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GEOM-SPIDER-EM: FASTER VARIANCE REDUCED STOCHASTIC EXPECTATION MAXIMIZATION FOR NONCONVEX FINITE-SUM OPTIMIZATION

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ABSTRACT

The Expectation Maximization (EM) algorithm is a key reference for inference in latent variable models; unfortunately, its computational cost is prohibitive in the large scale learning setting. In this paper, we propose an extension of the Stochastic Path-Integrated Differential Estimator EM (SPIDER-EM) and derive complexity bounds for this novel algorithm, designed to solve smooth nonconvex finite-sum optimization problems. We show that it reaches the same state of the art complexity bounds as SPIDER-EM; and provide conditions for a linear rate of convergence. Numerical results support our findings.

Index Terms— Large scale learning, Latent variable analysis, Expectation Maximization, Stochastic nonconvex optimization, Variance reduction.

1. INTRODUCTION

Intelligent processing of large data set and efficient learning of high-dimensional models require new optimization algorithms designed to be robust to big data and complex models era (see e.g. [1, 2, 3]). This paper is concerned with stochastic optimization of a nonconvex finite-sum smooth objective function

$$\operatorname{Argmin}_{\theta \in \Theta} F(\theta), \quad F(\theta) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\theta) + R(\theta), \quad (1)$$

when $\Theta \subseteq \mathbb{R}^d$ and F cannot be explicitly evaluated (nor its gradient). Many statistical learning problems can be cast into this framework, where n is the number of observations or examples, \mathcal{L}_i is a loss function associated to example $\#i$ (most of often a negated log-likelihood), and R is a penalty term promoting sparsity, regularity, etc.. Intractability of $F(\theta)$ might come from two sources. The first, referred to as *large scale learning* setting, is that the number n is very large so that the computations involving a sum over n terms should be either simply avoided or sparingly used during the run of the optimization algorithm (see e.g. [4] for an introduction to the bridge between large scale learning and stochastic approximation; see [5, 6] for applications to training of deep neural networks for signal and image processing; and more generally, empirical risk minimization in machine learning is a matter for (1)). The second is due to the presence of latent variables: for any i , the function \mathcal{L}_i as a (high-dimensional) integral over latent variables. Such a latent variable context is a classical statistical modeling; for example as a tool for solving inference in mixture models [7], for the definition of mixed models capturing variability among examples [8] or for

modeling hidden and/or missing variables (see e.g. applications in text modeling through latent Dirichlet allocation [9], in audio source separation [10, 11], in hyper-spectral imaging [12]).

In this contribution, we address the two levels of intractability in the case \mathcal{L}_i is of the form

$$\mathcal{L}_i(\theta) \stackrel{\text{def}}{=} -\log \int_{\mathcal{Z}} h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta)) \mu(dz). \quad (2)$$

This setting in particular covers the case when $\sum_{i=1}^n \mathcal{L}_i(\theta)$ is the negated log-likelihood of the observations (Y_1, \dots, Y_n) , the pairs observation/latent variable $\{(Y_i, Z_i), i \leq n\}$ are independent, and the distribution of the complete data (Y_i, Z_i) given by $(y_i, z) \mapsto h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta)) \mu(dz)$ is from the curved exponential family. Gaussian mixture models are typical examples, as well as mixtures of distributions from the curved exponential family.

In the framework (1)-(2), a Majorize-Minimization approach through the Expectation-Maximization (EM) algorithm [13] is standard; unfortunately, the computational cost of the batch EM can be prohibitive in the large scale learning setting. Different strategies were proposed to address this issue [14, 15, 16, 17, 18]: they combine mini-batches processing, Stochastic Approximation (SA) techniques (see e.g. [19, 20]) and variance reduction methods.

The first contribution of this paper is to provide a novel algorithm, the generalized Stochastic Path-Integrated Differential Estimator EM (\mathcal{G} -SPIDER-EM), which is among the variance reduced stochastic EM methods for nonconvex finite-sum optimization of the form (1)-(2); the generalizations allow a reduced computational cost without altering the convergence properties. The second contribution is the proof of complexity bounds, that is the number of parameter updates (M-step) and the number of conditional expectations evaluations (E-step), in order to reach ϵ -approximate stationary points; these bounds are derived for a specific form of \mathcal{G} -SPIDER-EM: we show that its complexity bounds are the same as those of SPIDER-EM, bounds which are state of the art ones and overpass all the previous ones. Linear convergence rates are proved under a Polyak-Łojasiewicz condition. Finally, numerical results support our findings and provide insights on how to implement \mathcal{G} -SPIDER-EM in order to inherit the properties of SPIDER-EM while reducing the computational cost.

Notations For two vectors $a, b \in \mathbb{R}^q$, $\langle a, b \rangle$ is the scalar product, and $\|\cdot\|$ the associated norm. For a matrix A , A^T is its transpose. For a positive integer n , set $[n]^* \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $[n] \stackrel{\text{def}}{=} \{0, \dots, n\}$. ∇f denotes the gradient of a differentiable function f . The minimum of a and b is denoted by $a \wedge b$. Finally, we use standard big O notation to leave out constants.

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2. EM-BASED METHODS IN THE EXPECTATION SPACE

We begin by formulating the model assumptions:

A1. $\Theta \subseteq \mathbb{R}^d$ is a convex set. (Z, \mathcal{Z}) is a measurable space and μ is a σ -finite positive measure on \mathcal{Z} . The functions $R : \Theta \rightarrow \mathbb{R}$, $\phi : \Theta \rightarrow \mathbb{R}^q$, $\psi : \Theta \rightarrow \mathbb{R}$, $s_i : Z \rightarrow \mathbb{R}^q$, $h_i : Z \rightarrow \mathbb{R}_+$ for all $i \in [n]^*$ are measurable. For any $\theta \in \Theta$ and $i \in [n]^*$, $|\mathcal{L}_i(\theta)| < \infty$.

For any $\theta \in \Theta$ and $i \in [n]^*$, define the posterior density of the latent variable Z_i given the observation Y_i :

$$p_i(z; \theta) \stackrel{\text{def}}{=} h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta) + \mathcal{L}_i(\theta)), \quad (3)$$

note that the dependence upon y_i follows through the index i in the above. Set

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} \int_Z s_i(z) p_i(z; \theta) \mu(dz), \quad \bar{s}(\theta) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n \bar{s}_i(\theta). \quad (4)$$

A2. The expectations $\bar{s}_i(\theta)$ are well defined for all $\theta \in \Theta$ and $i \in [n]^*$. For any $s \in \mathbb{R}^q$, $\text{Argmin}_{\theta \in \Theta} (\psi(\theta) - \langle s, \phi(\theta) \rangle + R(\theta))$ is a (non empty) singleton denoted by $\{\mathsf{T}(s)\}$.

