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Approximate Nash equilibria in large nonconvex aggregative games

Kang Liu*, Nadia Oudjane†, Cheng Wan‡

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Abstract

This paper shows the existence of $\mathcal{O}(\frac{1}{n^\gamma})$ -Nash equilibria in n -player noncooperative aggregative games where the players' cost functions depend only on their own action and the average of all the players' actions, and is lower semicontinuous in the former while γ -Hölder continuous in the latter. Neither the action sets nor the cost functions need to be convex. For an important class of aggregative games which includes congestion games with γ being 1, a proximal best-reply algorithm is used to construct an $\mathcal{O}(\frac{1}{n})$ -Nash equilibria with at most $\mathcal{O}(n^3)$ iterations. These results are applied in a numerical example of demand-side management of the electricity system. The asymptotic performance of the algorithm is illustrated when n tends to infinity.

Keywords. Shapley-Folkman lemma, aggregative games, nonconvex game, large finite game, ϵ -Nash equilibrium, proximal best-reply algorithm, congestion game

MSC Class Primary: 91A06; secondary: 90C26

1 Introduction

In this paper, players minimize their costs so that the definition of equilibria and equilibrium conditions are in the opposite sense of the usual usage where players maximize their payoffs. Players have actions in Euclidean spaces. If it is not precised, then “Nash equilibrium” means a pure-strategy Nash equilibrium.

This paper studies the approximation of pure-strategy Nash equilibria (PNE for short) in a specific class of finite-player noncooperative games, referred to as large nonconvex aggregative games. Recall that the cost functions of players in an aggregative game depend on their own action (i.e. pure-strategy) and the aggregate, i.e. weighted sum, of all the players' actions. In a large nonconvex aggregative game, players may have nonconvex action sets and nonconvex cost functions, and the number of players are relatively large so that the impact of each particular player on the aggregate behavior is relatively small, in a sense to be specified later.

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Background and motivation. The existence of pure-strategy Nash equilibrium has been proved only for some specific classes of finite-player noncooperative games.

When players have a finite number of actions, Mondrer and Shapley [29] shows that potential games where a so-called potential function exists admit PNEs and, as a matter of fact, every finite potential game is isomorphic to a finite congestion game introduced by Rosenthal [37], where players have equal weights and non-player-specific resource cost functions. Recall that in a congestion game, resources are shared among players, with each resource having a cost function of the aggregate load onto it. However, when players have player-specific weights and/or cost functions for each resource, a potential function no longer necessarily exists so that the existence of PNEs is not guaranteed, except in some particular cases (cf. Milchtaich [28]). In integer-splittable congestion games where the unequally integer-weighted players can split their weight into unit-weight blocks, the existence of PNE is shown for the case where a pure-strategy consists in a single resource and the non-player-specific resource cost functions are convex and monotone by Tran-Thanh et al. [41], and for the case where the non-player-specific resource cost functions are linear by Meyers [9]. For games with discrete (but not necessarily finitely many) strategies other than congestion games, Sagratella [38] proposes a branching method to compute the whole solution set of Nash equilibria. He proves the existence of PNEs for a particular class of such games and proposes a Jacobi-type algorithm which leads to one of the equilibria.

When players have a continuum of strategies, Rosen [36] shows that a Nash equilibrium exists if each player has a convex and compact action set in an Euclidean space and her cost function is continuous in the action profile and convex in her own action. In an atomic splittable congestion game where unequally weighted players can split their weight into arbitrary blocks, if the resource cost functions are continuous, then the conditions for Rosen's results to be valid are satisfied (cf. Orda, Rom and Shimkin [33]). However, when the players' action sets or their cost functions are nonconvex, there is no general result for the existence of Nash equilibrium.

This paper focuses on a particular class of finite-player noncooperative games which are aggregative games (Selten [39], Corchón [10], Jensen [22]), which includes congestion games. These games find practical applications in various fields in political science, economics, social biology, and engineering such as voting [31, 34], market competition [30], public good provision [2, 16], rent seeking [12], population dynamics [18], traffic analysis [11, 27], communications network control [26, 33] and electrical system management [19, 21]. In an aggregative game, for each particular player, the aggregate behavior of the other players matters instead of their respective identities. Another particularity of these applications is that the number of players are often very large so that the influence of a specific player's behavior on the others are not significant. However, in these real-life situations, the players' action sets and their cost functions are often nonconvex. Here is an example of flexible electric vehicle charging control whose convex version is studied by Jacquot et al. [20].

One day is divided into peak hours (e.g. 6 am–10 pm) and off-peak hours. The electricity production cost function for total flexible load ℓ^P and ℓ^{OP} at peak and off-peak hours are respectively $C^P(\ell^P) = \alpha_0^P \ell^P + \beta_0(\ell^P)^2$ and $C^{OP}(\ell^{OP}) = \alpha_0^{OP} \ell^{OP} + \beta_0(\ell^{OP})^2$, where $\alpha_0^P > \alpha_0^{OP} > 0$ and $\beta_0 > 0$. Player i 's action is denoted by $\ell_i = (\ell_i^P, \ell_i^{OP})$, where ℓ_i^P (resp. ℓ_i^{OP}) is the peak (resp. off-peak) consumption of player i . Player i 's electricity bill is then

defined by

$$b_i(\ell_i, \ell_{-i}) := \frac{C^P(\ell^P)}{\ell^P} \ell_i^P + \frac{C^{OP}(\ell^{OP})}{\ell^{OP}} \ell_i^{OP},$$

where $\ell^P = \sum_i \ell_i^P$, $\ell^{OP} = \sum_i \ell_i^{OP}$. Player i 's cost is then defined by

$$\phi_i(\ell_i, \ell_{-i}) = b_i(\ell_i, \ell_{-i}) + \gamma_i \|\ell_i - \ell_i^{ref}\|^2 \quad (1.1)$$

where γ_i indicates her sensitivity to the deviation from her preference ℓ_i^{ref} . In [20], the action set of player i is the convex compact set $S_i = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \underline{\ell}_i^P \leq \ell_i^P \leq \overline{\ell}_i^P, \underline{\ell}_i^{OP} \leq \ell_i^{OP} \leq \overline{\ell}_i^{OP}\}$, where e_i stands for the energy required by player i to charge a electric vehicle battery and $\underline{\ell}_i^P$ and $\overline{\ell}_i^P$ (resp. $\underline{\ell}_i^{OP}$ and $\overline{\ell}_i^{OP}$) are minimum and maximum power consumption for player i during peak (resp. off-peak) hours. However, for various reasons such as finite choices for charging power, or battery protection which demands that the charging must be interrupted as less as possible, etc., the players' action sets can be nonconvex. For example, in this paper a particular case where the nonconvex action set $S_i^{NC} = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \ell_i^P \in \{\underline{\ell}_i^P, \overline{\ell}_i^P\}\}$ is to be adopted for numerical simulation.

When the players' action sets or their cost functions are nonconvex, the existence of PNE is not guaranteed. Although mixed-strategy Nash equilibria always exist in the case of a finite number of strategies [32], pure strategies are often more relevant than mixed strategies (where players play pure strategies according to a certain probability distribution on their action set) for engineering applications. Besides, action sets are often not discrete. This is why the objective of the present paper is to find approximate PNEs that are reasonable substitutes, at least for practical purpose, for nonconvex aggregative games. More precisely, this paper aims to prove the existence and provide approximation schemes for $\epsilon(n)$ -Nash equilibria with $\epsilon(n)$ vanishing to zero, when the number of players, n , is sufficiently large.

Methodology and related work. The main idea is to use the Shapley-Folkman lemma in order to take advantage of the large number of players to circumvent the difficulty related to nonconvexities. Shapley and Folkman first derived their eponymous lemma in private communications before it was officially evoked and applied by Starr [40]. It states that, for n subsets S_1, \dots, S_n of \mathbb{R}^q , if $x \in \text{conv} \sum_{i=1}^n S_i = \sum_{i=1}^n \text{conv} S_i$, then there are points $x_i \in \text{conv} S_i$ such that $x = \sum_{i=1}^n x_i$ and $x_i \in S_i$ except for at most q values of i , where conv signifies the convex hull, and the sum over sets are to be understood as a Minkowski sum. Such a profile $(x_i)_i$ shall be called a ‘‘Shapley-Folkman disaggregation’’ of x in this paper. Roughly speaking, it means that the Minkowski sum of a finite number of sets in Euclidean spaces is close to convex when the number of sets is very large compared with their dimensions. It has been applied in nonconvex optimization for its convexification effect. Aubin and Ekeland [1] used the lemma to derive an upper bound on the duality gap in an additive, separable nonconvex optimization problem. Since, quite a few papers have extended or sharpened this result (cf. Ekeland and Temam [13], Bertsekas and coauthors [4, 7], Pappalardo [35], Wang [45], Kerdreux et al. [24], Bi and Tang [8]). These theoretical results have found applications in engineering, such as the large-scale unit commitment problem [3, 25] and optimization of Plug-in Electric Vehicles charging [44] in the electricity system, optimization of multicarrier communication systems [48], supply-chain management [43], and spatial graphical model estimation [15].

In the theories of economics and games, Starr [40] first applied the Shapley-Folkman lemma in the study of finite exchange economy where agents have nonconvex preference to show the existence of approximate equilibria. Later, Hildenbrand, Schmeidler and Zamir [46] further extends the existence results to the case where the agents' preferences are not complete either. However, as far as we know, there has not yet been applications of the Shapley-Folkman lemma in the theory of finite-player noncooperative games to approximate PNEs in the nonconvex case. The present paper aims to make a first attempt in this direction, and it takes important inspiration from [40].

Here is a brief sketch of the usage of the Shapley-Folkman lemma in this paper. Consider an n -player aggregative game where player i 's cost function can be written as $\theta_i(x_i, \frac{1}{n} \sum_{j=1}^n x_j)$. If θ_i is smooth in its second variable (i.e. the average term), the impact of $\frac{1}{n}x_i$ in the average term is very small on player i 's own cost when n is large. Therefore, approximately, a Nash equilibrium $x^* = (x_j^*)_j$, if it exists, can be seen as an action profile such that for each player i , x_i^* is a best reply to the average term $\xi^* := \frac{1}{n} \sum_{j=1}^n x_j^*$ or, more explicitly, x_i^* minimizes $\theta_i(\cdot, \xi^*)$ over her action set and, furthermore, the average of the best replies of all the players, $\frac{1}{n} \sum_{j=1}^n x_j^*$, happen to be ξ^* . In the nonconvex case where the existence of PNEs are not guaranteed, it is natural to consider the convexification of the players' action sets by their convex hulls and the convexification of their cost functions θ_i (with respect to x_i) by their Fenchel biconjugates, so as to obtain a PNE in the convexified game \tilde{x}^* . However, the recovery of an approximate PNE of the original nonconvex game from \tilde{x}^* is not immediate. The idea is to keep the average term $\tilde{\xi}^*$ while looking for x_i^* in the original nonconvex action set for each i such that, on the one hand, x_i^* is a best reply to $\tilde{\xi}^*$ in terms of the original nonconvex cost function $\theta(\cdot, \tilde{\xi}^*)$ whereas, on the other hand, their average (almost) coincides with $\tilde{\xi}^*$: $\frac{1}{n} \sum_{i=1}^n x_i^* \approx \tilde{\xi}^*$. The Shapley-Folkman lemma is effectively used to achieve this end.

