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# SEPARATION OF PERIODS OF QUARTIC SURFACES

PIERRE LAIREZ AND EMRE CAN SERTÖZ

**ABSTRACT.** The periods of a quartic surface  $X$  are complex numbers obtained by integrating a holomorphic 2-form over 2-cycles in  $X$ . For any quartic  $X$  defined over the algebraic numbers and any 2-cycle  $\gamma$  in  $X$ , we give a computable positive number  $c(\gamma)$  such that the associated period  $\alpha$  satisfies either  $|\alpha| \geq c(\gamma)$  or  $\alpha = 0$ . This makes it possible in principle to certify a part of the numerical computation of the Picard group of quartics and to study the Diophantine properties of periods of quartics.

## 1. INTRODUCTION

Periods are a countable set of complex numbers containing all the algebraic numbers as well as many of the transcendental constants of nature. In light of the ubiquity of periods in mathematics and the sciences, Kontsevich and Zagier (2001) ask for the development of an algorithm to check for the equality of two given periods. We solve this problem for periods coming from quartic surfaces by giving a computable separation bound, that is, a lower bound on the minimum distance between distinct periods.

Let  $f \in \mathbb{C}[w, x, y, z]_4$  be a homogeneous quartic polynomial defining a smooth quartic  $X_f$  in  $\mathbb{P}^3(\mathbb{C})$ . The periods of  $X_f$  are the integrals of a nowhere vanishing holomorphic 2-form on  $X_f$  over integral 2-cycles in  $X_f$ . The periods can also be given in the form of integrals of a rational function

$$(1) \quad \frac{1}{2\pi i} \oint_{\gamma} \frac{dx dy dz}{f(1, x, y, z)},$$

where  $\gamma$  is a 3-cycle in  $\mathbb{C}^3 \setminus X_f$ . The integral (1) depends only on the homology class of  $\gamma$ . These periods form a group under addition. The geometry of quartic surfaces dictates that there are only 21 independent 3-cycles in  $\mathbb{C}^3 \setminus X_f$ . These give 21 periods  $\alpha_1, \dots, \alpha_{21} \in \mathbb{C}$  such that the integral over any other 3-cycle is an integer linear combination of these periods.

It is possible to compute the periods to high precision (Sertöz 2019), typically to thousands of decimal digits, and to deduce from them interesting algebraic invariants such as the Picard group of  $X_f$  (Lairez and Sertöz 2019). This point of view has been fruitful for computing algebraic invariants for algebraic curves from their periods (Booker et al. 2016; Bruin et al. 2019; Costa et al. 2019; van Wamelen 1999).

For quartic surfaces, the computation of the Picard group reduces to computing the lattice in  $\mathbb{Z}^{21}$  of integer relations  $x_1\alpha_1 + \cdots + x_{21}\alpha_{21} = 0$ ,  $x_i \in \mathbb{Z}$ . A basis for this lattice can be guessed from approximate  $\alpha_i$ 's using lattice reduction algorithms. But is it possible to *prove* that all guessed relations are true relations? Previous work related to this question (Simpson 2008) require explicit construction of algebraic curves on  $X_f$ , which becomes challenging very quickly. Instead, we give a method of proving relations by checking them at a predetermined finite precision. At the moment, this is equally challenging, but we conjecture that the numerical approach can be made asymptotically faster, see §4.5 for details.

The Lefschetz theorem on  $(1, 1)$ -classes (§2.2) associates a divisor on  $X_f$  to any integer relation between the periods of  $X_f$ . In turn, the presence of a divisor imposes algebraic conditions on the coefficients of  $f$ . Such algebraic conditions define the *Noether–Lefschetz loci* on the space of quartic polynomials (§3). In addition to the degree computations of Maulik and Pandharipande (2013), we give height bounds on the polynomial equations defining the Noether–Lefschetz loci (Theorem 14). These lead to our main result (Theorem 17): Assume  $f$  has integer coefficients, then for  $x_i \in \mathbb{Z}$ ,

$$(2) \quad x_1\alpha_1 + \cdots + x_{21}\alpha_{21} = 0 \text{ or } |x_1\alpha_1 + \cdots + x_{21}\alpha_{21}| > 2^{-c^{\max_i |x_i|} 9}$$

for some constant  $c > 0$  depending only on  $f$ . The constant  $c$  is computable in rather simple terms and without prior knowledge of the Picard group of  $X_f$ . The result generalizes to  $f$  with algebraic coefficients (Theorem 19).

As a consequence of this separation bound, we apply a construction in the manner of Liouville (1851) and prove, for instance, that the number

$$(3) \quad \sum_{n \geq 0} (2 \uparrow\uparrow 3n)^{-1}$$

is not a quotient of two periods of a single quartic surface defined over  $\mathbb{Q}$ , where  $2 \uparrow\uparrow 3n$  denotes an exponentiation tower with  $3n$  twos (Theorem 20, with  $\theta_{i+1} = 2^{2^{\theta_i}}$ ).

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## 2. PERIODS AND DEFORMATIONS

**2.1. Construction of the period map.** For any non-zero homogeneous polynomial  $f$  in  $\mathbb{C}[w, x, y, z]$ , let  $X_f$  denote the surface in  $\mathbb{P}^3$  defined as the zero locus

of  $f$ . Let  $R \doteq \mathbb{C}[w, x, y, z]$  and let  $R_4 \subset R$  be the subspace of degree 4 elements. Let  $U_4 \subset R_4$  denote the dense open subset of all homogeneous polynomials  $f$  of degree 4 such that  $X_f$  is smooth. For our purposes, it will be useful to consider not only the periods of a single quartic surface  $X_f$  but also the *period map* to study the dependence of periods on  $f$ .

The topology of  $X_f$  does not depend on  $f$  as long as  $X_f$  is smooth: given two polynomials  $f$  and  $g \in U_4$ , we can connect them by a continuous path in  $U_4$  and the surface  $X_f$  deforms continuously along this path, giving a homeomorphism  $X_f \simeq X_g$ , which is uniquely defined up to isotopy. In particular, if we fix a base point  $b \in U_4$ , then for every  $f \in \tilde{U}_4$ , where  $\tilde{U}_4$  is a universal covering of  $U_4$ , we have a uniquely determined isomorphism of cohomology groups  $H^2(X_b, \mathbb{Z}) \simeq H^2(X_f, \mathbb{Z})$ . Let  $H_{\mathbb{Z}}$  denote the second cohomology group of  $X_b$ , which is isomorphic to  $\mathbb{Z}^{22}$  (e.g. Huybrechts 2016, §1.3.3).

An element of  $\tilde{U}_4$  can be viewed as a polynomial  $f \in U_4$  together with an identification of  $H^2(X_f, \mathbb{Z})$  with  $H_{\mathbb{Z}}$ . We often work locally around a given polynomial  $f$  and, in that case, we do not actively distinguish between  $U_4$  and its universal covering.

The group  $H_{\mathbb{Z}}$  is endowed with an even unimodular pairing

$$(4) \quad (x, y) \in H_{\mathbb{Z}} \times H_{\mathbb{Z}} \rightarrow x \cdot y \in \mathbb{Z},$$

given by the intersection form on cohomology. Through this pairing, the second homology and cohomology groups are canonically identified with one another. For K3 surfaces, such as smooth quartic surfaces in  $\mathbb{P}^3$ , the structure of the lattice  $H_{\mathbb{Z}}$  with its intersection form is explicitly known (ibid., Proposition 1.3.5). The fundamental class of a generic hyperplane section of  $X_f$  gives an element of  $H_{\mathbb{Z}}$  denoted by  $h$ .

Furthermore, the complex cohomology group  $H^2(X_f, \mathbb{C})$ , which is just  $H_{\mathbb{C}} \doteq H_{\mathbb{Z}} \otimes \mathbb{C}$ , is isomorphic to the corresponding de Rham cohomology  $H_{\text{dR}}^2(X_f, \mathbb{C})$  group as follows. Elements of  $H_{\text{dR}}^2(X_f, \mathbb{C})$  are represented by differential 2-forms. To a form  $\Omega$  one associates the element  $\Theta(\Omega)$  of  $H^2(X_f, \mathbb{C})$  given by the map

$$(5) \quad \Theta(\Omega): [\gamma] \in H_2(X_f, \mathbb{C}) \mapsto \int_{\gamma} \Omega \in \mathbb{C}.$$

The group  $H_{\text{dR}}^2(X_f, \mathbb{C})$  has a distinguished element  $\Omega_f$ , a nowhere vanishing holomorphic 2-form, described below. Every other holomorphic 2-form on  $X_f$  is a scalar multiple of  $\Omega_f$  (ibid., Example 1.1.3). Mapping  $\Omega_f$  to  $H_{\mathbb{C}}$  gives rise to the *period map*

$$(6) \quad \mathcal{P}: f \in \tilde{U}_4 \mapsto \omega_f \doteq \Theta(\Omega_f) \in H_{\mathbb{C}}.$$

The coordinates of the *period vector*  $\omega_f$ , in some fixed basis of  $H_{\mathbb{Z}}$ , generates the group of periods of  $X_f$ .

To make the connection clear, we first consider the tube map

$$(7) \quad T: H_2(X_f, \mathbb{Z}) \rightarrow H_3(\mathbb{P}^3 \setminus X_f, \mathbb{Z}),$$

constructed as follows (Griffiths 1969, §3). Let  $\varepsilon > 0$  be small enough. For  $x \in X$ , the normal  $\varepsilon$ -circle over  $x$  is the set of all points  $y \in \mathbb{P}^3$  such that  $d(y, X) = \varepsilon$  and  $x$  is the closest point to  $y$  in  $X$  (which is unique if  $\varepsilon$  is small enough), that is  $d(x, y) = \varepsilon$ . The union of all normal  $\varepsilon$ -circles over the points of an effective 2-cycle  $\gamma \in H_2(X, \mathbb{Z})$  is a 3-cycle in  $\mathbb{P}^3 \setminus X$ , denoted by  $T(\gamma)$ . The map  $T$  is a surjective morphism and its kernel is generated by the class of a hyperplane section of  $X_f$ . We choose  $\Omega_f$  so that the following identity holds

$$(8) \quad \int_{\gamma} \Omega_f = \frac{1}{2\pi i} \int_{T(\gamma)} \frac{dx dy dz}{f(1, x, y, z)}.$$

Therefore, in view of (5), the coefficients of  $\omega_f$  in a basis of  $H_{\mathbb{Z}}$  coincides with periods as defined in (1).

