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Jakub Janarek, Dominique Delande, Nicolas Cherroret, Jakub Zakrzewski. Quantum boomerang effect for interacting particles. *Physical Review A*, 2020, 102 (1), pp.013303. 10.1103/PhysRevA.102.013303 . hal-03021955

HAL Id: hal-03021955

<https://hal.science/hal-03021955>

Submitted on 24 Nov 2020

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Quantum boomerang effect for interacting particles

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(Dated: March 20, 2020)

When a quantum particle is launched with a finite velocity in a disordered potential, it may surprisingly come back to its initial position at long times and remain there forever. This phenomenon, dubbed “quantum boomerang effect”, was introduced in [Phys. Rev. A **99**, 023629 (2019)]. Interactions between particles, treated within the mean-field approximation, are shown to partially destroy the boomerang effect: the center of mass of the wave packet makes a U-turn, but does not completely come back to its initial position. We show that this phenomenon can be quantitatively interpreted using a single parameter, the average interaction energy.

I. INTRODUCTION

Anderson Localization (AL) [1], i.e. inhibition of transport in disordered media, has been the source of various, often counterintuitive phenomena discovered over the last 60 years [2]. Already in one dimension (1D) the fact that even the tiniest random disorder generically leads to a full localization of eigenstates is totally against a classical way of thinking. This effect is a clear manifestation of the inherently quantum nature of AL, typically explained as the effect of quantum wave interference. The phenomenon was observed in many experimental setups including light [3, 4], sound waves [5] as well as matter waves [6–10]. The closely related phenomenon of Aubry-André localization in quasi-periodic potentials [11] was also observed in a cold atomic setting [12]. Last years have also led to a number of studies of the many-body counterpart of AL [13, 14], the so called many-body localization (MBL) (for recent reviews see [15, 16]). It has already been observed in cold atomic experiments in quasi-periodic potentials [17, 18]. While studies of MBL have been very extensive, even its very existence has been questioned recently [19], provoking a vivid debate [20–22].

The physics of a single particle in a random potential, particularly in 1D, has much stronger foundations, while still bringing novel features such as studies of random (or quasi-random) potentials revealing the presence of mobility edge [23, 24], i.e. situations where localized and extended states appear at different energies. Even for a pure random standard case, one may find new counterintuitive phenomena as exemplified by the *quantum boomerang effect* [25]. As a classical boomerang returns to the initial position from which it was launched, the center of mass of a wave packet with a nonzero initial

velocity returns to its origin due to AL. The effect is quite general and occurs in Anderson localized multi-dimensional systems [25] whose Hamiltonians preserve time-reversal invariance.

The aim of this work is to investigate how interactions between particles affect the boomerang effect. Our study focuses on the limit of weak interactions, a regime where it was previously shown that AL of wave packets is replaced by a subdiffusive evolution at very long time [26–30]. Computing the temporal evolution of a many-body wave packet in a disordered potential is in general a formidable task, even in 1D. When interactions are sufficiently weak however, one may use a mean-field approximation and describe the dynamics with a one-dimensional Gross-Pitaevskii equation (GPE). Within this formalism, we provide an in depth numerical analysis of the quantum boomerang effect. For simplicity, we study the weak-disorder case, for which an analytic description of the boomerang effect in the absence of interaction is available. However, the conclusions are expected to be qualitatively identical when the disorder strength is increased. At short times, of the order of a few disorder scattering times, we observe that the dynamics of the wave packet is essentially not affected by interactions, with an initial ballistic flight followed by a U-turn of the center of mass, slowly returning towards its initial position. The main effect in this regime is a small modification of the scattering time and scattering mean free path due to interactions. At longer time, we observe that the center of mass, instead of returning to its initial position, stops at a finite distance from it, which increases with the interaction strength. We show that this phenomenon can be interpreted in terms of a break time, the time scale beyond which interference effects are typically destroyed by interactions. We finally show that in the presence of interactions, the boomerang effect can be quantitatively described in terms of a single scaling parameter, a variant of the interaction energy, computed at the break time.

The paper is organized as follows. After formulating the problem and introducing the main parameters in Sec. II, we numerically analyze the influence of interactions on the boomerang effect in Sec. III. In Sec. IV, we then

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perform several numerical studies which establish that the dynamics can be described by a single parameter. The appendix shows that the short-time dynamics can also be understood using the same parameter. We finally conclude and briefly discuss open questions.

II. THE MODEL

To study the boomerang effect in the presence of interactions, we use the one-dimensional, time dependent Gross-Pitaevskii equation (GPE) [31]:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[\frac{p^2}{2m} + V(x) + g|\psi(x,t)|^2 \right] \psi(x,t), \quad (1)$$

where $V(x)$ is a disordered potential, while g represents the interaction strength. Wave functions are normalized to unity, $\int |\psi(x,t)|^2 dx = 1$. Throughout this work we assume that the disordered potential is a Gaussian uncorrelated random variable, i.e.

$$\overline{V(x)} = 0, \quad \overline{V(x)V(x')} = \gamma \delta(x - x'), \quad (2)$$

where the overbar denotes the average over disorder realizations. The parameter γ measures the disorder strength and determines the characteristic time and length scales for scattering: the mean scattering time τ_0 (the typical time for a particle to be scattered by the disorder) and the corresponding mean scattering length ℓ_0 , usually called mean free path. The uncorrelated disorder model is sufficient to capture the main features of the boomerang effect, which depends only on a small set of well defined parameters, τ_0 , ℓ_0 and time t itself. In the Born approximation, τ_0 and ℓ_0 are given by [32]:

$$\tau_0 = \frac{\hbar^3 k_0}{2m\gamma}, \quad \ell_0 = \frac{\hbar k_0}{m} \tau_0 = \frac{\hbar^4 k_0^2}{2m^2 \gamma}. \quad (3)$$

Non-interacting 1D systems remain always strongly localized, i.e. their eigenstates decay exponentially over the localization length $\xi_{\text{loc}} = 2\ell_0$ [33].