EM is an iterative algorithm: given a current value $\tau_k \in \Theta$, the next value is $\tau_{k+1} \leftarrow \mathsf{T} \circ \bar{s}(\tau_k)$. It combines an expectation step which boils down to the computation of $\bar{s}(\tau_k)$, the conditional expectation of $s(z)$ under $p(\cdot; \tau_k)$; and a maximization step which corresponds to the computation of the map T . Equivalently, by using T which maps \mathbb{R}^q to Θ , it can be described in the expectation space (see [21]): given the current value $\bar{s}^k \in \bar{s}(\Theta)$, the next value is $\bar{s}^{k+1} \leftarrow \bar{s} \circ \mathsf{T}(\bar{s}^k)$.

In this paper, we see EM as an iterative algorithm operating in the expectation space. In that case, the fixed points of the EM operator $\bar{s} \circ \mathsf{T}$ are the roots of the function h

$$h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s. \quad (5)$$

EM possesses a Lyapunov function: in the parameter space, it is the objective function F where by definition of the EM sequence, it holds $F(\tau_{k+1}) \leq F(\tau_k)$; in the expectation space, it is $W \stackrel{\text{def}}{=} F \circ \mathsf{T}$, and $W(\bar{s}^{k+1}) \leq W(\bar{s}^k)$ holds. In order to derive complexity bounds, regularity assumptions are required on W :

A3. The functions ϕ , ψ and R are continuously differentiable on Θ^v , where Θ^v is a neighborhood of Θ . T is continuously differentiable on \mathbb{R}^q . The function F is continuously differentiable on Θ^v and for any $\theta \in \Theta$, $\nabla F(\theta) = -\nabla \phi(\theta)^T \bar{s}(\theta) + \nabla \psi(\theta) + \nabla R(\theta)$. For any $s \in \mathbb{R}^q$, $B(s) \stackrel{\text{def}}{=} \nabla(\phi \circ \mathsf{T})(s)$ is a symmetric $q \times q$ matrix and there exist $0 < v_{\min} \leq v_{\max} < \infty$ such that for all $s \in \mathbb{R}^q$, the spectrum of $B(s)$ is in $[v_{\min}, v_{\max}]$. For any $i \in [n]^*$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on \mathbb{R}^q with constant L_i . The function $s \mapsto \nabla(F \circ \mathsf{T})(s) = B(s) (\bar{s} \circ \mathsf{T}(s) - s)$ is globally Lipschitz on \mathbb{R}^q with constant $L_{\bar{W}}$.

A6 implies that W has globally Lipschitz gradient and $\nabla W(s) = -B(s)h(s)$ for some positive definite matrix $B(s)$ (see e.g. [21, Lemma 2]; see also [22, Propositions 1 and 2]). Note that this implies that $\nabla W(s^*) = 0$ iff $h(s^*) = 0$.

Unfortunately, in the large scale learning setting (when $n \gg 1$), EM can not be easily applied since each iteration involves n conditional expectations (CE) evaluations through $\bar{s} = n^{-1} \sum_{i=1}^n \bar{s}_i$. Incremental EM techniques have been proposed to address this issue: the most straightforward approach amounts to use a SA scheme with mean field h since. Upon noting that $h(s) = \mathbb{E}[\bar{s}_I \circ \mathsf{T}(s)] - s$

where I is a uniform random variable (r.v.) on $[n]^*$, the fixed points of the EM operator $\bar{s} \circ \mathsf{T}$ are those of the SA scheme

$$\hat{S}_{k+1} = \hat{S}_k + \gamma_{k+1} \left(b^{-1} \sum_{i \in \mathcal{B}_{k+1}} \bar{s}_i \circ \mathsf{T}(\hat{S}_k) - \hat{S}_k \right) \quad (6)$$

where $\{\gamma_k, k \geq 0\}$ is a deterministic positive step size sequence, and \mathcal{B}_{k+1} is sampled in $[n]^*$ independently from the past of the algorithm. This forms the basis of Online-EM proposed by [15] (see also [23]). Variance reduced versions were also proposed and studied: Incremental EM (i-EM) [14, 24], Stochastic EM with variance reduction (sEM-vr) [16], Fast Incremental EM [17, 22] (FIEM) and more recently, Stochastic Path-Integrated Differential Estimator (SPIDER-EM) [18].

As shown in [22, section 2.3], these algorithms can be seen as a combination of SA with *control variate*: upon noting that $h(s) = h(s) + \mathbb{E}[U]$ for any r.v. U such that $\mathbb{E}[U] = 0$, *control variates* within SA procedures replace (6) with

$$\hat{S}_{k+1} = \hat{S}_k + \gamma_{k+1} \left(b^{-1} \sum_{i \in \mathcal{B}_{k+1}} \bar{s}_i \circ \mathsf{T}(\hat{S}_k) + U_{k+1} - \hat{S}_k \right)$$

for a choice of U_{k+1} such that the new algorithm has better properties (for example, in terms of complexity - see the end of Section 3).

Lastly, we remark that A4–A6 are common assumptions satisfied by many statistical models such as the Gaussian Mixture Model; see [18] for a rigorous justification of these assumptions.

3. THE GEOM-SPIDER-EM ALGORITHM

Data: $k_{\text{out}} \in \mathbb{N}^*$; $\hat{S}_{\text{init}} \in \mathbb{R}^q$; $\xi_t \in \mathbb{N}^*$ for $t \in [k_{\text{out}}]^*$; $\gamma_{t,0} \geq 0$, $\gamma_{t,k} > 0$ for $t \in [k_{\text{out}}]^*$, $k \in [\xi_t]^*$.