However, the paper does not end there. For the theoretical results to be able to be applied in real-life engineering or economic issues, the computation of (approximate) PNEs is another major topic of the paper. Even though it is mentioned that one can recover an approximate PNE of the nonconvex game from a PNE of the convexified game, there are two difficulties now. Firstly, although the existence of Nash equilibria in general convex games is guaranteed, their computation is far from immediate. Convergent algorithms are known only for some special cases, such as for potential games or (strongly) monotone games. A common approach is to solve the variational inequality characterizing the Nash equilibria in such games (cf. Facchinei and Pang [14] and the references therein). However, the convexified game as described above can be rarely strongly monotone. This paper provides an algorithm, referred to as *proximal best-reply algorithm*, for a particular yet important class of aggregative games, which includes atomic splittable congestion games. The algorithm converges in polynomial time. Secondly, an exact PNE \tilde{x}^* of the convexified game cannot always be attained (in reasonable time). If only an approximate PNE \tilde{x}^* can be obtained, the question arising immediately is that whether the recovery of an approximate PNE x^* is still possible from this approximation. Our answer is yes if this approximate \tilde{x}^* is close enough to an exact PNE and it satisfies an additional condition called *disaggregatability*. Indeed, the proximal best-reply algorithm adopted for that important class of aggregative games is shown to find such an approximation in polynomial time.

Main contribution. The main contribution of the present paper is twofold.

Theoretically, Theorem 2.9 gives the existence of $\mathcal{O}(\frac{1}{n^\gamma})$ -Nash equilibrium for n -player nonconvex aggregative games where the players' cost functions depend only on their own action and the average of all the players' actions, and is lower semicontinuous in the former while γ -Hölder continuous in the latter. Neither the action sets nor the cost functions need to be convex.

Algorithmically, for an important class of n -player nonconvex aggregative games including congestion games with γ being 1 and where actions sets are compact subsets of Euclidean spaces, Theorem 3.4 ensures the performances of a proximal best reply algorithm which computes an $\mathcal{O}(\frac{1}{n})$ -Nash equilibrium with at most $\mathcal{O}(n^3)$ iterations. In the case where a ‘‘Shapley-Folkman disaggregation’’ is difficult to carry out, an extremely fast, easy and decentralized method is introduced to obtain an $\mathcal{O}(\frac{1}{\sqrt{n}})$ -mixed-strategy Nash equilibrium after the same number of iterations, as shown by Corollary 3.5.

Structure of the paper. The paper is organized as follows. Section 2 introduces n -player nonconvex aggregative games and their convexified game, then shows the existence of an $\mathcal{O}(\frac{1}{n^\gamma})$ -Nash equilibrium in such nonconvex games when the players' cost functions is lower semicontinuous in their own action while γ -Hölder continuous in the average of all the players' actions. Section 3 presents a proximal best-response algorithm to obtain an $\mathcal{O}(\frac{1}{n})$ -Nash equilibrium in a subclass of nonconvex aggregative games which includes congestion games with at most $\mathcal{O}(n^3)$ iterations as well as a decentralized randomized disaggregation method to obtain an $\mathcal{O}(\frac{1}{\sqrt{n}})$ -mixed-strategy Nash equilibrium in case that a centralized ‘‘Shapley-Folkman disaggregation’’ method is not at hand. In Section 4, a numerical simulation with the proximal best response algorithm is done for the previously introduced flexible electric vehicle charging problem. Section 5 concludes with some perspectives. The proofs of some lemmata are in the appendices.

Notations. In a Euclidean space, $\|\cdot\|$ denotes the l^2 -norm. For a point $x \in \mathbb{R}^d$ and a subset \mathcal{X} of \mathbb{R}^d , $d(x, \mathcal{X}) = \inf_{y \in \mathcal{X}} \{\|x - y\|\}$ is the distance from the point to the subset. For two subsets \mathcal{X} and \mathcal{Y} of \mathbb{R}^d , their Minkowski sum is the set $\{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}^+$, $B(x, r) = \{y \in \mathbb{R}^d \mid \|y - x\| \leq r\}$, the r -radial ball centered on x .

For a matrix $A \in \mathbb{R}^d \times \mathbb{R}^q$, $\|\cdot\|_2$ is the 2-norm of matrices: $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ where A^T is the transpose of A , $\lambda_{\max}(A^T A)$ stands for the largest eigenvalue of the matrix $A^T A$.

2 Existence of ϵ -Nash equilibrium in large nonconvex aggregative games

2.1 A nonconvex aggregative game and its convexification

Consider an n -player noncooperative game Γ . The players are indexed over $N = \{1, 2, \dots, n\}$. Each player $i \in N$ has an action set $\mathcal{X}_i \subset \mathbb{R}^d$, which is closed and bounded but not necessarily convex. Let $\tilde{\mathcal{X}}_i := \text{conv}(\mathcal{X}_i)$ be the convex hull of \mathcal{X}_i (which is also closed and bounded) and denote $\mathcal{X} = \prod_{i \in N} \mathcal{X}_i$, $\tilde{\mathcal{X}} := \prod_{i \in N} \tilde{\mathcal{X}}_i$, $\tilde{\mathcal{X}}_{-i} := \prod_{j \in N_{-i}} \tilde{\mathcal{X}}_j$ where $N_{-i} := N \setminus \{i\}$. Let constant $\Delta > 0$ be such that, for all $i \in N$, the compact set \mathcal{X}_i has diameter $|\mathcal{X}_i| := \max_{x_i, y_i \in \mathcal{X}_i} \|x_i - y_i\|$ that is not greater than Δ .

As usual, let x_{-i} denote the profile of actions of all the players except that of player i . Each player i has a real-valued cost function f_i defined on $\mathcal{X}_i \times \tilde{\mathcal{X}}_{-i}$ which has the following specific form:

$$f_i(x_i, x_{-i}) := \theta_i \left(x_i, \frac{1}{n} \sum_{j \in N} A_j x_j \right), \quad \text{for any } x_i \in \mathcal{X}_i, x_{-i} \in \tilde{\mathcal{X}}_{-i}, \quad (2.1)$$

where A_j 's are $q \times d$ matrices for all $j \in N$, and θ_i is a real-valued function defined on $\mathcal{X}_i \times \Omega$, with $\Omega \subset \mathbb{R}^q$ a neighborhood of $\{\frac{1}{n} \sum_{j \in N} A_j y_j \mid y_j \in \tilde{\mathcal{X}}_j, \forall j \in N\}$. Let constant $M > 0$ be such that $\|A_i\|_2 \leq M$ for each $i \in N$, where $\|\cdot\|_2$ is the 2-norm of matrices.

Remark that Γ is a generalization of *aggregative games*, which correspond to the specific case where, for each $j \in N$, A_j reduces to the identity matrix.

Definition 2.1. For a constant $\epsilon \geq 0$, an ϵ -Nash equilibrium $x^\epsilon \in \mathcal{X}$ in game Γ is a profile of actions of the n players such that, for each player $i \in N$,

$$f_i(x_i^\epsilon, x_{-i}^\epsilon) \leq f_i(x_i, x_{-i}^\epsilon) + \epsilon, \quad \text{for any } x_i \in \mathcal{X}_i.$$

If $\epsilon = 0$, then x^ϵ is a *Nash equilibrium*.

For nonconvex games (where either action sets or cost functions are not convex), the existence of Nash equilibria is not clear. In this paper, we show the existence of ϵ -Nash equilibria with the help of the convexified game associated with Γ defined as follows.

Definition 2.2. [5] The *convexified game* $\tilde{\Gamma}$ associated with Γ is a noncooperative game played by n players. Each player $i \in N$ has an action set $\tilde{\mathcal{X}}_i$ and a real-valued cost function \tilde{f}_i defined on $\tilde{\mathcal{X}}$ as follows: for all $x \in \tilde{\mathcal{X}}$,

$$\tilde{f}_i(x_i, x_{-i}) = \inf \left\{ \sum_{k=1}^{d+1} \alpha^k f_i(x_i^k, x_{-i}) \mid x_i = \sum_{k=1}^{d+1} \alpha^k x_i^k, \alpha \in \mathcal{S}_d \text{ and } \forall k, x_i^k \in \mathcal{X}_i \right\}, \quad (2.2)$$

where $\mathcal{S}_d := \{\alpha = (\alpha^k)_{k=1}^{d+1} \in \mathbb{R}^{d+1} \mid \forall k, \alpha^k \geq 0, \sum_{k=1}^{d+1} \alpha^k = 1\}$ denotes the probability simplex in dimension d .

Nash equilibria and ϵ -Nash equilibria are similarly defined for the convexified game $\tilde{\Gamma}$.

The remaining of this subsection is dedicated to some preliminary analysis of the convexified game.

First let us introduce two assumptions that will hold in this section.

Assumption 1. For any player $i \in N$, for any $x_{-i} \in \tilde{\mathcal{X}}_{-i}$, $f_i(\cdot, x_{-i})$ is lower semicontinuous (l.s.c.) with respect to its first variable on \mathcal{X}_i .

Assumption 2. For all $i \in N$, for all $x_i \in \mathcal{X}_i$, function $y \rightarrow \theta_i(x_i, y)$ is (H, γ) -Hölder on $y \in \Omega$, i.e.

$$|\theta_i(x_i, y') - \theta_i(x_i, y)| \leq H \|y' - y\|^\gamma, \quad (2.3)$$

with constants $H > 0, \gamma > 0$.

Lemma 2.3. For each $x_{-i} \in \tilde{\mathcal{X}}_{-i}$,

- (1) $\tilde{f}_i(x_i, x_{-i}) \leq f_i(x_i, x_{-i})$ for all $x_i \in \mathcal{X}_i$;
- (2) the infimum in (2.2) can be attained, i.e. it is in fact a minimum for all $x_i \in \mathcal{X}_i$;
- (3) function $\tilde{f}_i(\cdot, x_{-i})$ is l.s.c. and convex on $\tilde{\mathcal{X}}_i$, and $\text{conv}(\text{epi } f_i(\cdot, x_{-i})) = \text{epi } \tilde{f}_i(\cdot, x_{-i}) = \overline{\text{conv}}(\text{epi } f_i(\cdot, x_{-i}))$;
- (4) both $\tilde{f}_i(\cdot, x_{-i})$ and $f_i(\cdot, x_{-i})$ attain their minimum respectively on $\tilde{\mathcal{X}}_i$ and \mathcal{X}_i , and

$$\min_{\tilde{x}_i \in \tilde{\mathcal{X}}_i} \tilde{f}_i(\tilde{x}_i, x_{-i}) = \min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}). \quad (2.4)$$

In particular, if $\tilde{x}_i \in \arg \min_{y_i \in \tilde{\mathcal{X}}_i} \tilde{f}_i(y_i, \tilde{x}_{-i})$, then $W_i(\tilde{x}) \subset \arg \min_{y_i \in \mathcal{X}_i} f_i(y_i, \tilde{x}_{-i})$ where $W_i(\tilde{x})$ is defined in Definition 2.4.

Definition 2.4. Let $W_i(x_i, x_{-i}) := \bigcup_{\lambda \in \Lambda_i} W_i^\lambda(x_i, x_{-i})$ denote the collection of all groups of $d+1$ (not necessarily distinct) elements of \mathcal{X}_i that attain the minimum in the definition of $\tilde{f}_i(x_i, x_{-i})$ (cf. (2.2)), indexed by indices $\lambda \in \Lambda_i \subset \mathbb{N}$. Such a group, $W_i^\lambda(x_i, x_{-i})$, is called a *generator* of $(x_i, \tilde{f}_i(x_i, x_{-i}))$. In other words, for any $\lambda \in \Lambda_i$, $W_i^\lambda(x) = \{x_i^{1,\lambda}, \dots, x_i^{d+1,\lambda}\} \subset \mathcal{X}_i$ and there is $\alpha^\lambda \in \mathcal{S}_d$ such that $\sum_k \alpha^{k,\lambda} x_i^{k,\lambda} = x_i$ and $\sum_k \alpha^{k,\lambda} f(x_i^{k,\lambda}, x_{-i}) = \tilde{f}_i(x_i, x_{-i})$.