Following [25], as the initial state for time evolution we take a Gaussian wave packet with mean velocity $\hbar k_0/m$:

$$\psi(x,t=0) = \left(\frac{1}{\pi\sigma^2} \right)^{1/4} \exp(-x^2/2\sigma^2 + ik_0x), \quad (4)$$

where σ and k_0 are chosen such that the initial wave function is sharply peaked around $\hbar k_0$ in momentum space, i.e. $k_0\sigma \gg 1$. Moreover, we focus on the weak-disorder regime where the mean free path is much longer than the de Broglie wavelength, that is $k_0\ell_0 \gg 1$ [32]. In this regime, a full analytic theory, based on the Berezinskii diagrammatic technique [34] exists in the non-interacting limit. It makes it possible to express the average center-of-mass position (CMP):

$$\langle x(t) \rangle = \int x \overline{|\psi(x,t)|^2} dx, \quad (5)$$

as:

$$\frac{\langle x(t) \rangle}{\ell_0} = f\left(\frac{t}{\tau_0}\right), \quad (6)$$

where f is a universal function whose Taylor expansion at short time and asymptotic behavior at long time are exactly known [25]. In particular $\langle x(t) \rangle \approx t\ell_0/\tau_0$ at short time $t \ll \tau_0$, meaning that the initial motion is ballistic, and

$$\langle x(t) \rangle \approx 64\ell_0 \frac{\ln(t/4\tau_0)\tau_0^2}{t^2} \quad (7)$$

at long time $t \gg \tau_0$. The two assumptions of weak disorder, $k_0\ell_0 \gg 1$, and narrow wave packet in momentum space, $k_0\sigma \gg 1$, make it possible to have a well controlled non-interacting limit [35]. If they are not valid, the wave packet will contain a broad energy distribution, and consequently a distribution of scattering time (which depends on energy), making the analysis more tedious. However, the phenomena described below are expected to be very similar.

Another important ingredient of the model are interactions. Our interest is only in small values of g , which corresponds to a weak interaction regime [36–38].

In the following, we study numerically the propagation of the initial wave function $\psi(x,t=0)$ for different disorder realizations. It yields the averaged density profile $|\overline{\psi(x,t)}|^2$, from which we compute the CMP, Eq. (5). The numerical technique is as follows. The one-dimensional configuration space is discretized on a regular grid, over which the wave function is computed. The temporal propagation is performed using a split-step algorithm of step Δt , alternating propagations of the linear part of the GPE, $\exp[-i(p^2/2m + V)\Delta t/\hbar]$, and of the nonlinear part, which is simply a phase factor $\exp(-ig|\psi(x,t)|^2\Delta t/\hbar)$. The linear part of the evolution operator is expanded in a series of Chebyshev polynomials, as described in [39–42].

Throughout our work, we express lengths in units of $1/k_0$, times in $m/\hbar k_0^2$, and energies in $\hbar^2 k_0^2/m$. The interaction strength is expressed in units of $\hbar^2 k_0/m$, and the disorder strength γ in units of $\hbar^4 k_0^3/m^2$.

Numerical results have been obtained on a discretized grid of size $4000/k_0$ (sufficiently large for the wave function to be vanishingly small at the edges; open boundary conditions have been used) divided into 20000 points, so that discretization effects are negligible. We have used a disorder strength $\gamma = 0.1\hbar^4 k_0^3/m^2$, so that, in the Born approximation (3), one has $k_0\ell_0 = 5$, i.e. the disorder is weak. Note that, while the true mean free path and scattering time may slightly differ from their expressions (3) at the lowest order Born approximation, the dynamics of the non-interacting quantum boomerang effect is still given by Eq. (6), provided corrected values of τ_0 and ℓ_0 are used. We have performed calculations for various widths σ of the initial wave packet $\psi(x,t=0)$, as well as various values of the interaction strength g .

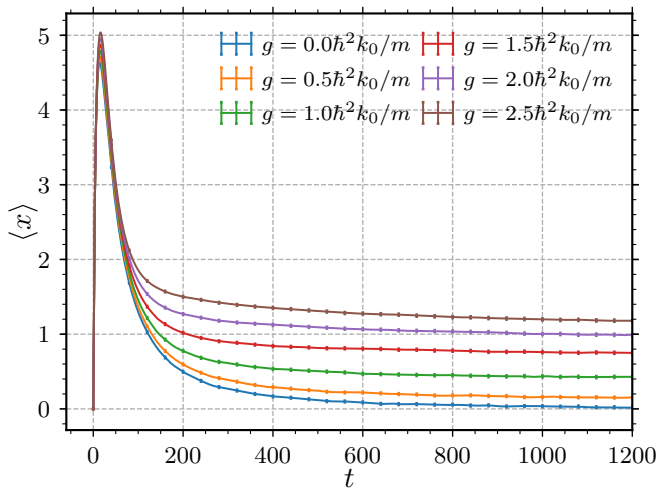


FIG. 1. CMP time evolution for an initial wave packet with $\sigma = 10/k_0$, for different values of the interaction strength g . Here, $\langle x \rangle$ is in units of $1/k_0$ and time t in units of $m/\hbar k_0^2$. All curves have been averaged over more than $5 \cdot 10^5$ disorder realizations. The short-time behavior remains almost unchanged, whereas the long-time evolution clearly depends on the interaction strength. The error bars represent statistical average errors. Center-of-mass trajectories among different disorder realizations are normally distributed, such that we use the standard error of the mean as estimator of the errors.

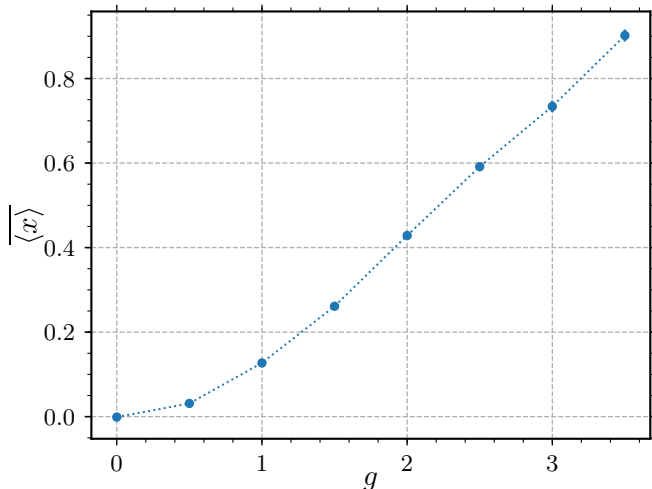


FIG. 2. Long-time average $\overline{\langle x \rangle}$ vs. interaction strength g for a wave packet with initial width $\sigma = 40/k_0$. $\overline{\langle x \rangle}$ is in units of $1/k_0$, and g is in units of $\hbar^2 k_0/m$. Data have been averaged over more than $5 \cdot 10^5$ disorder realizations.