Result: The SPIDER-EM sequence: $\{\hat{S}_{t,k}\}$

```

1  $\hat{S}_{1,0} = \hat{S}_{1,-1} = \hat{S}_{\text{init}}$ ;
2  $\mathsf{S}_{1,0} = \bar{s} \circ \mathsf{T}(\hat{S}_{1,-1}) + \mathcal{E}_1$ ;
3 for  $t = 1, \dots, k_{\text{out}}$  do
4   for  $k = 0, \dots, \xi_t - 1$  do
5     Sample a mini batch  $\mathcal{B}_{t,k+1}$  of size  $b$  in  $[n]^*$ ;
6      $\mathsf{S}_{t,k+1} = \mathsf{S}_{t,k} +$ 
7        $b^{-1} \sum_{i \in \mathcal{B}_{t,k+1}} (\bar{s}_i \circ \mathsf{T}(\hat{S}_{t,k}) - \bar{s}_i \circ \mathsf{T}(\hat{S}_{t,k-1}))$ 
8       ;
9      $\hat{S}_{t,k+1} = \hat{S}_{t,k} + \gamma_{t,k+1} (\mathsf{S}_{t,k+1} - \hat{S}_{t,k})$ 
10     $\hat{S}_{t+1,-1} = \hat{S}_{t,\xi_t}$ ;
11     $\mathsf{S}_{t+1,0} = \bar{s} \circ \mathsf{T}(\hat{S}_{t+1,-1}) + \mathcal{E}_{t+1}$ ;
12     $\hat{S}_{t+1,0} = \hat{S}_{t+1,-1} + \gamma_{t+1,0} (\mathsf{S}_{t+1,0} - \hat{S}_{t+1,-1})$ 
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Algorithm 1: The g-SPIDER-EM algorithm. The \mathcal{E}_t 's are introduced as a perturbation to the computation of $\bar{s} \circ \mathsf{T}(\hat{S}_{t,-1})$; they can be null.

The algorithm *generalized Stochastic Path-Integrated Differential Estimator Expectation Maximization* (g-SPIDER-EM) described by Algorithm 1 uses a new strategy when defining the approximation of $\bar{s} \circ \mathsf{T}(s)$ at each iteration. It is composed of nested loops: k_{out} outer loops, each of them formed with a possibly random number of inner loops. Within the t th outer loop, g-SPIDER-EM mimics the identity $\bar{s} \circ \mathsf{T}(\hat{S}_{t,k}) = \bar{s} \circ \mathsf{T}(\hat{S}_{t,k-1}) + \{\bar{s} \circ \mathsf{T}(\hat{S}_{t,k}) - \bar{s} \circ \mathsf{T}(\hat{S}_{t,k-1})\}$. More precisely, at iteration $k+1$, the approximation

$S_{t,k+1}$ of the full sum $\bar{s} \circ T(\hat{S}_{t,k})$ is the sum of the current approximation $S_{t,k}$ and of a Monte Carlo approximation of the difference (see Lines 5, 6, in Algorithm 1); the examples i in $\mathcal{B}_{t,k+1}$ used in the approximation of $\bar{s} \circ T(\hat{S}_{t,k})$ and those used for the approximation of $\bar{s} \circ T(\hat{S}_{t,k-1})$ are the same - which make the approximations correlated and favor a variance reduction when plugged in the SA update (Line 7). $\mathcal{B}_{t,k+1}$ is sampled with or without replacement; even when $\mathcal{B}_{t,k+1}$ collects independent examples sampled uniformly in $[n]^*$, we have $\mathbb{E}[S_{t,k+1} | \mathcal{F}_{t,k}] - \bar{s} \circ T(\hat{S}_{t,k}) = S_{t,k} - \bar{s} \circ T(\hat{S}_{t,k-1})$ where $\mathcal{F}_{t,k}$ is the sigma-field collecting the randomness up to the end of the outer loop $\#t$ and inner loop $\#k$: the approximation $S_{t,k+1}$ of $\bar{s} \circ T(\hat{S}_{t,k})$ is biased - a property which makes the theoretical analysis of the algorithm challenging. This approximation is reset (see Lines 2,9) at the end of an outer loop: in the "standard" SPIDER-EM, $S_{t,0} = \bar{s} \circ T(\hat{S}_{t,-1})$ is computed, but this "refresh" can be only partial, by computing an update on a (large) batch $\tilde{\mathcal{B}}_{t,0}$ (size \tilde{b}_t) of observations: $S_{t,0} = \tilde{b}_t^{-1} \sum_{i \in \tilde{\mathcal{B}}_{t,0}} \bar{s}_i \circ T(\hat{S}_{t,-1})$. Such a reset starts a so-called *epoch* (see Line 3). The number of inner loops ξ_t at epoch $\#t$ can be deterministic ξ_t ; or random, such as a uniform distribution on $[k_{\text{in}}]^*$ or a geometric distribution, and drawn prior the run of the algorithm.

Comparing \mathcal{G} -SPIDER-EM with SPIDER-EM [18], we notice that the former allows a perturbation \mathcal{E}_t when initializing $S_{t,0}$. This is important for computational cost reduction. Moreover, \mathcal{G} -SPIDER-EM considers epochs with time-varying length ξ_t which covers situations when it is random and chosen independently of the other sources of randomness (the errors \mathcal{E}_t , the batches $\mathcal{B}_{t,k+1}$). Hereafter, we provide an original analysis of an \mathcal{G} -SPIDER-EM, namely Geom-SPIDER-EM which corresponds to the case $\xi_t \leftarrow \Xi_t$, Ξ_t being a geometric r.v. on \mathbb{N}^* with success probability $1 - \rho_t \in (0, 1)$: $\mathbb{P}(\Xi_t = k) = (1 - \rho_t)\rho_t^{k-1}$ for $k \geq 1$ (hereafter, we will write $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$). Since Ξ_t is also the first success distribution in a sequence of independent Bernoulli trials, the geometric length could be replaced with: (i) at each iteration k of epoch t , sample a Bernoulli r.v. with a probability of success $(1 - \rho_t)$; (ii) when the coin comes up head, start a new epoch (see [25, 26] for similar ideas on stochastic gradient algorithms).

Let us establish complexity bounds for Geom-SPIDER-EM. We analyze a randomized terminating iteration Ξ^* [27] and discuss how to choose k_{out} , b and $\xi_1, \dots, \xi_{k_{\text{out}}}$ as a function of the batch size n and an accuracy $\epsilon > 0$ to reach ϵ -approximate stationarity i.e. $\mathbb{E}[\|h(\hat{S}_{\Xi^*})\|^2] \leq \epsilon$. To this end, we endow the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the sigma-fields $\mathcal{F}_{1,0} \stackrel{\text{def}}{=} \sigma(\mathcal{E}_1)$, $\mathcal{F}_{t,0} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t-1,\xi_t}, \mathcal{E}_t)$ for $t \geq 2$, and $\mathcal{F}_{t,k+1} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$ for $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$. For a r.v. $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$, set $\mathbb{E}_t[\phi(\Xi_t) | \mathcal{F}_{t,0}] \stackrel{\text{def}}{=} (1 - \rho_t) \sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k) | \mathcal{F}_{t,0}]$ for any bounded measurable function ϕ .