Remark 2.1. If $f_i(\cdot, x_{-i})$ is not l.s.c, the inclusion relationship in Lemma 2.3(2) can be strict, as shown respectively by the following two examples in dimension 1.

- $\mathcal{X} = \{0\} \cup \{\pm \frac{1}{z}\}_{z \in \mathbb{N}^*}$, $f(x) = |x|$ for $x \in \mathcal{X} \setminus \{0\}$, and $f(0) = 1$. Then, $\tilde{f}(x) = |x|$, for all $x \in \tilde{\mathcal{X}} = [-1, 1]$, and $\text{conv}(\text{epi } f) \subsetneq \text{epi } \tilde{f}$.
- $\mathcal{X} = [0, 1]$, $f(x) = 0$ for $x \neq 0$, and $f(0) = 1$. Then, $\tilde{f}(x) = f(x)$ for all $x \in [0, 1]$, and $\text{epi } \tilde{f} \subsetneq \overline{\text{conv}}(\text{epi } f)$.

Notation. Since $x_i \mapsto \tilde{f}_i(x_i, x_{-i})$ is convex and l.s.c on $\tilde{\mathcal{X}}_i$, its subdifferential exists and let it be denoted by $\partial_i \tilde{f}_i(\cdot, x_{-i})$. Then, for each $x_i \in \tilde{\mathcal{X}}_i$, $\partial_i \tilde{f}_i(x_i, x_{-i})$ is a nonempty convex subset of \mathbb{R}^d .

Proposition 2.5. *The convexified game $\tilde{\Gamma}$ has a Nash equilibrium.*

Proof. It results from Theorem 5.3 in Appendix B. □

Remark 2.2. Theorem 5.3 extends Rosen's theorem on the existence of Nash equilibria in games with convex continuous cost functions [36] to the case where the cost functions are only l.s.c. instead of being continuous with respect to the players' own actions. The following example shows that even the continuity of f_i on \mathcal{X}_i cannot guarantee the continuity of \tilde{f}_i on $\tilde{\mathcal{X}}_i$ so that Rosen's theorem is not sufficient here.

Consider $d = 3$, $\mathcal{X}_i = T \cup B \cup S$ where $T = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = 1\}$, $B = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = -1\}$, $S = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | x^1 = 1, x^2 = 0, -1 \leq x^3 \leq 1\}$; f_i is independent of x_{-i} , and $f_i(x) = 0$ for $x \in T \cup B$, $f_i(x) = |x^3| - 1$ for $x \in S$. Then, for all $x \in \{(x^1, x^2, x^3) \in \mathbb{R}^3 | (x^1)^2 + (x^2)^2 = 1, x^3 = 0\} \subset \partial \tilde{\mathcal{X}}_i$, $\tilde{f}_i(x) = 0$ except for $x^* = (1, 0, 0)$, but $f_i(x^*) = f_i(x^*) = -1$.

2.2 η -disaggregability

As explained in the Introduction, once a Nash equilibrium \tilde{x} of the convexified game $\tilde{\Gamma}$ is found, the next step is to find a feasible action profile x^* of the original nonconvex game Γ such that $\frac{1}{n} \sum_i A_i x_i^* \approx \frac{1}{n} \sum_i A_i \tilde{x}_i$ and x_i^* is (almost) a best response to $\frac{1}{n} \sum_i A_i \tilde{x}_i$, in the sense that x_i^* (almost) minimizes $\theta_i(\cdot, \frac{1}{n} \sum_i A_i \tilde{x}_i)$ on X_i . Indeed, as is shown below, such a feasible action profile x^* exists, with x_i^* in $W_i(\tilde{x})$ for each i , thanks to the Shapley-Folkman lemma.

However, from an algorithmic point of view, a Nash equilibrium is not always easy or fast to compute for the convexified $\tilde{\Gamma}$ even if its existence is guaranteed. The question that naturally arises then is whether the idea above is still valid if \tilde{x} is only an approximate Nash equilibrium of $\tilde{\Gamma}$. More explicitly, is there still a feasible action profile x^* of Γ such that $\frac{1}{n} \sum_i A_i x_i^* \approx \frac{1}{n} \sum_i A_i \tilde{x}_i$ and x_i^* is (almost) a best response to $\frac{1}{n} \sum_i A_i \tilde{x}_i$, while x_i^* 's are in $W_i(\tilde{x})$? The answer is YES, if the approximate Nash equilibrium \tilde{x} of the convexified game $\tilde{\Gamma}$ satisfies more demanding conditions.

Definition 2.6. In game $\tilde{\Gamma}$, a point $\tilde{x} \in \tilde{\mathcal{X}}$ is said to satisfy the η -disaggregatable condition with $(\lambda_i)_i \in (\Lambda_i)_i$ if, for each player i , $f_i(x_i, \tilde{x}_{-i}) \leq \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) + \eta$ for all $x_i \in W_i^{\lambda_i}(\tilde{x}_i, \tilde{x}_{-i})$.

Note that a Nash equilibrium of $\tilde{\Gamma}$ satisfies the 0-disaggregatable condition with any $(\lambda_i)_i \in (\Lambda_i)_i$.

Lemma 2.7. For any profile $\tilde{x} \in \tilde{\mathcal{X}}$, for any player i , for all $x_i \in W_i(\tilde{x}_i, \tilde{x}_{-i})$,

$$(1) f_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(x_i, \tilde{x}_{-i});$$

$$(2) \text{ for any } h \in \partial_i \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}),$$

$$f_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(x_i, \tilde{x}_{-i}) = \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) + \langle h, x_i - \tilde{x}_i \rangle. \quad (2.5)$$

The following lemma provides a sufficient condition for the η -disaggregatability of \tilde{x} .

Lemma 2.8. For any action profile $\tilde{x} \in \tilde{\mathcal{X}}$, for any player i , if there is $\lambda_i \in \Lambda_i$ and $h \in \partial \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i})$, such that,

$$\langle h, x_i - \tilde{x}_i \rangle \geq -\eta \|x_i - \tilde{x}_i\|, \quad \forall x_i \in \text{conv} W_i^{\lambda_i}(\tilde{x}_i, \tilde{x}_{-i}). \quad (2.6)$$

then,

$$|f_i(x_i, \tilde{x}_{-i}) - \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i})| \leq \eta \|x_i - \tilde{x}_i\|, \quad \text{for all } x_i \in W_i^{\lambda_i}(\tilde{x}_i, \tilde{x}_{-i}).$$

In particular, \tilde{x} satisfies the $\eta\Delta$ -disaggregatable condition with $(\lambda_i)_i$.

2.3 Approximate Nash equilibrium in the large nonconvex aggregative game Γ

This subsection is dedicated to the construction of an $\mathcal{O}(\frac{1}{n^\gamma})$ -Nash equilibrium of the original nonconvex game Γ .

Theorem 2.9. Suppose that $\tilde{x} \in \tilde{\mathcal{X}}$ is an ϵ -Nash equilibrium in $\tilde{\Gamma}$ which satisfies the η -disaggregatable condition with $(\lambda_i)_i$. Let $x^* \in \mathcal{X}$ be such that

$$x^* \in \operatorname{argmin}_{x_i \in W_i^{\lambda_i}(\tilde{x}), i \in N} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i x_i \right\|^2. \quad (2.7)$$

Then, x^* is a $\tilde{\epsilon}$ -Nash equilibrium of the nonconvex game Γ , where $\tilde{\epsilon} = \epsilon + \eta + 2H \left(\frac{(q+1)M\Delta}{n} \right)^\gamma$.

In particular, this shows the existence of $2H \left(\frac{(q+1)M\Delta}{n} \right)^\gamma$ -Nash equilibrium of Γ , by taking \tilde{x} to be a Nash equilibrium of $\tilde{\Gamma}$, which exists according to Proposition 2.5.

Proof. For each $i \in N$, define a set $E_i(\tilde{x}) := A_i W_i^{\lambda_i}(\tilde{x})$ in \mathbb{R}^q . Since $\tilde{x}_i \in \operatorname{conv}(W_i^{\lambda_i}(\tilde{x}))$, one has $\sum_{i \in N} A_i \tilde{x}_i \in \sum_{i \in N} \operatorname{conv}(E_i(\tilde{x})) = \operatorname{conv}\left(\sum_{i \in N} E_i(\tilde{x})\right)$ by the linearity of A_i 's. According to the Shapley-Folkman lemma, there exists $e_i \in \operatorname{conv}(E_i(\tilde{x}))$ for each $i \in N$, and a subset $I \subset N$ with $|I| \leq q$, such that: (i) $\sum_{i \in N} A_i \tilde{x}_i = \sum_{i \in N} e_i$ and (ii) $e_i \in E_i(\tilde{x})$ for all $i \notin I$. Thus, for all $i \notin I$, there exists $\bar{x}_i \in W_i^{\lambda_i}(\tilde{x})$, such that $e_i = A_i \bar{x}_i$. For all $i \in I$, take arbitrarily $\bar{x}_i \in W_i^{\lambda_i}(\tilde{x})$. Then,

$$\begin{aligned} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i x_i^* \right\| &\leq \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i \bar{x}_i \right\| = \left\| \sum_{i \in N} e_i - \sum_{i \in N} A_i \bar{x}_i \right\| = \left\| \sum_{i \in I} A_i (\tilde{x}_i - \bar{x}_i) \right\| \\ &\leq qM\Delta. \end{aligned} \quad (2.8)$$

Now, for all i , $x_i^* \in W_i^{\lambda_i}(\tilde{x})$, so that it satisfies

$$f_i(x_i^*, \tilde{x}_{-i}) \leq \tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) \leq \tilde{f}_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta \leq f_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta, \quad \text{for all } x_i \in \mathcal{X}_i, \quad (2.9)$$

according to Lemma 2.7.(1), Lemma 2.8.(3) and Lemma 2.3.(1).

Recall that $f_i(x) = \theta_i(x_i, \frac{1}{n} \sum_{j \in N} A_j x_j)$. Hence, for any $x_i \in \mathcal{X}_i$

$$\begin{aligned} f_i(x_i, \tilde{x}_{-i}) &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j \tilde{x}_j\right) \\ &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^* + \frac{1}{n} A_i (x_i^* - \tilde{x}_i) + \frac{1}{n} \sum_{j \in N} A_j (\tilde{x}_j - x_j^*)\right) \\ &= \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^* + \delta_i\right) - \theta_i\left(x_i, \frac{1}{n} A_i x_i + \frac{1}{n} \sum_{j \in N_{-i}} A_j x_j^*\right) + f_i(x_i, x_{-i}^*), \end{aligned}$$

where $\delta_i := \frac{1}{n} A_i (x_i^* - \tilde{x}_i) + \frac{1}{n} \sum_{j \in N} A_j (\tilde{x}_j - x_j^*)$.