III. THE BOOMERANG EFFECT WITH INTERACTIONS

A. Role of the interaction strength

In Fig. 1 we present results obtained for a wave packet with initial width $\sigma = 10/k_0$ for various interaction strengths g . Similarly to the non-interacting case, af-

ter the initial ballistic motion of the center of mass, we observe a subsequent retro-reflection towards the origin, that is a boomerang effect. However, the long-time behavior is affected by interactions. For all non-zero studied values of g , the center of mass does not return to the origin but saturates at some finite value. In other words, the boomerang effect is only partial [43]. This can be understood as follows. For $g = 0$, the disorder is static and full Anderson localization sets in; the complicated interference between multiply scattered paths leads to full localization at infinite time, and also to full return of the center of mass to its initial position. For non-zero but small g , the nonlinear term in Eq. (1), $g|\psi(r, t)|^2$, plays the role of a small additional effective potential which is *time-dependent*, thereby adding a fluctuating phase along each scattering path. This breaks the interference between multiple scattering paths and thus destroys both Anderson localization [26–30] and the full boomerang effect at long time. For all studied widths of the wave packet, namely $\sigma = 5/k_0$, $\sigma = 10/k_0$, $\sigma = 20/k_0$, and $\sigma = 40/k_0$ we observe a similar saturation effect. Of course, the phase scrambling progressively develops over time. The characteristic break time over which it kills coherent transport and boomerang effect is discussed in detail in the sequel of this paper. Note that the interpretation of the effect of interactions in terms of a decoherence mechanism may be questioned at very long time, where thermalization comes into play and affects the momentum distribution as well as the dynamics of the system [44, 45]. In our case, both disorder and interactions are small, so that thermalization takes place at times significantly longer than the ones considered here. Neglecting thermalization nevertheless restricts our analysis to the regime where the long-time CMP $\langle x \rangle \ll \ell_0$ (see Eq. (8) below), which constrains the maximum value of g .

To study in detail the long-time evolution, we run numerical simulations up to time $t_{\max} \approx 2500\tau_0$. From these simulations we calculate the long-time average of the CMP, $\overline{\langle x \rangle}$, defined in the following way:

$$\overline{\langle x \rangle} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \langle x(t) \rangle dt, \quad (8)$$

where we choose $t_1 \approx 1200\tau_0$, $t_2 \approx 2500\tau_0$. The results are essentially independent of these bounds, provided they are much longer than τ_0 . This definition provides us with a very good estimate of the infinite-time CMP. Figure 2 shows the dependence of $\overline{\langle x \rangle}$ on the interaction strength g . For small values of g , the CMP dependence is quadratic in g , $\overline{\langle x \rangle} \propto g^2$, and becomes approximately linear for larger g . In the following, we will mostly concentrate on the quadratic regime of small interactions, leaving the more difficult case of larger g for future studies.

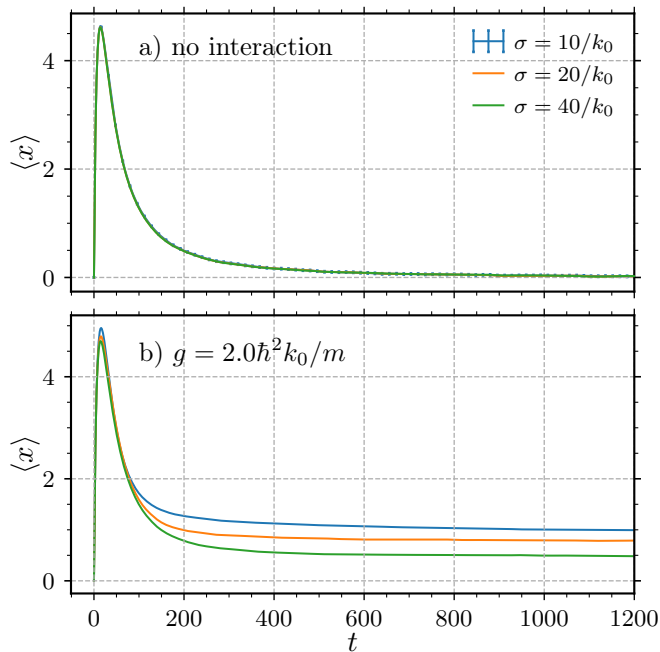


FIG. 3. a) Comparison of non-interacting CMP $\langle x(t) \rangle$ for wave packets of widths $\sigma = 10/k_0, 20/k_0, 40/k_0$. All three curves overlap, indicating that $\langle x(t) \rangle$ is σ -independent in the non-interacting limit. b) Same as a) but for non-zero interaction strength $g = 2.0\hbar^2 k_0/m$. Here, the saturation point of $\langle x(t) \rangle$ is higher for initially narrower wave packets. The center of mass $\langle x(t) \rangle$ is in units of $1/k_0$ and time t in units of $m/\hbar k_0^2$. Error bars are only shown in panel a) for $\sigma = 10/k_0$ to indicate their order of magnitude.

B. Role of the wave-packet width

Another important parameter is the width σ of the initial wave packet. In Fig. 3, we show $\langle x(t) \rangle$ for different σ values. While, for $g = 0$, it follows the analytic prediction (7) *independently* of σ , for interacting particles the long-time behavior strongly depends on σ . A simple qualitative explanation is that the destruction of the boomerang effect is controlled by the nonlinear term $g|\psi(x)|^2$ in Eq. (1). This term is larger for a spatially narrow wave packet, so that interference between scattered waves are suppressed at shorter time, giving a higher $\langle x \rangle$ value.