The image  $\mathcal{D}$  of the period map  $\mathcal{P}$  is called the *period domain*. It admits a simple description:

$$(9) \quad \mathcal{D} \doteq \mathcal{P}(\tilde{U}_4) = \{w \in H_{\mathbb{C}} \setminus \{0\} \mid w \cdot h = 0, w \cdot w = 0, w \cdot \bar{w} > 0\},$$

where “ $\cdot$ ” denotes the intersection form on  $H_{\mathbb{Z}}$  and  $h$  the fundamental class of a hyperplane section, as introduced above (Huybrechts 2016, Chapter 6). Moreover, by the local Torelli theorem for K3 surfaces (ibid., Proposition 6.2.8), the map  $\mathcal{P}$  is a submersion; its derivative at any point of  $\tilde{U}_4$  is surjective.

**2.2. The Lefschetz (1,1)-theorem.** The linear integer relations between the periods of a quartic surface  $X_f$  are in correspondence with formal linear combinations of algebraic curves in  $X_f$ . Let  $C \subset X_f$  be an algebraic curve. Its fundamental class is the element  $[C]$  of  $H_{\mathbb{Z}}$  obtained as the Poincaré dual of the homology class of  $C$ . The Picard group  $\text{Pic}(X_f)$  of  $X_f$  is the sublattice of  $H_{\mathbb{Z}}$  spanned by the fundamental classes of algebraic curves.

It follows from the definition that for any class  $[\Omega] \in H_{\text{dR}}^2(X_f)$  of a differential 2-form on  $X_f$ ,

$$(10) \quad [C] \cdot \Theta(\Omega) = \int_C \Omega.$$

Moreover, if  $\Omega$  is a holomorphic 2-form, then  $\int_C \Omega = 0$  because the restriction of  $\Omega$  to the complex 1-dimensional subvariety  $C$  vanishes. In particular  $[C] \cdot \omega_f = 0$ . It turns out that this condition characterizes the elements of  $\text{Pic}(X_f)$ .

More precisely, let  $H^{1,1}(X_f) \subset H_{\mathbb{C}}$  denote the space orthogonal to  $\omega_f$  and  $\bar{\omega}_f$ , the conjugate of  $\omega_f$ , with respect to the intersection form. This space is a direct summand in the Hodge decomposition of  $H^2(X_f, \mathbb{C})$ .

The Lefschetz (1,1)-theorem (Griffiths and Harris 1978, p. 163) asserts that the lattice of integer relations coincide with the Picard group:

$$(11) \quad \text{Pic}(X_f) = H_{\mathbb{Z}} \cap H^{1,1}(X_f).$$

Noting that for any  $\gamma \in H_{\mathbb{Z}}$ ,  $\bar{\gamma} = \gamma$ , where  $\bar{\gamma}$  denotes the complex conjugate, we have  $\bar{\omega}_f \cdot \bar{\gamma} = \bar{\omega}_f \cdot \gamma$ , so that (11) becomes

$$(12) \quad \text{Pic}(X_f) = \{\gamma \in H_{\mathbb{Z}} \mid \gamma \cdot \omega_f = 0\}.$$

**2.3. A deformation argument.** Let  $\gamma_1, \dots, \gamma_{22}$  be a basis of  $H_{\mathbb{Z}}$ . The space  $H_{\mathbb{R}}$  (resp.  $H_{\mathbb{C}}$ ) is endowed with the coefficient wise Euclidean (resp. Hermitian) norm

$$(13) \quad \left\| \sum_{i=1}^{22} x_i \gamma_i \right\|^2 = \sum_{i=1}^{22} |x_i|^2.$$

For  $\gamma \in H_{\mathbb{Z}}$ , if  $|\gamma \cdot \omega_f|$  is small enough, then  $\gamma$  is close to being an integer relation between the periods of  $X_f$ . We want to argue that, in this case,  $\gamma$  is a genuine integer relation between the periods of  $X_g$  for some polynomial  $g \in U_4$  close to  $f$ .

Recall  $f, g \in \tilde{U}_4$  means  $f$  and  $g$  are smooth quartics with second cohomology identified with  $H_{\mathbb{Z}}$ . The space  $\tilde{U}_4$  inherits a metric from  $U_4$  so that  $\tilde{U}_4 \rightarrow U_4$  is locally isometric. The metric on  $U_4 \subset R_4 \simeq \mathbb{C}^{35}$  is induced by an inner product. The choice of an inner product will change the distances but this is absorbed into the constants in the statements below.

Let  $f \in \tilde{U}_4$  be fixed. For any  $g \in R_4$  and  $t \in \mathbb{C}$  small enough, the polynomials  $f + tg \in R_4$  lift canonically to  $\tilde{U}_4$ . For any  $\gamma \in H_{\mathbb{C}}$  we consider the map

$$(14) \quad \phi_{\gamma, g}(t) \doteq \gamma \cdot \mathcal{P}(f + tg)$$

which is well-defined and analytic in a neighbourhood of 0 in  $\mathbb{C}$ .

**Lemma 1.** *There is a constant  $C > 0$ , depending only on  $f$ , such that for any  $\gamma \in H_{\mathbb{C}}$  satisfying  $\gamma \cdot h = 0$  and  $|\gamma \cdot \bar{\omega}_f| \|\omega_f\| \leq \frac{1}{2} \|\gamma\| (\omega_f \cdot \bar{\omega}_f)$ , there is a monomial  $m \in R_4$  for which  $|\phi'_{\gamma, m}(0)| \geq C \|\gamma\|$ .*

*Proof.* Observe that  $\phi'_{\gamma, m}(0) = \gamma \cdot d_f \mathcal{P}(m)$ . It follows that any constant  $C$  satisfying the following inequality would work, provided the infimum is not zero,

$$(15) \quad C < \inf_{\|\gamma\|=1} \max_m |\gamma \cdot d_f \mathcal{P}(m)|,$$

with the infimum taken over  $\gamma$  satisfying  $h \cdot \gamma = 0$  and  $|\gamma \cdot \bar{\omega}_f| \|\omega_f\| \leq \frac{1}{2} (\omega_f \cdot \bar{\omega}_f)$ . If the infimum is zero, it is realized by some  $\gamma$  of norm one that annihilates  $d_f \mathcal{P}(m)$  for each monomial  $m$ . It follows that  $\gamma$  is orthogonal (with respect to the intersection product) to the tangent space  $T_{\omega_f} \mathcal{D}$  of  $\mathcal{D}$  at  $\omega_f$ . By (9),

$$(16) \quad T_{\omega_f} \mathcal{D} = \{w \in H_{\mathbb{C}} \mid w \cdot h = w \cdot \omega_f = 0\}.$$

It follows that  $\gamma = ah + b\omega_f$  for some  $a, b \in \mathbb{C}$ . The condition  $\gamma \cdot h = 0$  implies  $a = 0$  (note that  $\omega_f \cdot h = 0$  because  $\omega_f \in \mathcal{D}$ ). Since  $\|\gamma\| = 1$  we have  $|\gamma \cdot \bar{\omega}_f| = \|\omega_f\|^{-1} (\omega_f \cdot \bar{\omega}_f)$  which is a contradiction.  $\square$

The next statement is proved using the following result of Smale (1986). Let  $\phi$  be an analytic function on a maximal open disc around 0 in  $\mathbb{C}$  with  $\phi'(0) \neq 0$ .

We define

$$(17) \quad \gamma_{\text{Smale}}(\phi) \doteq \sup_{k \geq 2} \left| \frac{1}{k!} \frac{\phi^{(k)}(0)}{\phi'(0)} \right|^{\frac{1}{k-1}} \quad \text{and} \quad \beta_{\text{Smale}}(\phi) \doteq \left| \frac{\phi(0)}{\phi'(0)} \right|.$$

If  $\beta_{\text{Smale}}(\phi)\gamma_{\text{Smale}}(\phi) \leq \frac{1}{34}$ , then there is a  $t \in \mathbb{C}$  such that  $|t| \leq 2\beta_{\text{Smale}}(\phi)$  and  $\phi(t) = 0$  (Smale 1986; see also Blum et al. 1998, Chapter 8, Theorem 2).

**Proposition 2.** *For any  $f \in \tilde{U}_4$ , there exists  $C_f$  and  $\varepsilon_f > 0$  such that for all  $\varepsilon < \varepsilon_f$  the following holds. For any  $\gamma \in H_{\mathbb{R}}$ , if  $\gamma \cdot h = 0$  and  $|\gamma \cdot \omega_f| \leq \varepsilon \|\gamma\|$  then there is a monomial  $m \in R_4$  and  $t \in \mathbb{C}$  such that  $|t| \leq C_f \varepsilon$  and  $\gamma \cdot \omega_{f+tm} = 0$ .*

*Proof.* Let  $\gamma \in H_{\mathbb{R}}$  such that  $\gamma \cdot h = 0$  and

$$(18) \quad |\gamma \cdot \omega_f| \leq \left( \frac{\omega_f \cdot \bar{\omega}_f}{2\|\omega_f\|} \right) \|\gamma\|.$$

Since  $\gamma$  has real coefficients, we have  $|\gamma \cdot \omega_f| = |\gamma \cdot \bar{\omega}_f|$  and we may apply Lemma 1 to obtain a monomial  $m$  and a constant  $C$  such that

$$(19) \quad |\phi'_{\gamma,m}(0)| \geq C\|\gamma\|.$$

It follows in particular that

$$(20) \quad \beta_{\text{Smale}}(\phi_{\gamma,m}) \leq \frac{|\gamma \cdot \omega_f|}{C\|\gamma\|}.$$

Moreover, for any  $k \geq 2$ , and using  $C \leq 1$ ,

$$(21) \quad \left| \frac{1}{k!} \frac{\phi_{\gamma,m}^{(k)}(0)}{\phi'_{\gamma,m}(0)} \right|^{\frac{1}{k-1}} \leq C^{-1} \left| \frac{\phi_{\gamma,m}^{(k)}(0)}{\|\gamma\|} \right|^{\frac{1}{k-1}} = C^{-1} \left| \frac{\gamma}{\|\gamma\|} \cdot d_f^k \mathcal{P}(m, \dots, m) \right|^{\frac{1}{k-1}}$$

$$(22) \quad \leq C^{-1} \left\| \frac{1}{k!} d_f^k \mathcal{P} \right\|^{\frac{1}{k-1}},$$

where  $\|\cdot\|$  is the operator norm defined as

$$(23) \quad \left\| \frac{1}{k!} d_f^k \mathcal{P} \right\| \doteq \sup_{\gamma \in H_{\mathbb{C}}} \sup_{h_1, \dots, h_k} \frac{\left| \gamma \cdot \frac{1}{k!} d_f^k \mathcal{P}(h_1, \dots, h_k) \right|}{\|\gamma\| \|h_1\| \cdots \|h_k\|},$$

with supremum taken over  $h_1, \dots, h_k \in \mathbb{C}[w, x, y, z]_4$ . It follows that

$$(24) \quad \gamma_{\text{Smale}}(\phi_{\gamma,m}) \leq C^{-1} \sup_{k \geq 2} \left\| \frac{1}{k!} d_f^k \mathcal{P} \right\|^{\frac{1}{k-1}}.$$

Let  $\Gamma$  denote the right-hand side of (24). By Smale's theorem, together with (20) and (24), if  $|\gamma \cdot \omega_f| \leq \frac{1}{34} C^2 \Gamma^{-1} \|\gamma\|$ , then there is a  $t \in \mathbb{C}$  such that  $|t| \leq 2C^{-1} |\gamma \cdot \omega_f|$  and  $\gamma \cdot \mathcal{P}(f + tm) = 0$ . The claim follows with  $C_f \doteq 2C^{-1}$  and

$$(25) \quad \varepsilon_f \doteq \min \left( \frac{1}{34} C^2 \Gamma^{-1}, \frac{\omega_f \cdot \bar{\omega}_f}{2\|\omega_f\|} \right). \quad \square$$

The constants  $C_f$  and  $\varepsilon_f$  are actually computable with simple algorithms. The constant from Lemma 1 is not hard to get with elementary linear algebra. It only remains to compute an upper bound for  $\Gamma$ . We address this issue in §2.4.