By (2.8), $\|\delta_i\| \leq \frac{(q+1)M\Delta}{n}$. Using now Assumption 2 yields that, for any $x_i \in \mathcal{X}_i$,

$$|f_i(x_i, x_{-i}^*) - f_i(x_i, \tilde{x}_{-i})| \leq H \left(\frac{(q+1)M\Delta}{n} \right)^\gamma.$$

Injecting this result in (2.9) yields

$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*) + \epsilon + \eta + 2H \left(\frac{(q+1)M\Delta}{n} \right)^\gamma, \quad \forall x_i \in \mathcal{X}_i, \forall i \in N. \quad (2.10)$$

□

2.4 A decentralized randomized ‘‘Shapley-Folkman disaggregation’’

In the proof of Theorem 2.9, the Shapley-Folkman lemma is used to show the existence $(e_i)_i$, which can be called a ‘‘Shapley-Folkman disaggregation’’ of \tilde{x} or, more precisely, $\sum_i A_i \tilde{x}_i$. Nevertheless, the Shapley-Folkman lemma is not constructive and there has been few practical algorithms proposed in the literature to find such a ‘‘Shapley-Folkman disaggregation’’ (see Udell and Boyd [42] for a particular setting of optimization) and the disaggregation is done in a centralized way for the n sets $E_i(\tilde{x}) = A_i W_i^{\lambda_i}(\tilde{x})$. This is the reason why optimization problem (2.7) is introduced as a constructive solution. Note that \bar{x} and x^* are not ‘‘Shapley-Folkman disaggregations’’ strictly speaking, but feasible action profiles for the original nonconvex game while $\sum_i A_i \bar{x}$ and $\sum_i A_i x_i^*$ are close enough to $\sum_i A_i \tilde{x}$. For simplification, they are also called ‘‘Shapley-Folkman disaggregation’’ of \tilde{x} .

Although this paper does not aim to propose a generic algorithm for the resolution of discrete optimization program (2.7), let us introduce a fast, easy and decentralized method which finds a feasible mixed-strategy profile that is an $\tilde{\epsilon}$ -mixed strategy Nash equilibrium of the original nonconvex game, and such that the error $\tilde{\epsilon}$ is decreasing in n though less fast than x^* in Theorem 2.9.

Lemma 2.10. *Suppose that $\tilde{x} \in \tilde{\mathcal{X}}$ is an ϵ -Nash equilibrium in $\tilde{\Gamma}$ satisfying the η -disaggregatable condition with $(\lambda_i)_i$, where $W_i^{\lambda_i}(\tilde{x}) = \{x_i^1, x_i^2, \dots, x_i^{l_i}\}$ with $1 \leq l_i \leq d + 1$ and $\tilde{x}_i = \sum_{l=1}^{l_i} \alpha_i^l x_i^l$, where $\alpha \in \mathcal{S}_{l_i-1}$. Each player i plays a mixed strategy independently, i.e. a random action X_i following distribution $\tilde{\mu}_i$ over \mathcal{X}_i defined by $\mathbb{P}(X_i = x_i^l) = \alpha_i^l$. In other words,*

$$\tilde{\mu}_i = \sum_{l=1}^{l_i} \alpha_i^l \delta_{x_i^l}, \quad (2.11)$$

where $\delta_{x_i^l}$ stands for the Dirac distribution on x_i^l . Then,

$$\mathbb{E} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\| \leq \sqrt{n} M \Delta.$$

Proof. By the independence of X_i , $A_i X_i$ are independent of each other. From the definition of $\tilde{\mu}_i$, $\mathbb{E}(A_i X_i) = A_i \tilde{x}_i$. Therefore,

$$\left(\mathbb{E} \left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\| \right)^2 \leq \mathbb{E} \left[\left\| \sum_{i \in N} A_i \tilde{x}_i - \sum_{i \in N} A_i X_i \right\|^2 \right] = \sum_{i \in N} \text{Var}(A_i X_i) \leq n M^2 \Delta^2,$$

where the first inequality is by Jensen’s inequality. \square

Proposition 2.11. *Suppose that $\tilde{x} \in \tilde{\mathcal{X}}$ is an ϵ -Nash equilibrium in $\tilde{\Gamma}$ satisfying the η -disaggregatable condition with $(\lambda_i)_i$, and each player i plays a mixed strategy $\tilde{\mu}_i$ as introduced in Lemma 2.10. Then, for $\gamma \leq 1$, $\tilde{\mu} = (\tilde{\mu}_i)_i$ is a $\tilde{\epsilon}$ -mixed-strategy Nash equilibrium of the nonconvex game Γ , where $\tilde{\epsilon} = \epsilon + \eta + 2H \left(\frac{(\sqrt{n}+1)M\Delta}{n} \right)^\gamma$, in the sense that*

$$\mathbb{E}[f_i(X_i, X_{-i})] \leq \mathbb{E}[f_i(x_i, X_{-i})] + \tilde{\epsilon}, \quad \forall x_i \in \mathcal{X}_i,$$

where X_i follows $\tilde{\mu}_i$ independently.

Proof. By same arguments as in the proof of Theorem 2.9, one has

$$|f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i})| \leq H \|\delta_i(X)\|^\gamma,$$

where $\delta_i(X) := \frac{1}{n}A_i(X_i - \tilde{x}_i) + \frac{1}{n}\sum_{j \in N} A_j(\tilde{x}_j - X_j)$. By Lemma 2.10,

$$\mathbb{E}\|\delta_i(X)\| \leq \frac{1 + \sqrt{n}}{n}M\Delta.$$

Besides, since X_i takes values in $W_i^{\lambda_i}(\tilde{x})$,

$$\begin{aligned} f_i(X_i, X_{-i}) &= f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(X_i, \tilde{x}_{-i}) \\ &\leq f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(x_i, \tilde{x}_{-i}) + \epsilon + \eta \\ &= f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i}) + f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i}) + f_i(x_i, X_{-i}) + \epsilon + \eta, \end{aligned}$$

so that

$$\begin{aligned} f_i(X_i, X_{-i}) - f_i(x_i, X_{-i}) &\leq |f_i(X_i, X_{-i}) - f_i(X_i, \tilde{x}_{-i})| + |f_i(x_i, \tilde{x}_{-i}) - f_i(x_i, X_{-i})| + \epsilon + \eta \\ &\leq 2H(\delta_i(X))^\gamma + \epsilon + \eta. \end{aligned}$$

Therefore,

$$\mathbb{E}[f_i(X_i, X_{-i}) - f_i(x_i, X_{-i})] \leq 2H \mathbb{E}[\|\delta_i(X)\|^\gamma] + \epsilon + \eta \leq 2H \left(\frac{(\sqrt{n} + 1)M\Delta}{n} \right)^\gamma + \epsilon + \eta.$$

□

3 Computing ϵ -equilibria for large nonconvex congestion games

3.1 Nonconvex generalized congestion game

An extensively studied class of aggregative games are congestion games.

Consider a generalized congestion game where each player $i \in N$ has a action set $\mathcal{X}_i \subset \mathbb{R}^d$ and a cost function of the following form:

$$\begin{aligned} f_i(x_i, x_{-i}) &= \left\langle g\left(\frac{1}{n}\sum_{j \in N} a_j x_j\right), x_i \right\rangle + h_i\left(\frac{1}{n}\sum_{j \in N} a_j x_j\right) + r_i(x_i) \\ &= \sum_{t=1}^d g_t\left(\frac{1}{n}\sum_{j \in N} a_j x_{j,t}\right) x_{i,t} + h_i\left(\frac{1}{n}\sum_{j \in N} a_j x_j\right) + r_i(x_i), \end{aligned} \tag{3.1}$$

Suppose that the following assumptions hold on \mathcal{X}_i , $(a_j)_{j \in N} \in \mathbb{R}^n$, g_t 's, h_i 's and r_i 's.

Assumption 3.

- There exist constants $m > 0$ and $M > 0$, such that $m \leq a_i \leq M$ for all $i \in N$.
- For $t = 1, \dots, d$, function $g_t : \mathbb{R} \rightarrow \mathbb{R}$ is L_{g_t} -Lipschitz continuous and nondecreasing on a neighborhood of $[D_1, D_2]$, where constants D_1 and D_2 are such that $D_1 \leq \min_{t=1, \dots, d; x \in \tilde{\mathcal{X}}} \frac{1}{n} \sum_{j \in N} a_j x_{j,t} \leq \max_{t=1, \dots, d; x \in \tilde{\mathcal{X}}} \frac{1}{n} \sum_{j \in N} a_j x_{j,t} \leq D_2$.

- For each $i \in N$, function $h_i : \mathbb{R}^d \rightarrow \mathbb{R}$ is L_{h_i} -Lipschitz continuous on $[D_1, D_2]^d$.
- Players' local cost functions $r_i : \mathbb{R}^d \rightarrow \mathbb{R}$ are uniformly bounded, i.e. there exists constant $B_r > 0$ such that, for all $i \in N$ and all $x_i \in \mathcal{X}_i$, $|r_i(x_i)| \leq B_r$.

Notation. Let constant $\Delta = \max\{\max_{i \in N} \max_{x_i \in \tilde{\mathcal{X}}_i} \|x_i\|, \max_{i \in N} |\tilde{\mathcal{X}}_i|\}$. Let $L_g = \max_{1 \leq t \leq d} L_{g_t}$, $L_h = \max_{i \in N} L_{h_i}$, $B_g = \max_{1 \leq t \leq d, D_1 \leq s \leq D_2} |g_t(s)|$.

The convexification of Γ is rather complicated to compute in the general case. Let us first introduce an auxiliary game which is very close to Γ but whose convexification is easier to obtain.

Fix arbitrarily $x_i^+ \in \mathcal{X}_i$ for each player $i \in N$. Auxiliary game $\bar{\Gamma}$ is defined as follows: the player set and each player's action set are the same as in Γ , but player i 's cost function is, for all $x_i \in \mathcal{X}_i$ and all $x_{-i} \in \tilde{\mathcal{X}}_{-i}$,

$$\bar{f}_i(x_i, x_{-i}) := \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j + \frac{1}{n} a_i x_i^+\right), x_i \right\rangle + r_i(x_i).$$

The following lemma shows why Γ can be approximated by $\bar{\Gamma}$.

Lemma 3.1. For auxiliary game $\bar{\Gamma}$,

- (1) Assumptions 1 and 2 are verified with $H = L_g \Delta$ and $\gamma = 1$;
- (2) an ϵ -Nash equilibrium of $\bar{\Gamma}$ is an $(\epsilon + \frac{L_h M \Delta}{n} + \frac{2L_g M \Delta^2}{n})$ -Nash equilibrium of Γ .

For any fixed $x_{-i} \in \tilde{\mathcal{X}}_{-i}$, $\bar{f}_i(\cdot, x_{-i})$ is composed by a linear function of x_i and a local function of x_i . By abuse of notation, let us still use \bar{f}_i to denote its convexification on $\tilde{\mathcal{X}}_i$. More explicitly,

$$\bar{f}_i(x_i, x_{-i}) := \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j + \frac{1}{n} a_i x_i^+\right), x_i \right\rangle + \tilde{r}_i(x_i), \quad (3.2)$$

where \tilde{r}_i is the convexification of r_i defined on $\tilde{\mathcal{X}}_i$ in the same way as \bar{f}_i .

By abuse of notation, let $\tilde{\Gamma}$ denote the convexification of $\bar{\Gamma}$ on $\tilde{\mathcal{X}}$.

3.2 A proximal best-reply algorithm

This subsection presents a proximal best-reply algorithm based on the block coordination proximal algorithm introduced by Xu and Yin [47] to construct an $\mathcal{O}(\frac{1}{n})$ -Nash equilibrium of $\tilde{\Gamma}$ that satisfies the $\mathcal{O}(\frac{1}{n})$ -disaggregatable condition.

