

Consider  $d = 3$ ,  $\mathcal{X}_i = T \cup B \cup S$  where  $T = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = 1\}$ ,  $B = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = -1\}$ ,  $S = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | x^1 = 1, x^2 = 0, -1 \leq x^3 \leq 1\}$ ;  $f_i$  is independent of  $x_{-i}$ , and  $f_i(x) = 0$  for  $x \in T \cup B$ ,  $f_i(x) = |x^3| - 1$  for  $x \in S$ . Then, for all  $x \in \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = 0\} \subset \partial \tilde{\mathcal{X}}_i$ ,  $\tilde{f}_i(x) = 0$  except for  $x^* = (1, 0, 0)$ , but  $\tilde{f}_i(x^*) = f_i(x^*) = -1$ .

## 2.2 Existence and construction of an $\epsilon$ -PNE of the nonconvex game

The following proposition shows the existence of  $\epsilon$ -PNE in the nonconvex game  $\Gamma$  and its construction from an exact PNE of the convexified game  $\tilde{\Gamma}$ .

**Proposition 2.4** (Existence of  $\epsilon$ -PNE). *Under Assumption 1, the nonconvex game  $\Gamma$  admits an  $\epsilon$ -PNE, where  $\epsilon = 2H(\frac{(\sqrt{q}+1)M\Delta}{n})^\gamma$ .*

*In particular, suppose that  $\tilde{x} \in \tilde{\mathcal{X}}$  is a PNE in  $\tilde{\Gamma}$  (which exists according to Proposition 2.3), and  $Z(i, \tilde{x})$  an arbitrary generator, for each player  $i$ , then  $x^* \in \mathcal{X}$  such that*

$$x^* \in \operatorname{argmin}_{x_i \in Z(i, \tilde{x}), i \in N} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i x_i \right\|^2, \quad (2.4)$$

*is such an  $\epsilon$ -PNE of the nonconvex game  $\Gamma$ .*

*Sketch of the proof:* By the definition of PNE in  $\tilde{\Gamma}$ ,  $\tilde{x}_i$  is a best response to  $\tilde{x}_{-i}$  in terms of  $\tilde{f}_i$ . By Lemma 5.5, all the points in  $Z(i, \tilde{x})$  are also best replies to  $\tilde{x}_{-i}$  in terms of  $f_i$ .

We then use the Shapley-Folkman lemma (Lemma 5.4) to disaggregate  $\frac{1}{n} \sum_i A_i \tilde{x}_i$  over the sets  $Z(i, \tilde{x})$  to obtain a feasible profile  $x^*$ .

Finally, we can show that  $\frac{1}{n} \sum_i A_i x_i^* \approx \frac{1}{n} \sum_i A_i \tilde{x}_i$  and  $x_i^*$  is (almost) a best response to  $\frac{1}{n} \sum_i A_i x_i^*$ .

From an algorithmic point of view, a PNE is not always easy or fast to compute for the convexified  $\tilde{\Gamma}$ , even though its existence is guaranteed. Even when we have a convergent algorithm, the outputs of the algorithm at each iteration provide only approximations of the exact PNE which may constitute  $\epsilon$ -PNE but rarely exact PNE. Then, the question that naturally arises is whether the idea above is still valid if  $\tilde{x}$  is only an  $\epsilon$ -PNE of  $\tilde{\Gamma}$ , i.e.  $\tilde{x}_i$  is an  $\epsilon$ -best response to  $\tilde{x}_{-i}$  in terms of  $\tilde{f}_i$ . More explicitly, are points in  $Z(i, \tilde{x})$  still  $\epsilon$ -best replies to  $\tilde{x}_{-i}$  in terms of  $f_i$ ? The answer is YES, if the  $\epsilon$ -PNE  $\tilde{x}$  of the convexified game  $\tilde{\Gamma}$  satisfies a more demanding condition, introduced by the following definition.

**Definition 2.5** (Stability condition). In game  $\tilde{\Gamma}$ , a point  $\tilde{x} \in \tilde{\mathcal{X}}$  is said to satisfy the  $\eta$ -stability condition with respect to  $(Z(i, \tilde{x}))_i$  if, for each player  $i$ ,  $\tilde{f}_i(x_i, \tilde{x}_{-i}) \leq \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) + \eta$  for all  $x_i \in Z(i, \tilde{x})$ .

A point  $\tilde{x}$  is said to satisfy the  $\eta$ -stability condition if it satisfies the  $\eta$ -stability condition with respect to a certain generator profile  $(Z(i, \tilde{x}))_i$ .

A point  $\tilde{x}$  is said to satisfy the full  $\eta$ -stability condition if it satisfies the  $\eta$ -stability condition with respect to any generator profile  $(Z(i, \tilde{x}))_i$ .

The stability condition of  $\tilde{x}$  with respect to  $(Z(i, \tilde{x}))_i$  means that, for each player  $i$ , her cost is only slightly increased if her choice is unilaterally perturbed within the convex hull of the generator  $Z(i, \tilde{x})$ .

According to Lemma 5.5, a PNE of  $\tilde{\Gamma}$  satisfies the full 0-stability condition.

A sufficient condition for the  $\eta$ -stability of  $\tilde{x}$  is given by Lemma 2.6.

**Lemma 2.6.** Under Assumption 1, for any action profile  $\tilde{x} \in \tilde{\mathcal{X}}$ , for any player  $i$ , if there is a generator  $Z(i, \tilde{x})$  and  $h \in \partial_i \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i})$ , such that,

$$\langle h, x_i - \tilde{x}_i \rangle \geq -\eta \|x_i - \tilde{x}_i\|, \quad \forall x_i \in \text{conv } Z(i, \tilde{x}). \quad (2.5)$$

then,

$$|\tilde{f}_i(x_i, \tilde{x}_{-i}) - \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i})| \leq \eta \|x_i - \tilde{x}_i\|, \quad \text{for all } x_i \in Z(i, \tilde{x}).$$

In particular,  $\tilde{x}$  satisfies the  $\eta\Delta$ -stability condition with respect to  $(Z(i, \tilde{x}))_i$ .

The following theorem shows how to construct an  $\mathcal{O}(\frac{1}{n^\gamma})$ -PNE of the original nonconvex game  $\Gamma$ , when we know an  $\epsilon$ -PNE of the convexified game  $\tilde{\Gamma}$  satisfying the  $\eta$ -stability condition and the associated generator profile.

**Theorem 2.7** (Construction of  $\epsilon$ -PNE of  $\Gamma$ ). Under Assumption 1, suppose that  $\tilde{x} \in \tilde{\mathcal{X}}$  is an  $\epsilon$ -PNE in  $\tilde{\Gamma}$  which satisfies the  $\eta$ -stability condition with respect to a specific generator profile  $(Z(i, \tilde{x}))_i$ . Let  $x^* \in \mathcal{X}$  be such that

$$x^* \in \underset{x_i \in Z(i, \tilde{x}), i \in N}{\operatorname{argmin}} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i x_i \right\|^2. \quad (2.6)$$

Then,  $x^*$  is a  $\tilde{\epsilon}$ -PNE of the nonconvex game  $\Gamma$ , where  $\tilde{\epsilon} = \epsilon + \eta + 2H(\frac{(\sqrt{\eta}+1)M\Delta}{n})^\gamma$ .

## 2.3 A distributed randomized “Shapley-Folkman disaggregation”

Once an exact PNE or an  $\epsilon$ -PNE satisfying the  $\eta$ -stability condition,  $\tilde{x}$ , of the convexified game  $\tilde{\Gamma}$  is obtained, as well as the associated generator profile  $(Z(i, \tilde{x}))_i$ , we would like to find an  $\tilde{\epsilon}$ -PNE of the nonconvex game  $\Gamma$ , whose existence is shown by Proposition 2.4 and Theorem 2.7. However, solving (2.6) is generally hard (cf. Udell and Boyd [45] for such a “Shapley-Folkman disaggregation” in a particular setting of optimization). In this section, we present a method to compute an  $\tilde{\epsilon}$ -mixed-strategy Nash equilibrium (MNE) in a distributed way, based on the known  $\epsilon$ -PNE of  $\tilde{\Gamma}$ , its associated generator profile and coefficients. The algorithm is qualified “distributed” because, for each player  $i$ , it computes her mixed-strategy  $\mu_i$  from the information of  $\tilde{x}_i$ , the generator  $Z(i, \tilde{x})$  and the coefficients  $\alpha(i, x)$  only.

**Proposition 2.8.** *Under Assumption 1, suppose that  $\tilde{x} \in \tilde{\mathcal{X}}$  is an  $\epsilon$ -PNE of  $\tilde{\Gamma}$  satisfying the  $\eta$ -stability condition with respect to  $(Z(i, \tilde{x}))_i$ , and each player  $i$  plays a mixed strategy  $\tilde{\mu}_i$  independently, i.e. a random action  $X_i$  following distribution  $\tilde{\mu}_i$  over  $\mathcal{X}_i$ , defined by  $\mathbb{P}(X_i = x_i^l) = \alpha_i^l$ , where  $x_i^l \in Z(i, x)$  for  $l = 1, \dots, d+1$ , and  $(\alpha_i^l)_{l=1}^{d+1} = \alpha(i, x)$  is the corresponding coefficient. Then, for  $\gamma \leq 1$ ,  $\tilde{\mu} = (\tilde{\mu}_i)_i$  is a  $\tilde{\epsilon}$ -MNE of the nonconvex game  $\Gamma$ , where  $\tilde{\epsilon} = \epsilon + \eta + 2H(\frac{(\sqrt{n}+1)M\Delta}{n})^\gamma$ , in the sense that*

$$\mathbb{E}[f_i(X_i, X_{-i})] \leq \mathbb{E}[f_i(x_i, X_{-i})] + \tilde{\epsilon}, \quad \forall x_i \in \mathcal{X}_i.$$

**Remark 2.3.** Note that this is not an algorithm for the players to attain an  $\epsilon$ -PNE/MNE in the nonconvex game in a decentralized adaptation/learning process, but a distributed, randomized disaggregation algorithm to recover an  $\tilde{\epsilon}$ -MNE of  $\Gamma$  from a known  $\epsilon$ -PNE of  $\tilde{\Gamma}$  satisfying the  $\eta$ -stability condition and its generators.