**Corollary 3.** *For any  $f \in \tilde{U}_4$ , any  $\varepsilon < \varepsilon_f$ , and any  $\gamma \in H_{\mathbb{Z}}$ , if  $|\gamma \cdot \omega_f| \leq \frac{1}{4}\varepsilon$  then there exists a monomial  $m \in R_4$  and  $t \in \mathbb{C}$  such that  $|t| \leq C_f \varepsilon$  and  $\gamma \in \text{Pic}(X_{f+tm})$ .*

*Proof.* We may assume that  $\gamma \cdot \omega_f \neq 0$  (otherwise choose any  $m$  and  $t = 0$ ). Let  $\gamma' = \gamma - \frac{1}{4}(\gamma \cdot h)h$ . Since  $h \cdot h = 4$ , we have  $\gamma' \cdot h = 0$ . Moreover  $\gamma' \cdot \omega_f = \gamma \cdot \omega_f \neq 0$ . In particular,  $\gamma' \neq 0$  and since  $\gamma' \in \frac{1}{4}H_{\mathbb{Z}}$ , we have  $\|\gamma'\| \geq \frac{1}{4}$  and then

$$(26) \quad |\gamma' \cdot \omega_f| \leq 4\|\gamma'\| |\gamma \cdot \omega_f| \leq \varepsilon\|\gamma'\|,$$

and Proposition 2 applies.  $\square$

**2.4. Effective bounds for the higher derivatives of the period map.** In the proof of Proposition 2, only the quantity  $\Gamma$  is not clearly computable. We show in this section how to compute an upper bound for  $\Gamma$  using the Griffiths–Dwork reduction. We follow here Griffiths (1969).

Firstly, as a variant of (8) avoiding dehomogenization, we write

$$(27) \quad \mathcal{P}(f) = \left( \frac{1}{2\pi i} \int_{T(\gamma_i)} \frac{\text{Vol}}{f} \right)_{1 \leq i \leq 22}$$

where Vol is the projective volume form

$$(28) \quad \text{Vol} \doteq w dx dy dz - x dw dy dz + y dw dx dz - z dw dx dy.$$

For any  $k > 0$  and  $a \in R_{4k-4}$ , we denote

$$(29) \quad \int \frac{a \text{Vol}}{f^k} \doteq \left( \frac{1}{2\pi i} \int_{T(\gamma_i)} \frac{a \text{Vol}}{f^k} \right)_{1 \leq i \leq 22} \in H_{\mathbb{C}}.$$

For any  $h \in R_4$  close enough to 0, we have the power series expansion

$$(30) \quad \int \frac{\text{Vol}}{f+h} = \sum_{k \geq 1} (-1)^{k-1} \int \frac{h^{k-1} \text{Vol}}{f^k}.$$

**Proposition 4.** *For any  $k \geq 3$ , there is a linear map  $G_k: R_{4k-4} \rightarrow R_8$  such that*

$$\int \frac{a}{f^k} \text{Vol} = \int \frac{G_k(a)}{f^3} \text{Vol}.$$

*Moreover, there is a computable constant  $C$ , which depends only on  $f$ , such that for any  $k \geq 3$ ,  $\|G_k\| \leq C^{k-3}$ , where  $R$  is endowed with the 1-norm (55).*

Before we begin the proof of proposition, let us show that this is enough to bound  $\Gamma$ . Let  $A: a \in R_8 \mapsto \int \frac{a}{f^3} \text{Vol} \in H_{\mathbb{C}}$ , then, using (30) we obtain

$$(31) \quad \int \frac{\text{Vol}}{f+h} = \sum_{k \geq 1} (-1)^{k-1} A(G_k(h^{k-1})),$$

and it follows that

$$(32) \quad \frac{1}{k!} d_f^k \mathcal{P}(h_1, \dots, h_k) = (-1)^k A(G_{k+1}(h_1 \cdots h_k)).$$



In particular,

$$(33) \quad \left\| \frac{1}{k!} d_f^k \mathcal{P}(h_1, \dots, h_k) \right\| \leq \|A\| \|G_{k+1}\| \|h_1 \cdots h_n\|_1$$

$$(34) \quad \leq \|A\| \|G_{k+1}\| \|h_1\|_1 \cdots \|h_n\|_1,$$

and therefore  $\left\| \frac{1}{k!} d_f^k \mathcal{P} \right\| \leq \|A\| C^{k+1}$ , and it follows

$$(35) \quad \Gamma \leq C \max(\|A\| C^2, 1)$$

2.4.1. *Proof of Proposition 4.* Let  $R = \mathbb{C}[w, x, y, z]$ . We define two families of maps for this proof. First, for  $d \geq 12$ , a multivariate division map  $Q_d: R_d \rightarrow R_{d-3}^4$ , such that for any  $a \in R_d$ ,

$$(36) \quad a = \sum_{i=0}^3 Q_d(a)_i \partial_i f.$$

Note that such a map exists as soon as  $d \geq 12$  by a theorem due to Macaulay (see Lazard 1977, Corollaire, p. 169). The choice of  $Q_d$  is not unique. We fix  $Q_{12}$  arbitrarily and define  $Q_d(a)$ , for  $d > 12$  and  $a \in R_{12}$ , as follows. Write  $a = \sum_{i=0}^3 x_i a_i$ , in such a way that the terms of the sum have disjoint monomial support, and define

$$(37) \quad Q_d(a) = \sum_{i=0}^3 x_i Q_{d-1}(a_i).$$

It is easy to check that this definition satisfies (36).

Second, for  $k \geq 3$ , we define  $G_k: R_{4k-4} \rightarrow R_8$  as follows. Begin with  $G_3 = \text{id}$  and then define  $G_k$  for  $k \geq 4$  inductively as follows. For  $a \in R_{4k-4}$  we write  $(b_0, \dots, b_3) = Q_{4k-4}(a)$  and define

$$(38) \quad G_k(a) \doteq G_{k-1} \left( \frac{1}{k-1} (\partial_0 b_0 + \cdots + \partial_3 b_3) \right).$$

This map is the Griffiths–Dwork reduction, and it satisfies

$$(39) \quad \int_{\gamma} \frac{a\Omega}{f^k} = \int_{\gamma} \frac{G_k(a)\Omega}{f^3}.$$

**Lemma 5.** *For any  $d \geq 12$ ,  $\|Q_d\| \leq \|Q_{12}\|$ , where  $R$  is endowed with the 1-norm and  $R^4$  with the norm  $\|(f_0, \dots, f_3)\|_1 \doteq \|f_0\|_1 + \cdots + \|f_3\|_1$ .*

*Proof.* For any  $a \in R_d$ ,

$$(40) \quad \|Q_d(a)\|_1 = \sum_{i=0}^3 \|Q_d(a)_i\|_1 \leq \sum_{i=0}^3 \sum_{j=0}^3 \|x_j Q_{d-1}(a_j)_i\|_1$$

$$(41) \quad = \sum_{i=0}^3 \sum_{j=0}^3 \|Q_{d-1}(a_j)_i\|_1 = \sum_j \|Q_{d-1}(a_j)\|_1$$

$$(42) \quad \leq \|Q_{d-1}\| \sum_j \|a_j\|_1 = \|Q_{d-1}\| \|a\|_1,$$

using, for the last equality, that the terms  $a_j$  have disjoint monomial support.  $\square$

**Lemma 6.** *For any  $k \geq 3$ ,  $\|G_k\| \leq (4 \|Q_{12}\|)^{k-3}$ , where  $R$  is endowed with the 1-norm.*

*Proof.* We proceed by induction on  $k$  (the base case  $k = 3$  is trivial since  $G_3 = \text{id}$ ). Let  $a \in R_{4k-4}$  and  $(b_0, \dots, b_3) = Q_{4k-4}(a)$ . By (38), we have

$$(43) \quad \|G_k(a)\|_1 \leq \frac{\|G_{k-1}\|}{k-1} (\|\partial_0 b_0\|_1 + \dots + \|\partial_3 b_3\|_1).$$

By induction hypothesis,  $\|G_{k-1}\| \leq (4 \|Q_{12}\|)^{k-4}$  and moreover  $\|\partial_i b_i\|_1 \leq (4k-7)\|b_i\|_1$ , since each  $b_i$  has degree  $4k-7$ . It follows that

$$(44) \quad \|G_k(a)\|_1 \leq (4 \|Q_{12}\|)^{k-4} \frac{4k-7}{k-1} (\|b_0\|_1 + \dots + \|b_3\|_1).$$

Next, we note that  $\|b_0\|_1 + \dots + \|b_3\|_1 = \|Q_{4k-4}(a)\|_1$  and, by Lemma 5,  $\|Q_{4k-4}(a)\| \leq \|Q_{12}\|$ . Therefore

$$(45) \quad \|G_k(a)\|_1 \leq (4 \|Q_{12}\|)^{k-3} \|a\|_1,$$

and the claim follows.  $\square$

### 3. THE NOETHER–LEFSCHETZ LOCUS

**3.1. Basic properties.** We define the Noether–Lefschetz locus for quartic surfaces and review a few classical properties, especially algebraicity, with a view towards Theorem 14 about the degree and the height of the equations defining the components of the Noether–Lefschetz locus.

**3.1.1. Definition.** The Noether–Lefschetz locus of quartics  $\mathcal{NL}$  is the set of all  $f \in U_4$  such that the rank of  $\text{Pic}(X_f)$  is at least 2. Equivalently, in view of (12),  $\mathcal{NL}$  is the set of quartic polynomials  $f$  whose primitive periods (1) are  $\mathbb{Z}$ -linearly dependent.