Algorithm 1: Proximal best-reply algorithm for $\tilde{\Gamma}$

Initialization: choose initial point $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \tilde{\mathcal{X}}$

for $k = 1, 2, \dots$ **do**

for $i = 1, 2, \dots, n$ **do**

$$x_i^k = \operatorname{argmin}_{x_i \in \tilde{\mathcal{X}}_i} \left\langle g \left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1} \right), x_i - x_i^{k-1} \right\rangle + \frac{a_i L_g}{2n} \|x_i - x_i^{k-1}\|^2 + \tilde{r}_i(x_i) \quad (3.3)$$

end

if stopping criterion is satisfied **then**

 | return $(x_1^k, x_2^k, \dots, x_n^k)$. Break.

end

end

Lemma 3.2. Let $(x^k)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1 with some initial point $x^0 \in \tilde{\mathcal{X}}$. Then,

(1) $\sum_{k=1}^{\infty} \|x^{k-1} - x^k\|^2 \leq \frac{2n^2}{m^2 L_g} C$, where $C = (d\Delta L_g + 2B_r)M$;

(2) for any $K \in \mathbb{N}^*$, there exists some $k^* \leq K$, such that $\|x^{k^*-1} - x^{k^*}\| \leq \frac{\sqrt{2Cn}}{m\sqrt{L_g K}}$.

Proposition 3.3. For $K \in \mathbb{N}^*$, there is $k^* \leq K$ such that x^{k^*} is an $\omega(K, n)\Delta$ -Nash equilibrium of game $\tilde{\Gamma}$ which is $\omega(K, n)\Delta$ -disaggregatable, where

$$\omega(K, n) = \frac{\sqrt{2CL_g M}}{m} \sqrt{\frac{n}{K}} + \frac{2L_g M \Delta}{n}, \quad (3.4)$$

where $C = (d\Delta L_g + 2B_r)M$.

In particular, if constant $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$ for some constant $\delta > 0$, then, there exists some $k^* \leq K$ such that x^{k^*} is an $L_g M \Delta (n^{-\delta} + 2\Delta n^{-1})$ -Nash equilibrium of game $\tilde{\Gamma}$ satisfying $L_g M \Delta (n^{-\delta} + 2\Delta n^{-1})$ -disaggregatable condition.

Proof. First, notice that vector function $\zeta : \tilde{\mathcal{X}} \rightarrow \mathbb{R}^d, x \mapsto \zeta(x) = g(\frac{1}{n} \sum_{j \in N} a_j x_j)$ is $\frac{L_g M}{\sqrt{n}}$ -Lipschitz continuous, i.e. $\|\zeta(x) - \zeta(y)\| \leq \frac{L_g M}{\sqrt{n}} \|x - y\|$, for all $x, y \in \tilde{\mathcal{X}}$. Indeed, $\|\zeta(x) - \zeta(y)\|^2 = \sum_{t=1}^d |g_t(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}) - g_t(\frac{1}{n} \sum_{j \in N} a_j y_{j,t})|^2 \leq \sum_{t=1}^d |L_{g_t}| \sum_{j \in N} a_j |x_{j,t} - y_{j,t}|^2 \leq \sum_{t=1}^d \left(\frac{L_{g_t}^2}{n^2} \sum_{j=1}^n a_j^2 \sum_{j \in N} (x_{j,t} - y_{j,t})^2 \right) \leq \frac{L_g^2 M^2}{n} \|x - y\|^2$, where the first inequality is because g_t is L_{g_t} -Lipschitz, while the second results from the Cauchy-Schwarz inequality.

Next, suppose that sequence $(x^k)_{k \in \mathbb{N}}$ is generated by Algorithm 1 with some initial point $x^0 \in \tilde{\mathcal{X}}$. Let us show that, if $\|x^{k-1} - x^k\| \leq u_k$, then, x^k satisfies the $\eta(u_k)\Delta$ -disaggregatable condition and, furthermore, it is an $\eta(u_k)\Delta$ -Nash equilibrium of game $\tilde{\Gamma}$, where $\eta(u_k) = \frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}$.

Since $\|x^k - x^{k-1}\| \leq u_k$, one has $\|(x_1^k, \dots, x_{i-1}^k, x_i^k, x_{i+1}^k, \dots, x_n^k) - (x_1^{k-1}, \dots, x_{i-1}^{k-1}, x_i^{k-1}, x_{i+1}^{k-1}, \dots, x_n^{k-1})\| \leq u_k$. This, the Lipschitz continuity of ζ on $\tilde{\mathcal{X}}$ and the Lipschitz continuity

of g in x_i imply that

$$\begin{aligned}
& \left\| g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^0\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^{k-1} + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right) \right\| \\
& \leq \left\| g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^0\right) - g\left(\frac{1}{n} \sum_{j \in N} a_j x_j^k\right) \right\| \\
& \quad + \left\| g\left(\frac{1}{n} \sum_{j \in N} a_j x_j^k\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^{k-1} + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right) \right\| \\
& \leq \frac{L_g M \Delta}{n} + \frac{L_g M u_k}{\sqrt{n}}.
\end{aligned} \tag{3.5}$$

The first order condition of optimality of the optimization problem (3.3) is: there exists some p_i in the subdifferential of $\tilde{r}_i(x_i^k)$ at x_i^k , denoted by $\partial \tilde{r}_i(x_i^k)$, such that for all $x_i \in \tilde{\mathcal{X}}_i$,

$$\left\langle g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right) + \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}) + p_i, x_i - x_i^k \right\rangle \geq 0. \tag{3.6}$$

Then,

$$\begin{aligned}
& \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) + p_i, x_i - x_i^k \right\rangle \\
& = \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right), x_i - x_i^k \right\rangle \\
& \quad + \left\langle g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} \sum_{j \geq i} a_j x_j^{k-1}\right) + \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}) + p_i, x_i - x_i^k \right\rangle \\
& \quad - \left\langle \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}), x_i - x_i^k \right\rangle \\
& \geq \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right) - g\left(\frac{1}{n} \sum_{j < i} a_j x_j^k + \frac{1}{n} a_i x_i^k + \frac{1}{n} \sum_{j > i} a_j x_j^{k-1}\right), x_i - x_i^k \right\rangle \\
& \quad - \left\langle \frac{a_i L_g}{n} (x_i^k - x_i^{k-1}), x_i - x_i^k \right\rangle \\
& \geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{L_g M \Delta}{n} + \frac{a_i L_g \Delta}{n}\right) \|x_i - x_i^k\| \\
& \geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}\right) \|x_i - x_i^k\| = -\eta(u_k) \|x_i - x_i^k\|,
\end{aligned}$$

where the first inequality is due to (3.6), while the second one by (3.5) and the Cauchy-Schwarz inequality. Then, according to Lemma 2.8, x^k satisfies the $\eta(u_k)\Delta$ -disaggregatable condition for game $\tilde{\Gamma}$, where $\eta(u_k) = \frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}$.

Furthermore, since \tilde{r}_i is convex on $\tilde{\mathcal{X}}_i$,

$$\begin{aligned}\bar{f}_i(x_i, x_{-i}^k) - \bar{f}_i(x^k) &= \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right), x_i - x_i^k \right\rangle + \tilde{r}_i(x_i) - \tilde{r}_i(x_i^k) \\ &\geq \left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j x_j^k + \frac{1}{n} a_i x_i^+\right), x_i - x_i^k \right\rangle + \langle p_i, x_i - x_i^k \rangle \\ &\geq -\left(\frac{L_g M u_k}{\sqrt{n}} + \frac{2L_g M \Delta}{n}\right) \|x_i - x_i^k\|.\end{aligned}$$

Thus, x^k is an an $\eta(u_k)\Delta$ -Nash equilibrium of game $\tilde{\Gamma}$.

For any $K \in \mathbb{N}^*$, there exists some $k^* \leq K$ such that $\|x^{k^*-1} - x^{k^*}\| \leq \frac{\sqrt{2Cn}}{m\sqrt{L_g K}}$ according to Lemma 3.2(2). The conclusion is immediate by taking $\omega(K, n) = \eta\left(\frac{\sqrt{2Cn}}{m\sqrt{L_g K}}\right)$. \square

Theorem 3.4. *For constant $\delta > 0$ and integer $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$, let $x^* \in \mathcal{X}$ be the pure-strategy profile generated by (2.7), where \tilde{x} is replaced by x^{k^*} in Proposition 3.3. Then, x^* is a $(2L_g M \Delta (n^{-\delta} + \frac{(g+4)\Delta}{n}) + \frac{L_h M \Delta}{n})$ -Nash equilibrium of the nonconvex game Γ .*

Proof. Proposition 3.3 shows that x^{k^*} is an approximate equilibrium of game $\tilde{\Gamma}$ (convexification of the nonconvex auxiliary game $\bar{\Gamma}$). Then, Theorem 2.9 is applied to show that (the ‘‘Shapley-Folkman disaggregation’’ of x^{k^*}) x^* is an approximate equilibrium of the nonconvex auxiliary $\bar{\Gamma}$. The use of Theorem 2.9 is justified by Lemma 3.1(1). Finally, Lemma 3.1(2) is evoked to show that x^* is an approximate equilibrium of the original nonconvex game Γ . \square

Remark 3.1. For each fixed n , Algorithm 1 converges to a so-called *block coordinatewise minimum point* of $G(x) = \sum_{t=1}^d G_t\left(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}\right) + \sum_{j \in N} \frac{a_j}{n} \tilde{r}_j(x_j)$, where G_t is a primitive function of g_t . (For more details, see Xu and Yin [47].) However, a block coordinatewise minimum point is not a Nash equilibrium of the convexification of the auxiliary game $\tilde{\Gamma}$. Indeed, by Definitions 2.3 and 2.4 in Xu and Yin [47], a block coordinatewise minimum point x^∞ of G satisfies that, for each i , there exists some p_i in $\partial \tilde{r}_i(x_i^\infty)$ such that

$$\left\langle g\left(\frac{1}{n} \sum_j a_j x_j^\infty\right) + p_i, x_i - x_i^\infty \right\rangle \geq 0, \quad \forall x_i \in \tilde{\mathcal{X}}_i,$$

while a Nash equilibrium \bar{x} of the convexification of the auxiliary game $\tilde{\Gamma}$ satisfies that, for each i , there exists some p_i in $\partial \tilde{r}_i(\bar{x}_i)$ such that

$$\left\langle g\left(\frac{1}{n} \sum_{j \neq i} a_j \bar{x}_j + \frac{1}{n} a_i x_i^+\right) + p_i, x_i - \bar{x}_i \right\rangle \geq 0, \quad \forall x_i \in \tilde{\mathcal{X}}_i.$$

Even if \mathcal{X}_i 's and g_t 's are convex sets and functions, and h_i 's are neglected, x^∞ is still not a Nash equilibrium \tilde{x} of the convexified game of Γ which satisfies that, for each i , there exists some p_i in $\partial \tilde{r}_i(\tilde{x}_i)$ such that

$$\left\langle g\left(\frac{1}{n} \sum_j a_j \tilde{x}_j\right) + \frac{a_i}{n} \left(g_t\left(\frac{1}{n} \sum_j a_j \tilde{x}_j\right)\right)_t + p_i, x_i - \tilde{x}_i \right\rangle \geq 0, \quad \forall x_i \in \tilde{\mathcal{X}}_i.$$

In particular, it means that G_t is not a potential function of the convexified game $\tilde{\Gamma}$ in the sense of Monderer and Shapley [29].

Nevertheless, it is easy to see that, thanks to the Lipschitz continuity of g , x^∞ is an $\mathcal{O}(\frac{1}{n})$ -Nash equilibrium of both $\bar{\Gamma}$ and $\tilde{\Gamma}$. However, there is no need to wait for the algorithm to converge to x^∞ , because x^{k^*} is sufficient to generate an approximate Nash equilibrium of Γ with an error bound of the same magnitude as x^∞ can achieve.