Theorem 1. Assume A4 to A6. For any $t \in [k_{\text{out}}]^*$, let $\rho_t \in (0, 1)$ and $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$. Run Algorithm 1 with $\gamma_{t,k+1} = \gamma_t > 0$ and $\xi_t \leftarrow \Xi_t$ for any $t \in [k_{\text{out}}]^*$, $k \geq 0$. Then, for any $t \in [k_{\text{out}}]^*$,

$$\begin{aligned} & \frac{v_{\min} \gamma_t}{2(1 - \rho_t)} \mathbb{E}_t[\|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0}] \\ & \leq W(\hat{S}_{t,0}) - \mathbb{E}_t[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0}] + \frac{v_{\max} \gamma_t}{2(1 - \rho_t)} \|\mathcal{E}_t\|^2 \\ & + \frac{v_{\max} \gamma_t \gamma_{t,0}^2}{2(1 - \rho_t)} \frac{L^2}{b} \|\Delta \hat{S}_{t,0}\|^2 + \mathcal{N}_t \mathbb{E}_t[\|\Delta \hat{S}_{t,\Xi_t}\|^2 | \mathcal{F}_{t,0}] ; \end{aligned}$$

where $\Delta \hat{S}_{t,\xi} \stackrel{\text{def}}{=} S_{t,\xi} - \hat{S}_{t,\xi-1}$, $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i$, and

$$\mathcal{N}_t \stackrel{\text{def}}{=} \frac{\gamma_t}{2(1 - \rho_t)} \left(v_{\min} - \gamma_t L_{\hat{W}} - \frac{v_{\max} L^2 \rho_t}{(1 - \rho_t) b} \gamma_t^2 \right).$$

Theorem 1 is the key result from which our conclusions are drawn; its proof is adapted from [18, section 8] (also see [28]).

Let us discuss the rate of convergence and the complexity of Geom-SPIDER-SA in the case: for any $t \in [k_{\text{out}}]^*$, the mean number of inner loops is $(1 - \rho_t)^{-1} = k_{\text{in}}$, $\gamma_{t,0} = 0$ and $\gamma_t = \alpha/L$ for $\alpha > 0$ satisfying

$$v_{\min} - \alpha \frac{L_{\hat{W}}}{L} - \alpha^2 v_{\max} \frac{k_{\text{in}}}{b} \left(1 - \frac{1}{k_{\text{in}}} \right) > 0.$$

Linear rate. When $\Xi \sim \mathcal{G}^*(1 - \rho)$, we have

$$\rho \mathbb{E}[D_{\Xi}] \leq \rho \mathbb{E}[D_{\Xi}] + (1 - \rho) D_0 = \mathbb{E}[D_{\Xi-1}] \quad (7)$$

for any positive sequence $\{D_k, k \geq 0\}$, Theorem 1 implies

$$\begin{aligned} \mathbb{E}_t[\|h(\hat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0}] & \leq \frac{2L}{v_{\min} \alpha (k_{\text{in}} - 1)} (W(\hat{S}_{t,0}) - \min W) \\ & + \frac{v_{\max}}{v_{\min}} \frac{k_{\text{in}}}{k_{\text{in}} - 1} \|\mathcal{E}_t\|^2. \end{aligned} \quad (8)$$

Hence, when $\|\mathcal{E}_t\| = 0$ and W satisfies a Polyak-Łojasiewicz condition [29], i.e.

$$\exists \tau > 0, \forall s \in \mathbb{R}^q, \quad W(s) - \min W \leq \tau \|\nabla W(s)\|^2 \quad (9)$$

then (8) yields

$$\mathcal{H}_t \stackrel{\text{def}}{=} \mathbb{E}_t[\|h(\hat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0}] \leq \frac{2L\tau v_{\max}^2}{v_{\min} \alpha (k_{\text{in}} - 1)} \|h(\hat{S}_{t-1,\xi_{t-1}})\|^2,$$

thus establishing a linear rate of the algorithm along the path $\{\hat{S}_{t,\Xi_t}, t \in [k_{\text{out}}]^*\}$ as soon as k_{in} is large enough:

$$\mathbb{E}[\mathcal{H}_t] \leq \left(\frac{2L\tau v_{\max}^2}{v_{\min} \alpha (k_{\text{in}} - 1)} \right)^t \|h(\hat{S}_{\text{init}})\|^2.$$

Even if the Polyak-Łojasiewicz condition (9) is quite restrictive, the above discussion gives the intuition of the *lock-in* phenomenon which often happens at convergence: a linear rate of convergence is observed when the path is trapped in a neighborhood of its limiting point, which may be the consequence that locally, the Polyak-Łojasiewicz condition holds (see figure 1 in Section 4).

Complexity for ϵ -approximate stationarity. From Theorem 1, Eq. (7) and $\hat{S}_{t,\Xi_t} = \hat{S}_{t+1,0}$ (here $\gamma_{t,0} = 0$), it holds

$$\frac{v_{\min} \alpha (k_{\text{in}} - 1)}{2L} \mathbb{E}[\mathcal{H}_t] \leq \mathbb{E}[W(\hat{S}_{t,0}) - W(\hat{S}_{t+1,0})].$$

Therefore,

$$\frac{1}{k_{\text{out}}} \sum_{t=1}^{k_{\text{out}}} \mathbb{E}[\mathcal{H}_t] \leq \frac{2L (W(\hat{S}_{\text{init}}) - \min W)}{v_{\min} \alpha (k_{\text{in}} - 1) k_{\text{out}}}. \quad (10)$$

Eq. (10) establishes that in order to obtain an ϵ -approximate stationary point, it is sufficient to stop the algorithm at the end of the epoch $\#T$, where T is sampled uniformly in $[k_{\text{out}}]^*$ with $k_{\text{out}} = O(\epsilon^{-1} L/k_{\text{in}})$ - and return \hat{S}_{T,ξ_T} . To do such, the mean number of conditional expectations evaluations is $\mathcal{K}_{\text{CE}} \stackrel{\text{def}}{=} n + nk_{\text{out}} +$

$2bk_{\text{in}}k_{\text{out}}$; and the mean number of optimization steps is $\mathcal{K}_{\text{Opt}} \stackrel{\text{def}}{=} k_{\text{out}} + k_{\text{in}}k_{\text{out}}$. By choosing $k_{\text{in}} = O(\sqrt{n})$ and $b = O(\sqrt{n})$, we have $\mathcal{K}_{\text{CE}} = O(L\sqrt{n}\epsilon^{-1})$ and $\mathcal{K}_{\text{Opt}} = O(L\epsilon^{-1})$. Similar randomized terminating strategies were proposed in the literature: their optimal complexity in terms of conditional expectations evaluations is $O(\epsilon^{-2})$ for Online-EM [15], $O(\epsilon^{-1}n)$ for i-EM [14], $O(\epsilon^{-1}n^{2/3})$ for SEM-VF [16, 17], $O(\{\epsilon^{-1}n^{2/3}\} \wedge \{\epsilon^{-3/2}\sqrt{n}\})$ for FIEM [17, 22] and $O(\epsilon^{-1}\sqrt{n})$ for SPIDER-EM - see [18, section 6] for a comparison of the complexities \mathcal{K}_{CE} and \mathcal{K}_{Opt} of these incremental EM algorithms. Hence, Geom-SPIDER-EM has the same complexity bounds as SPIDER-EM, and they are optimal among the class of incremental EM algorithms.

4. NUMERICAL ILLUSTRATION