Although the boomerang effect is affected by a change of either g or σ in the interacting case, one may guess that the CMP is not an independent function of these two parameters. Indeed, closer investigation reveals that similar “trajectories” of $\langle x(t) \rangle$ can be achieved for different combinations of g and σ . In particular, the same values of $\langle \bar{x} \rangle$ can be obtained from different initial states, provided g is properly adjusted. This property is illustrated in Fig. 4, where we have computed $\langle x(t) \rangle$ for various values of σ and have adjusted g so that the curves fall on top of each other.

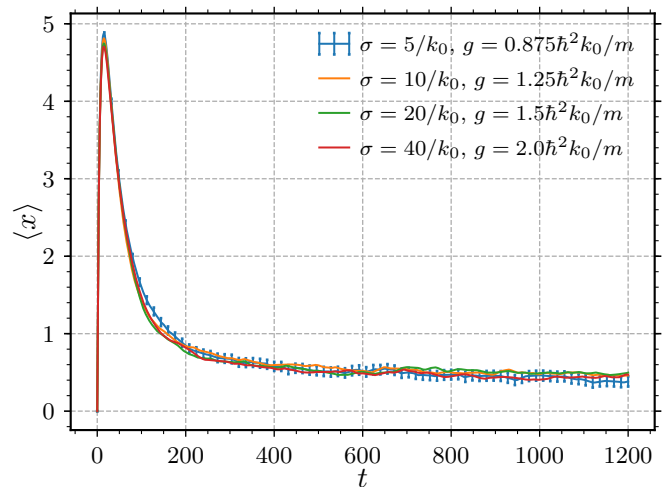


FIG. 4. CMP $\langle x(t) \rangle$ vs. time for different initial states chosen such that all of them saturate around the value $\langle \bar{x} \rangle = 0.4/k_0$. Results have been averaged over 16000 disorder realizations. The CMP is shown in units of $1/k_0$ and time is in units of $m/\hbar k_0^2$. Error bars are shown only for $\sigma = 5/k_0$.

IV. UNIVERSAL SCALING OF THE INTERACTING BOOMERANG EFFECT

From the results of the previous section, it is natural to ask whether or not the interacting boomerang effect, and more specifically its long-time average $\langle \bar{x} \rangle$ – Eq. (8) – can be described in terms of a *single* parameter, in the spirit of scaling approaches well-known in the context of Anderson localization of non-interacting particles [46]. We now address this question.

A. Break time

Before attempting to rescale the CMP, let us introduce an important parameter that will turn useful in the following. We recall that in the non-interacting limit, the CMP at long time is given by [25]:

$$\langle x(t) \rangle = 64\ell_0 \frac{\ln(t/4\tau_0)\tau_0^2}{t^2}. \quad (9)$$

If we neglect the logarithmic part, $\langle x(t) \rangle$ decays as t^{-2} . It suggests that one may identify a characteristic time scale connected with weak interactions which is inversely proportional to g . In our analysis we dub this time scale the *break time* t_b and define it by the relation:

$$t_b(g) : \langle x(t_b) \rangle_{g=0} = \langle \bar{x} \rangle(g), \quad (10)$$

where for the left-hand-side we use the analytical prediction of the non-interacting theory, Eq. (9). A given time scale in quantum mechanics corresponds to some energy scale, here the *break energy*:

$$E_b = \frac{2\pi\hbar}{t_b}, \quad (11)$$

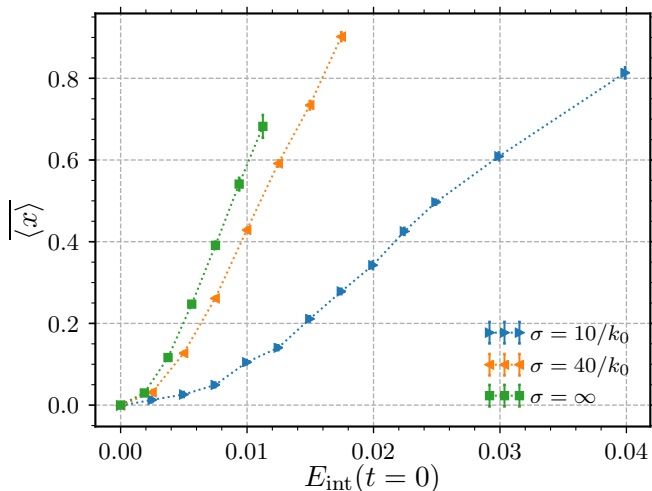


FIG. 5. Dependence of the long-time average $\overline{\langle x \rangle}$, Eq. (8), on the initial interaction energy. For wave packets $E_{\text{int}}(t=0) = g/(2\sqrt{2\pi}\sigma)$, while $E_{\text{int}}(t=0) = g\rho_0/2$ in the plane-wave limit $\sigma \rightarrow \infty$. The interaction energy is expressed in units of $\hbar^2 k_0^2/m$, and the CMP in units of $1/k_0$. Error bars represent standard deviation of the averaged points.

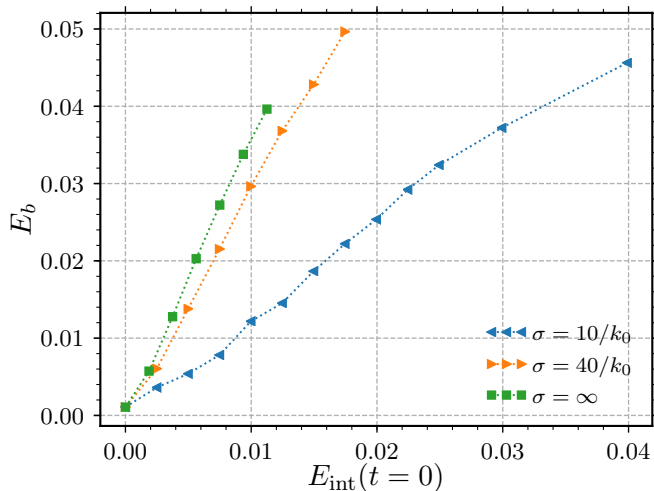


FIG. 6. Break energy E_b vs. initial interaction energy $E_{\text{int}}(t=0)$. The break energy is defined by $E_b = 2\pi\hbar/t_b$, where the break time t_b , calculated according to Eq. (10), is the characteristic time beyond which the boomerang effect disappears. Energies are in units of $\hbar^2 k_0^2/m$.

which will turn out to be a key parameter in our rescaling of the CMP.