Finally, note that $\tilde{\Gamma}$ is getting closer to a potential game when n tends to ∞ .

In the case where a ‘‘Shapley-Folkman’’ disaggregation of x^{k^*} is not easy to obtain, one can use the decentralized randomized disaggregation method introduced in Section 2.4 to obtain immediately an approximate mixed-strategy Nash equilibrium, though the quality of approximation is less good than a ‘‘Shapley-Folkman’’ disaggregation.

Corollary 3.5. *For constant $\delta > 0$ and integer $K \geq \frac{2C}{m^2 L_g} n^{1+2\delta} + 1$, let $\tilde{\mu} = (\tilde{\mu}_i)_i$ be a profile of independent mixed strategies defined as in Lemma 2.10, where \tilde{x} is replaced by x^{k^*} in Proposition 3.3. Then, $\tilde{\mu}$ is a $(2L_g M \Delta (n^{-\delta} + \frac{(\sqrt{n+4})\Delta}{n}) + \frac{L_h M \Delta}{n})$ -mixed-strategy Nash equilibrium of the nonconvex game Γ .*

4 Numerical example

This section applies Algorithm 1 to game (1.1) to a specific problem in the field of electric systems taking the form of (3.1) in dimension $d = 1$. The asymptotic performance of the algorithm for large n is illustrated. To avoid rescaling cost functions for each n , *relative ϵ -Nash equilibria* defined below are considered instead of ϵ -Nash equilibria defined by Definition 2.1.

Definition 4.1. For a constant $\epsilon \geq 0$, a *relative ϵ -Nash equilibrium* $x^\epsilon \in \mathcal{X}$ in game Γ is a profile of actions of the n players such that, for each player $i \in N$,

$$f_i(x_i^\epsilon, x_{-i}^\epsilon) - \inf_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) \leq \epsilon \left(\sup_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) - \inf_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}^\epsilon) \right).$$

If $\epsilon = 0$, then x^ϵ is a Nash equilibrium.

First, game (1.1) is reformulated with uni-dimensional actions. For simplification, suppose that all the players have the same type of EV (Nissan Leaf 2018) with battery capacity e , and two charging rate levels p_{\min} and p_{\max} . The total consumption of player i is denoted by e_i and determined by a parameter τ_i as follows: $e_i = (1 - \tau_i)e = \ell_i^P + \ell_i^{OP}$, where $\tau_i \in [0, 1]$ signifies the player’s remaining proportion of energy in her battery when arriving at home. Let $x_i := \frac{\ell_i^P}{e}$ denote player i ’s strategy in the following reformulation of game (1.1):

$$\tilde{f}_i^{(n)}(x_i, x_{-i}) = \tilde{b}_i^{(n)}(x_i, x_{-i}) + \tilde{\gamma}_i \|x_i - x_i^{ref}\|^2, \quad (4.1)$$

where $\tilde{\gamma}_i$ indicates how player i values the deviation from her preferred consumption profile

and is uniformly set to be ne for simplification, and

$$\begin{aligned}\tilde{b}_i^{(n)}(x_i, x_{-i}) &= (\alpha_0^P + \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j) \ell_i^P + (\alpha_0^{OP} + \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) (1 - x_j)) \ell_i^{OP} \\ &= e(1 - \tau_i) \left[(\alpha_0^P - \alpha_0^{OP} - \beta_0 ne + 2\beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j) x_i \right. \\ &\quad \left. + \alpha_0^{OP} + \beta_0 ne - \beta_0 ne \frac{1}{n} \sum_j (1 - \tau_j) x_j \right].\end{aligned}$$

The nonconvex action set of player i , introduced in Section 1 as $S_i^{NC} = \{\ell_i = (\ell_i^P, \ell_i^{OP}) \mid \ell_i^P + \ell_i^{OP} = e_i, \ell_i^P \in \{\underline{\ell}_i^P, \overline{\ell}_i^P\}\}$, is now translated into $\mathcal{X}_i = \{\underline{x}_i, \overline{x}_i\} \subset [0, 1]$, where \underline{x}_i and \overline{x}_i correspond respectively to charging at p_{min} and p_{max} .

By extracting the common factor $ne(1 - \tau_i)$, player i 's cost function becomes

$$f_i^{(n)}(x_i, x_{-i}) := \left\langle g^{(n)}\left(\frac{1}{n} \sum_{j=1}^n (1 - \tau_j) x_j\right), x_i \right\rangle + h^{(n)}\left(\frac{1}{n} \sum_{j=1}^n (1 - \tau_j) x_j\right) + \frac{r_i^{(n)}(x_i)}{1 - \tau_i}, \quad (4.2)$$

where $g^{(n)}(y) := \frac{\alpha_0^P - \alpha_0^{OP}}{n} + \beta_0 e(2y - 1)$, $h^{(n)}(y) := \frac{\alpha_0^{OP}}{n} + \beta_0 e(1 - y)$, and $r_i^{(n)}(y) := \|y - x_i^{ref}\|^2$ for $y \in \mathbb{R}$, where $\alpha_0^P = -4.17 + 0.59 \times 12n$ (€/kWh), $\alpha_0^{OP} = -4.17 + 0.59 \times 8n$ (€/kWh), $\beta_0 = 0.295$ (€/kWh²) according to Jacquot et al. [20].

Simulation parameters The peak hours are between 6 am and 10 pm while the remaining of the day is off-peak hours. The battery capacity of Nissan Leaf 2018 is $e = 40$ kWh. The discrete action set of player i is determined as follows. The players' arrival time at home is independently generated according to a *Von Mises* distribution with parameter $\kappa = 1$ between 5 pm and 7 pm. Their departure time is independently generated according to a *Von Mises* distribution with parameter $\kappa = 1$ between 7 am and 9 am. The remaining energy proportion τ_i is independently generated according to a *Beta* distribution with parameter $\beta(2, 5)$. Once a player arrives at home, she starts charging at one of the two alternative levels, $p_{min} = 3.7$ kW or $p_{max} = 7$ kW. This power level is maintained until the energy need e_i is reached. Indeed the arrival and departure times parameters are such that the problem is always feasible i.e. the energy need e_i can always be reached during the charging period by choosing power level p_{max} . Players are all assumed to prefer charging as fast as possible, so that $x_i^{ref} = \overline{x}_i$ for all i . Fifty instances of the problem are considered for the numerical test. They are obtained by independent simulations of those parameters (players' arrival and departure time and remaining energies).

Algorithm 1 is applied to EV charging game $\Gamma^{(n)}$ (4.2) for $n = 2^s$, $s = 1, \dots, 15$. For each game $\Gamma^{(n)}$, for each iteration k of the algorithm, relative error $\epsilon^{(n),k}$ is defined by

$$\begin{aligned}\epsilon^{(n),k} &:= \min \left\{ \epsilon \geq 0 \mid f_i^{(n)}(x_i^{(n),k}, x_{-i}^{(n),k}) - \inf_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) \right. \\ &\quad \left. \leq \epsilon \left(\sup_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) - \inf_{x_i \in \mathcal{X}_i} f_i^{(n)}(x_i, x_{-i}^{(n),k}) \right) \right\},\end{aligned}$$

where $x^{(n),k}$ denotes the k^{th} iterate of Algorithm 1 applied to game $\Gamma^{(n)}$.

Figure 1 represents the minimum current error at iteration K i.e. $error^{(n),K} := \min\{\epsilon^{(n),k}, k \leq K\}$ averaged over fifty instances of the problem for different numbers of players n and different values of iteration number K of the algorithm.

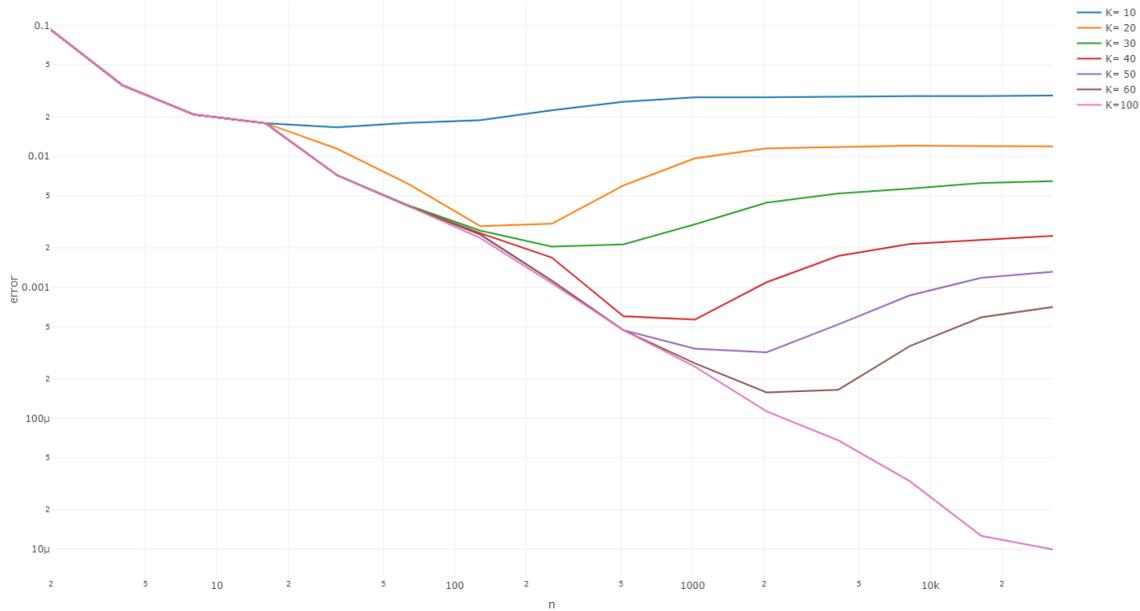


Figure 1: Log-log chart of epsilon- n (player number) relation with k iterations.

According to Proposition 3.3, when the iteration number K is fixed, due to the domination term of $\frac{2L_g M \Delta}{n}$ in (3.4) when n is small, $error^{(n),K}$ first decreases linearly in n before reaching a certain threshold. After that, $\frac{\sqrt{2CL_g M}}{m} \sqrt{\frac{n}{K}}$ dominates the error value so that $error^{(n),K}$ may increase in n . The threshold itself increases with the iteration number K . Figure 1 is consistent with this analysis.

5 Perspectives

In Section 2.4, a decentralized disaggregating method is introduced to obtain a randomized “Shapley-Folkman disaggregation” for the case $\gamma \leq 1$. It is extremely fast and easy to carry out: once an (approximate) Nash equilibrium \tilde{x} is obtained for the convexified game, each player i chooses randomly one feasible action that is in $W_i^{\lambda_i}(\tilde{x})$, the generator of \tilde{x}_i , according to a suitable distribution law.

This method renders an $\mathcal{O}(\frac{1}{\sqrt{n}^\gamma})$ -mixed strategy Nash equilibrium, with the error vanishing when the number of players going to infinity. However, even if an $\mathcal{O}(\frac{1}{n^\gamma})$ -PNE can be difficult to obtain by an exact “Shapley-Folkman disaggregation”, especially if a large, centralized program is involved, for example, to solve (2.7), it would be desirable to find other algorithms to find better approximations of Nash equilibria of the nonconvex game. Decentralized and randomized algorithms are appealing because they can be faster to carry out, needing less coordination hence more tractable, and taking advantage of the law of large numbers when n is large.

Appendix A: Proofs of Lemmata 2.3, 2.7, 2.8, 3.1 and 3.2

For Lemmata 2.3, 2.7, 2.8, in order to simplify the notations, i and $x_{-i} \in \tilde{\mathcal{X}}_{-i}$ are arbitrarily fixed. Index i and the parameter x_{-i} are thus omitted in $f_i, \tilde{f}_i, \mathcal{X}_i, \tilde{\mathcal{X}}_i$ and W_i .