We perform experiments on the MNIST dataset, which consists of $n = 6 \times 10^4$ images of handwritten digits, each with 784 pixels. We pre-process the data as detailed in [22, section 5]: 67 uninformative pixels are removed from each image, and then a principal component analysis is applied to further reduce the dimension; we keep the 20 principal components of each observation. The learning problem consists in fitting a Gaussian mixture model with $g = 12$ components: θ collects the weights of the mixture, the expectations of the components (i.e. g vectors in \mathbb{R}^{20}) and a full 20×20 covariance matrix; here, $R = 0$ (no penalty term). All the algorithms start from an initial value $\hat{S}_{\text{init}} = \bar{s} \circ T(\theta_{\text{init}})$ such that $-F(\theta_{\text{init}}) = -58.3$, and their first two epochs are Online-EM. The first epoch with a variance reduction technique is epoch #3; on Fig. 1, the plot starts at epoch #2.

The proposed Geom-SPIDER-EM is run with a constant step size $\gamma_{t,k} = 0.01$ (and $\gamma_{t,0} = 0$); $k_{\text{out}} = 148$ epochs (which are preceded with 2 epochs of Online-EM); a mini batch size $b = \sqrt{n}$. Different strategies are considered for the initialization $S_{t,0}$ and the parameter of the geometric r.v. Ξ_t . In full-geom, $k_{\text{in}} = \sqrt{n}/2$ so that the mean total number of conditional expectations evaluations per outer loop is $2bk_{\text{in}} = n$; and $\mathcal{E}_t = 0$ which means that $S_{t,0}$ requires the computation of the full sum \bar{s} over n terms. In half-geom, k_{in} is defined as in full-geom, but for all $t \in [k_{\text{out}}]^*$, $S_{t,0} = (2/n) \sum_{i \in \mathcal{B}_{t,0}} \bar{s}_i \circ T(\hat{S}_{t-1})$ where $\mathcal{B}_{t,0}$ is of cardinality $n/2$; therefore $\mathcal{E}_t \neq 0$. In quad-geom, a quadratic growth is considered both for the mean of the geometric random variables: $\mathbb{E}[\xi_t] = \min(n, \max(20t^2, n/50))/(2b)$; and for the size of the mini batch when computing $S_{t,0}$: $S_{t,0} = \tilde{b}_t^{-1} \sum_{i \in \mathcal{B}_t} \bar{s}_i \circ T(\hat{S}_{t-1})$ with $\tilde{b}_t = \min(n, \max(20t^2, n/50))$. The g-SPIDER-EM with a constant number of inner loops $\xi_t = k_{\text{in}} = n/(2b)$ is also run for comparison: different strategies for $S_{t,0}$ are considered, the same as above (it corresponds to full-ctt, half-ctt and quad-ctt on the plots). Finally, in order to illustrate the benefit of the variance reduction, a pure Online-EM is run for 150 epochs, one epoch corresponding to \sqrt{n} updates of the statistics \hat{S} , each of them requiring a mini batch \mathcal{B}_{k+1} of size \sqrt{n} (see Eq.(6)).

The algorithms are compared through an estimation of the quantile of order 0.5 of $\|h(\hat{S}_{t,\Xi_t})\|^2$ over 30 independent realizations. It is plotted versus the number of epochs t in Fig. 1 and the number of conditional expectations (CE) evaluations in Fig. 2. They are also compared through the objective function F along the path; the mean value over 30 independent paths is displayed versus the number of CE, see Fig. 3.

We first observe that Online-EM has a poor convergence rate, thus justifying the interest of variance reduction techniques as shown in Fig. 1. Having a persistent bias along iterations when defining $S_{t,0}$

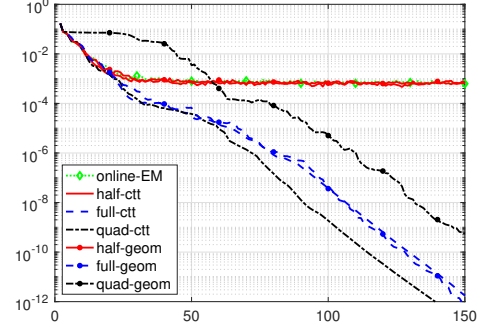


Fig. 1. Quantile 0.5 of $\|h(\hat{S}_{t,\Xi_t})\|^2$ vs the number of epochs

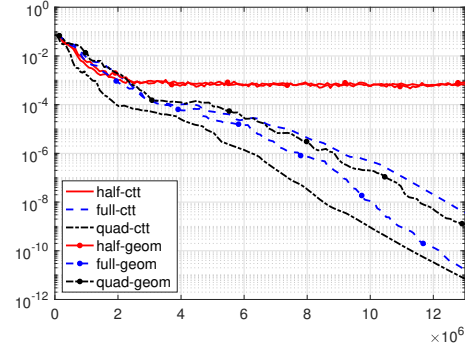


Fig. 2. Quantile 0.5 of $\|h(\hat{S}_{t,\Xi_t})\|^2$ vs the number of CE evaluations

i.e. considering $\tilde{b}_t \neq n$ and therefore $\mathcal{E}_t \neq 0$, is also a bad strategy as seen in Fig. 1, 2 for half-ctt and half-geom. For the four other SPIDER-EM strategies, we observe a linear convergence rate in Fig. 1, 2. The best strategy, both in terms of CE evaluations and in terms of efficiency given a number of epochs, is quad-ctt: a constant and deterministic number of inner loops ξ_t combined with an increasing accuracy when computing $S_{t,0}$; therefore, during the first iterations, it is better to reduce the computational cost of the algorithm by considering $\tilde{b}_t \ll n$. When $\mathcal{E}_t = 0$ (i.e. $\tilde{b}_t = n$ so the computational cost of $S_{t,0}$ is maximal), it is possible to reduce the total CE computational cost of the algorithm by considering a random number of inner loops (see full-geom and full-ctt on Fig. 1, 2). Finally, the strategy which consists in increasing both \tilde{b}_t and the number of inner loops, does not look the best one (see quad-ctt and quad-geom on Fig. 1 to Fig. 3).

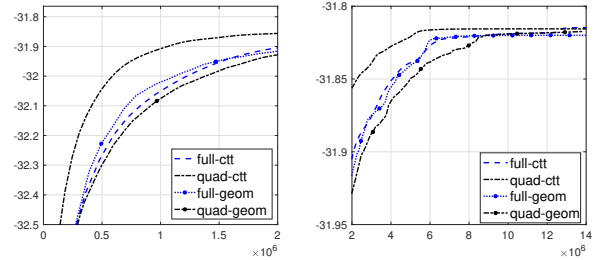


Fig. 3. (Left) $-F$ vs CE, until $2e6$. (Right) $-F$ vs CE, after $2e6$.

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**Supplementary material,
paper “GEOM-SPIDER-EM: faster variance
reduced stochastic Expectation Maximization
for nonconvex finite-sum optimization”**

6. PROOF OF THEOREM 1

Let $\{\mathcal{E}_t, t \in [k_{\text{out}}]^*\}$ and $\{\mathcal{B}_{t,k+1}, t \in [k_{\text{out}}]^*, k \in [\xi_t - 1]\}$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define the filtrations $\mathcal{F}_{1,0} \stackrel{\text{def}}{=} \sigma(\mathcal{E}_1)$, $\mathcal{F}_{t,0} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t-1,0}, \mathcal{E}_t)$ for $t \geq 2$, and $\mathcal{F}_{t,k+1} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$ for $t \in [k_{\text{out}}]^*, k \in [\xi_t - 1]$.

For $\rho_t \in (0, 1)$, set

$$\bar{\mathbb{E}}_t[\phi(\Xi_t)|\mathcal{F}_{t,0}] \stackrel{\text{def}}{=} (1 - \rho_t) \sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k)|\mathcal{F}_{t,0}],$$

for any measurable positive function ϕ .

A4. $\Theta \subseteq \mathbb{R}^d$ is a convex set. (Z, \mathcal{Z}) is a measurable space and μ is a σ -finite positive measure on \mathcal{Z} . The functions $R : \Theta \rightarrow \mathbb{R}$, $\phi : \Theta \rightarrow \mathbb{R}^q$, $\psi : \Theta \rightarrow \mathbb{R}$, $s_i : Z \rightarrow \mathbb{R}^q$, $h_i : Z \rightarrow \mathbb{R}_+$ for all $i \in [n]^*$ are measurable. For any $\theta \in \Theta$ and $i \in [n]^*$, $|\mathcal{L}_i(\theta)| < \infty$.