B. Rescaling of the CMP: first attempt

A closer inspection of Fig. 4 reveals that, when trying to superimpose the CMP curves, broader wave packets require to adjust the interaction strength to larger values. A first natural candidate to represent the CMP is thus

the interaction energy,

$$E_{\text{int}}(t) = \frac{g}{2} \int |\psi(x,t)|^4 dx. \quad (12)$$

We recall that the total energy, conserved by the GPE, is the sum of the non-interacting part and the interaction energy: $E_{\text{tot}} = \langle p^2 \rangle / 2m + \langle V \rangle + E_{\text{int}}$. A related important quantity is the interaction energy at initial time, $E_{\text{int}}(t=0) = g/(2\sqrt{2\pi}\sigma)$. Notice that while the CMP curves in Fig. 4 are obtained from quite different values of g and σ , they are all associated with comparable interaction energies at time $t=0$. This is a very clear hint that $E_{\text{int}}(t=0)$ is a crucial parameter for describing the impact of interactions on the boomerang effect. We can thus try to rescale the results by plotting them vs. the initial interaction energy. This is done for $\overline{\langle x \rangle}$ in Fig. 5, and for the break energy – see Eq. (11) – in Fig. 6. In these plots we also show, for comparison, the formal limit $\sigma \rightarrow \infty$ of infinitely large wave packets, where $\psi_0(x) \rightarrow \sqrt{\rho_0} \exp(ik_0 x)$ reduces to a plane wave with $E_{\text{int}}(t=0) = g\rho_0/2$. In this limit, the sole parameter $g\rho_0$ controls the boomerang effect. Note that despite the flatness of the density profile in the plane-wave limit, in practice it is still possible to study the boomerang effect by measuring an effective CMP defined as:

$$\langle x(t) \rangle_{\sigma=\infty} \equiv \frac{1}{m} \int_0^t \langle p(t') \rangle dt'. \quad (13)$$

We have verified numerically, in particular, that in the non-interacting case the definition (13) agrees with results for wave packets of finite width, thereupon with the theoretical prediction (7).

The curves in Figs. 5 and 6, which correspond to different values of σ , are qualitatively similar, despite the fact that the g values are widely different. This suggests that the interaction energy is indeed an important parameter. Moreover, note that the break energy E_b is comparable (within a factor 4) to the initial interaction energy. In particular, because $E_{\text{int}}(t=0)$ is proportional to the interaction strength g for all initial states, at small values of $E_{\text{int}}(t=0)$ we see the expected linear dependence of E_b with $E_{\text{int}}(t=0)$. Nevertheless, it is clear that plots based on $E_{\text{int}}(t=0)$ do not fall on the same universal curve: $\overline{\langle x \rangle}$ and E_b deviate from each other as σ is changed, approaching the upper limit curve $\sigma = \infty$ as σ increases (this limit is represented by green square symbols in Figs. 5 and 6).

C. Nonlinear energy at break time

The reason why $E_{\text{int}}(t=0)$ does not allow for a universal rescaling of the CMP stems from the fact that the interaction energy $E_{\text{int}}(t)$ varies significantly from $t=0$ onwards. This evolution is shown in Fig. 7 for two finite values of σ and for the plane-wave limit $\sigma = \infty$. The figure reveals that the time evolution of the interaction

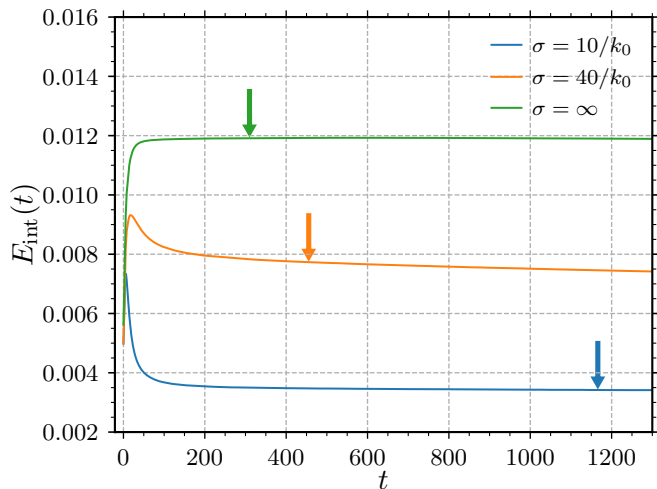


FIG. 7. Interaction energy vs. time for different initial states and interaction strengths. The plot shows data for wave packets with $[\sigma = 10/k_0, g = 0.25\hbar^2 k_0/m]$, $[\sigma = 40/k_0, g = 1.0\hbar^2 k_0/m]$, and $[\sigma = \infty, g = 45.0\hbar^2 k_0/m, \rho_0 g = 0.01125\hbar^2 k_0^2/m]$. E_{int} is in units of $\hbar^2 k_0^2/m$ and time t in units of $m/\hbar k_0^2$. Initial states are chosen so to clearly emphasize the specklelization phenomenon. It is, however, generically present for any interaction strength. In the plane-wave limit, specklelization doubles E_{int} , which then remains stationary. At finite σ , specklelization is not full and, due to the wave-packet dilution, E_{int} decays in time. From $t = t_b$ onwards, however, the decay is very slow (the location of t_b is indicated as arrows).

energy generally consists of two stages. In a first stage, which takes place on a time scale of a few scattering times, the interaction energy roughly doubles. This can be understood by noticing that $E_{\text{int}}(t)$ depends on the fourth moment of the field, see Eq. (12), which obeys:

$$\overline{|\psi(x,t)|^4} = \overline{|\psi(x,t)|^2}^2 + \text{Var}(|\psi(x,t)|^2). \quad (14)$$

In the plane-wave limit where the profile is flat, the initial density variance is zero. During the temporal evolution however, the density develops fluctuations depending on the realization of the disorder, which makes the interaction energy increase. The factor 2 enhancement is obtained by assuming that, after a few scattering times, $\psi(x,t)$ is a complex Gaussian random variable. The variance in Eq. (14) is then $\overline{|\psi(x,t)|^2}^2$, so that $\overline{|\psi(x,t)|^4}$ is doubled. Due to similarities of this phenomenon with optical speckles in scattering media [47] we dub it *specklelization* of the interaction energy. It implies that $E_{\text{int}}(t \gg \tau_0) = 2E_{\text{int}}(t = 0)$ in the limit $\sigma = \infty$. For wave packets of finite size σ , the effect is also present, albeit slightly smaller [48].