The set  $\mathcal{NL}$  is locally the union of smooth analytic hypersurfaces in  $U_4$ . To see this, let  $\widetilde{\mathcal{NL}}$  be the lift of  $\mathcal{NL}$  in the universal covering  $\widetilde{U}_4$  of  $U_4$ . Recall  $\mathcal{P}: \widetilde{U}_4 \rightarrow \mathcal{D}$  is the period map. The Lefschetz (1,1)-theorem implies

$$(46) \quad \widetilde{\mathcal{NL}} = \bigcup_{\gamma \in H_{\mathbb{Z}} \setminus \mathbb{Z}h} \mathcal{P}^{-1} \{w \in \mathcal{D} \mid w \cdot \gamma = 0\}.$$

That is,  $\widetilde{\mathcal{NL}}$  is the pullback of smooth hyperplane sections of  $\mathcal{D}$ . Since  $\mathcal{P}$  is a submersion,  $\widetilde{\mathcal{NL}}$  is the union of smooth analytic hypersurfaces. It follows that  $\mathcal{NL}$  is locally the union of smooth analytic hypersurfaces.

We break  $\mathcal{NL}$  into algebraic pieces as follows. For any integers  $d$  and  $g$ , let  $\mathcal{NL}_{d,g}$  be the set

$$(47) \quad \mathcal{NL}_{d,g} = \{f \in U_4 \mid \exists \gamma \in \text{Pic}(X_f) \setminus \mathbb{Z}h : \gamma \cdot h = d \text{ and } \gamma \cdot \gamma = 2g - 2\},$$

By replacing  $\gamma$  by  $\gamma + h$  or  $-\gamma$ , we observe that

$$(48) \quad \mathcal{NL}_{d,g} = \mathcal{NL}_{d+4,g+d+2} = \mathcal{NL}_{-d,g}.$$

In particular,  $\mathcal{NL}_{d,g}$  is equal to some  $\mathcal{NL}_{d',g'}$  with  $d' > 0$  and  $g' \geq 0$ , so that

$$(49) \quad \mathcal{NL} = \bigcup_{d>0} \bigcup_{g \geq 0} \mathcal{NL}_{d,g}.$$

For  $\gamma \in H_{\mathbb{Z}}$ , let  $\Delta(\gamma) = (h \cdot \gamma)^2 - 4\gamma \cdot \gamma$ . It is the opposite of the discriminant of the lattice generated by  $h$  and  $\gamma$  in  $H_{\mathbb{Z}}$ , with respect to the intersection product (and it is zero if  $\gamma \in \mathbb{Z}h$ ). It follows from the Hodge index theorem (see Hartshorne 1977, Theorem V.1.9) that for any  $f \in U_4$  and any  $\gamma \in \text{Pic}(X_f)$ ,  $\Delta(\gamma) \geq 0$ , with equality if and only if  $\gamma \in \mathbb{Z}h$ . If  $\gamma \cdot h = d$  and  $\gamma \cdot \gamma = 2g - 2$ , then  $\Delta(\gamma) = d^2 - 8g + 8$ . We obtain therefore that for any  $d > 0$  and  $g \geq 0$ ,

$$(50) \quad \mathcal{NL}_{d,g} = \begin{cases} \{f \in U_4 \mid \exists \gamma \in \text{Pic}(X_f) : \gamma \cdot h = d \text{ and } \gamma \cdot \gamma = 2g - 2\} & \text{if } d^2 > 8g - 8 \\ \emptyset & \text{otherwise.} \end{cases}$$

It is in fact more natural to introduce, for  $\Delta > 0$ , the following locus

$$(51) \quad \mathcal{NL}_{\Delta} \doteq \{f \in U_4 \mid \exists \gamma \in \text{Pic}(X_f) : \Delta(\gamma) = \Delta\}$$

$$(52) \quad = \bigcup_{\substack{d>0 \\ d^2 \equiv \Delta \pmod{8}}} \mathcal{NL}_{d, \frac{d^2 - \Delta}{8} + 1}.$$

Due to (48),  $\mathcal{NL}_{\Delta}$  reduces to a single  $\mathcal{NL}_{d,g}$ . Namely,

$$(53) \quad \mathcal{NL}_{\Delta} = \begin{cases} \mathcal{NL}_{4t, 2t^2 + \frac{8-\Delta}{8}}, & \text{if } \Delta \equiv 0 \pmod{8}, \\ \mathcal{NL}_{4t+1, 2t^2+t + \frac{9-\Delta}{8}}, & \text{if } \Delta \equiv 1 \pmod{8}, \\ \mathcal{NL}_{4t+2, 2t^2+2t + \frac{12-\Delta}{8}}, & \text{if } \Delta \equiv 4 \pmod{8}, \\ \emptyset & \text{otherwise,} \end{cases}$$

where  $t = \lceil \frac{1}{4}\sqrt{\Delta} \rceil$ . Conversely, each  $\mathcal{NL}_{d,g} = \mathcal{NL}_{d^2-8g+8}$ .

**3.1.2. Algebraicity.** For any  $d > 0$  and  $g \geq 0$ , the set  $\mathcal{NL}_{d,g}$  is either empty or an algebraic hypersurface in  $U_4$ . This is a classical result (e.g. Voisin 2003, Theorem 3.32) which we recall here to obtain an explicit algebraic description of  $\mathcal{NL}_{d,g}$ .

**Lemma 7.** *For any  $f \in U_4$ ,  $d > 0$  and  $g \geq 0$  we have:  $f \in \mathcal{NL}_{d,g}$  if and only if  $X_f$  contains an effective divisor with Hilbert polynomial  $t \mapsto dt + 1 - g$ .*

*Proof.* Assume that  $X_f$  contains an effective divisor  $C$  with Hilbert polynomial  $t \mapsto td + 1 - g$ . Since  $X_f$  is smooth,  $C$  is a locally principal divisor and gives an element  $\gamma$  of  $\text{Pic } X_f$ . The integer  $d$  is the degree of  $C$ , so it is the number of points in the intersection with a generic hyperplane, that is  $d = \gamma \cdot h$ . Moreover,  $g$  is the arithmetic genus of  $C$ , which is determined by  $2g - 2 = \gamma \cdot \gamma$  (Hartshorne 1977, Ex. III.5.3(b) and V.1.3(a)). So  $f \in \mathcal{NL}_{d,g}$ .

Conversely, let  $f \in \mathcal{NL}_{d,g}$ . By definition, there is a divisor  $C$  on  $X_f$  such that its class  $\gamma$  in  $\text{Pic } X_f$  satisfies  $\gamma \cdot h = d$  and  $\gamma \cdot \gamma = 2g - 2$ . From the Riemann–Roch theorem for surfaces (ibid., p. V.1.6) we get:

$$\dim H^0(X, \mathcal{O}_X(C)) + \dim H^0(X, \mathcal{O}_X(-C)) \geq \frac{1}{2}\gamma \cdot \gamma + 2 = g + 1 > 0$$

so that either  $C$  or  $-C$  must be linearly equivalent to an effective divisor. Since  $\gamma \cdot h > 0$ ,  $-C$  can not be effective and therefore  $C$  must be. As above, the Hilbert polynomial of  $C$  is given by  $t \mapsto dt + 1 - g$ .  $\square$

In light of Lemma 7, the algebraicity of  $\mathcal{NL}_{d,g}$  is proved by using the Hilbert scheme  $\mathcal{H}_{d,g}$ . The Hilbert scheme  $\mathcal{H}_{d,g}$  of degree  $d$  and genus  $g$  curves in  $\mathbb{P}^3$  is a projective scheme that parametrizes all the subschemes of  $\mathbb{P}^3$  whose Hilbert polynomial is  $t \mapsto dt + 1 - g$ .

The Hilbert scheme  $\mathcal{H}_{d,g}$  may contain components that are not desirable for our purposes. For example  $\mathcal{H}_{3,0}$ , which contains twisted cubics in  $\mathbb{P}^3$ , contains two irreducible components (Piene and Schlessinger 1985): a 12-dimensional component that is the closure of the space of all smooth cubic rational curves in  $\mathbb{P}^3$ ; and a 15-dimensional component parametrizing the union of a plane cubic curve with a point in  $\mathbb{P}^3$ . We would be only interested in the first, not in the second component. So we introduce  $\mathcal{H}'_{d,g}$ , the union of components of  $\mathcal{H}_{d,g}$  obtained by removing the components that does not correspond to locally-complete-intersection pure-dimensional subschemes of  $\mathbb{P}^3$ .

When  $d^2 > 8g - 8$ , Lemma 7 can be rephrased as

$$(54) \quad \mathcal{NL}_{d,g} = \text{proj}_1 \{ (f, C) \in U_4 \times \mathcal{H}'_{d,g} \mid C \subset X_f \},$$

where  $\text{proj}_1$  denotes the projection  $U_4 \times \mathcal{H}'_{d,g} \rightarrow U_4$ . Since  $\mathcal{H}'_{d,g}$  is a projective variety, and the condition  $C \subset X_f$  is algebraic, this shows that  $\mathcal{NL}_{d,g}$  is a closed subvariety of  $U_4$  (for more details about this construction, see Voisin 2003, §3.3).

We note furthermore that  $\mathcal{NL}_{d,g}$  is clearly invariant under the action of the Galois group of algebraic numbers. Therefore, it can be defined over the rational numbers.

As a consequence, for any nonnegative integers  $d$  and  $g$ , there is a squarefree primitive homogeneous polynomial  $\text{NL}_{d,g} \in \mathbb{Z}[u_1, \dots, u_{35}]$  in the 35 coefficients of the general quartic polynomial that is unique up to sign and whose zero locus is  $\mathcal{NL}_{d,g}$  in  $U_4$ . Similarly, we define  $\text{NL}_\Delta$  upto sign as the unique squarefree primitive polynomial vanishing exactly on  $\mathcal{NL}_\Delta$ .

**3.2. Height of multiprojective varieties.** The mainstay of our results is a bound on the degree and size of the coefficients of the polynomials  $\text{NL}_{d,g}$ . The determination of these bounds is based on (54) and involves the theory of heights of multiprojective varieties as developed by D'Andrea et al. (2013), and, before them, Bost et al. (1991), Krick et al. (2001), Philippon (1995), and Rémond (2001a,b), among others. We recall here the results that we need, following D'Andrea et al. (2013).