A5. The expectations $\bar{s}_i(\theta)$ are well defined for all $\theta \in \Theta$ and $i \in [n]^*$. For any $s \in \mathbb{R}^q$, $\text{Argmin}_{\theta \in \Theta} (\psi(\theta) - \langle s, \phi(\theta) \rangle + R(\theta))$ is a (non empty) singleton denoted by $\{\mathsf{T}(s)\}$.

A6. The functions ϕ , ψ and R are continuously differentiable on Θ^v , where Θ^v is a neighborhood of Θ . T is continuously differentiable on \mathbb{R}^q . The function F is continuously differentiable on Θ^v and for any $\theta \in \Theta$, $\nabla F(\theta) = -\nabla \phi(\theta)^T \bar{s}(\theta) + \nabla \psi(\theta) + \nabla R(\theta)$. For any $s \in \mathbb{R}^q$, $B(s) \stackrel{\text{def}}{=} \nabla(\phi \circ \mathsf{T})(s)$ is a symmetric $q \times q$ matrix and there exist $0 < v_{\min} \leq v_{\max} < \infty$ such that for all $s \in \mathbb{R}^q$, the spectrum of $B(s)$ is in $[v_{\min}, v_{\max}]$. For any $i \in [n]^*$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on \mathbb{R}^q with constant L_i . The function $s \mapsto \nabla(F \circ \mathsf{T})(s) = B(s)(\bar{s} \circ \mathsf{T}(s) - s)$ is globally Lipschitz on \mathbb{R}^q with constant $L_{\bar{W}}$.

Lemma 2. Let $\rho \in (0, 1)$ and $\{D_k, k \geq 0\}$ be real numbers such that $\sum_{k \geq 0} \rho^k |D_k| < \infty$. Let $\xi \sim \mathcal{G}^*(1 - \rho)$. Then $\mathbb{E}[D_{\xi-1}] = \rho \mathbb{E}[D_{\xi}] + (1 - \rho)D_0 = \mathbb{E}[D_{\xi}] + (1 - \rho)(D_0 - \mathbb{E}[D_{\xi}])$.

Proof. By definition of ξ ,

$$\begin{aligned} \mathbb{E}[D_{\xi}] &= (1 - \rho) \sum_{k \geq 1} \rho^{k-1} D_k \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 1} \rho^k D_k \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 0} \rho^k D_k - \rho^{-1}(1 - \rho)D_0 \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 1} \rho^{k-1} D_{k-1} - \rho^{-1}(1 - \rho)D_0 \\ &= \rho^{-1} \mathbb{E}[D_{\xi-1}] - \rho^{-1}(1 - \rho)D_0. \end{aligned}$$

This yields $\rho \mathbb{E}[D_{\xi}] = \mathbb{E}[D_{\xi-1}] - (1 - \rho)D_0$ and concludes the proof. \square

Lemma 3. For any $t \in [k_{\text{out}}]^*$, $k \in [\xi_t]^*$, $\mathcal{B}_{t,k}$ and $\mathcal{F}_{t,k-1}$ are independent. In addition, for any $s \in \mathbb{R}^q$, $\mathsf{b}^{-1} \mathbb{E}[\sum_{i \in \mathcal{B}_{t,k}} \bar{s}_i \circ \mathsf{T}(s)] =$

$\bar{s} \circ \mathsf{T}(s)$. Finally, assume that for any $i \in [n]^*$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz with constant L_i . Then for any $s, s' \in \mathbb{R}^q$,

$$\begin{aligned} \mathbb{E} \left[\left\| \mathsf{b}^{-1} \sum_{i \in \mathcal{B}_{t,k}} \{ \bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s') \} - \bar{s} \circ \mathsf{T}(s) + \bar{s} \circ \mathsf{T}(s') \right\|^2 \right] \\ \leq \frac{1}{\mathsf{b}} (L^2 \|s - s'\|^2 - \|\bar{s} \circ \mathsf{T}(s) - \bar{s} \circ \mathsf{T}(s')\|^2), \end{aligned}$$

where $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Proof. See [18, Lemma 4]; the proof holds true when $\mathcal{B}_{t,k}$ is sampled with or without replacement. \square

Proposition 4. For any $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$,

$$\mathbb{E}[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}] - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) = \mathsf{S}_{t,k} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1}),$$

and

$$\mathbb{E}[\mathsf{S}_{t,k+1} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k})|\mathcal{F}_{t,0}] = \mathcal{E}_t.$$

Proof. Let $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$. By Lemma 3,

$$\mathbb{E}[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}] = \mathsf{S}_{t,k} + \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1}).$$

By definition of $\mathsf{S}_{t,0}$ and of the filtrations, $\mathsf{S}_{t,0} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,-1}) = \mathcal{E}_t \in \mathcal{F}_{t,0}$. The proof follows by induction on k . \square

Proposition 5. Assume that for any $i \in [n]^*$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz with constant L_i . For any $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$,

$$\begin{aligned} \mathbb{E}[\|\mathsf{S}_{t,k+1} - \mathbb{E}[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}]\|^2|\mathcal{F}_{t,k}] \\ \leq \frac{1}{\mathsf{b}} (L^2 \|\hat{\mathsf{S}}_{t,k} - \hat{\mathsf{S}}_{t,k-1}\|^2 - \|\bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1})\|^2) \\ \leq \frac{L^2}{\mathsf{b}} \gamma_{t,k}^2 \|\mathsf{S}_{t,k} - \hat{\mathsf{S}}_{t,k-1}\|^2, \end{aligned}$$

where $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Proof. Let $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$. By Lemma 3, Proposition 4, the definition of $\mathsf{S}_{t,k+1}$ and of the filtration $\mathcal{F}_{t,k}$,

$$\begin{aligned} \mathsf{S}_{t,k+1} - \mathbb{E}[\mathsf{S}_{t,k+1}|\mathcal{F}_{t,k}] \\ = \mathsf{S}_{t,k+1} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) - \mathsf{S}_{t,k} + \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1}) \\ = \mathsf{b}^{-1} \sum_{i \in \mathcal{B}_{t,k+1}} \{ \bar{s}_i \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) - \bar{s}_i \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1}) \} \\ - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k}) + \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1}). \end{aligned}$$

We then conclude by Lemma 3 for the first inequality; and by using the definition of $\hat{\mathsf{S}}_{t,k}$ for the second one. \square