Besides, when  $\tilde{x}$  is an exact PNE of  $\tilde{\Gamma}$ , all the generators of  $\tilde{x}$  can be used in this algorithm. On the contrary, when  $\tilde{x}$  is only an  $\epsilon$ -PNE of  $\tilde{\Gamma}$ , we need a specific profile of generators  $(Z(i, x))_i$ , with respect to which  $\tilde{x}$  satisfies the  $\eta$ -stability condition. It is not evident to find such a generator profile. However, in the next section, we provide an algorithm to compute, for a specific class of games, an  $\epsilon$ -PNE of  $\tilde{\Gamma}$  satisfying the full  $\eta$ -stability condition (i.e. with respect to any profile of generators of  $\tilde{x}$ ). In that case, any profile of generators can be used in the algorithm of Proposition 2.8.

Finally, note that the estimated error  $\tilde{\epsilon}$  for the distributed randomized approximate MNE in Proposition 2.8 is larger than the estimated error  $\tilde{\epsilon}$  for the approximate PNE  $x^*$  in Theorem 2.7.

## 3 Computing $\epsilon$ -equilibria for large nonconvex congestion games

### 3.1 Nonconvex generalized congestion game

An extensively studied class of sum-aggregative games are congestion games. In this section, we present an iterative algorithm to compute an  $\omega(K, n)\Delta$ -PNE of the convexification of a specific congestion game, where  $\omega(K, n)$  tends to zero when both the number of players  $n$  and the number of iterations  $K$  tend to  $+\infty$ , while  $\frac{n}{K}$  tends to zero. Note that any algorithm returning an approximate PNE of the convexified game will not necessarily ensure that it

verifies the stability condition. The proposed algorithm is of particular interest because we can show that the iterates provide an  $\omega(K, n)\Delta$ -PNE of the convexified game which satisfies the full  $\omega(K, n)\Delta$ -stability condition (cf. Proposition 3.1). Then, taking  $K \sim \mathcal{O}(n^3)$ , Theorem 3.2 shows that one can recover a  $\mathcal{O}(\frac{1}{n})$ -PNE of the original nonconvex congestion game from this  $\omega(K, n)\Delta$ -PNE of the convexified game.

Consider a generalized congestion game where each player  $i \in N$  has an action set  $\mathcal{X}_i \subset \mathbb{R}^d$  and a cost function of the following form:

$$\begin{aligned} f_i(x_i, x_{-i}) &= \left\langle g\left(\frac{1}{n} \sum_{j \in N} a_j x_j\right), x_i \right\rangle + h_i\left(\frac{1}{n} \sum_{j \in N} a_j x_j\right) + r_i(x_i) \\ &= \sum_{t=1}^d g_t\left(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}\right) x_{i,t} + h_i\left(\frac{1}{n} \sum_{j \in N} a_j x_j\right) + r_i(x_i), \end{aligned} \quad (3.1)$$

Suppose that the following assumptions hold on  $\mathcal{X}_i$ ,  $(a_j)_{j \in N} \in \mathbb{R}^n$ ,  $g_t$ 's,  $h_i$ 's and  $r_i$ 's.

**Assumption 2.**

- There exist constants  $m > 0$  and  $M > 0$ , such that  $m \leq a_i \leq M$  for all  $i \in N$ .
- For  $t = 1, \dots, d$ , function  $g_t : \mathbb{R} \rightarrow \mathbb{R}$  is  $L_{g_t}$ -Lipschitz continuous and nondecreasing on a neighborhood of  $[D_1, D_2]$ , where constants  $D_1$  and  $D_2$  are such that  $D_1 \leq \min_{t=1, \dots, d; x \in \tilde{\mathcal{X}}} \frac{1}{n} \sum_{j \in N} a_j x_{j,t} \leq \max_{t=1, \dots, d; x \in \tilde{\mathcal{X}}} \frac{1}{n} \sum_{j \in N} a_j x_{j,t} \leq D_2$ .
- For each  $i \in N$ , function  $h_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $L_{h_i}$ -Lipschitz continuous on  $[D_1, D_2]^d$ .
- Players' local cost functions  $r_i : \mathbb{R}^d \rightarrow \mathbb{R}$  are uniformly bounded, i.e. there exists constant  $B_r > 0$  such that, for all  $i \in N$  and all  $x_i \in \mathcal{X}_i$ ,  $|r_i(x_i)| \leq B_r$ .

**Notation.** Let constant  $\Delta := \max\{\max_{i \in N} \max_{x_i \in \tilde{\mathcal{X}}_i} \|x_i\|, \max_{i \in N} |\tilde{\mathcal{X}}_i|\}$ . Let  $L_g := \max_{1 \leq t \leq d} L_{g_t}$ ,  $L_h := \max_{i \in N} L_{h_i}$ ,  $B_g := \max_{1 \leq t \leq d, D_1 \leq s \leq D_2} |g_t(s)|$ .

The convexification of  $\Gamma$  is rather complicated to compute in the general case. Let us first introduce an auxiliary game which is very close to  $\Gamma$  but whose convexification is easier to obtain.

Fix arbitrarily  $x_i^+ \in \mathcal{X}_i$  for each player  $i \in N$ . The auxiliary game  $\bar{\Gamma}$  is defined as follows: the player set and each player's action set are the same as in  $\Gamma$ , but player  $i$ 's cost function is, for all  $x_i \in \mathcal{X}_i$  and all  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ ,

$$\bar{f}_i(x_i, x_{-i}) := \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j + \frac{1}{n} a_i x_i^+\right), x_i \right\rangle + r_i(x_i).$$

The original game  $\Gamma$  can be approximated by the auxiliary game  $\bar{\Gamma}$  because their equilibria are very close to each other as Lemma 5.8 in Appendix B shows.

For any fixed  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ ,  $\bar{f}_i(\cdot, x_{-i})$  is composed of a linear function of  $x_i$  and a local function of  $x_i$ . By abuse of notation, let us still use  $\bar{f}_i$  to denote its convexification on  $\tilde{\mathcal{X}}_i$ . More explicitly,

$$\bar{f}_i(x_i, x_{-i}) := \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j + \frac{1}{n} a_i x_i^+\right), x_i \right\rangle + \tilde{r}_i(x_i), \quad (3.2)$$

where  $\tilde{r}_i$  is the convexification of  $r_i$  defined on  $\tilde{\mathcal{X}}_i$  in the same way as  $\tilde{f}_i$ .

By abuse of notation, let  $\tilde{\Gamma}$  denote the convexification of  $\Gamma$  on  $\tilde{\mathcal{X}}$ .

### 3.2 A gradient-proximal algorithm

This subsection presents a gradient-proximal algorithm based on the block coordination proximal algorithm introduced by Xu and Yin [49] to construct an  $\mathcal{O}(\frac{1}{n})$ -PNE of  $\tilde{\Gamma}$  that satisfies the full  $\mathcal{O}(\frac{1}{n})$ -stability condition.

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**Algorithm 1:** Gradient-proximal algorithm for  $\tilde{\Gamma}$

---

**Initialization:** choose initial point  $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \tilde{\mathcal{X}}$   
**for**  $k = 1, 2, \dots$  **do**  
    **for**  $i = 1, 2, \dots, n$  **do**  
        
$$x_i^k = \underset{x_i \in \tilde{\mathcal{X}}_i}{\operatorname{argmin}} \left\langle g \left( \frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1} \right), x_i - x_i^{k-1} \right\rangle + \frac{a_i L_g}{2n} \|x_i - x_i^{k-1}\|^2 + \tilde{r}_i(x_i) \quad (3.3)$$
  
    **end**  
    **if** *stopping criterion is satisfied* **then**  
        | return  $(x_1^k, x_2^k, \dots, x_n^k)$ . Break.  
    **end**  
**end**

---

**Remark 3.1.** This is a decentralized-coordinated type algorithm. The coordinator needs to know the current choices of the players and  $(a_i)_i$  to compute  $g(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1})$ . The value of the vector  $g(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1})$  is sent to player  $i$  by the coordinator in iteration  $k$ , when it is her turn to compute. No detailed information of the other players' choices is revealed. Receiving this value, player  $i$  uses her local information, i.e.  $a_i$  and  $\tilde{r}_i$ , to update her choice according to (3.3), then sends it to the coordinator.

**Proposition 3.1.** *Under Assumption 2, for  $K \in \mathbb{N}^*$ , there is  $k^* \leq K$  such that  $x^{k^*}$  is an  $\omega(K, n)\Delta$ -PNE of game  $\tilde{\Gamma}$  which satisfies the full  $\omega(K, n)\Delta$ -stability condition, where*

$$\omega(K, n) = \frac{\sqrt{2CL_g M}}{m} \sqrt{\frac{n}{K}} + \frac{2L_g M \Delta}{n}, \quad (3.4)$$

where  $C = (d\Delta L_g + 2B_r)M$ .

*In particular, if constant  $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$  for some constant  $\delta > 0$ , then, there exists some  $k^* \leq K$  such that  $x^{k^*}$  is an  $L_g M \Delta (n^{-\delta} + 2\Delta n^{-1})$ -PNE of game  $\tilde{\Gamma}$  satisfying the full  $L_g M \Delta (n^{-\delta} + 2\Delta n^{-1})$ -stability condition.*

**Theorem 3.2.** *Under Assumption 2, for constant  $\delta > 0$  and integer  $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$ , let  $x^* \in \mathcal{X}$  be the pure-strategy profile generated by (2.6), where  $\tilde{x}$  is replaced by  $x^{k^*}$  in Proposition 3.1. Then,  $x^*$  is a  $(2L_g M \Delta (n^{-\delta} + \frac{(q+4)\Delta}{n}) + \frac{L_h M \Delta}{n})$ -PNE of the nonconvex game  $\Gamma$ .*

In the case where a “Shapley-Folkman” disaggregation of  $x^{k*}$  is not easy to obtain, one can use the distributed randomized disaggregation method introduced in Section 2.3 to immediately obtain a  $\tilde{\epsilon}$ -MNE, where  $\tilde{\epsilon}$  is given by the following corollary. However, the quality of approximation is less good than a “Shapley-Folkman” disaggregation.