**3.2.1. Heights of polynomials.** Let  $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} \in \mathbb{C}[x_1, \dots, x_n]$ . We recall the following different measures of height of  $f$ :

$$(55) \quad \|f\|_1 \doteq \sum_{\alpha} |c_{\alpha}|,$$

$$(56) \quad \|f\|_{\sup} \doteq \sup_{|x_1|=\dots=|x_n|=1} |f(\mathbf{x})|,$$

$$(57) \quad m(f) \doteq \int_{[0,1]^n} \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| dt_1 \cdots dt_n.$$

**Lemma 8** (D’Andrea et al. 2013, Lemma 2.30). *For any homogeneous polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ ,*

$$\exp(m(f)) \leq \|f\|_{\sup} \leq \|f\|_1 \leq \exp(m(f))(n+1)^{\deg f}.$$

**3.2.2. The extended Chow ring.** The extended Chow ring (ibid., Definition 2.50) is a tool to track a measure of height of multiprojective varieties when performing intersections and projections. We present here a very brief summary. Bold letters refer to multi-indices and all varieties are considered over  $\mathbb{Q}$ . Let  $\mathbf{n} \in \mathbb{N}^r$  and let  $\mathbb{P}^{\mathbf{n}}$  be the multiprojective space  $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ .

An *algebraic cycle* is a finite  $\mathbb{Z}$ -linear combination  $\sum_V n_V V$  of irreducible subvarieties of  $\mathbb{P}^{\mathbf{n}}$ . The *irreducible components* of an algebraic cycle as above are the irreducible varieties  $V$  such that  $n_V \neq 0$ . An algebraic cycle is *equidimensional* if all its irreducible components have the same dimension. An algebraic cycle is *effective* if  $n_V \geq 0$  for all  $V$ . The *support* of  $X$ , denoted by  $\text{supp } X$ , is the union of the irreducible components of  $X$ .

Let  $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$  be the extended Chow ring, namely

$$(58) \quad A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z}) \doteq \mathbb{R}[\eta, \theta_1, \dots, \theta_m] / (\eta^2, \theta_1^{n_1+1}, \dots, \theta_m^{n_m+1}),$$

where  $\theta_i$  is the class of the pullback of a hyperplane from  $\mathbb{P}^{n_i}$  and  $\eta$  is used to keep track of heights of varieties. For two elements  $a$  and  $b$  of this ring, we write  $a \leq b$  when the coefficients of  $b - a$  in the monomial basis are nonnegative.

To an algebraic cycle  $X$  of  $\mathbb{P}^{\mathbf{n}}$  we associate an element  $[X]_{\mathbb{Z}}$  of  $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$  (ibid., Definition 2.50). If  $X$  is effective, then  $[X]_{\mathbb{Z}} \geq 0$ . The coefficients of the terms in  $[X]_{\mathbb{Z}}$  for monomials not involving  $\eta$  record the usual multi-degrees of  $X$ . The terms involving  $\eta$  record mixed canonical heights of  $X$ . The definition of these heights is based on the heights of various Chow forms associated to  $X$  (ibid., §2.3). For the computations in this paper, we only need the following results.

Let  $f \in \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_r]$  be a nonzero multihomogeneous polynomial with respect to the group of variables  $\mathbf{x}_1, \dots, \mathbf{x}_r$ . We assume that  $f$  is *primitive*, that is, the g.c.d. of the coefficients of  $f$  is 1. The element associated in  $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$  to the hypersurface  $V(f) \subset \mathbb{P}^{\mathbf{n}}$  is (ibid., Proposition 2.53(2))

$$(59) \quad [V(f)]_{\mathbb{Z}} = m(f)\eta + \deg_{\mathbf{x}_1}(f)\theta_1 + \cdots + \deg_{\mathbf{x}_r}(f)\theta_r.$$

To such a polynomial  $f$ , we also associate (ibid., Eq. (2.57))

$$(60) \quad [f]_{\sup} \doteq \log(\|f\|_{\sup})\eta + \deg_{\mathbf{x}_1}(f)\theta_1 + \cdots + \deg_{\mathbf{x}_r}(f)\theta_r.$$

**3.2.3. Arithmetic Bézout theorem.** Let  $X$  be an effective cycle and  $H$  a hypersurface in  $\mathbb{P}^{\mathbf{n}}$ . They *intersect properly* if no irreducible component of  $X$  is in  $H$ . When  $X$  and  $H$  intersect properly, one defines an intersection product  $X \cdot H$ , that

is an effective cycle supported on  $X \cap H$ . If  $X$  is equidimensional of dimension  $r$ , then  $X \cdot H$  is equidimensional of dimension  $r - 1$ .

The following statement is an arithmetic Bézout bound that not only bounds the degree, as with the classical Bézout bound, but also the height of an intersection.

**Theorem 9** ([ibid.](#), Theorem 2.58). *Let  $X$  be an effective equidimensional cycle on  $\mathbb{P}^n$  and  $f \in \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_m]$ . If  $X$  and  $V(f)$  intersect properly, then  $[X \cdot V(f)]_{\mathbb{Z}} \leq [X]_{\mathbb{Z}} \cdot [f]_{\text{sup}}$ .*

This theorem can be applied (as in [ibid.](#), Corollary 2.61) to bound the height of the irreducible components of a variety in terms of its defining equations.

**Proposition 10.** *Let  $Z \subset \mathbb{P}^n$  be an equidimensional variety and let  $X$  be  $V(f_1, \dots, f_s) \cap Z$ , where  $f_i$  is a multihomogeneous polynomial of multidegree at most  $\mathbf{d}$  and sup-norm at most  $L$ . Let  $X_r$  be the union of all the irreducible components of  $X$  of codimension  $r$  in  $Z$ . Then*

$$[X_r]_{\mathbb{Z}} \leq [Z]_{\mathbb{Z}} \left( \log(sL)\eta + \sum_{i=1}^m d_i \theta_i \right)^r.$$

*Proof.* Let  $(y_{ij})$  be a new group of variables, with  $1 \leq i \leq r$  and  $1 \leq j \leq s$ . Let  $g_i \doteq \sum_{j=1}^s y_{ij} f_j$  and  $X' \doteq V(g_1, \dots, g_r)$  in  $\mathbb{P}^k \times Z$ , with  $k = rs - 1$ . We first claim that  $\mathbb{P}^k \times X_r$  is a union of components of  $X'$ . Indeed, let  $\xi_0$  be the generic point of  $\mathbb{P}^k$  and  $\xi_1$  be the generic point of a component  $Y$  of  $X_r$ , so that  $\xi = (\xi_0, \xi_1)$  is the generic point of the component  $\mathbb{P}^k \times Y$  of  $\mathbb{P}^k \times X_r$ . Since  $X$  has codimension  $r$  at  $\xi_1$ , the generic linear combinations  $g_1, \dots, g_r$  form a regular sequence at  $\xi$  (in other words, they form a regular sequence at  $\xi_1$  for generic values of the  $v_{ij}$ ). Therefore,  $X'$  has codimension  $r$  at  $\xi$ . Since  $\mathbb{P}^k \times Y \subseteq X'$ , it follows that  $\mathbb{P}^k \times Y$  is a component of  $X'$ .

Let  $X'_r$  be the union of the components of codimension  $r$  of  $X'$ . The argument above shows that  $[\mathbb{P}^k \times X_r]_{\mathbb{Z}} \leq [X'_r]_{\mathbb{Z}}$ . Besides, by repeated application of ([ibid.](#), Corollary 2.61),

$$(61) \quad [X'_r]_{\mathbb{Z}} \leq [\mathbb{P}^k \times Z]_{\mathbb{Z}} \prod_{i=1}^r [g_i]_{\text{sup}}.$$

We compute, using (59) that

$$(62) \quad [g_i]_{\text{sup}} \leq \log(sL)\eta + \theta_0 + \sum_{i=1}^s d_i \theta_i.$$

Finally, we note that  $[\mathbb{P}^k \times X_r]_{\mathbb{Z}} = [X_r]_{\mathbb{Z}}$  and  $[\mathbb{P}^k \times Z]_{\mathbb{Z}} = [Z]_{\mathbb{Z}}$  ([ibid.](#), Proposition 2.51.3 and 2.66).  $\square$

**Proposition 11.** *Let  $X$  be an equidimensional closed subvariety of  $\mathbb{P}^k \times \mathbb{P}^n$  and let  $Y \subset \mathbb{P}^n$  be the projection of  $X$ . If  $Y$  is equidimensional, then*

$$\theta_0^k[Y]_{\mathbb{Z}} \leq \theta_0^{\dim X - \dim Y}[X]_{\mathbb{Z}} \in A^*(\mathbb{P}^k \times \mathbb{P}^n; \mathbb{Z}),$$

where  $\theta_0$  is the variable attached to  $\mathbb{P}^k$  in the extended Chow ring of  $\mathbb{P}^k \times \mathbb{P}^n$ .

*Proof.* We will argue by induction on  $r \doteq \dim X - \dim Y$ . When  $r = 0$ , this is (D'Andrea et al. 2013, Proposition 2.64).

Suppose now that  $r > 0$  and  $X$  is irreducible. Let  $\mathbb{Q}[\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_m]$  denote the multihomogeneous coordinate ring of  $\mathbb{P}^k \times \mathbb{P}^n$ . There is an  $i$ ,  $0 \leq i \leq k$ , such that  $H \doteq V(y_i) \subset \mathbb{P}^k \times \mathbb{P}^n$  intersects  $X$  properly (otherwise  $X$  would be included in all  $V(y_i)$  and would be empty). Since the fibers of  $X \rightarrow Y$  are positive dimensional,  $H$  intersects each fiber. In particular, the set-theoretical projections of  $X$  and  $X \cap H$  coincide. As  $X$  is irreducible, so is  $Y$ . In particular, there is an irreducible component  $X' \subset X \cap H$  that maps to  $Y$ . By induction hypothesis applied to  $X'$ ,  $\theta_0^k[Y]_{\mathbb{Z}} \leq \theta_0^{\dim X' - \dim Y}[X']_{\mathbb{Z}}$ . Moreover,  $[X']_{\mathbb{Z}} \leq [X]_{\mathbb{Z}}[y_i]_{\text{sup}}$ , and, in view of (60),  $[y_i]_{\text{sup}} = \theta_0$ . The claim follows.

If  $X$  is reducible, then we apply the inequality above to each of the irreducible components of  $Y$  together with an irreducible component of  $X$  mapping onto that component.  $\square$

**3.3. Explicit equations for the Noether–Lefschetz loci.** Following Gotzmann (1978), Bayer (1982), and the exposition of Lella (2012), we describe the equations defining the Hilbert schemes of curves in  $\mathbb{P}^3$ . An explicit description of the Noether–Lefschetz loci  $\mathcal{NL}_{d,g}$  follows.

**3.3.1. The Hilbert schemes of curves.** For  $d > 0$  and  $g \geq 0$  let  $\mathcal{H}_{d,g}$  be the Hilbert scheme of curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^3$ . It parametrizes subschemes of  $\mathbb{P}^3$  with Hilbert polynomial  $p(m) \doteq dm + 1 - g$ . Smooth curves in  $\mathbb{P}^3$  of degree  $d$  and genus  $g$ , in particular, have Hilbert polynomial  $p(m)$ . Let  $R = \mathbb{C}[w, x, y, z]$  be the homogeneous coordinate ring of  $\mathbb{P}^3$ . For  $m \geq 0$ , let  $R_m$  denote the  $m$ th homogeneous part of  $R$  and let  $q(m) = \dim R_m - p(m)$ .