Proposition 6. Assume that for any $i \in [n]^*$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz with constant L_i . For any $t \in [k_{\text{out}}]^*$, $k \in [\xi_t - 1]$,

$$\begin{aligned} \mathbb{E}[\|\mathsf{S}_{t,k+1} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k})\|^2|\mathcal{F}_{t,k}] \\ \leq \frac{L^2}{\mathsf{b}} \gamma_{t,k}^2 \|\mathsf{S}_{t,k} - \hat{\mathsf{S}}_{t,k-1}\|^2 + \|\mathsf{S}_{t,k} - \bar{s} \circ \mathsf{T}(\hat{\mathsf{S}}_{t,k-1})\|^2, \end{aligned}$$

where $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Proof. By definition of the conditional expectation, we have for any r.v. $\phi(V)$

$$\mathbb{E} [\|U - \phi(V)\|^2 | V] = \mathbb{E} [\|U - \mathbb{E}[U|V]\|^2 | V] + \|\mathbb{E}[U|V] - \phi(V)\|^2.$$

The proof follows from this equality and Propositions 4 and 5. \square

Corollary 7. Assume that for any $i \in [n]^*$, $\bar{s}_i \circ \mathbf{T}$ is globally Lipschitz with constant L_i . For any $t \in [k_{\text{out}}]^*$, let $\rho_t \in (0, 1)$ and $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$. For any $t \in [k_{\text{out}}]^*$,

$$\begin{aligned} & \mathbb{E}_t \left[(\gamma_{t,\Xi_t} - \rho_t \gamma_{t,\Xi_t+1}) \|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \frac{L^2 \rho_t}{b} \mathbb{E}_t \left[\gamma_{t,\Xi_t+1} \gamma_{t,\Xi_t}^2 \|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2(1 - \rho_t)}{b} \gamma_{t,1} \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2 + (1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2, \end{aligned}$$

where $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Proof. Let $t \in [k_{\text{out}}]^*$ and $k \in [\xi_t - 1]$. From Proposition 6 and since $\mathcal{F}_{t,0} \subseteq \mathcal{F}_{t,k}$ for $k \in [\xi_t - 1]$, we have

$$\begin{aligned} & \mathbb{E} \left[\|S_{t,k+1} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[\|S_{t,k} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,k-1})\|^2 + \frac{L^2}{b} \gamma_{t,k}^2 \|S_{t,k} - \hat{S}_{t,k-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Multiply by $\gamma_{t,k+1}$ and apply with $k = \xi_t - 1$:

$$\begin{aligned} & \mathbb{E} \left[\gamma_{t,\xi_t} \|S_{t,\xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[\gamma_{t,\xi_t} \|S_{t,\xi_t-1} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2}{b} \mathbb{E} \left[\gamma_{t,\xi_t} \gamma_{t,\xi_t-1}^2 \|S_{t,\xi_t-1} - \hat{S}_{t,\xi_t-2}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

This implies

$$\begin{aligned} & \mathbb{E}_t \left[\gamma_{t,\Xi_t} \|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[\gamma_{t,\Xi_t} \|S_{t,\Xi_t-1} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2}{b} \mathbb{E}_t \left[\gamma_{t,\Xi_t} \gamma_{t,\Xi_t-1}^2 \|S_{t,\Xi_t-1} - \hat{S}_{t,\Xi_t-2}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} & \mathbb{E}_t \left[\gamma_{t,\Xi_t} \|S_{t,\Xi_t-1} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & = \rho_t \mathbb{E}_t \left[\gamma_{t,\Xi_t+1} \|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \gamma_{t,1} \mathbb{E}_t \left[\|S_{t,0} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,-1})\|^2 | \mathcal{F}_{t,0} \right]; \end{aligned}$$

by definition of $S_{t,0}$ and $\mathcal{F}_{t,0}$, the last term is equal to $(1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2$. By Lemma 2, we have

$$\begin{aligned} & \mathbb{E}_t \left[\gamma_{t,\Xi_t} \gamma_{t,\Xi_t-1}^2 \|S_{t,\Xi_t-1} - \hat{S}_{t,\Xi_t-2}\|^2 | \mathcal{F}_{t,0} \right] \\ & = \rho_t \mathbb{E}_t \left[\gamma_{t,\Xi_t+1} \gamma_{t,\Xi_t}^2 \|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \gamma_{t,1} \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2. \end{aligned}$$

This concludes the proof. \square

Lemma 8. For any $h, s, S \in \mathbb{R}^q$ and any $q \times q$ symmetric matrix B , it holds

$$-2 \langle Bh, S \rangle = -\langle BS, S \rangle - \langle Bh, h \rangle + \langle B\{h - S\}, h - S \rangle.$$

Proposition 9. Assume A4 to A6. For any $t \in [k_{\text{out}}]^*$ and $k \in [\xi_t - 1]$,

$$\begin{aligned} & \mathbb{E} \left[W(\hat{S}_{t,k+1}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \gamma_{t,k+1} \mathbb{E} \left[\|h(\hat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[W(\hat{S}_{t,k}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \gamma_{t,k+1} \mathbb{E} \left[\|S_{t,k+1} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t,k+1}}{2} (v_{\min} - \gamma_{t,k+1} L_{\dot{W}}) \mathbb{E} \left[\|S_{t,k+1} - \hat{S}_{t,k}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Proof. Since W is continuously differentiable with $L_{\dot{W}}$ -Lipschitz gradient, then for any $s, s' \in \mathbb{R}^q$,

$$W(s') - W(s) \leq \langle \nabla W(s), s' - s \rangle + \frac{L_{\dot{W}}}{2} \|s' - s\|^2.$$

Set $s' = s + \gamma S$ where $\gamma > 0$ and $S \in \mathbb{R}^q$. Since $\nabla W(s) = -B(s)h(s)$ and $B(s)$ is symmetric, apply Lemma 8 with $h \leftarrow h(s)$, $B \leftarrow B(s)$ and $S = (s' - s)/\gamma$; this yields

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma}{2} \langle B(s)S, S \rangle - \frac{\gamma}{2} \langle B(s)h(s), h(s) \rangle \\ & + \frac{\gamma}{2} \langle B(s)\{h(s) - S\}, h(s) - S \rangle + \frac{L_{\dot{W}}}{2} \gamma^2 \|S\|^2. \end{aligned}$$

Since $\|a\|^2 v_{\min} \leq \langle B(s)a, a \rangle \leq v_{\max} \|a\|^2$ for any $a \in \mathbb{R}^q$, we have

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma v_{\min}}{2} \|S\|^2 - \frac{\gamma v_{\min}}{2} \|h(s)\|^2 \\ & + \frac{\gamma v_{\max}}{2} \|h(s) - S\|^2 + \frac{L_{\dot{W}}}{2} \gamma^2 \|S\|^2. \end{aligned}$$