**Proposition 3.3.** *Under Assumption 2, for constant  $\delta > 0$  and integer  $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$ , let  $\tilde{\mu} = (\tilde{\mu}_i)_i$  be a profile of independent mixed strategies defined as in Lemma 5.7, where  $\tilde{x}$  is replaced by  $x^{k*}$  in Proposition 3.1. Then,  $\tilde{\mu}$  is a  $(2L_g M \Delta (n^{-\delta} + \frac{(\sqrt{n}+4)\Delta}{n}) + \frac{L_h M \Delta}{n})$ -MNE of the nonconvex game  $\Gamma$ .*

## 4 Numerical example

Here is an example of flexible electric vehicle charging control whose convex version is studied by Jacquot et al. [20].

One day is divided into peak hours (e.g. 6 am–10 pm) and off-peak hours. The electricity production cost function for total flexible load  $\ell^P$  and  $\ell^{OP}$  at peak and off-peak hours are respectively  $C^P(\ell^P) = \alpha_0^P \ell^P + \beta_0 (\ell^P)^2$  and  $C^{OP}(\ell^{OP}) = \alpha_0^{OP} \ell^{OP} + \beta_0 (\ell^{OP})^2$ , where  $\alpha_0^P > \alpha_0^{OP} > 0$  and  $\beta_0 > 0$ . Player  $i$ ’s action is denoted by  $\ell_i = (\ell_i^P, \ell_i^{OP})$ , where  $\ell_i^P$  (resp.  $\ell_i^{OP}$ ) is the peak (resp. off-peak) consumption of player  $i$ . Player  $i$ ’s electricity bill is then defined by

$$b_i(\ell_i, \ell_{-i}) := \frac{C^P(\ell^P)}{\ell^P} \ell_i^P + \frac{C^{OP}(\ell^{OP})}{\ell^{OP}} \ell_i^{OP},$$

where  $\ell^P = \sum_i \ell_i^P$ ,  $\ell^{OP} = \sum_i \ell_i^{OP}$ . Player  $i$ ’s cost is then defined by

$$\phi_i(\ell_i, \ell_{-i}) = b_i(\ell_i, \ell_{-i}) + \gamma_i \|\ell_i - \ell_i^{ref}\|^2 \quad (4.1)$$

where  $\gamma_i$  indicates her sensitivity to the deviation from her preference  $\ell_i^{ref}$ . In [20], the action set of player  $i$  is the convex compact set  $S_i = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \underline{\ell}_i^P \leq \ell_i^P \leq \overline{\ell}_i^P, \underline{\ell}_i^{OP} \leq \ell_i^{OP} \leq \overline{\ell}_i^{OP}\}$ , where  $e_i$  stands for the energy required by player  $i$  to charge an electric vehicle battery and  $\underline{\ell}_i^P$  and  $\overline{\ell}_i^P$  (resp.  $\underline{\ell}_i^{OP}$  and  $\overline{\ell}_i^{OP}$ ) are the minimum and maximum power consumption for player  $i$  during peak (resp. off-peak) hours. However, for various reasons such as finite choices for charging power, or battery protection which demands that the charging must be interrupted as infrequently as possible, the players’ action sets can be nonconvex. For example, in this paper a particular case where the nonconvex action set  $S_i^{NC} = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \ell_i^P \in \{\underline{\ell}_i^P, \overline{\ell}_i^P\}\}$  is adopted for numerical simulation.

Let us apply Algorithm 1 to this game. The asymptotic performance of the algorithm for large  $n$  is illustrated. To avoid rescaling cost functions for each  $n$ , *multiplicatively  $\epsilon$ -PNE* defined below are considered instead of additively  $\epsilon$ -PNE defined by Definition 2.1.

**Definition 4.1** (Multiplicatively  $\epsilon$ -PNE). For a constant  $\epsilon \geq 0$ , a *multiplicatively  $\epsilon$ -PNE*  $x^\epsilon \in \mathcal{X}$  in game  $\Gamma$  is a profile of actions of the  $n$  players such that, for each player  $i \in N$ ,

$$f_i(x_i^\epsilon, x_{-i}^\epsilon) - \inf_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) \leq \epsilon \left( \sup_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) - \inf_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) \right).$$

If  $\epsilon = 0$ , then  $x^\epsilon$  is a PNE.

For  $x \in \mathcal{X}$ ,  $\epsilon(x) := \min\{\epsilon \geq 0 \mid x \text{ is a multiplicatively } \epsilon\text{-PNE}\}$  is called the *relative error* of  $x$ .

First, game (4.1) is reformulated with uni-dimensional actions. For simplification, suppose that all the players have the same type of EV (Nissan Leaf 2018) with battery capacity  $e$ , and two charging rate levels  $p_{\min}$  and  $p_{\max}$ . The total consumption of player  $i$  is denoted by  $e_i$  and determined by a parameter  $\tau_i$  as follows:  $e_i = (1 - \tau_i)e = \ell_i^P + \ell_i^{OP}$ , where  $\tau_i \in [0, 1]$  signifies the player's remaining proportion of energy in her battery when arriving at home. Let  $x_i := \frac{\ell_i^P}{e}$  denote player  $i$ 's strategy in the following reformulation of game (4.1):

$$\tilde{f}_i^{(n)}(x_i, x_{-i}) = \tilde{b}_i^{(n)}(x_i, x_{-i}) + \tilde{\gamma}_i \|x_i - x_i^{ref}\|^2, \quad (4.2)$$

where  $\tilde{\gamma}_i$  indicates how player  $i$  values the deviation from her preferred consumption profile and is uniformly set to be  $ne$  for simplification, and

$$\begin{aligned} \tilde{b}_i^{(n)}(x_i, x_{-i}) &= (\alpha_0^P + \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j) \ell_i^P + (\alpha_0^{OP} + \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) (1 - x_j)) \ell_i^{OP} \\ &= e(1 - \tau_i) \left[ (\alpha_0^P - \alpha_0^{OP} - \beta_0 ne + 2\beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j) x_i \right. \\ &\quad \left. + \alpha_0^{OP} + \beta_0 ne - \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j \right]. \end{aligned}$$

The nonconvex action set of player  $i$ , introduced in Section 1 as  $S_i^{NC} = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \ell_i^P \in \{\ell_i^P, \overline{\ell_i^P}\}\}$ , is now translated into  $\mathcal{X}_i = \{\underline{x}_i, \bar{x}_i\} \subset [0, 1]$ , where  $\underline{x}_i$  and  $\bar{x}_i$  correspond respectively to charging at  $p_{\min}$  and  $p_{\max}$ .

By extracting the common factor  $ne(1 - \tau_i)$ , player  $i$ 's cost function becomes

$$f_i^{(n)}(x_i, x_{-i}) := \left\langle g^{(n)}\left(\frac{1}{n} \sum_{j=1}^n (1 - \tau_j) x_j\right), x_i \right\rangle + h^{(n)}\left(\frac{1}{n} \sum_{j=1}^n (1 - \tau_j) x_j\right) + \frac{r_i^{(n)}(x_i)}{1 - \tau_i}, \quad (4.3)$$

where  $g^{(n)}(y) := \frac{\alpha_0^P - \alpha_0^{OP}}{n} + \beta_0 e(2y - 1)$ ,  $h^{(n)}(y) := \frac{\alpha_0^{OP}}{n} + \beta_0 e(1 - y)$ , and  $r_i^{(n)}(y) := \|y - x_i^{ref}\|^2$  for  $y \in \mathbb{R}$ , where  $\alpha_0^P = -4.17 + 0.59 \times 12n$  (€/kWh),  $\alpha_0^{OP} = -4.17 + 0.59 \times 8n$  (€/kWh),  $\beta_0 = 0.295$  (€/kWh<sup>2</sup>) according to Jacquot et al. [20].

**Simulation parameters** The peak hours are between 6 am and 10 pm while the remaining hours of the day are off-peak. The battery capacity of Nissan Leaf 2018 is  $e = 40$  kWh. The discrete action set of player  $i$  is determined as follows. The players' arrival time at home is independently generated according to a *Von Mises* distribution with parameter  $\kappa = 1$  between 5 pm and 7 pm. Their departure time is independently generated according to a *Von Mises* distribution with parameter  $\kappa = 1$  between 7 am and 9 am. The proportion  $\tau_i$  of energy in the battery when a player arrives at home is independently generated according to a *Beta* distribution with parameter  $\beta(2, 5)$ . Once a player arrives at home, she starts charging at one of the two alternative levels,  $p_{\min} = 3.7$  kW or  $p_{\max} = 7$  kW. This power level is maintained until the energy requirement  $e_i$  is reached. The arrival and departure time parameters are defined such that the problem is always feasible i.e. the energy requirement  $e_i$  can always be reached during the charging period by choosing power level  $p_{\max}$ . Players are all assumed to prefer charging as fast as possible, so that  $x_i^{ref} = \bar{x}_i$  for all  $i$ . Fifty instances of the problem are considered for the numerical test. They are obtained

by independent simulations of those parameters (players' arrival and departure times and remaining energy when arriving at home).