The Hilbert scheme  $\mathcal{H}_{d,g}$  can be realized in a Grassmannian variety as follows. A subscheme  $X$  of  $\mathbb{P}^3$  is uniquely defined by a saturated homogeneous ideal  $I$  of  $R$ . If the Hilbert polynomial of  $X$  is  $p$ , then  $I$  is the saturation of the ideal generated by the degree  $r$  slice  $I_r \doteq I \cap R_r$  (Gotzmann 1978; Bayer 1982, §II.10), where

$$(63) \quad r = \binom{d}{2} + 1 - g,$$

is the Gotzmann number of  $p$  (Bayer 1982, §II.1.17). For practical reasons, we need  $r \geq 4$ , so we define instead

$$(64) \quad r = \max \left( \binom{d}{2} + 1 - g, 4 \right).$$

So  $X$  is entirely determined by  $I_r$ , which is a  $q(r)$ -dimensional subspace of  $R_r$ .

Let  $\mathbb{G}$  be the Grassmannian variety of  $q(r)$ -dimensional subspaces of  $R_r$ . As a set, one can construct  $\mathcal{H}_{d,g}$  as the subset of all  $\Xi \in \mathbb{G}$  such that the ideal generated by  $\Xi$  in  $R$  defines a subscheme of  $\mathbb{P}^3$  with Hilbert polynomial  $p$ . In fact,  $\mathcal{H}_{d,g}$  is a subvariety that is defined by the following condition (ibid., §VI.1):

$$(65) \quad \mathcal{H}_{d,g} = \{\Xi \in \mathbb{G} \mid \dim(R_1 \Xi) \leq q(r+1)\},$$

where  $R_1$  is the space of linear forms in  $w, x, y, z$ , so that  $R_1 \Xi$  is a subspace of  $R_{r+1}$ .

Several authors gave explicit equations for  $\mathcal{H}_{d,g}$  in the Plücker coordinates (Bayer 1982; Brachat et al. 2016; Gotzmann 1978; Grothendieck 1961). We will prefer here a more direct path that avoids the Plücker embedding.

**3.4. Equations for the relative Hilbert scheme.** Define the relative Hilbert scheme of curves inside quartic surfaces

$$(66) \quad \mathcal{H}_{d,g}(4) \doteq \{(f, C) \in \mathbb{P}(R_4) \times \mathcal{H}_{d,g} \mid C \subset V(f)\},$$

for each  $d > 0$ ,  $g \geq 0$ .

We define the following auxiliary spaces to better describe (66). First, define the following ambient space

$$(67) \quad \mathcal{A} \doteq \mathbb{P}(R_4) \times \mathbb{P}\left(\text{End}(\mathbb{C}^{q(r)-N_{r-4}}, R_r)\right) \times \mathbb{P}\left(\text{End}(R_{r+1}, \mathbb{C}^{p(r+1)})\right).$$

Second, let  $\mathcal{B} = \{(f, \phi, \psi) \in \mathcal{A}\}$  be the set of all triples satisfying the conditions

- (i)  $R_{r-3}f \subseteq \ker \psi$ ,
- (ii)  $R_1 \text{im}(\phi) \subseteq \ker \psi$ ,
- (iii)  $\text{im} \phi \cap R_{r-4}f = 0$ ,
- (iv)  $\phi$  and  $\psi$  are full rank.

Finally, we denote by  $\overline{\mathcal{B}}$  the Zariski closure of  $\mathcal{B}$ .

**Lemma 12.** *The map  $\mathcal{B} \rightarrow \mathcal{H}_{d,g}(4)$  defined by  $(f, \phi, \psi) \mapsto (f, R_{r-4}f + \text{im} \phi)$  is well defined and surjective.*

*Proof.* Let  $(f, \phi, \psi) \in \mathcal{B}$  and let  $\Xi = R_{r-4}f + \text{im} \phi$ . Constraint (iv) implies that  $\text{im} \phi$  has dimension  $q(r) - N_{r-4}$ . Together with Constraint (iii), we have  $\dim \Xi = q(r)$ . Moreover, Constraint (iv) implies that  $\ker \psi$  has dimension  $q(r+1)$ . In particular Since  $R_1 \Xi = R_{r-3}f + R_1 \text{im} \phi$ , Constraints (i) and (ii) implies that  $R_1 \Xi$  has dimension at most  $q(r+1)$ . So,  $\Xi \in \mathcal{H}_{d,g}(4)$ . Since  $R_{r-4}f \subseteq \Xi$ , the polynomial  $f$  is in the saturation of the ideal generated by  $\Xi$ . Hence,  $(f, \Xi) \in \mathcal{H}_{d,g}(4)$ .

Conversely, let  $(f, \Xi) \in \mathcal{H}_{d,g}(4)$ , then  $R_{r-4}f \subset \Xi$  and there is a full rank map  $\phi: \mathbb{C}^{q(r)-N_{r-4}} \rightarrow R_r$  such that  $\text{im} \phi$  complements  $R_{r-4}f$  in  $\Xi$ . Furthermore,  $\dim R_1 \Xi \leq q(r+1)$ , because  $\Xi \in \mathcal{H}_{d,g}$ , so there is a full rank map  $\psi: R_{r+1} \rightarrow \mathbb{C}^{p(r+1)}$  such that  $R_1 \Xi \subseteq \ker \psi$ . So  $(f, \phi, \psi)$  is the image of  $(f, \phi, \psi) \in \mathcal{B}$ .  $\square$

**Lemma 13.** *For any  $a \geq 0$ , let  $\overline{\mathcal{B}}_a$  be the union of the codimension  $a$  components of  $\overline{\mathcal{B}}$ . Then*

$$[\overline{\mathcal{B}}_a]_{\mathbb{Z}} \leq (15 \log(d+2) \eta + \theta_1 + \theta_2 + \theta_3)^a$$



*Proof.* Let  $\mathcal{B}'$  be the closed set defined by the constraints (i) and (ii). The constraints (iii) and (iv) are open, so any component of  $\overline{\mathcal{B}}$  is a component of  $\mathcal{B}'$ . In particular  $[\overline{\mathcal{B}}_a]_{\mathbb{Z}} \leq [\mathcal{B}'_a]_{\mathbb{Z}}$ .

Constraint (i) is expressed with  $p(r+1)N_{r-3}$  polynomial equations of multidegree  $(1, 0, 1)$  (w.r.t.  $f$ ,  $\phi$  and  $\psi$  respectively). Namely,  $\psi(mf) = 0$  for every monomial  $m$  in  $R_{r-3}$ . Each  $p(r+1)$  components of the equation  $\psi(mf) = 0$  involves a sum of 35 terms (since  $f$ , as a quartic polynomial, contains only 35 terms) with coefficients 1. So the 1-norm of these constraints is at most 35 (which is also at most  $N_r$ , since  $r \geq 4$ ).

Constraint (ii) is expressed with  $4p(r+1)(q(r) - N_{r-4})$  polynomial equations of multidegree  $(0, 1, 1)$ . Namely,  $\psi(v\phi(e)) = 0$  for any basis vector  $e$  and any variable  $v \in \{w, x, y, z\}$ . Each  $p(r+1)$  component of the equation  $\psi(v\phi(e)) = 0$  involves a sum of  $N_r$  terms with coefficients 1. So the 1-norm of these constraints is at most  $N_r$ .

The claim is then a consequence of Proposition 10, with  $s = p(r+1)N_{r-3} + 4p(r+1)(q(r) - N_{r-4})$  and  $L = N_r$ . We check routinely, with Mathematica, that  $sL \leq (d+2)^{15}$ .  $\square$

**Theorem 14.** *There is an absolute constant  $A > 0$  such that for any  $d > 0$  and  $g \geq 0$  we have*

$$\deg(\mathrm{NL}_{d,g}) \leq A^{d^9} \text{ and } \|\mathrm{NL}_{d,g}\|_1 \leq 2^{A^{d^9}}.$$

*Proof.* We assume  $\mathcal{NL}_{d,g}$  is non-empty, since these inequalities are trivially satisfied if  $\mathcal{NL}_{d,g} = \emptyset$  with  $\mathrm{NL}_{d,g} = 1$ . Let  $P_2 \doteq \mathbb{P}(\mathrm{End}(\mathbb{C}^{q(r)-N_{r-4}}, R_r))$  and  $P_3 \doteq \mathbb{P}(\mathrm{End}(R_{r+1}, \mathbb{C}^{p(r+1)}))$  denote the second and third factors of  $\mathcal{A}$ . Let  $\alpha \doteq (q(r) - N_{r-4})N_r - 1$  and  $\beta \doteq p(r+1)N_{r+1} - 1$  denote the dimensions of  $P_2$  and  $P_3$  respectively. Let  $\mathcal{E}$  be the projection of  $\overline{\mathcal{B}}$  on  $\mathbb{P}(R_4) \times P_2$ . The fibers of the map  $\overline{\mathcal{B}} \rightarrow \mathcal{E}$  are projective subspaces of  $P_3$  since Constraints (i) and (ii) are linear in  $\psi$ . The dimension of these fibers are  $\beta' \doteq p(r+1)^2 - 1$ . So, by Proposition 11,

$$(68) \quad \theta_3^\beta[\mathcal{E}]_{\mathbb{Z}} \leq \theta_3^{\beta'}[\overline{\mathcal{B}}]_{\mathbb{Z}}.$$

Next, the map  $\mathcal{B} \rightarrow \mathcal{H}_{d,g}(4)$  factors through  $\mathcal{E}$  and the fibers of the corresponding map  $\mathcal{E} \rightarrow \mathcal{H}_{d,g}(4)$  have dimension  $\alpha' \doteq (q(r) - N_{r-4})q(r) - 1$ . Finally, let  $e$  be the dimension of the fibers of the map  $\mathcal{H}_{d,g}(4) \rightarrow \mathcal{NL}_{d,g}$ . (If this dimension is not generically constant, we work one component at a time.) Once again, by Proposition 11, we obtain

$$(69) \quad \theta_2^\alpha[\mathcal{NL}_{d,g}]_{\mathbb{Z}} \leq \theta_2^{\alpha'+e}[\mathcal{E}]_{\mathbb{Z}}.$$

Since  $[\mathcal{NL}_{d,g}]_{\mathbb{Z}} = m(\mathrm{NL}_{d,g})\eta + \deg(\mathrm{NL}_{d,g})\theta_1$ , taking  $L = 15 \log(d+2)$ , we get

$$(70) \quad \deg \mathrm{NL}_{d,g} \leq \text{coeff of } \theta_1 \theta_2^{\alpha-\alpha'-e} \theta_3^{\beta-\beta'} \text{ in } (L\eta + \theta_1 + \theta_2 + \theta_3)^{\alpha+\beta-\alpha'-\beta'-e+1}$$

$$(71) \quad \leq 3^{\alpha+\beta-\alpha'-\beta'-e+1}.$$

The exponent is a polynomial in  $d$  and  $g$ . Unless  $d^2 \geq 8g - 8$ ,  $\mathcal{NL}_{d,g}$  is empty. So, we may bound the exponent with a polynomial only in  $d$ , which turns out to be of degree 9. Therefore,  $\deg \mathcal{NL}_{d,g} \leq A d^9$  for some constant  $A > 0$ .