In the plane-wave limit $\sigma = \infty$, the interaction energy remains constant once the specklelization process has ended. For finite σ on the contrary, a second stage occurs, where E_{int} decreases in time, see Fig. 7. This decrease is due to the spreading of the wave packet, which

becomes more and more dilute. The spreading is initially fast, and then slows down very quickly.

The time evolution of E_{int} makes a detailed rescaling analysis of the boomerang effect very complex. The curves in Fig. 7, however, suggest the simple idea of using as scaling parameter the interaction energy *at the break time* t_b , instead of the initial interaction energy. Although at finite σ such a rescaling can only be approximate, since wave packets keep evolving slowly in time beyond t_b (indicated as arrows in Fig. 7), we show below that it provides quite satisfactory results.

Before applying this strategy, a final adjustment must be performed. The quantum boomerang being a *dynamical* effect governed by the GPE (1), its evolution is not strictly governed by the interaction energy, $g|\psi(r,t)|^4/2$, but rather by the *nonlinear energy*, defined as

$$E_{\text{NL}}(t) = 2E_{\text{int}}(t) = g \int \overline{|\psi(x,t)|^4} dx. \quad (15)$$

In the appendix, we discuss the dynamical behavior of the system at short time and show analytically and numerically that it is indeed E_{NL} , rather than E_{int} , which governs the evolution.

D. Rescaling of the boomerang effect

We can now re-analyze the boomerang effect using the nonlinear energy at the break time, $E_{\text{NL}}(t = t_b)$, as a control parameter of interactions. We show in Fig. 8 the long-time average $\overline{\langle x \rangle}$ of the CMP as a function of the nonlinear energy calculated at the break time. In contrast with Fig. 5, now all points collapse on a single universal curve. As expected, in the regime of small nonlinear energy, $\overline{\langle x \rangle}$ shows a quadratic dependence.

In Fig. 9 we also compare the break energy $E_b = 2\pi\hbar/t_b$ with the nonlinear energy at the break time t_b for wave packets of size $\sigma = 10/k_0, 40/k_0$ and $\sigma = \infty$, for various values of g . There is a compelling evidence that these quantities are very similar. A small difference is observed in the plane-wave limit, which we attribute to the slow residual decay of $\langle x \rangle$ due to prethermalization processes, which lead to an underestimation of the break energy. Such a good agreement shows that a simple model involving a single parameter, the nonlinear energy at the break time, captures quantitatively the essential features of a complex dynamical process like the quantum boomerang effect for interacting particles. As discussed in the appendix, the same parameter turns out to also control the short-time dynamics and the correction of the scattering time due to interactions.

V. CONCLUSION

We have analyzed the quantum boomerang phenomenon in the presence of interactions on the basis of the one-dimensional Gross-Pitaevskii equation. We have

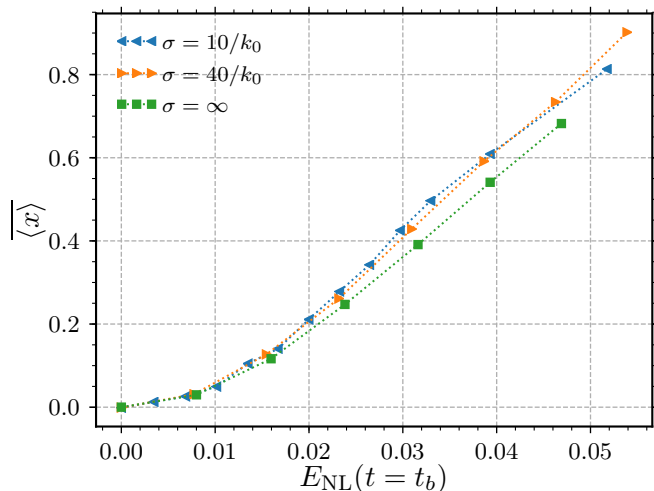


FIG. 8. Long-time averages $\overline{\langle x \rangle}$ of the CMP for different initial states vs. the nonlinear energy computed at the break time, $E_{\text{NL}}(t = t_b)$. $\overline{\langle x \rangle}$ is expressed in units of $1/k_0$, while energy is in units of $\hbar^2 k_0^2/m$. All results have been averaged over more than $5 \cdot 10^5$ disorder realizations. In this representation, data fall on the same single curve.

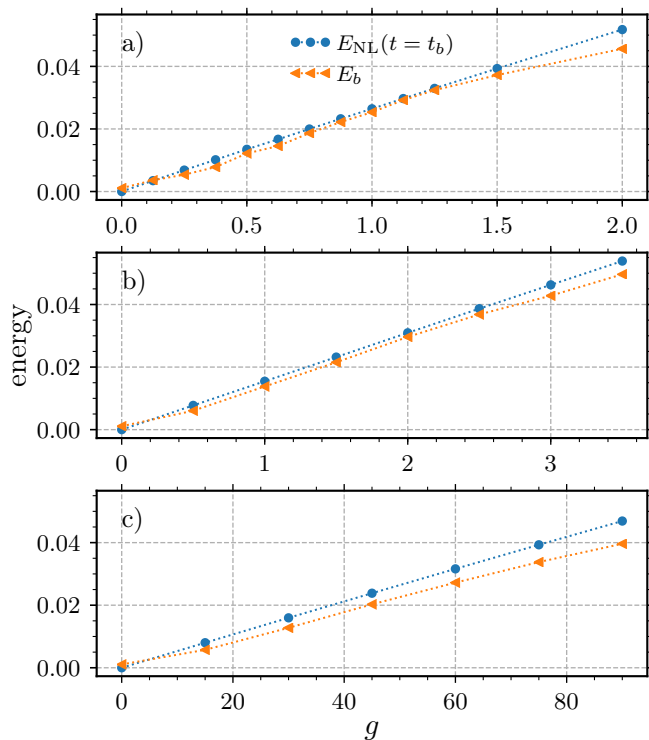


FIG. 9. Comparison of break energy, E_b , and nonlinear energy at break time, $E_{\text{NL}}(t = t_b)$, for increasing values of the interaction strength. Results are shown for wave packets of size a) $\sigma = 10/k_0$, b) $\sigma = 40/k_0$ and c) $\sigma = \infty$. Energy is in units of $\hbar^2 k_0^2/m$, interaction strength g in units of $\hbar^2 k_0/m$. All results have been averaged over $5 \cdot 10^5$ disorder realizations.