Algorithm 1 is applied to EV charging game  $\Gamma^{(n)}$  (4.3) for  $n = 2^s$ ,  $s = 1, \dots, 15$ . For each game  $\Gamma^{(n)}$ , for each iteration  $k$  of the algorithm, let  $x^{(n),k}$  denote the  $k^{th}$  iterate of Algorithm 1 applied to game  $\Gamma^{(n)}$ . Then, the relative error  $\epsilon^{(n),k}$  of  $x^{(n),k}$  is given by

$$\epsilon^{(n),k} := \min \left\{ \epsilon \geq 0 \mid f_i^{(n)}(x_i^{(n),k}, x_{-i}^{(n),k}) - \inf_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) \right. \\ \left. \leq \epsilon \left( \sup_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) - \inf_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) \right) \right\}.$$

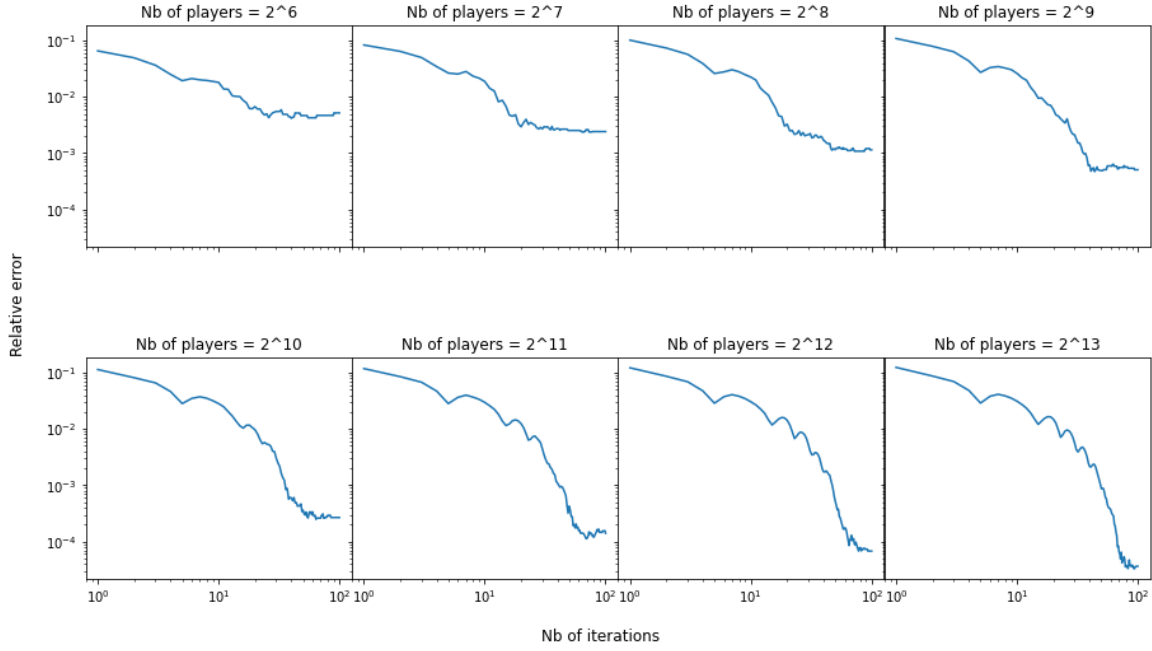


Figure 1: Log-log chart of relative error  $\epsilon^{(n),k}$  (averaged over fifty instances of the problem) as a function of the number of iterations  $k$  (for a fixed number of players  $n = 2^6, 2^7, \dots, 2^{13}$ ).

As Figure 1 shows, the relative error decreases with the number of iterations to a certain limit. This limiting relative error decreases with the number of players  $n$ . This observation is consistent with equation (3.4) in Proposition 3.1. For Figure 2, according to Proposition 3.1, when the iteration number  $k$  is fixed, due to the domination term of  $\frac{2L_g M \Delta}{n}$  in equation (3.4) when  $n$  is small,  $\epsilon^{(n),k}$  first decreases linearly in  $n$  before reaching a certain threshold. After that,  $\frac{\sqrt{2CL_g M}}{m} \sqrt{\frac{n}{k}}$  dominates the relative error value so that  $\epsilon^{(n),k}$  may increase in  $n$ . The threshold itself increases with the iteration number  $k$ . This is exactly what Figure 2 shows.

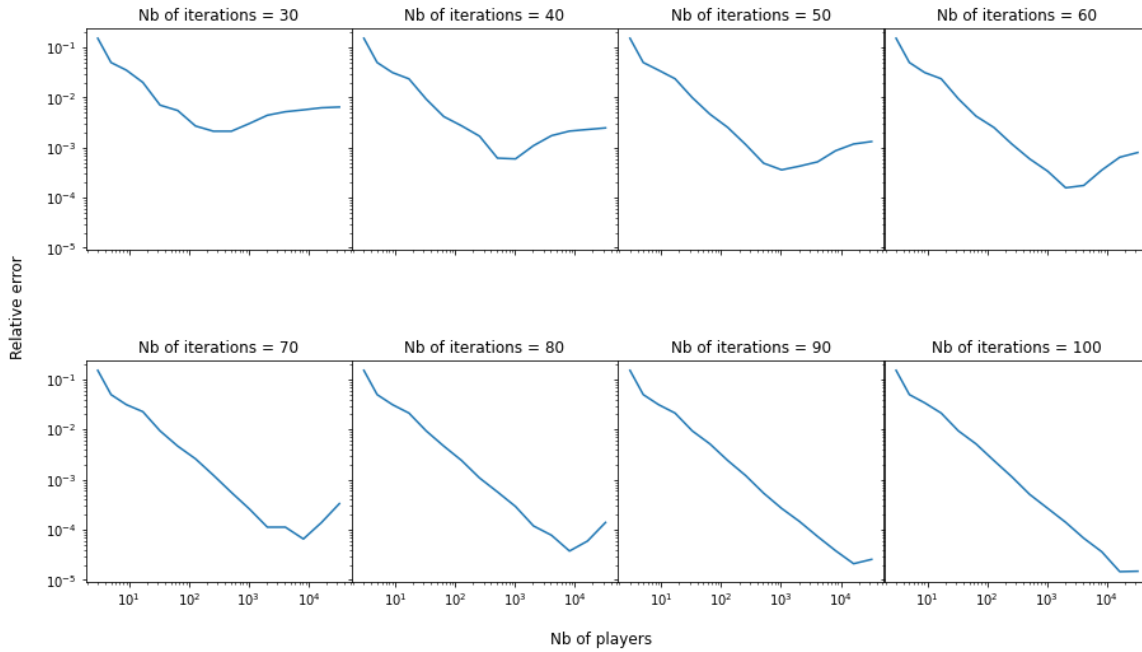


Figure 2: Log-log chart of relative error  $\epsilon^{(n),k}$  (averaged over fifty instances of the problem) as a function of the number of players  $n$  (for a fixed number of iterations  $k = 30, 40, \dots, 90, 100$ ).

## 5 Perspectives

**Distributed and randomized “Shapley-Folkman disaggregation”.** In Section 2.3, a distributed disaggregating method is introduced to obtain a randomized “Shapley-Folkman disaggregation” for the case  $\gamma \leq 1$ . It is extremely fast and easy to carry out: once an  $\epsilon$ -PNE  $\tilde{x}$  is obtained for the convexified game as well as the profile of generators  $(Z(i, \tilde{x}))_i$ , each player  $i$  chooses randomly one feasible action that is in  $Z(i, \tilde{x})$ , according to the distribution law  $\alpha(i, x)$ . This method renders an  $\mathcal{O}(\frac{1}{\sqrt{n^\gamma}})$ -MNE, with the error vanishing when the number of players going to infinity. However, even if an  $\mathcal{O}(\frac{1}{n^\gamma})$ -PNE can be difficult to obtain by an exact “Shapley-Folkman disaggregation”, especially if a large, centralized program is involved, for example, to solve (2.6), it would be desirable to find other algorithms to find better approximations of Nash equilibria of the nonconvex game. Distributed and randomized algorithms are appealing because they can be faster to carry out, needing less coordination hence more tractable, and taking advantage of the law of large numbers when  $n$  is large.

**Aggregation and disaggregation of clusters.** In the setting of power grid management, flexible agents can be regrouped into clusters, and each cluster is commanded by a so-called aggregator. The EV charging game considered in this paper then takes place between the relatively few aggregators instead of the individuals. This “aggregate game” is different from the EV charging game in the paper, as the individuals are no longer autonomous but are commanded by their respective aggregators in their choice of charging

behavior. One can build an aggregate model for each aggregator by defining his action set as the set of aggregate actions of the individuals in his cluster, and his cost function as an aggregate of the individuals' costs. When the clusters are large, it is possible to show, with the help of the Shapley-Folkman Lemma, that the aggregators' action sets and cost functions are almost convex. Then, the game admits an  $\epsilon$ -PNE (via Rosen's existence theorem), and its computation could be relatively easier owing to the small number of players. However, each aggregator then has to reconstruct for each individual under his control a feasible action consistent with their aggregate action at the equilibrium of this "aggregate game". When the constraints of each flexible individuals are non-convex, this aggregation/disaggregation approach can be rather difficult to implement. An original technique based on the Shapley-Folkman Lemma is proposed in Hreinsson et al. 2021 [19], within the optimization framework, with applications to the management of consumption flexibilities in power systems.

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## Appendix A: PNE in l.s.c. convex games

**Lemma 5.1.** *Let  $R$  be a nonempty convex compact set in  $\mathbb{R}^n$ . If real valued function  $\rho(x, y)$  defined on  $R \times R$  is continuous in  $x$  on  $R$  for any fixed  $y$  in  $R$ , l.s.c. in  $(x, y)$  on  $R \times R$ , and convex in  $y$  on  $R$  for any fixed  $x$  in  $R$ , then the set-valued map  $\zeta : R \rightarrow R$ ,  $x \mapsto \zeta(x) = \operatorname{argmin}_{z \in R} \rho(x, z)$  has a fixed point.*

*Proof.* The Kakutani fixed-point theorem [23] will be applied for the proof. First, let us show that  $\zeta$  is a Kakutani map, i.e. (i)  $\Gamma$  is upper semicontinuous (u.s.c.) in set map sense and (ii) for all  $x \in R$ ,  $\zeta(x)$  is non-empty, compact and convex.

(i) Fix  $x \in R$ . On the one hand, since  $\rho(x, y)$  is convex w.r.t  $y$ ,  $\zeta(x)$  is convex. On the other hand,  $\rho(x, y)$  is l.s.c in  $y$ , while  $R$  is compact, hence  $\rho(x, y)$  can attain its minimum w.r.t  $y$  and  $\zeta(x)$  is thus nonempty. Besides, since  $\rho$  is l.s.c.,  $\zeta(x) = \{y | \rho(x, y) \leq \min_{z \in R} \rho(x, z)\}$  is a closed subset of compact set  $R$ , hence it is compact.

(ii) Recall that the set-valued map  $\zeta$  is u.s.c. if, for any open set  $w \subset R$ , set  $\{x \in R | \zeta(x) \subset w\}$  is open.