Proof of Lemma 2.3.

The Lemma is a particular case of more general results well-known in Convex Analysis that have been shown in various work, such as [17, Lemma X.1.5.3]. Let us provide a proof for this particular case here for the sake of completeness.

(1) For $x \in X$, in the definition of $\tilde{f}(x)$, take $x^k = x$, $\alpha^k = \frac{1}{d+1}$ for all k . By definition, $\tilde{f}(x) \leq \sum_{k=1}^{d+1} \alpha^k f(x^k) = f(x)$.

(2) Suppose that $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$ is a minimizing sequence for $\tilde{f}(\tilde{x})$, i.e. $\tilde{f}(\tilde{x}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} f(x^{k,n})$, with $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$ satisfying conditions in (2.2). Since $(\alpha^{1,n}) \in [0, 1]$ for all n , so that it has a convergent subsequence $\alpha^{1, \phi_1(n)}$ which converges to some α^1 . Consider sequence $\alpha^{2, \phi_1(n)}$ which has a subsequence $\alpha^{2, \phi_2(n)}$ converging to some α^2 . Note that $\phi_2(n)$ is a subsequence of $\phi_1(n)$. Repeat this operation $d+1$ times and obtain subsequences $\phi_1(n), \dots, \phi_{d+1}(n)$ such that $\alpha^{k, \phi_k(n)}$ converges to α^k , for $k = 1, \dots, d+1$. Consider $x^{1, \phi_{d+1}(n)}$ which is in the compact set X . It has a convergent subsequence $x^{1, \phi_{d+2}(n)}$ converging to $x^1 \in \mathcal{X}$. Again, take a subsequence $\phi_{d+3}(n)$ such that $x^{2, \phi_{d+3}(n)}$ converges to x^2 , and so on. Finally, one obtains a subsequence $\phi_{2d+2}(n)$ of \mathbb{N} such that

$$\tilde{f}(\tilde{x}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k, \phi_{2d+2}(n)} f(x^{k, \phi_{2d+2}(n)}), \quad (5.1)$$

$$\alpha^k = \lim_{n \rightarrow \infty} \alpha^{k, \phi_{2d+2}(n)}, \quad \alpha^k \in [0, 1], \quad k = 1, 2, \dots, d+1, \quad (5.2)$$

$$\sum_{k=1}^{d+1} \alpha^k = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k, \phi_{2d+2}(n)} = 1, \quad (5.3)$$

$$x^k = \lim_{n \rightarrow \infty} x^{k, \phi_{2d+2}(n)}, \quad x^k \in \mathcal{X}, \quad k = 1, 2, \dots, d+1, \quad (5.4)$$

$$\sum_{k=1}^{d+1} \alpha^k x^k = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k, \phi_{2d+2}(n)} x^{k, \phi_{2d+2}(n)} = \lim_{n \rightarrow \infty} \tilde{x} = \tilde{x}. \quad (5.5)$$

Then,

$$\begin{aligned} \sum_{k=1}^{d+1} \alpha^k f(x^k) &\leq \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k, \phi_{2d+2}(n)} f(x^{k, \phi_{2d+2}(n)}) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k, \phi_{2d+2}(n)} f(x^{k, \phi_{2d+2}(n)}) \\ &= \tilde{f}(\tilde{x}) \leq \sum_{k=1}^{d+1} \alpha^k f(x^k). \end{aligned}$$

where the first inequality is due to (5.4), the second equality due to (5.2), the third equality due to (5.1) and the fourth inequality due to (5.3), (5.5) and (2.2). This shows that $\tilde{f}(\tilde{x}) = \sum_{k=1}^{d+1} \alpha^k f(x^k)$, i.e. $(\alpha^k, x^k)_{k=1}^{d+1}$, is a minimizer.

(3) On the one hand, for all $(x, y) \in \text{conv}(\text{epi } f)$, by Caratheodory theorem [6, Proposition 1.2.1], there exists $(x^k, y^k) \in \text{epi } f$, $k = 1, \dots, d+1$ such that $(x, y) = \sum_{k=1}^{d+1} \alpha^k (x^k, y^k)$, with

$\alpha \in \mathcal{S}_d$. Hence, $y^k \geq f(x^k)$, $y = \sum_{k=1}^{d+1} \alpha^k y^k \geq \sum_{k=1}^{d+1} \alpha^k f(x^k) \geq \tilde{f}(x)$. This shows that $(x, y) \in \text{epi } \tilde{f}$. Therefore, $\text{conv}(\text{epi } f) \subset \text{epi } \tilde{f}$. Recall that f is l.s.c. hence $\text{epi } f$ is a closed set hence so is $\text{conv}(\text{epi } f)$. Thus, $\overline{\text{conv}}(\text{epi } f) \subset \text{epi } \tilde{f}$.

On the other hand, for all $(x, y) \in \text{epi } \tilde{f}$, $y \geq \tilde{f}(x)$. Let $((\alpha^{k,n})_k, (x^{k,n})_k)_{n \in \mathbb{N}}$ be the minimizing sequence for $\tilde{f}(x)$, i.e. $\tilde{f}(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} f(x^{k,n})$, with $\alpha^{k,n}, x^{k,n}$ satisfying conditions in (2.2). Then, $y = \lim_{n \rightarrow \infty} \sum_{k=1}^{d+1} \alpha^{k,n} (f(x^{k,n}) + \frac{\delta}{d+1})$, where $\delta = y - \tilde{f}(x) \geq 0$. Denote $y^n = \sum_{k=1}^{d+1} \alpha^{k,n} (f(x^{k,n}) + \frac{\delta}{d+1})$. Then, $(x, y^n) \in \text{conv}(\text{epi } f)$, and $\lim_{n \rightarrow \infty} (x, y^n) = (x, y)$. This means that $(x, y) \in \overline{\text{conv}}(\text{epi } f)$ and, therefore, $\text{epi } \tilde{f} \subset \overline{\text{conv}}(\text{epi } f)$.

In conclusion, $\text{epi } \tilde{f}(\cdot) = \overline{\text{conv}}(\text{epi } f(\cdot))$, which implies that the epigraph of \tilde{f} is closed and convex. Thus, \tilde{f} is l.s.c. and convex on $\tilde{\mathcal{X}}$.

(4) By the lower semicontinuity of \tilde{f} and f on compact sets $\tilde{\mathcal{X}}$ and X , their minima can be attained. The equality (2.4) is thus clear by the definition in (2.2). \square

Proof of Lemma 2.7. Let $\{x^1, \dots, x^{d+1}\} \subset \mathcal{X}$ a generator of $(\tilde{x}, \tilde{f}(\tilde{x}))$ and $\alpha \in \mathcal{S}_d$ their corresponding weights.

(1) Suppose that there is k such that $f(x^k) > \tilde{f}(x^k)$. Then, there exists $(y^l)_l$ in \mathcal{X} and $\beta \in \mathcal{S}_d$ such that $x^k = \sum_l \beta^l y^l$ and $\tilde{f}(x^k) = \sum_l \beta^l f(y^l) < f(x^k)$. In consequence, $\tilde{f}(\tilde{x}) = \sum_m \alpha^m f(x^m) > \sum_{m \neq k} \alpha^m f(x^m) + \sum_l \alpha^k \beta^l f(y^l)$, while $\sum_{m \neq k} \alpha^m x^m + \sum_l \alpha^k \beta^l y^l = \tilde{x}$ and $\sum_{m \neq k} \alpha^m + \sum_l \alpha^k \beta^l = 1$, contradicting the definition of $\tilde{f}(\tilde{x})$.

(2) By the definition of subdifferential, one has

$$\tilde{f}(x^k) \geq \tilde{f}(\tilde{x}) + \langle h, x^k - \tilde{x} \rangle, \quad \forall k = 1, \dots, d+1. \quad (5.6)$$

Multiplying (5.6) by α^k for each k and adding the $d+1$ inequalities yield

$$\sum_{k=1}^{d+1} \alpha^k \tilde{f}(x^k) \geq \tilde{f}(\tilde{x}) + \langle h, \sum_k \alpha^k x^k - \tilde{x} \rangle \Leftrightarrow \tilde{f}(\tilde{x}) \geq \tilde{f}(\tilde{x}). \quad (5.7)$$

If, for at least one k , the inequality in (5.6) is strict, then the inequalities in (5.7) are strict as well, which is absurd. Therefore, for each k , $\tilde{f}(x^k) = \tilde{f}(\tilde{x}) + \langle h, x^k - \tilde{x} \rangle$. \square

Proof of Lemma 2.8. First note that \tilde{x} is in $\text{ri}(\text{conv } W^\lambda(\tilde{x}))$, the relative interior of $\text{conv } W^\lambda(\tilde{x})$. Hence, for $t > 0$ small enough, $\tilde{x} \pm t(x - \tilde{x})$ is in $\text{ri}(\text{conv } W(\tilde{x})) \subset \tilde{\mathcal{X}}$. By (2.6), $\langle h, \tilde{x} \pm t(x - \tilde{x}) - \tilde{x} \rangle \geq -\eta \|\tilde{x} \pm t(x - \tilde{x}) - \tilde{x}\|$, which yields $|\langle h, x - \tilde{x} \rangle| \leq \eta \|x - \tilde{x}\|$. Then, by Lemma 2.7, $|\tilde{f}(x) - \tilde{f}(\tilde{x})| = |\langle h, x - \tilde{x} \rangle| \leq \eta \|x - \tilde{x}\|$. \square

Proof of Lemma 3.1. (1) First show that, for any fixed $x_i \in \mathcal{X}_i$, function $\theta_i(x_i, y) := \langle g(y + \frac{\alpha_i}{n}(x_i^0 - x_i)), x_i \rangle + \ell_i(x_i)$ is $L_g \Delta$ -Lipschitz in y on Ω . For this, fix $x_i \in \mathcal{X}_i$. For any y and y'

in Ω ,

$$\begin{aligned}
|\theta_i(x_i, y') - \theta_i(x_i, y)|^2 &= \left| \left\langle g\left(y' + \frac{a_i}{n}(x_i^0 - x_i)\right) - g\left(y + \frac{a_i}{n}(x_i^0 - x_i)\right), x_i \right\rangle \right|^2 \\
&\leq \left\| g\left(y' + \frac{a_i}{n}(x_i^0 - x_i)\right) - g\left(y + \frac{a_i}{n}(x_i^0 - x_i)\right) \right\|^2 \Delta^2 \\
&= \sum_{t=1}^d \left(g_t\left(y'_t + \frac{a_i}{n}(x_{i,t}^0 - x_{i,t})\right) - g_t\left(y_t + \frac{a_i}{n}(x_{i,t}^0 - x_{i,t})\right) \right)^2 \Delta^2 \\
&\leq L_g^2 \Delta^2 \sum_{t=1}^d (y'_t - y_t)^2 \\
&= L_g^2 \Delta^2 \|y' - y\|^2,
\end{aligned}$$

where the first inequality results from the Cauchy-Schwarz inequality, while the second one is because g_t is L_{g_t} -Lipschitz.