found that weak interactions do not destroy the quantum boomerang effect, in the sense that the center of mass of a wave packet launched with a finite velocity is still retro-reflected after a few scattering times, slowly returning towards its initial position. The boomerang effect is only partial though, as the quantum particle does not return to its initial position but stops on the way back. We have interpreted this phenomenon as a consequence of the destruction of interference between multiple scattering paths induced by a time-dependent nonlinear phase acquired along a path. To characterize this phenomenon, we have introduced a break time – and the corresponding break energy – the characteristic time beyond which the destruction of interference prevents the wave packet to further move back to its initial position. We have finally shown that different initial states and interaction strengths can all be described by means of a single parameter, the nonlinear energy estimated at the break time.

Our analysis is limited to the regime of weak disorder and weak interactions. For stronger interactions, the break time is likely to decrease until it becomes comparable to the scattering time. Whether a single parameter also controls this regime is an interesting question left for future studies. Another important question – especially if one envisions experiments with ultra-cold atoms – is to know whether the observed softening of the boomerang effect due to interactions remains valid beyond the mean-field description. Studying the full many-body boomerang effect is a challenging task.

ACKNOWLEDGMENTS

We kindly thank PL-Grid Infrastructure for providing computational resources. JJ and JZ acknowledge support of National Science Centre (Poland) under the project OPUS11 2016/21/B/ST2/01086. We also acknowledge support of Polish-French bilateral project Polonium.

Appendix A: Short time behavior

We have seen that interactions modify the long-time dynamics of the center of mass position, and that a change of either parameters σ or g can be encompassed in the use of the nonlinear energy. In this appendix, we show that the very same concept – the nonlinear time-dependent energy – also accurately describes the short-time dynamics of the system, through a change of the real part of the self-energy and of the scattering time.

1. Self energy in interacting systems

The self energy is a key concept in the description of disordered quantum systems. It is a complex valued function of the state energy E and its momentum $\hbar k_0$ describ-

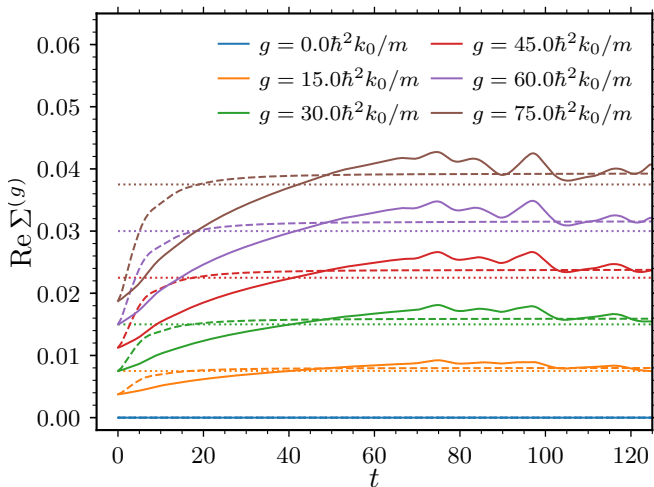


FIG. 10. Numerically calculated real parts of self energy $\Sigma^{(g)}$ (solid lines) for plane waves vs. time for several values of the interaction strength. In the plot we additionally show nonlinear energies $E_{\text{NL}}(t)$ (dashed lines), and indicate as dotted lines the value $2g\rho_0$. Energies are expressed in units of $\hbar^2 k_0^2/m$ and time in units of $m/\hbar k_0^2$.

ing the disorder-induced energy shift and the exponential decay of average Green's functions in configuration space. It is noted $\Sigma_E(k_0)$ and defined by:

$$\overline{G}_E^R(k_0) = \frac{1}{E - E_0 - \Sigma_E(k_0)}, \quad (\text{A1})$$

where \overline{G}^R denotes the averaged retarded Green's function and $E_0 = \hbar^2 k_0^2/2m$ the disorder-free energy. The self-energy vanishes in the absence of disorder and is much smaller than E_0 in the weak-disorder limit (by a factor $1/k_0 \ell_0$) [32].

The evolution operator is the temporal Fourier transform of the Green's function. If the self-energy is a smooth function of E , one obtains for the evolution of a plane wave:

$$\begin{aligned} \overline{\langle \psi_0 | \psi(t) \rangle} &= e^{-i(E_{k_0} + \Sigma_E(k_0))t/\hbar} \\ &= e^{-i(E_{k_0} + \text{Re } \Sigma_{E_0}(k_0))t/\hbar} e^{\text{Im } \Sigma_{E_0} t/\hbar}, \end{aligned} \quad (\text{A2})$$

so that $\text{Re } \Sigma$ is an energy shift and $-\text{Im } \Sigma$ the decay rate induced by the disorder. At the Born approximation, we have in 1D:

$$\Sigma_{E_0}^{(0)}(k_0) = -\frac{i\hbar}{2\tau_0}, \quad (\text{A3})$$

where the superscript (0) refers to zero interactions.

In the presence of interactions, the situation is in general much more complicated. Because of the nonlinearity of the GPE, the notion of evolution operator no longer exists and the overlap Eq. (A2) has no reason to be an exponential function of time. However, it is possible to define an effective self-energy using Eq. (A2), the left-hand-side of the equation being computed numerically

from the solution $|\psi(t)\rangle$ of the GPE. The obtained self-energy $\Sigma_{E_0}^{(g)}(k_0)$ depends on time.