Let us first show by contradiction that, for arbitrary  $x_0 \in R$ , for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $z \in B(x_0, \delta)$ ,  $\zeta(z) \subset \zeta(x_0) + B(0, \epsilon)$ . If it is not true, then there exists  $\epsilon_0 > 0$  and, for all  $n \in \mathbb{N}^*$ , point  $z_n \in B(x_0, \frac{1}{n})$  such that there exists  $y_n \in \zeta(z_n)$  with  $d(y_n, \zeta(x_0)) > \epsilon_0$ . Since sequence  $\{y_n\}$  is in the compact set  $R$ , it has a subsequence  $y_{\phi(n)}$  converging to some  $\bar{y}$  in  $R$ , and  $d(\bar{y}, \zeta(x_0)) \geq \epsilon_0$ . Then, for all  $y \in R$ ,

$$\rho(x_0, \bar{y}) \leq \liminf_{n \rightarrow \infty} \rho(z_{\phi(n)}, y_{\phi(n)}) \leq \lim_{n \rightarrow \infty} \rho(z_{\phi(n)}, y) = \rho(x_0, y),$$

where the first inequality is by the lower semicontinuity of  $\rho$  in  $(x, y)$ , the second inequality is by the definition of  $\zeta(z_{\phi(n)})$ , while the third equality is by the continuity of  $\rho$  in  $x$ . This shows that  $\bar{y} \in \zeta(x_0)$ , in contradiction with the fact that  $d(\bar{y}, \zeta(x_0)) \geq \epsilon_0$ .

Now fix arbitrarily an open set  $w \subset R$  and some  $x_0 \in R$  such that  $\zeta(x_0) \subset w$ . Since  $\zeta(x_0)$  is compact while  $w$  is open, there exists  $\epsilon > 0$  such that  $\zeta(x_0) + B(0, \epsilon) \subset w$ . According to the result of the previous paragraph, for this particular  $\epsilon$ , there exists  $\delta > 0$  such that  $\zeta(z) \subset \zeta(x_0) + B(0, \epsilon) \subset w$  for all  $z \in B(x_0, \delta)$ . This means  $B(x_0, \delta) \subset \{x \in R \mid \zeta(x) \subset w\}$ . As a result, the set  $\{x \in R \mid \zeta(x) \subset w\}$  is open.

Finally, according to the Kakutani fixed-point theorem, there exists  $\tilde{x} \in R$  such that  $\tilde{x} \in \zeta(\tilde{x})$ .  $\square$

**Definition 5.2.** A family of real-valued functions  $\{f(\cdot, y) : \mathcal{X} \rightarrow \mathbb{R} \mid y \in \mathcal{Y}\}$  indexed by  $y$ , with  $\mathcal{X} \subset \mathbb{R}^{d_1}$  and  $\mathcal{Y} \subset \mathbb{R}^{d_2}$ , is *uniformly equicontinuous* if, for all  $\epsilon > 0$ , there exists  $\delta$  such that, for all  $y \in \mathcal{Y}$ ,  $\|f(x_1, y) - f(x_2, y)\| \leq \epsilon$  whenever  $\|x_1 - x_2\| \leq \delta$ .

**Theorem 5.3** (Existence of PNE in l.s.c. convex games). *In an  $n$ -player game  $\Gamma$  where for each player  $i \in \{1, \dots, n\}$ , if the following three properties hold:*

- (1) *her action set  $\mathcal{X}_i$  is a convex compact subset of  $\mathbb{R}^d$ ;*
  - (2) *her cost function  $f_i(x_i, x_{-i}) : \mathcal{X}_i \times \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R}$  is convex and l.s.c. in  $x_i \in \mathcal{X}_i$  for any fixed  $x_{-i} \in \prod_{j \neq i} \mathcal{X}_j$ ;*
  - (3) *the family of functions  $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$  are uniformly equicontinuous,*
- then  $\Gamma$  admits a PNE.*

*Proof.* Define function  $\rho(x, y) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  by  $\rho(x, y) = \sum_{i=1}^n f_i(y_i, x_{-i})$ , where  $\mathcal{X} = \prod_i \mathcal{X}_i$ . It is easy to see that a fixed point of the set-valued map  $\zeta : \mathcal{X} \rightarrow \mathcal{X}$ ,  $x \mapsto \zeta(x) = \arg \min_{z \in R} \rho(x, z)$  is a Nash equilibrium of game  $\Gamma$ .

In order to apply Lemma 5.1, one needs to show that: (i)  $\rho(x, y)$  is continuous in  $x$  for each fixed  $y$ ; (ii)  $\rho(x, y)$  is l.s.c. in  $(x, y)$ ; (iii)  $\rho(x, y)$  is convex in  $y$  for each fixed  $x$ .

Results (i) and (iii) are straightforward by the definition of  $\rho$ .

For (ii), first note that, by the uniform equicontinuity of  $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$  for each  $i$  and the fact that  $n$  is finite,  $\{\rho(\cdot, y), y \in R\}$  is uniformly equicontinuous. Let  $(x^k, y^k)$  be a sequence in  $\mathcal{X} \times \mathcal{X}$  indexed by  $k$  which converges to  $(x, y) \in \mathcal{X} \times \mathcal{X}$ . Then,

$$\begin{aligned} \lim_{k \rightarrow \infty} (\rho(x^k, y^k) - \rho(x, y)) &= \lim_{k \rightarrow \infty} (\rho(x^k, y^k) - \rho(x, y^k) + \rho(x, y^k) - \rho(x, y)) \\ &= \lim_{k \rightarrow \infty} (\rho(x, y^k) - \rho(x, y)) \\ &\geq 0, \end{aligned}$$

where the second equality is due to the uniform equicontinuity of  $\{\rho(\cdot, y), y \in \mathcal{X}\}$  and the last inequality is because  $\rho(x, y)$  is l.s.c. in  $y$  for any fixed  $x$ .  $\square$

**Remark 5.1.** The property (3) is weaker than the condition that  $f_i$  is continuous on  $\mathcal{X}$ . Indeed, since  $\mathcal{X}$  is compact,  $f_i(x_i, x_{-i})$  is uniformly continuous on  $\mathcal{X}_i \times \prod_{j \neq i} \mathcal{X}_j$  which implies the equicontinuity of  $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$ . In other words, Rosen's theorem on the existence of convex continuous games with compact convex actions sets is a corollary of Theorem 5.3.

## Appendix B: Other proofs and lemmata

**Lemma 5.4** (Shapley-Folkman Lemma [43]). *For  $n$  compact subsets  $S_1, \dots, S_n$  of  $\mathbb{R}^q$ , let  $x \in \text{conv} \sum_{i=1}^n S_i = \sum_{i=1}^n \text{conv} S_i$ , where  $\text{conv}$  signifies the convex hull, and the sum over sets are to be understood as a Minkowski sum. Then,*

- *there is a point  $x_i \in \text{conv} S_i$  for each  $i$ , such that  $x = \sum_{i=1}^n x_i$ , and  $x_i \in S_i$  except for at most  $q$  values of  $i$ ;*
- *there is a point  $y_i \in S_i$  for each  $i$ , such that  $\|x - \sum_{i=1}^n y_i\|_{\mathbb{R}^q} \leq \sqrt{\min\{q, n\}}d$ , where  $d$  denotes the maximal diameter of  $S_i$ .*

In the proofs of Lemmata 5.5, 5.6, 2.6, in order to simplify the notations,  $i$  and  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$  are arbitrarily fixed. Index  $i$  and the parameter  $x_{-i}$  are thus omitted in  $f_i, \tilde{f}_i, \mathcal{X}_i, \tilde{\mathcal{X}}_i$  and  $Z(i, \cdot)$ .

**Lemma 5.5.** *Under Assumption 1, for each  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ ,*

- (1)  $\tilde{f}_i(x_i, x_{-i}) \leq f_i(x_i, x_{-i})$  for all  $x_i \in \mathcal{X}_i$ ;
- (2) the infimum in (2.2) can be attained, i.e. it is in fact a minimum for all  $x_i \in \mathcal{X}_i$ ;
- (3) function  $\tilde{f}_i(\cdot, x_{-i})$  is l.s.c. and convex on  $\tilde{\mathcal{X}}_i$ , and  $\text{conv}(\text{epi } f_i(\cdot, x_{-i})) = \text{epi } \tilde{f}_i(\cdot, x_{-i}) = \overline{\text{conv}}(\text{epi } f_i(\cdot, x_{-i}))$ ;
- (4) both  $\tilde{f}_i(\cdot, x_{-i})$  and  $f_i(\cdot, x_{-i})$  attain their minimum respectively on  $\tilde{\mathcal{X}}_i$  and  $\mathcal{X}_i$ , and

$$\min_{\tilde{x}_i \in \tilde{\mathcal{X}}_i} \tilde{f}_i(\tilde{x}_i, x_{-i}) = \min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}) . \quad (5.1)$$

*In particular, if  $\tilde{x}_i \in \arg \min_{y_i \in \tilde{\mathcal{X}}_i} \tilde{f}_i(y_i, \tilde{x}_{-i})$ , then  $Z(i, \tilde{x}) \subset \arg \min_{y_i \in \mathcal{X}_i} f_i(y_i, \tilde{x}_{-i})$  where  $Z(i, \tilde{x})$  is an arbitrary generator for  $(i, \tilde{x})$  defined in Definition 2.2.*

*Proof of Lemma 5.5.*

The Lemma is a particular case of more general results well-known in Convex Analysis that have been shown in various work, such as [16, Lemma X.1.5.3]. Let us provide a proof for this particular case here for the sake of completeness.