Similarly,

$$(72) \quad m(\mathcal{NL}_{d,g}) \leq \text{coeff of } \eta \theta_2^{\alpha-\alpha'-e} \theta_3^{\beta-\beta'} \text{ in } (L\eta + \theta_1 + \theta_2 + \theta_3)^{\alpha+\beta-\alpha'-\beta'-e+1}$$

$$(73) \quad \leq (\alpha + \beta - \alpha' - \beta' - e + 1) L 3^{\alpha+\beta-\alpha'-\beta'-e}$$

$$(74) \quad \leq 2^{O(d^9)}.$$

By D'Andrea et al. (2013, Lemma 2.30.3),

$$(75) \quad \|\mathcal{NL}_{d,g}\|_1 \leq \exp(m(\mathcal{NL}_{d,g})) 36^{\deg \mathcal{NL}_\Delta},$$

and this implies the claim, for some other constant  $A > 0$ .  $\square$

For the following, we write  $a \uparrow b$  for  $a^b$ . This is a right-associative operation.

**Corollary 15.** *There is an absolute constant  $A > 0$  such that for any  $\Delta > 0$ ,*

$$\deg(\mathcal{NL}_\Delta) \leq A \uparrow \Delta \uparrow \frac{9}{2} \text{ and } \|\mathcal{NL}_\Delta\|_1 \leq 2 \uparrow A \uparrow \Delta \uparrow \frac{9}{2}.$$

*In fact, one can obtain the following explicit bounds*

$$\deg(\mathcal{NL}_\Delta) \leq 3^{(\Delta+20)^{9/2}} \text{ and } \log_2 \|\mathcal{NL}_\Delta\|_1 \leq (\Delta + 60)^5 3^{(\Delta+20)^{9/2}}.$$

*Proof.* The first statement follows directly from (53) and Theorem 14 using a different  $A$ . The second statement is found by carrying out the arguments in the proof of Theorem 14 with the help of a computer algebra system.  $\square$

**3.5. How good are these bounds?** We can compare our degree bounds for  $\mathcal{NL}_\Delta$  to the exact degrees computed by Maulik and Pandharipande (2013), from which it actually follows that

$$(76) \quad \deg \mathcal{NL}_\Delta = O(\Delta^{\frac{19}{2}}).$$

This sharper bound does not directly imply a sharper bound on the height of  $\mathcal{NL}_\Delta$  but it suggests the following conjecture. This would improve subsequently Theorems 17 and 20. In particular, Equation (2) would be exponential in the size of the coefficients, as opposed to being doubly exponential.

**Conjecture 16.** *There is a constant  $c > 0$  such that for any  $\Delta > 0$ ,*

$$\|\mathcal{NL}_\Delta\|_1 \leq c \uparrow \Delta \uparrow \frac{19}{2}.$$

Now we turn to the details of (76). Following Maulik and Pandharipande (2013) (but replacing  $q$  by  $q^8$ ), consider the following power series

$$(77) \quad A \doteq \sum_{n \in \mathbb{Z}} q^{n^2}, \quad B \doteq \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}, \quad \Psi = 108 \sum_{n > 0} q^{8n^2},$$

and  $\Theta$  defined by

$$(78) \quad \begin{aligned} 2^{22}\Theta \doteq & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 - 20007A^{16}B^5 \\ & - 169092A^{15}B^6 - 120636A^{14}B^7 - 621558A^{13}B^8 - 292796A^{12}B^9 \\ & - 1038366A^{11}B^{10} - 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & - 361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} - 4812A^4B^{17} - 1881A^3B^{18} \\ & - 27A^2B^{19} + B^{21}. \end{aligned}$$

From (Maulik and Pandharipande 2013, Corollary 2), we have, for any  $\Delta > 0$ ,

$$(79) \quad \deg \text{NL}_\Delta \leq \text{coefficient of } q^\Delta \text{ in } \Theta - \Psi.$$

In fact, this is an equality when the components of  $\mathcal{NL}_\Delta$  are given appropriate multiplicities. Let  $\Theta[k]$  denote the coefficient of  $q^k$  in  $\Theta$ . By (79), we only need to bound  $\Theta[\Delta]$  in order to bound  $\deg \text{NL}_\Delta$ . To do so, replace every negative sign in the definition of  $\Theta$  by a positive sign, including those in  $B$ , to obtain the *coefficientwise* inequality

$$(80) \quad \Theta \leq 6 \left( \sum_{n \in \mathbb{Z}} q^{n^2} \right)^{21}.$$

The coefficient of  $q^k$  in  $\left( \sum_{n \in \mathbb{Z}} q^{n^2} \right)^{21}$  is

$$(81) \quad r_{21}(k) \doteq \# \left\{ (a_1, \dots, a_{21}) \in \mathbb{Z}^{21} \mid \sum_i a_i^2 = k \right\}.$$

The asymptotic bound  $r_d(k) = O(x^{\frac{d}{2}-1})$ , for  $d > 4$ , is well known (e.g. Krätzel 2000, Satz 5.8).

#### 4. SEPARATION BOUND

We now state and prove the main results. Recall that  $a \uparrow b = a^b$  is right associative and for  $\gamma \in H_{\mathbb{Z}}$  we defined the discriminant  $\Delta(\gamma)$  as  $(\gamma \cdot h)^2 - 4\gamma \cdot \gamma$ .

**Theorem 17.** *For any  $f \in \mathbb{Z}[w, x, y, z] \cap U_4$  there is a computable constant  $c > 1$  such that for any  $\gamma \in H^2(X_f, \mathbb{Z})$ , if  $\gamma \cdot \omega_f \neq 0$ , then*

$$|\gamma \cdot \omega_f| > \left( 2 \uparrow c \uparrow \Delta(\gamma) \uparrow \frac{9}{2} \right)^{-1}.$$

**4.1. Multiplicity of Noether–Lefschetz loci.** The multiplicity of some nonzero polynomial  $F \in \mathbb{C}[x_1, \dots, x_s]$  at a point  $p \in \mathbb{C}^s$  is the unique integer  $k$  such that all partial derivatives of  $F$  of order  $< k$  vanish at  $p$  and some partial derivative of order  $k$  does not. It is denoted by  $\text{mult}_p F$ .

The multiplicity of  $\text{NL}_\Delta$  at some  $f \in U_4$  is related to the elements of  $\text{Pic}(X_f)$  with discriminant  $\Delta$ . For  $\Delta > 0$ , let  $E_\Delta$  be a set of representatives of the equivalence classes of the relation  $\sim$  on  $H_{\mathbb{Z}}$  defined by

$$(82) \quad \gamma \sim \gamma' \text{ if } \exists a \in \mathbb{Q}^*, b \in \mathbb{Q} : \gamma' = a\gamma + bh.$$

**Lemma 18.** *For any  $f \in U_4$  and any  $\Delta > 0$ ,*

$$\text{mult}_f \text{NL}_\Delta = \#(\text{Pic } X_f \cap E_\Delta).$$

*Proof.* Let  $\widetilde{\mathcal{NL}}_\Delta$  be the lift of  $\mathcal{NL}_\Delta$  in  $\widetilde{U}_4$ . Arguing as in §3.1.1,  $\widetilde{\mathcal{NL}}_\Delta$  is the union of smooth analytic hypersurfaces:

$$(83) \quad \widetilde{\mathcal{NL}}_\Delta = \bigcup_{\eta \in E_\Delta} \mathcal{P}^{-1} \{w \in \mathcal{D} \mid w \cdot \eta = 0\}.$$

Then the same holds locally for  $\mathcal{NL}_\Delta$ .

For any  $f \in U_4$  it follows from the smoothness of branches of  $\mathcal{NL}_\Delta$  that  $\text{mult}_f \text{NL}_\Delta$  is exactly the number of branches meeting at  $f$ . The branches meeting at  $f$  are described by the elements of  $\text{Pic } X_f$  with discriminant  $\Delta$ . Two elements  $\gamma$  and  $\gamma'$  describe the same branch (that is the same hyperplane section of  $\mathcal{D}$ ) if and only if  $\gamma' \sim \gamma$ . So  $\text{mult}_f \text{NL}_\Delta$  is exactly the number of equivalence classes in  $\{\gamma \in \text{Pic } X_f \mid \Delta(\gamma) = \Delta\}$  for this relation.  $\square$

**4.2. Proof of Theorem 17.** We first apply Corollary 3. Let  $\varepsilon = 4|\gamma \cdot \omega_f|$ . The corollary gives constants  $C_f > 0$  and  $\varepsilon_f > 0$  (depending only on  $f$ ) such that if  $\varepsilon < \varepsilon_f$  then there exists a monomial  $m \in R_4$  and  $t \in \mathbb{C}$  such that

$$(84) \quad |t| \leq C_f \varepsilon$$

and

$$(85) \quad \gamma \in \text{Pic } X_{f+tm}.$$

Assume that  $\varepsilon < \varepsilon_f$  and let  $t$  and  $m$  be as above. As  $u$  varies, the number  $\#(\text{Pic}(X_{f+um}) \cap E_\Delta)$  has a strict local maximum at  $u = t$ . By Lemma 18, so does  $\text{mult}_{f+um} \text{NL}_{\Delta(\gamma)}$ . In particular, there is some higher-order partial derivative of  $\text{NL}_\Delta$  which vanishes at  $f + tm$  but not at  $f + um$ , for  $u$  close to but not equal to  $t$ . Let  $\alpha \in \mathbb{N}^{35}$  be the multi-index for which

$$(86) \quad P \doteq \frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} \text{NL}_\Delta}{\partial u^\alpha} \in \mathbb{Z}[u_1, \dots, u_{35}]$$

is this derivative. For a monomial  $u^\beta \doteq u_1^{\beta_1} \cdots u_{35}^{\beta_{35}}$  we have

$$(87) \quad \frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} u^\beta}{\partial u^\alpha} = \prod_{i=1}^{35} \binom{\beta_i}{\alpha_i} u^{\beta - \alpha}.$$