(2) It is easy to see that $|f_i(x_i, x_{-i}) - h_i(\frac{1}{n} \sum_{j \in N} a_j x_j) - \bar{f}_i(x_i, x_{-i})| \leq \frac{L_g M \Delta^2}{n}$ for all $x_i \in \mathcal{X}_i$ and all $x_{-i} \in \tilde{\mathcal{X}}_{-i}$. Hence, if $\bar{x} \in X$ is an ϵ -Nash equilibrium of $\bar{\Gamma}$, then, for each i , for any $x_i \in \mathcal{X}_i$,

$$\begin{aligned}
f_i(\bar{x}_i, \bar{x}_{-i}) &\leq \bar{f}_i(\bar{x}_i, \bar{x}_{-i}) + h_i\left(\frac{1}{n} \sum_{j \in N} a_j \bar{x}_j\right) + \frac{L_g M \Delta^2}{n} \\
&\leq \bar{f}_i(x_i, \bar{x}_{-i}) + \epsilon + h_i\left(\frac{1}{n} a_i x_i + \frac{1}{n} \sum_{j \neq i} a_j \bar{x}_j\right) + \frac{L_h M \Delta}{n} + \frac{L_g M \Delta^2}{n} \\
&\leq f_i(x_i, \bar{x}_{-i}) + \epsilon + \frac{L_h M \Delta}{n} + \frac{2L_g M \Delta^2}{n},
\end{aligned}$$

where the second inequality is due to the definition of ϵ -Nash equilibrium and the Lipschitz continuity of h_i . \square

Proof of Lemma 3.2. Consider the following two real-valued functions defined on $\tilde{\mathcal{X}}$:

$$G_0(x) := \sum_{t=1}^d G_t\left(\frac{1}{n} \sum_{j \in N} a_j x_{j,t}\right), \quad G(x) := G_0(x) + \sum_{j \in N} \frac{a_j}{n} \tilde{r}_j(x_j), \quad (5.8)$$

where G_t is a primitive function of g_t , which exists thanks to Assumption 3.

Note that function G_0 is convex and differentiable on a neighborhood of $\tilde{\mathcal{X}}$, and convex function \tilde{r}_j is uniformly bounded on $\tilde{\mathcal{X}}_j$ for all $j \in N$ with the same bound B_ℓ , according to Assumption 3. Besides, it is easy to see that, for any i and fixed $x_{-i} \in \tilde{\mathcal{X}}_{-i}$, $\nabla_i G_0(x_i, x_{-i}) := \frac{\partial G_0(x_i, x_{-i})}{\partial x_i} = \frac{a_i}{n} g\left(\frac{1}{n} a_i x_i + \frac{1}{n} \sum_{j \neq i} a_j x_j\right)$ is $\frac{a_i^2 L_g}{n^2}$ -Lipschitz continuous on $\tilde{\mathcal{X}}_i$.

Therefore, Assumptions 1 and 2 in [47] are verified. One can thus apply Lemma 2.2 in [47] and obtains

$$\sum_{i \in N} \frac{a_i^2 L_g}{2n^2} \|x_i^k - x_i^{k+1}\|^2 \leq G(x^k) - G(x^{k+1}),$$

so that

$$\|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} (G(x^k) - G(x^{k+1})).$$

In consequence,

$$\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} (G(x^0) - G_{min}), \quad (5.9)$$

where G_{min} , defined as $\inf_{\{x \in \tilde{\mathcal{X}}\}} G(x)$, exists and is finite, because G is l.s.c. on compact set $\tilde{\mathcal{X}}$. Suppose that G_{min} is attained at $\underline{x} \in \tilde{\mathcal{X}}$, then

$$\begin{aligned} G(x^0) - G_{min} &= G(x^0) - G(\underline{x}) \\ &= \sum_{t=1}^d \int_{\frac{1}{n} \sum_{j \in N} a_j \underline{x}_{j,t}}^{\frac{1}{n} \sum_{j \in N} a_j x_{j,t}^0} g_t(s) ds + \sum_{j \in N} \frac{a_j}{n} (\tilde{r}_j(x_j^0) - \tilde{r}_j(\underline{x}_j)) \\ &\leq dM\Delta B_g + 2MB, \end{aligned} \quad (5.10)$$

where the last inequality is due to mean value theorem and Assumption 3. Combining (5.9) and (5.10) yields $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|^2 \leq \frac{2n^2}{m^2 L_g} C$. This immediately implies

$$\sum_{k=1}^K \|x^{k-1} - x^k\|^2 \leq \frac{2n^2}{m^2 L_g} C.$$

The second result of the lemma is then straightforward. \square

Appendix B: Nash equilibria in l.s.c. convex games

Lemma 5.1. *Let R be a nonempty convex compact set in \mathbb{R}^n . If real valued function $\rho(x, y)$ defined on $R \times R$ is continuous in x on R for any fixed y in R , l.s.c. in (x, y) on $R \times R$, and convex in y on R for any fixed x in R , then the set-valued map $\zeta : R \rightarrow R$, $x \mapsto \zeta(x) = \arg \min_{z \in R} \rho(x, z)$ has a fixed point.*

Proof. The Kakutani fixed-point theorem [23] will be applied for the proof. First, let us show that ζ is a Kakutani map, i.e. (i) Γ is upper semicontinuous (u.s.c.) in set map sense and (ii) for all $x \in R$, $\zeta(x)$ is non-empty, compact and convex.

(i) Fix $x \in R$. On the one hand, since $\rho(x, y)$ is convex w.r.t y , $\zeta(x)$ is convex. On the other hand, $\rho(x, y)$ is l.s.c in y , while R is compact, hence $\rho(x, y)$ can attain its minimum w.r.t y and $\zeta(x)$ is thus nonempty. Besides, since ρ is l.s.c., $\zeta(x) = \{y | \rho(x, y) \leq \min_{z \in R} \rho(x, z)\}$ is a closed subset of compact set R , hence it is compact.

(ii) Recall that the set-valued map ζ is u.s.c. if, for any open set $w \subset R$, set $\{x \in R | \zeta(x) \subset w\}$ is open.

Let us first show by contradiction that, for arbitrary $x_0 \in R$, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $z \in B(x_0, \delta)$, $\zeta(z) \subset \zeta(x_0) + B(0, \epsilon)$. If it is not true, then there exists $\epsilon_0 > 0$ and, for all $n \in \mathbb{N}^*$, point $z_n \in B(x_0, \frac{1}{n})$ such that there exists $y_n \in \zeta(z_n)$ with $d(y_n, \zeta(x_0)) > \epsilon_0$. Since sequence $\{y_n\}$ is in the compact set R , it has a subsequence $y_{\phi(n)}$ converging to some \bar{y} in R , and $d(\bar{y}, \zeta(x_0)) \geq \epsilon_0$. Then, for all $y \in R$,

$$\rho(x_0, \bar{y}) \leq \liminf_{n \rightarrow \infty} \rho(z_{\phi(n)}, y_{\phi(n)}) \leq \liminf_{n \rightarrow \infty} \rho(z_{\phi(n)}, y) = \rho(x_0, y),$$

where the first inequality is by the lower semicontinuity of ρ in (x, y) , the second inequality is by the definition of $\zeta(z_{\phi(n)})$, while the third equality is by the continuity of ρ in x . This shows that $\bar{y} \in \zeta(x_0)$, in contradiction with the fact that $d(\bar{y}, \zeta(x_0)) \geq \epsilon_0$.

Now fix arbitrarily an open set $w \subset R$ and some $x_0 \in R$ such that $\zeta(x_0) \subset w$. Since $\zeta(x_0)$ is compact while w is open, there exists $\epsilon > 0$ such that $\zeta(x_0) + B(0, \epsilon) \subset w$. According to the result of the previous paragraph, for this particular ϵ , there exists $\delta > 0$ such that $\zeta(z) \subset \zeta(x_0) + B(0, \epsilon) \subset w$ for all $z \in B(x_0, \delta)$. This means $B(x_0, \delta) \subset \{x \in R \mid \zeta(x) \subset w\}$. As a result, the set $\{x \in R \mid \zeta(x) \subset w\}$ is open.

Finally, according to the Kakutani fixed-point theorem, there exists $\tilde{x} \in R$ such that $\tilde{x} \in \zeta(\tilde{x})$. \square

Definition 5.2. A family of real-valued function $\{f(\cdot, y) : \mathcal{X} \rightarrow \mathbb{R} \mid y \in \mathcal{Y}\}$ indexed by y , with $\mathcal{X} \subset \mathbb{R}^{d_1}$ and $\mathcal{Y} \subset \mathbb{R}^{d_2}$, is *uniformly equicontinuous* if, for all $\epsilon > 0$, there exists δ such that, for all $y \in \mathcal{Y}$, $\|f(x_1, y) - f(x_2, y)\| \leq \epsilon$ whenever $\|x_1 - x_2\| \leq \delta$.

Theorem 5.3 (Existence of Nash equilibrium in l.s.c. convex games). *In an n -player game Γ where for each player $i \in \{1, \dots, n\}$, if the following three properties hold:*

- (1) *her action set \mathcal{X}_i is a convex compact subset of \mathbb{R}^d ;*
 - (2) *her cost function $f_i(x_i, x_{-i}) : \mathcal{X}_i \times \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R}$ is convex and l.s.c. in $x_i \in \mathcal{X}_i$ for any fixed $x_{-i} \in \prod_{j \neq i} \mathcal{X}_j$;*
 - (3) *the family of functions $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$ are uniformly equicontinuous,*
- then Γ has a Nash equilibrium.*

Proof. Define function $\rho(x, y) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ by $\rho(x, y) = \sum_{i=1}^n f_i(y_i, x_{-i})$, where $\mathcal{X} = \prod_i \mathcal{X}_i$. It is easy to see that a fixed point of the set-valued map $\zeta : \mathcal{X} \rightarrow \mathcal{X}$, $x \mapsto \zeta(x) = \arg \min_{z \in R} \rho(x, z)$ is a Nash equilibrium of game Γ .

In order to apply Lemma 5.1, one needs to show that: (i) $\rho(x, y)$ is continuous in x for each fixed y ; (ii) $\rho(x, y)$ is l.s.c. in (x, y) ; (iii) $\rho(x, y)$ is convex in y for each fixed x .

Results (i) and (iii) are straightforward by the definition of ρ .

For (ii), first note that, by the uniform equicontinuity of $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$ for each i and the fact that n is finite, $\{\rho(\cdot, y), y \in R\}$ is uniformly equicontinuous. Let (x^k, y^k) be a sequence in $\mathcal{X} \times \mathcal{X}$ indexed by k which converges to $(x, y) \in \mathcal{X} \times \mathcal{X}$. Then,

$$\begin{aligned} \underline{\lim}_{k \rightarrow \infty} (\rho(x^k, y^k) - \rho(x, y)) &= \underline{\lim}_{k \rightarrow \infty} (\rho(x^k, y^k) - \rho(x, y^k) + \rho(x, y^k) - \rho(x, y)) \\ &= \underline{\lim}_{k \rightarrow \infty} (\rho(x, y^k) - \rho(x, y)) \\ &\geq 0, \end{aligned}$$

where the second equality is due to the uniform equicontinuity of $\{\rho(\cdot, y), y \in \mathcal{X}\}$ and the last inequality is because $\rho(x, y)$ is l.s.c. in y for any fixed x . \square

Remark 5.1. The property (3) is weaker than the condition that f_i is continuous on \mathcal{X} . Indeed, since \mathcal{X} is compact, $f_i(x_i, x_{-i})$ is uniformly continuous on $\mathcal{X}_i \times \prod_{j \neq i} \mathcal{X}_j$ which implies the equicontinuity of $\{f_i(x_i, \cdot) : \prod_{j \neq i} \mathcal{X}_j \rightarrow \mathbb{R} \mid x_i \in \mathcal{X}_i\}$. In other words, Rosen's theorem on the existence of convex continuous games with compact convex actions sets is a corollary of Theorem 5.3.

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