- (1) For  $x \in X$ , in the definition of  $\tilde{f}(x)$ , take  $x^k = x$ ,  $\alpha^k = \frac{1}{d+1}$  for all  $k$ . By definition,  $\tilde{f}(x) \leq \sum_{k=1}^{d+1} \alpha^k f(x^k) = f(x)$ .
- (2) Suppose that  $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$  is a minimizing sequence for  $\tilde{f}(\tilde{x})$ , i.e.  $\tilde{f}(\tilde{x}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} f(x^{k,n})$ , with  $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$  satisfying conditions in (2.2). Since  $(\alpha^{1,n}) \in [0, 1]$  for all  $n$ , so that it has a convergent subsequence  $\alpha^{1, \phi_1(n)}$  which converges to some  $\alpha^1$ . Consider sequence  $\alpha^{2, \phi_1(n)}$  which has a subsequence  $\alpha^{2, \phi_2(n)}$  converging to some  $\alpha^2$ . Note that  $\phi_2(n)$  is a subsequence of  $\phi_1(n)$ . Repeat this operation  $d+1$  times and obtain subsequences  $\phi_1(n), \dots, \phi_{d+1}(n)$  such that  $\alpha^{k, \phi_k(n)}$  converges to  $\alpha^k$ , for  $k = 1, \dots, d+1$ . Consider  $x^{1, \phi_{d+1}(n)}$  which is in the compact set  $X$ . It has a convergent subsequence  $x^{1, \phi_{d+2}(n)}$

converging to  $x^1 \in \mathcal{X}$ . Again, take a subsequence  $\phi_{d+3}(n)$  such that  $x^{2,\phi_{d+3}(n)}$  converges to  $x^k$ , and so on. Finally, one obtains a subsequence  $\phi_{2d+2}(n)$  of  $\mathbb{N}$  such that

$$\tilde{f}(\tilde{x}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,\phi_{2d+2}(n)} f(x^{k,\phi_{2d+2}(n)}), \quad (5.2)$$

$$\alpha^k = \lim_{n \rightarrow \infty} \alpha^{k,\phi_{2d+2}(n)}, \quad \alpha^k \in [0, 1], \quad k = 1, 2, \dots, d+1, \quad (5.3)$$

$$\sum_{k=1}^{d+1} \alpha^k = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,\phi_{2d+2}(n)} = 1, \quad (5.4)$$

$$x^k = \lim_{n \rightarrow \infty} x^{k,\phi_{2d+2}(n)}, \quad x^k \in \mathcal{X}, \quad k = 1, 2, \dots, d+1, \quad (5.5)$$

$$\sum_{k=1}^{d+1} \alpha^k x^k = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,\phi_{2d+2}(n)} x^{k,\phi_{2d+2}(n)} = \lim_{n \rightarrow \infty} \tilde{x} = \tilde{x}. \quad (5.6)$$

Then,

$$\begin{aligned} \sum_{k=1}^{d+1} \alpha^k f(x^k) &\leq \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,\phi_{2d+2}(n)} f(x^{k,\phi_{2d+2}(n)}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,\phi_{2d+2}(n)} f(x^{k,\phi_{2d+2}(n)}) \\ &= \tilde{f}(\tilde{x}) \leq \sum_{k=1}^{d+1} \alpha^k f(x^k). \end{aligned}$$

where the first inequality is due to (5.5), the second equality due to (5.3), the third equality due to (5.2) and the fourth inequality due to (5.4), (5.6) and (2.2). This shows that  $\tilde{f}(\tilde{x}) = \sum_{k=1}^{d+1} \alpha^k f(x^k)$ , i.e.  $(\alpha^k, x^k)_{k=1}^{d+1}$ , is a minimizer.

(3) On the one hand, for all  $(x, y) \in \text{conv}(\text{epi } f)$ , by Caratheodory theorem [6, Proposition 1.2.1], there exists  $(x^k, y^k) \in \text{epi } f$ ,  $k = 1, \dots, d+1$  such that  $(x, y) = \sum_{k=1}^{d+1} \alpha^k (x^k, y^k)$ , with  $\alpha \in \mathcal{S}_d$ . Hence,  $y^k \geq f(x^k)$ ,  $y = \sum_{k=1}^{d+1} \alpha^k y^k \geq \sum_{k=1}^{d+1} \alpha^k f(x^k) \geq \tilde{f}(x)$ . This shows that  $(x, y) \in \text{epi } \tilde{f}$ . Therefore,  $\text{conv}(\text{epi } f) \subset \text{epi } \tilde{f}$ . Recall that  $f$  is l.s.c. hence  $\text{epi } f$  is a closed set hence so is  $\text{conv}(\text{epi } f)$ . Thus,  $\overline{\text{conv}}(\text{epi } f) \subset \text{epi } \tilde{f}$ .

On the other hand, for all  $(x, y) \in \text{epi } \tilde{f}$ ,  $y \geq \tilde{f}(x)$ . Let  $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$  be the minimizing sequence for  $\tilde{f}(x)$ , i.e.  $\tilde{f}(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} f(x^{k,n})$ , with  $\alpha^{k,n}, x^{k,n}$  satisfying conditions in (2.2). Then,  $y = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} (f(x^{k,n}) + \frac{\delta}{d+1})$ , where  $\delta = y - \tilde{f}(x) \geq 0$ . Denote  $y^n = \sum_{k=1}^{d+1} \alpha^{k,n} (f(x^{k,n}) + \frac{\delta}{d+1})$ . Then,  $(x, y^n) \in \text{conv}(\text{epi } f)$ , and  $\lim_{n \rightarrow \infty} (x, y^n) = (x, y)$ . This means that  $(x, y) \in \overline{\text{conv}}(\text{epi } f)$  and, therefore,  $\text{epi } \tilde{f} \subset \overline{\text{conv}}(\text{epi } f)$ .

In conclusion,  $\text{epi } \tilde{f}(\cdot) = \overline{\text{conv}}(\text{epi } f(\cdot))$ , which implies that the epigraph of  $\tilde{f}$  is closed and convex. Thus,  $\tilde{f}$  is l.s.c. and convex on  $\tilde{\mathcal{X}}$ .

(4) By the lower semicontinuity of  $\tilde{f}$  and  $f$  on compact sets  $\tilde{\mathcal{X}}$  and  $X$ , their minima can be attained. The equality (5.1) is thus clear by the definition in (2.2).  $\square$

**Remark 5.2.** If  $f_i(\cdot, x_{-i})$  is not l.s.c, the inclusion relationship in Lemma 5.5(2) can be strict, as shown respectively by the following two examples of dimension 1.

- $\mathcal{X} = \{0\} \cup \{\pm \frac{1}{z}\}_{z \in \mathbb{N}^*}$ ,  $f(x) = |x|$  for  $x \in \mathcal{X} \setminus \{0\}$ , and  $f(0) = 1$ . Then,  $\tilde{f}(x) = |x|$ , for all  $x \in \tilde{\mathcal{X}} = [-1, 1]$ , and  $\text{conv}(\text{epi } f) \subsetneq \text{epi } \tilde{f}$ .

- $\mathcal{X} = [0, 1]$ ,  $f(x) = 0$  for  $x \neq 0$ , and  $f(0) = 1$ . Then,  $\tilde{f}(x) = f(x)$  for all  $x \in [0, 1]$ , and  $\text{epi } \tilde{f} \subsetneq \overline{\text{conv}}(\text{epi } f)$ .

**Lemma 5.6.** *Under Assumption 1, for any profile  $\tilde{x} \in \tilde{\mathcal{X}}$ , for any player  $i$ , for all  $x_i \in Z(i, \tilde{x})$ ,*

$$(1) \ f_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(x_i, \tilde{x}_{-i});$$

$$(2) \ \text{for any } h \in \partial_i \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}),$$

$$f_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) + \langle h, x_i - \tilde{x}_i \rangle. \quad (5.7)$$

*Proof of Lemma 5.6.* Let  $\{x^1, \dots, x^{d+1}\} \subset \mathcal{X}$  a generator of  $(\tilde{x}, \tilde{f}(\tilde{x}))$  and  $\alpha \in \mathcal{S}_d$  their corresponding weights.

(1) Suppose that there is  $k$  such that  $f(x^k) > \tilde{f}(x^k)$ . Then, there exists  $(y^l)_l$  in  $\mathcal{X}$  and  $\beta \in \mathcal{S}_d$  such that  $x^k = \sum_l \beta^l y^l$  and  $\tilde{f}(x^k) = \sum_l \beta^l f(y^l) < f(x^k)$ . In consequence,  $\tilde{f}(\tilde{x}) = \sum_m \alpha^m f(x^m) > \sum_{m \neq k} \alpha^m f(x^m) + \sum_l \alpha^k \beta^l f(y^l)$ , while  $\sum_{m \neq k} \alpha^m x^m + \sum_l \alpha^k \beta^l y^l = \tilde{x}$  and  $\sum_{m \neq k} \alpha^m + \sum_l \alpha^k \beta^l = 1$ , contradicting the definition of  $\tilde{f}(\tilde{x})$ .

(2) By the definition of subdifferential, one has

$$\tilde{f}(x^k) \geq \tilde{f}(\tilde{x}) + \langle h, x^k - \tilde{x} \rangle, \quad \forall k = 1, \dots, d+1. \quad (5.8)$$

Multiplying (5.8) by  $\alpha^k$  for each  $k$  and adding the  $d+1$  inequalities yield

$$\sum_{k=1}^{d+1} \tilde{f}(x^k) \geq \tilde{f}(\tilde{x}) + \langle h, \sum_k \alpha^k x^k - \tilde{x} \rangle \Leftrightarrow \tilde{f}(\tilde{x}) \geq \tilde{f}(\tilde{x}). \quad (5.9)$$

If, for at least one  $k$ , the inequality in (5.8) is strict, then the inequalities in (5.9) are strict as well, which is absurd. Therefore, for each  $k$ ,  $\tilde{f}(x^k) = \tilde{f}(\tilde{x}) + \langle h, x^k - \tilde{x} \rangle$ .  $\square$

*Proof of Lemma 2.6.* First note that  $\tilde{x}$  is in  $\text{ri}(\text{conv } Z(\tilde{x}))$ , the relative interior of  $\text{conv } Z(\tilde{x})$ . Hence, for  $t > 0$  small enough,  $\tilde{x} \pm t(x - \tilde{x})$  is in  $\text{ri}(\text{conv } Z(\tilde{x})) \subset \tilde{\mathcal{X}}$ . By (2.5),  $\langle h, \tilde{x} \pm t(x - \tilde{x}) - \tilde{x} \rangle \geq -\eta \|\tilde{x} \pm t(x - \tilde{x}) - \tilde{x}\|$ , which yields  $|\langle h, x - \tilde{x} \rangle| \leq \eta \|x - \tilde{x}\|$ . Then, by Lemma 5.6,  $|\tilde{f}(x) - \tilde{f}(\tilde{x})| = |\langle h, x - \tilde{x} \rangle| \leq \eta \|x - \tilde{x}\|$ .  $\square$