Since  $\binom{\beta_i}{\alpha_i} \leq 2^{\beta_i}$ , it follows that

$$(88) \quad \left\| \frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} \text{NL}_\Delta}{\partial u^\alpha} \right\|_1 \leq 2^{\deg \text{NL}_\Delta} \|\text{NL}_\Delta\|_1.$$

Let  $Q \in \mathbb{Z}[x]$  be the integer polynomial  $Q(x) \doteq P(f + xm)$ . By construction  $Q \neq 0$  and  $Q(t) = 0$ . Clearly  $\deg Q \leq \deg \text{NL}_\Delta$ , and we check that

$$(89) \quad \|Q\|_1 \leq \|P\|_1 (\|f\|_1 + 1)^{\deg P}.$$

and then

$$(90) \quad \|Q\|_1 \leq 2^{\deg \text{NL}_\Delta} \|\text{NL}_\Delta\|_1 (\|f\|_1 + 1)^{\deg \text{NL}_\Delta}.$$

From Corollary 15, we find a constant  $c$  depending only on  $f$  such that

$$(91) \quad \deg Q \leq c \uparrow \Delta \uparrow \frac{9}{2} \text{ and } \|Q\|_1 \leq 2 \uparrow c \uparrow \Delta \uparrow \frac{9}{2}.$$

We write  $Q = \sum_{i=0}^{\deg Q} q_i x^i$ . Let  $k$  be the smallest integer such that  $q_k \neq 0$ . Since  $Q(t) = 0$ , it follows that

$$(92) \quad |q_k t^k| \leq \sum_{i=k+1}^{\deg Q} |q_i t^i|$$

If  $\varepsilon < C_f^{-1}$ , we have  $|t| < 1$ , by (84), and it follows that  $|q_k t^k| \leq |t^{k+1}| \|Q\|_1$ . Since  $q_k \in \mathbb{Z}$  and nonzero, it follows that

$$(93) \quad |t| \geq \frac{1}{\|Q\|_1}.$$

By (84), this leads to

$$(94) \quad \varepsilon \geq (2 \uparrow c \uparrow \Delta \uparrow \frac{9}{2})^{-1},$$

for some other constant  $c$  which depends only on  $f$ . Recall that (94) holds with the assumption that  $\varepsilon \leq \varepsilon_f$  and  $\varepsilon < C_f^{-1}$ . However, we can choose  $c$  large enough so that the right-hand side of (94) is smaller than  $\varepsilon_f$  and  $C_f^{-1}$ . Then (94) holds unconditionally, concluding the proof of Theorem 17.  $\square$

**4.3. Quartics with algebraic coefficients.** Let  $K \subset \mathbb{C}$  be a number field of degree  $D = [K : \mathbb{Q}]$ . Suppose that  $f \in K[w, x, y, z]_4 \cap U_4$  has coefficients that are algebraic integers in  $K$ . Let  $H > 0$  be an upper bound for the absolute logarithmic Weil height for the coefficient vector of  $f$  (Waldschmidt 2000, p.77).

**Theorem 19.** *Let  $f$  and  $H, D > 0$  be as above. Then there is a computable constant  $c > 1$  such that for any  $\gamma \in H^2(X_f, \mathbb{Z})$ , if  $\gamma \cdot \omega_f \neq 0$ , we have*

$$|\gamma \cdot \omega_f| > (2 \uparrow c \uparrow \Delta(\gamma) \uparrow \frac{9}{2})^{-D(1+H)}.$$

*Proof.* The proof of Theorem 17 carries through, with the sole exception that  $Q(x)$  no longer has integer coefficients. If  $q_k$  is the first non-zero coefficient of  $Q(x)$ , then  $q_k$  is an algebraic integer defined by a polynomial expression  $\tilde{q}_k(f)$  in coefficients of  $f$  with  $\tilde{q}_k$  having integer coefficients. Therefore, the number “1” in (93) must be replaced by a suitable lower bound on the norm of  $q_k$ . For this, we use Liouville’s inequality (ibid., Proposition 3.14):

$$(95) \quad |q_k| \geq \|\tilde{q}_k\|_1^{-D+1} e^{-DH \deg \tilde{q}_k}.$$

It is easy to see that  $\deg \tilde{q}_k \leq \deg \text{NL}_\Delta$  and  $\|\tilde{q}_k\|_1 \leq 2^{\deg \text{NL}_\Delta} \|\text{NL}_\Delta\|_1$ , the latter can be bounded by  $\|Q\|_1$ . The result follows for some other  $c$ .  $\square$

**4.4. Numbers à la Liouville.** Let  $(\theta_i)_{i \geq 0}$  be a sequence of positive integers such that  $\theta_i$  is a strict divisor of  $\theta_{i+1}$  for all  $i \geq 0$  (in particular  $\theta_i \geq 2^i$ .) Consider

the number

$$L_\theta \doteq \sum_{i=0}^{\infty} \theta_i^{-1}.$$

As a corollary to the separation bound obtained in Theorem 17, the following result states that  $L_\theta$  is not a ratio of periods of quartic surfaces when  $\theta$  grows too fast.

**Theorem 20.** *If  $|\theta_{i+1}| \geq 2 \uparrow 2 \uparrow \theta_i \uparrow 10$ , for all  $i$  large enough, then  $L_\theta$  is not equal to  $\frac{\gamma_1 \cdot \omega_f}{\gamma_2 \cdot \omega_f}$  for any  $\gamma_1, \gamma_2 \in H_{\mathbb{Z}}$  and any  $f \in U_4$  with rational coefficients.*

*Proof.* Let  $l_k = \sum_{i=0}^k \theta_i^{-1}$ . Since  $\theta_i$  divides  $\theta_{i+1}$ , we can write  $l_k = \frac{u_k}{\theta_k}$  for some integer  $u_k$ . And since the divisibility is strict,  $\theta_i \geq 2^i$  and  $u_k \leq 2\theta_k$ . Moreover

$$(96) \quad 0 < L_\theta - l_k \leq 2\theta_{k+1}^{-1},$$

using  $\theta_{k+i+1} \geq 2^i \theta_{k+1}$ , for any  $i \geq 0$ . Assume now that  $L_\theta = \frac{\gamma_1 \cdot \omega_f}{\gamma_2 \cdot \omega_f}$  for some  $\gamma_1, \gamma_2 \in H_{\mathbb{Z}}$  and some  $f \in U_4$  with rational coefficients. Then, with

$$(97) \quad \gamma_k \doteq \theta_k \gamma_1 - u_k \gamma_2,$$

we check that  $\Delta(\gamma_k) = O(\theta_k^2)$  and that

$$(98) \quad 0 < |\gamma_k \cdot \omega_f| = |\theta_k| |\gamma_2 \cdot \omega_f| |L_\theta - l_k| \leq C \frac{\theta_k}{\theta_{k+1}},$$

for some constant  $C$ . By Theorem 17, we obtain therefore

$$(99) \quad (2 \uparrow c \uparrow \theta_k \uparrow 9)^{-1} \leq C \frac{\theta_k}{\theta_{k+1}},$$

for some constant  $c > 0$  which depends only on  $f$ . This contradicts the assumption on the growth of  $\theta$ .  $\square$

**4.5. Computational complexity.** Given a polynomial  $f \in \mathbb{Z}[w, x, y, z] \cap U_4$  and a cohomology class  $\gamma \in H^2(X_f, \mathbb{Z})$ , we can decide if  $\gamma \in \text{Pic}(X_f)$  (that is  $\gamma \cdot \omega_f = 0$ ) as follows:

- (a) Compute the constant  $c$  in Theorem 17;
- (b) Let  $\varepsilon = (2 \uparrow c \uparrow \Delta(\gamma) \uparrow \frac{9}{2})^{-1}$  and compute an approximation  $s \in \mathbb{C}$  of the period  $\gamma \cdot \omega_f$  such that  $|s - \gamma \cdot \omega_f| < \frac{1}{2}\varepsilon$ .

Then  $\gamma$  is in  $\text{Pic}(X_f)$  if and only if  $|s| < \frac{1}{2}\varepsilon$ .

Computing the Picard group itself is an interesting application of this procedure. Algorithms for computing the Picard group of  $X_f$ , or even just the rank of it, break the problem into two: a part gives larger and larger lattices inside  $\text{Pic}(X_f)$  while the other part gets finer and finer upper bounds on the rank of  $\text{Pic}(X_f)$  (Charles 2014; Hassett et al. 2013; Poonen et al. 2015). The computation stops when the two parts meet. Approximations from the inside are based on finding sufficiently many elements of  $\text{Pic}(X_f)$ . So while deciding the membership of  $\gamma$  in  $\text{Pic}(X_f)$  can be solved by computing  $\text{Pic}(X_f)$  first, it makes sense not to

assume prior knowledge of the Picard group and to study the complexity of deciding membership as  $\Delta(\gamma) \rightarrow \infty$ , with  $f$  fixed.

Step (a) does not depend on  $\gamma$ , so only the complexity of Step (b) matters, that is the numerical approximation of  $\gamma \cdot \omega_f$ . This approximation amounts to numerically solving a Picard–Fuchs differential equation (Sertöz 2019) and the complexity is  $(\log \frac{1}{\varepsilon})^{1+o(1)}$  (Beeler et al. 1972; Hoeven 2001; Mezzarobba 2010, 2016). With the value of  $\varepsilon$  in Step (b), we have a complexity bound of  $\exp(\Delta(\gamma)^{O(1)})$  for deciding membership.

For the sake of comparison, we may speculate about an approach that would decide the membership of  $\gamma$  in  $\text{Pic}(X_f)$  by trying to construct an explicit algebraic divisor on  $X_f$  whose cohomology class is equal to  $\gamma$ . It would certainly need to decide the existence of a point satisfying some algebraic conditions in some Hilbert scheme  $\mathcal{H}_{d,g}$ , with  $d = O(\Delta^{\frac{1}{2}})$  and  $g = O(\Delta)$  (see §3.1.1). Embedding  $\mathcal{H}_{d,g}$  (or some fibration over it, as we did in §3.4) in some affine chart of a projective space of dimension  $d^{O(1)}$  will lead to a complexity of  $\exp(\Delta(\gamma)^{O(1)})$  for deciding membership in this way.

However, if Conjecture 16 holds true, then the complexity of the numerical approach for deciding membership would reduce to  $\Delta(\gamma)^{O(1)}$ .

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PIERRE LAIREZ, INRIA, FRANCE

Email address: [pierre.lairez@inria.fr](mailto:pierre.lairez@inria.fr)

URL: [pierre.lairez.fr](http://pierre.lairez.fr)

EMRE CAN SERTÖZ, MAX PLANCK INSTITUTE FOR MATHEMATICS, BONN, GERMANY

Email address: [emre@sertoz.com](mailto:emre@sertoz.com)

URL: [emresertoz.com](http://emresertoz.com)