To analyze the impact of interactions on the self energy, we introduce its nonlinear part $\Sigma^{(g)}$, defined as:

$$\Sigma^{(g)} = \Sigma_{E_0}^{(g)}(k_0) - \Sigma_{E_0}^{(0)}(k_0), \quad (\text{A4})$$

where both $\Sigma_{E_0}^{(g)}(k_0)$ and $\Sigma_{E_0}^{(0)}(k_0)$ are calculated numerically. The real part of this quantity is plotted in Fig. 10 in the plane-wave limit $\sigma = \infty$. $\Sigma^{(g)}$ increases over a few mean scattering times and then saturates at roughly twice its initial value.

It is easy to compute $\Sigma^{(g)}$ at $t = 0$ from the GPE. The result is:

$$\Sigma^{(g)}(t = 0) = E_{\text{NL}}(t = 0) = g\rho_0, \quad (\text{A5})$$

where the first equality is valid for any initial state, while the second holds only for a plane wave. At time longer than the scattering time, the speckelization phenomenon described in Sec. IV C is responsible for a doubling of the nonlinear energy. It is thus very natural, and fully confirmed by the numerical results in Fig. (10) as well as by a theoretical approach [38] to have:

$$\Sigma^{(g)}(t \gg \tau_0) = E_{\text{NL}}(t \gg \tau_0) = 2g\rho_0. \quad (\text{A6})$$

This close connection between the nonlinear energy and the nonlinear part of the self-energy also exists at intermediate times, as shown in Fig. 10, where we also plot numerically computed nonlinear energies. After an initial growth, both $\Sigma^{(g)}(t)$ and $E_{\text{NL}}(t)$ saturate around $2g\rho_0$ and follow a close evolution (even though the growth rate of $\text{Re } \Sigma^{(g)}$ is slightly lower than for the nonlinear energy). Altogether, it suggests that $\text{Re } \Sigma^{(g)}$ and E_{NL} may have a similar status for the problem of interacting disordered systems. This corroborates the conclusion of Sec. IV C, since $\text{Re } \Sigma$ is typically involved in the calculation of any observable, in particular of the CMP.

2. Modification of the scattering time

We now show that the nonlinear energy, Eq. (15), which governs the long-time behavior of the quantum boomerang effect, also controls the change in the mean scattering time.

As described in Sec. III, during the first part of the time evolution, precisely in the range $t < 20 - 30\tau_0$, see Fig. 1, the CMP is essentially not modified by interactions. The only difference between the interacting and non-interacting cases is that the mean scattering time and mean free path are increased. We have used the theoretical prediction for $\langle x(t) \rangle$ [25] in the non-interacting limit, which depends on τ and ℓ , to fit the data in the interacting case in the short-time region, and thus access the dependence of τ and ℓ on the interaction strength g . Fits have been performed including weights inversely proportional to the square of the statistical errors in the

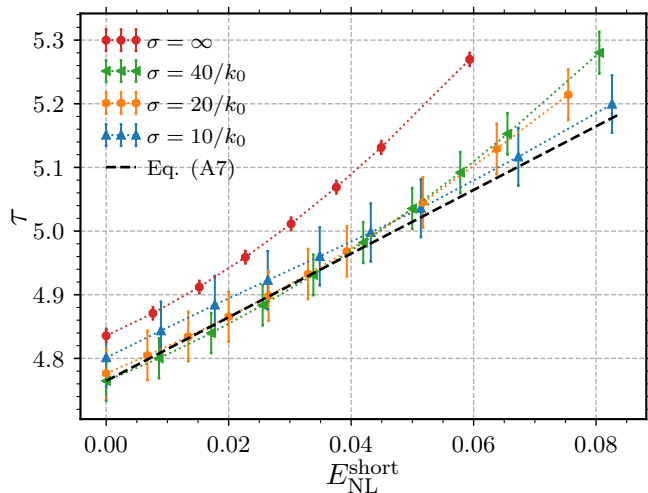


FIG. 11. Fitted values of the mean scattering time τ for wave packets of different initial widths vs. the nonlinear energy averaged over the fit time window, $E_{\text{NL}}^{\text{short}}$. Time is in units of $m/\hbar k_0^2$ and energy in units of $\hbar^2 k_0^2/m$. The black dashed line shows the prediction of Eq. (A7), where the wave packet with $\sigma = 40/k_0$ is used for τ_0 .

time window $t \in [0, t_{\text{fit}}]$. Our choice is $t_{\text{fit}} = 20\tau_0$. This value is chosen such that the fits enclose the whole ballistic motion and the beginning of the retro-reflection. The value of t_{fit} slightly influences the fitted parameters, but the changes are smaller than the error originating from the fitting procedure.

The fits return both τ and ℓ for a given interaction strength g . This allows us to calculate the average veloc-

ity $v = \ell/\tau$. From our data, we observe that the average velocity remains almost unaffected across all studied interaction values, although it is a little higher than the predicted value $\hbar k_0/m$ at the Born approximation. This apparent discrepancy is caused by higher order corrections to the Born approximation and is of the order of $1/k_0 \ell_0 \ll 1$. We can thus restrict the analysis to the mean scattering time τ only.

The fitted values of τ are shown in Fig. 11 as a function of the nonlinear energy averaged over the fit time window, $E_{\text{NL}}^{\text{short}}$. To explain these curves, we expand the mean scattering time $\tau(E \simeq E_0 + E_{\text{NL}})$ to leading order in $E_{\text{NL}} \ll E_0$, using the Born approximation, Eq. (3). This yields:

$$\tau \simeq \tau_0 + \frac{\hbar}{2k_0\gamma} E_{\text{NL}}. \quad (\text{A7})$$

The linear increase of τ is well visible in Fig. 11. The fact that curves at different σ are slightly shifted upwards is due to the (small) dependence of the g -independent part, τ_0 , on σ , which shows up beyond the Born approximation. Eq. (A7) is shown in Fig. 11, where for τ_0 we use the numerical value of the scattering time for $\sigma = 40/k_0$ and $g = 0$. The agreement between Eq. (A7) and the data is very good. At larger values of $E_{\text{NL}}^{\text{short}}$, the curves start to deviate from the linear behavior, bending upwards. This effect is smaller for decreasing wave-packet widths σ . The observed change corresponds to a relatively strong boomerang breakdown, so that the use of the non-interacting theoretical prediction for the fit becomes less reliable.

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