*Proof of Theorem 2.7.* For each  $i \in N$ , define a set  $E_i(\tilde{x}) := A_i Z(i, \tilde{x})$  in  $\mathbb{R}^q$ . Since  $\tilde{x}_i \in \text{conv}(Z(i, \tilde{x}))$ , one has  $\sum_{i \in N} A_i \tilde{x}_i \in \sum_{i \in N} \text{conv}(E_i(\tilde{x})) = \text{conv}\left(\sum_{i \in N} E_i(\tilde{x})\right)$  by the linearity of  $A_i$ 's. According to the Shapley-Folkman lemma, there exists  $e_i \in \text{conv}(E_i(\tilde{x}))$  for each  $i \in N$ , and a subset  $I \subset N$  with  $|I| \leq q$ , such that: (i)  $\sum_{i \in N} A_i \tilde{x}_i = \sum_{i \in N} e_i$  and (ii)  $e_i \in E_i(\tilde{x})$  for all  $i \notin I$ . Thus, for all  $i \notin I$ , there exists  $\bar{x}_i \in Z(i, \tilde{x})$ , such that  $e_i = A_i \bar{x}_i$ . For all  $i \in I$ , take arbitrarily  $\bar{x}_i \in Z(i, \tilde{x})$ . Then,

$$\begin{aligned} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i \bar{x}_i \right\| &\leq \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i \bar{x}_i \right\| = \left\| \sum_{i \in N} e_i - \sum_{i \in N} A_i \bar{x}_i \right\| = \left\| \sum_{i \in I} A_i (\tilde{x}_i - \bar{x}_i) \right\| \\ &\leq \sqrt{q} M \Delta. \end{aligned} \quad (5.10)$$

Now, for all  $i$ ,  $x_i^* \in Z(i, \tilde{x})$ , so that it satisfies

$$f_i(x_i^*, \tilde{x}_{-i}) \leq \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) \leq \tilde{f}_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta \leq f_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta, \quad \text{for all } x_i \in \mathcal{X}_i, \quad (5.11)$$

according to Lemma 5.6.(1), Lemma 2.6 and Lemma 5.5.(1).

Recall that  $f_i(x) = \theta_i(x_i, \frac{1}{n} \sum_{j \in N} A_j x_j)$ . Hence, for any  $x_i \in \mathcal{X}_i$

$$\begin{aligned} f_i(x_i, \tilde{x}_{-i}) &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j \tilde{x}_j\right) \\ &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^* + \frac{1}{n} A_i (x_i^* - \tilde{x}_i) + \frac{1}{n} \sum_{j \in N} A_j (\tilde{x}_j - x_j^*)\right) \\ &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^* + \delta_i\right) - \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^*\right) + f_i(x_i, x_{-i}^*), \end{aligned}$$

where  $\delta_i := \frac{1}{n} A_i (x_i^* - \tilde{x}_i) + \frac{1}{n} \sum_{j \in N} A_j (\tilde{x}_j - x_j^*)$ .

By (5.10),  $\|\delta_i\| \leq \frac{(\sqrt{q}+1)M\Delta}{n}$ . Using now Assumption 1 yields that, for any  $x_i \in \mathcal{X}_i$ ,

$$|f_i(x_i, x_{-i}^*) - f_i(x_i, \tilde{x}_{-i})| \leq H \left( \frac{(\sqrt{q}+1)M\Delta}{n} \right)^\gamma.$$

Injecting this result in (5.11) yields

$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*) + \epsilon + \eta + 2H \left( \frac{(\sqrt{q}+1)M\Delta}{n} \right)^\gamma, \quad \forall x_i \in \mathcal{X}_i, \quad \forall i \in N. \quad (5.12)$$

□

**Lemma 5.7.** *Under Assumption 1, suppose that  $\tilde{x} \in \tilde{\mathcal{X}}$  is an  $\epsilon$ -PNE in  $\tilde{\Gamma}$  satisfying the  $\eta$ -stability condition with respect to  $(Z(i, \tilde{x}))_i$ , where  $Z(i, \tilde{x}) = \{x_i^1, x_i^2, \dots, x_i^{l_i}\}$  with  $1 \leq l_i \leq d+1$  and  $\tilde{x}_i = \sum_{l=1}^{l_i} \alpha_i^l x_i^l$ , where  $\alpha \in \mathcal{S}_{l_i-1}$ . Each player  $i$  plays a mixed strategy independently, i.e. a random action  $X_i$  following distribution  $\tilde{\mu}_i$  over  $\mathcal{X}_i$  defined by  $\mathbb{P}(X_i = x_i^l) = \alpha_i^l$ . In other words,*

$$\tilde{\mu}_i = \sum_{l=1}^{l_i} \alpha_i^l \delta_{x_i^l}, \quad (5.13)$$

where  $\delta_{x_i^l}$  stands for the Dirac distribution on  $x_i^l$ . Then,

$$\mathbb{E} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\| \leq \sqrt{n} M \Delta.$$

*Proof.* By the independence of  $X_i$ ,  $A_i X_i$  are independent of each other. From the definition of  $\tilde{\mu}_i$ ,  $\mathbb{E}(A_i X_i) = A_i \tilde{x}_i$ . Therefore,

$$\left( \mathbb{E} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\| \right)^2 \leq \mathbb{E} \left[ \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\|^2 \right] = \sum_{i \in N} \text{Var}(A_i X_i) \leq n M^2 \Delta^2,$$

where the first inequality is by Jensen's inequality. □

*Proof of Proposition 2.8.* By same arguments as in the proof of Theorem 2.7, one has

$$|f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i})| \leq H \|\delta_i(X)\|^\gamma,$$

where  $\delta_i(X) := \frac{1}{n}A_i(X_i - \tilde{x}_i) + \frac{1}{n}\sum_{j \in N} A_j(\tilde{x}_j - X_j)$ . By Lemma 5.7,

$$\mathbb{E}\|\delta_i(X)\| \leq \frac{1 + \sqrt{n}}{n} M\Delta.$$

Besides, since  $X_i$  takes values in  $Z(i, \tilde{x})$ ,

$$\begin{aligned} f_i(X_i, X_{-i}) &= f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(X_i, \tilde{x}_{-i}) \\ &\leq f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta \\ &= f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i}) + f_i(x_i, X_{-i}) + \epsilon + \eta, \end{aligned}$$

so that

$$\begin{aligned} f_i(X_i, X_{-i}) - f_i(x_i, X_{-i}) &\leq |f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i})| + |f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i})| + \epsilon + \eta \\ &\leq 2H(\delta_i(X))^\gamma + \epsilon + \eta. \end{aligned}$$

Therefore,

$$\mathbb{E}[f_i(X_i, X_{-i}) - f_i(x_i, X_{-i})] \leq 2H \mathbb{E}[\|\delta_i(X)\|^\gamma] + \epsilon + \eta \leq 2H \left( \frac{(\sqrt{n} + 1)M\Delta}{n} \right)^\gamma + \epsilon + \eta.$$

□

**Lemma 5.8.** *Under Assumption 2, , for the auxiliary game  $\bar{\Gamma}$ ,*

(1) *Assumption 1 are verified with  $H = L_g\Delta$  and  $\gamma = 1$ ;*

(2) *an  $\epsilon$ -PNE of  $\bar{\Gamma}$  is an  $(\epsilon + \frac{L_h M\Delta}{n} + \frac{2L_g M\Delta^2}{n})$ -PNE of  $\Gamma$ .*

*Proof of Lemma 5.8.* (1) First show that, for any fixed  $x_i \in \mathcal{X}_i$ , function  $\theta_i(x_i, y) := \langle g(y + \frac{a_i}{n}(x_i^0 - x_i)), x_i \rangle + \ell_i(x_i)$  is  $L_g\Delta$ -Lipschitz in  $y$  on  $\Omega$ . For this, fix  $x_i \in \mathcal{X}_i$ . For any  $y$  and  $y'$  in  $\Omega$ ,

$$\begin{aligned} |\theta_i(x_i, y') - \theta_i(x_i, y)|^2 &= \left| \left\langle g\left(y' + \frac{a_i}{n}(x_i^0 - x_i)\right) - g\left(y + \frac{a_i}{n}(x_i^0 - x_i)\right), x_i \right\rangle \right|^2 \\ &\leq \left\| g\left(y' + \frac{a_i}{n}(x_i^0 - x_i)\right) - g\left(y + \frac{a_i}{n}(x_i^0 - x_i)\right) \right\|^2 \Delta^2 \\ &= \sum_{t=1}^d \left( g_t\left(y'_t + \frac{a_i}{n}(x_{i,t}^0 - x_{i,t})\right) - g_t\left(y_t + \frac{a_i}{n}(x_{i,t}^0 - x_{i,t})\right) \right)^2 \Delta^2 \\ &\leq L_g^2 \Delta^2 \sum_{t=1}^d (y'_t - y_t)^2 \\ &= L_g^2 \Delta^2 \|y' - y\|^2, \end{aligned}$$

where the first inequality results from the Cauchy-Schwarz inequality, while the second one is because  $g_t$  is  $L_{g_t}$ -Lipschitz.

(2) It is easy to see that  $|f_i(x_i, x_{-i}) - h_i(\frac{1}{n} \sum_{j \in N} a_j x_j) - \bar{f}_i(x_i, x_{-i})| \leq \frac{L_g M \Delta^2}{n}$  for all  $x_i \in \mathcal{X}_i$  and all  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ . Hence, if  $\bar{x} \in X$  is an  $\epsilon$ -PNE of  $\bar{\Gamma}$ , then, for each  $i$ , for any  $x_i \in \mathcal{X}_i$ ,

$$\begin{aligned} f_i(\bar{x}_i, \bar{x}_{-i}) &\leq \bar{f}_i(\bar{x}_i, \bar{x}_{-i}) + h_i\left(\frac{1}{n} \sum_{j \in N} a_j \bar{x}_j\right) + \frac{L_g M \Delta^2}{n} \\ &\leq \bar{f}_i(x_i, \bar{x}_{-i}) + \epsilon + h_i\left(\frac{1}{n} a_i x_i + \frac{1}{n} \sum_{j \neq i}^n a_j \bar{x}_j\right) + \frac{L_h M \Delta}{n} + \frac{L_g M \Delta^2}{n} \\ &\leq f_i(x_i, \bar{x}_{-i}) + \epsilon + \frac{L_h M \Delta}{n} + \frac{2L_g M \Delta^2}{n}, \end{aligned}$$

where the second inequality is due to the definition of  $\epsilon$ -PNE and the Lipschitz continuity of  $h_i$ .  $\square$

**Lemma 5.9.** *Under Assumption 2, let  $(x^k)_{k \in \mathbb{N}}$  be the sequence generated by Algorithm 1 with some initial point  $x^0 \in \tilde{\mathcal{X}}$ . Then,*

- (1)  $\sum_{k=1}^{\infty} \|x^{k-1} - x^k\|^2 \leq \frac{2n^2}{m^2 L_g} C$ , where  $C = (d\Delta L_g + 2B_r)M$ ;
- (2) for any  $K \in \mathbb{N}^*$ , there exists some  $k^* \leq K$ , such that  $\|x^{k^*-1} - x^{k^*}\| \leq \frac{\sqrt{2C} n}{m\sqrt{L_g K}}$ .

*Proof of Lemma 5.9.* Consider the following two real-valued functions defined on  $\tilde{\mathcal{X}}$ :

$$G_0(x) := \sum_{t=1}^d G_t\left(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}\right), \quad G(x) := G_0(x) + \sum_{j \in N} \frac{a_j}{n} \tilde{r}_j(x_j), \quad (5.14)$$

where  $G_t$  is a primitive function of  $g_t$ , which exists thanks to Assumption 2.

Note that function  $G_0$  is convex and differentiable on a neighborhood of  $\tilde{\mathcal{X}}$ , and convex function  $\tilde{r}_j$  is uniformly bounded on  $\tilde{\mathcal{X}}_j$  for all  $j \in N$  with the same bound  $B_\ell$ , according to Assumption 2.

Besides, it is easy to see that, for any  $i$  and fixed  $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ ,  $\nabla_i G_0(x_i, x_{-i}) := \frac{\partial G_0(x_i, x_{-i})}{\partial x_i} = \frac{a_i}{n} g\left(\frac{1}{n} a_i x_i + \frac{1}{n} \sum_{j \neq i} a_j x_j\right)$  is  $\frac{a_i^2 L_g}{n^2}$ -Lipschitz continuous on  $\tilde{\mathcal{X}}_i$ .

Therefore, Assumptions 1 and 2 in [49] are verified. One can thus apply Lemma 2.2 in [49] and obtains

$$\sum_{i \in N} \frac{a_i^2 L_g}{2n^2} \|x_i^k - x_i^{k+1}\|^2 \leq G(x^k) - G(x^{k+1}),$$

so that

$$\|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} (G(x^k) - G(x^{k+1})).$$

In consequence,

$$\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} (G(x^0) - G_{\min}), \quad (5.15)$$

where  $G_{\min}$ , defined as  $\inf_{\{x \in \tilde{\mathcal{X}}\}} G(x)$ , exists and is finite, because  $G$  is l.s.c. on compact set  $\tilde{\mathcal{X}}$ . Suppose that  $G_{\min}$  is attained at  $\underline{x} \in \tilde{\mathcal{X}}$ , then

$$\begin{aligned} G(x^0) - G_{\min} &= G(x^0) - G(\underline{x}) \\ &= \sum_{t=1}^d \int_{\frac{1}{n} \sum_{j \in N} a_j \underline{x}_{j,t}}^{\frac{1}{n} \sum_{j \in N} a_j x_{j,t}^0} g_t(s) ds + \sum_{j \in N} \frac{a_j}{n} (\tilde{r}_j(x_j^0) - \tilde{r}_j(\underline{x}_j)) \\ &\leq dM\Delta B_g + 2MB, \end{aligned} \quad (5.16)$$

where the last inequality is due to mean value theorem and Assumption 2. Combining (5.15) and (5.16) yields  $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} C$ . This immediately implies

$$\sum_{k=1}^K \|x^{k-1} - x^k\|^2 \leq \frac{2n^2}{m^2 L_g} C.$$

The second result of the lemma is then straightforward.  $\square$

*Proof of Proposition 3.1.* First, notice that vector function  $\zeta : \tilde{\mathcal{X}} \rightarrow \mathbb{R}^d, x \mapsto \zeta(x) = g(\frac{1}{n} \sum_{j \in N} a_j x_j)$  is  $\frac{L_g M}{\sqrt{n}}$ -Lipschitz continuous, i.e.  $\|\zeta(x) - \zeta(y)\| \leq \frac{L_g M}{\sqrt{n}} \|x - y\|$ , for all  $x, y \in \tilde{\mathcal{X}}$ . Indeed,  $\|\zeta(x) - \zeta(y)\|^2 = \sum_{t=1}^d |g_t(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}) - g_t(\frac{1}{n} \sum_{j \in N} a_j y_{j,t})|^2 \leq \sum_{t=1}^d \left| \frac{L_{g_t}}{n} \sum_{j \in N} a_j (x_{j,t} - y_{j,t}) \right|^2 \leq \sum_{t=1}^d \left( \frac{L_{g_t}^2}{n^2} \sum_{j=1}^n a_j^2 \sum_{j \in N} (x_{j,t} - y_{j,t})^2 \right) \leq \frac{L_g^2 M^2}{n} \|x - y\|^2$ , where the first inequality is because  $g_t$  is  $L_{g_t}$ -Lipschitz, while the second results from the Cauchy-Schwarz inequality.

Next, suppose that sequence  $(x^k)_{k \in \mathbb{N}}$  is generated by Algorithm 1 with some initial point  $x^0 \in \tilde{\mathcal{X}}$ . Let us show that, if  $\|x^{k-1} - x^k\| \leq u_k$ , then,  $x^k$  satisfies the full  $\eta(u_k)\Delta$ -stability condition and, furthermore, it is an  $\eta(u_k)\Delta$ -PNE of game  $\tilde{\Gamma}$ , where  $\eta(u_k) = \frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}$ .

Since  $\|x^k - x^{k-1}\| \leq u_k$ , one has  $\|(x_1^k, \dots, x_{i-1}^k, x_i^k, x_{i+1}^k, \dots, x_n^k) - (x_1^{k-1}, \dots, x_{i-1}^{k-1}, x_i^{k-1}, x_{i+1}^{k-1}, \dots, x_n^{k-1})\| \leq u_k$ . This, the Lipschitz continuity of  $\zeta$  on  $\tilde{\mathcal{X}}$  and the Lipschitz continuity of  $g$  in  $x_i$  imply that

$$\begin{aligned} &\left\| g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^0\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^{k-1} + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right) \right\| \\ &\leq \left\| g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^0\right) - g\left(\frac{1}{n} \sum_{j \in N} a_j x_j^k\right) \right\| \\ &\quad + \left\| g\left(\frac{1}{n} \sum_{j \in N} a_j x_j^k\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^{k-1} + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right) \right\| \\ &\leq \frac{L_g M \Delta}{n} + \frac{L_g M u_k}{\sqrt{n}}. \end{aligned} \quad (5.17)$$

The first order condition of optimality of the optimization problem (3.3) is: there exists some  $p_i$  in the subdifferential of  $\tilde{r}_i(x_i^k)$  at  $x_i^k$ , denoted by  $\partial \tilde{r}_i(x_i^k)$ , such that for all  $x_i \in \tilde{\mathcal{X}}_i$ ,

$$\left\langle g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right) + \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}) + p_i, x_i - x_i^k \right\rangle \geq 0. \quad (5.18)$$

Then,

$$\begin{aligned}
& \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) + p_i, x_i - x_i^k \right\rangle \\
&= \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right), x_i - x_i^k \right\rangle \\
&\quad + \left\langle g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right) + \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}) + p_i, x_i - x_i^k \right\rangle \\
&\quad - \left\langle \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}), x_i - x_i^k \right\rangle \\
&\geq \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^k + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right), x_i - x_i^k \right\rangle \\
&\quad - \left\langle \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}), x_i - x_i^k \right\rangle \\
&\geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{L_g M \Delta}{n} + \frac{a_i L_g \Delta}{n}\right) \|x_i - x_i^k\| \\
&\geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}\right) \|x_i - x_i^k\| = -\eta(u_k) \|x_i - x_i^k\|,
\end{aligned}$$

where the first inequality is due to (5.18), while the second one by (5.17) and the Cauchy-Schwarz inequality. Then, according to Lemma 2.6,  $x^k$  satisfies the full  $\eta(u_k)\Delta$ -stability condition for game  $\tilde{\Gamma}$ , where  $\eta(u_k) = \frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}$ .

Furthermore, since  $\tilde{r}_i$  is convex on  $\mathcal{X}_i$ ,

$$\begin{aligned}
\bar{f}_i(x_i, x_{-i}^k) - \bar{f}_i(x^k) &= \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right), x_i - x_i^k \right\rangle + \tilde{r}_i(x_i) - \tilde{r}_i(x_i^k) \\
&\geq \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right), x_i - x_i^k \right\rangle + \langle p_i, x_i - x_i^k \rangle \\
&\geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}\right) \|x_i - x_i^k\|.
\end{aligned}$$

Thus,  $x^k$  is an  $\eta(u_k)\Delta$ -PNE of game  $\tilde{\Gamma}$ .

For any  $K \in \mathbb{N}^*$ , there exists some  $k^* \leq K$  such that  $\|x^{k^*-1} - x^{k^*}\| \leq \frac{\sqrt{2Cn}}{m\sqrt{L_g K}}$  according to Lemma 5.9(2). The conclusion is immediate by taking  $\omega(K, n) = \eta\left(\frac{\sqrt{2Cn}}{m\sqrt{L_g K}}\right)$ .  $\square$

*Proof of Theorem 3.2.* Proposition 3.1 shows that  $x^{k^*}$  is an approximate PNE of game  $\tilde{\Gamma}$  (convexification of the nonconvex auxiliary game  $\bar{\Gamma}$ ). Then, Theorem 2.7 is applied to show that (the ‘‘Shapley-Folkman disaggregation’’ of  $x^{k^*}$ )  $x^*$  is an approximate PNE of the nonconvex auxiliary  $\bar{\Gamma}$ . The use of Theorem 2.7 is justified by Lemma 5.8(1). Finally, Lemma 5.8(2) is evoked to show that  $x^*$  is an approximate PNE of the original nonconvex game  $\Gamma$ .  $\square$

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