Applying this inequality with $s \leftarrow \hat{S}_{t,k}$, $\gamma \leftarrow \gamma_{t,k+1}$, $S \leftarrow S_{t,k+1} - \hat{S}_{t,k}$ (which yields $s + \gamma S = \hat{S}_{t,k+1}$), and then the conditional expectation yield the result. \square

Proposition 10. Assume A4 to A6. For any $t \in [k_{\text{out}}]^*$

$$\begin{aligned} & W(\hat{S}_{t+1,0}) - W(\hat{S}_{t+1,-1}) \\ & \leq -\frac{\gamma_{t+1,0} v_{\min}}{2} \|h(\hat{S}_{t+1,-1})\|^2 + \frac{v_{\max} \gamma_{t+1,0}}{2} \|\mathcal{E}_{t+1}\|^2 \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \|S_{t+1,0} - \hat{S}_{t+1,-1}\|^2. \end{aligned}$$

Proof. As in the proof of Proposition 9, we write for any $s, s' \in \mathbb{R}^q$,

$$W(s') - W(s) \leq \langle \nabla W(s), s' - s \rangle + \frac{L_{\dot{W}}}{2} \|s' - s\|^2.$$

With Lemma 8, this yields when $s' = s + \gamma S$ for $\gamma > 0$ and $S \in \mathbb{R}^q$

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma}{2} (v_{\min} - \gamma L_{\dot{W}}) \|S\|^2 - \frac{\gamma v_{\min}}{2} \|h\|^2 \\ & + \frac{v_{\max} \gamma}{2} \|h - S\|^2. \end{aligned}$$

Apply this inequality with $\gamma \leftarrow \gamma_{t+1,0}$, $s \leftarrow \hat{S}_{t+1,-1}$, $S \leftarrow S_{t+1,0} - \hat{S}_{t+1,-1}$ and $h \leftarrow h(s)$ (which yields $s + \gamma S = \hat{S}_{t+1,0}$). \square

Theorem 11. Assume A4 to A6. For any $t \in [k_{\text{out}}]^*$, let $\rho_t \in (0, 1)$ and $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$. Finally, choose $\gamma_{t,k+1} = \gamma_t > 0$ for any $k \geq 0$. For any $t \in [k_{\text{out}}]^*$,

$$\begin{aligned} & \frac{v_{\min}\gamma_t}{2(1-\rho_t)} \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \leq W(\hat{S}_{t,0}) - \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2(1-\rho_t)} \frac{L^2}{b} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2(1-\rho_t)} \gamma_{t,1} \|\mathcal{E}_t\|^2 \\ & - \frac{\gamma_t}{2(1-\rho_t)} \left(v_{\min} - \gamma_t L_{\dot{W}} - \frac{v_{\max} L^2 \rho_t}{(1-\rho_t)b} \gamma_t^2 \right) \cdots \\ & \times \mathbb{E}_t \left[\|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Proof. Apply Proposition 9 with $k \leftarrow \xi_t - 1$ and then set $\xi_t \leftarrow \Xi_t$; this yields

$$\begin{aligned} & \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \mathbb{E}_t \left[\gamma_{t,\Xi_t} \|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t-1}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \mathbb{E}_t \left[\gamma_{t,\Xi_t} \|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \mathbb{E}_t \left[\frac{\gamma_{t,\Xi_t}}{2} (v_{\min} - \gamma_{t,\Xi_t} L_{\dot{W}}) \|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Since $\Xi_t \geq 1$ and $\gamma_{t,k} = \gamma_t$ for any $k \geq 1$, we have

$$\begin{aligned} & \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \gamma_t \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t-1}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \gamma_t \mathbb{E}_t \left[\|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_t}{2} (v_{\min} - \gamma_t L_{\dot{W}}) \mathbb{E}_t \left[\|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

By Lemma 2, it holds

$$\begin{aligned} & \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] = \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t-1}) | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \left(\mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] - W(\hat{S}_{t,0}) \right) \end{aligned}$$

Furthermore, by Corollary 7 applied with $\gamma_{t,\Xi_t} = \gamma_{t,\Xi_t+1} = \gamma_t$

$$\begin{aligned} & (1 - \rho_t) \gamma_t \mathbb{E}_t \left[\|S_{t,\Xi_t} - \bar{s} \circ \mathbf{T}(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \frac{L^2 \rho_t}{b} \gamma_t^3 \mathbb{E}_t \left[\|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2(1-\rho_t)}{b} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2 + (1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2, \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{v_{\min}}{2} \gamma_t \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq (1 - \rho_t) W(\hat{S}_{t,0}) - (1 - \rho_t) \mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \frac{L^2 \rho_t}{(1-\rho_t)b} \gamma_t^3 \mathbb{E}_t \left[\|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \frac{L^2}{b} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2} \gamma_t \|\mathcal{E}_t\|^2 \\ & - \frac{\gamma_t}{2} (v_{\min} - \gamma_t L_{\dot{W}}) \mathbb{E}_t \left[\|S_{t,\Xi_t} - \hat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

This concludes the proof. \square

Corollary 12 (of Theorem 11). For any $t \in [k_{\text{out}}]^*$,

$$\begin{aligned} & \left(\frac{\gamma_t \rho_t}{(1-\rho_t)} + \gamma_{t+1,0} \right) \frac{v_{\min}}{2} \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq W(\hat{S}_{t,0}) - \mathbb{E}_t \left[W(\hat{S}_{t+1,0}) | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \mathbb{E} \left[\|S_{t+1,0} - \hat{S}_{t+1,-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2(1-\rho_t)} \frac{L^2}{b} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \hat{S}_{t,-1}\|^2 \\ & + \frac{v_{\max}}{2(1-\rho_t)} \gamma_t \|\mathcal{E}_t\|^2 + \frac{v_{\max} \gamma_{t+1,0}}{2} \mathbb{E} \left[\|\mathcal{E}_{t+1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Proof. Let $t \in [k_{\text{out}}]^*$. By Proposition 10, since $\hat{S}_{t,\xi_t} = \hat{S}_{t+1,-1}$ we have

$$\begin{aligned} & -\mathbb{E} \left[W(\hat{S}_{t,\xi_t}) | \mathcal{F}_{t,0} \right] \leq -\mathbb{E} \left[W(\hat{S}_{t+1,0}) | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0} v_{\min}}{2} \mathbb{E} \left[\|h(\hat{S}_{t,\xi_t})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max} \gamma_{t+1,0}}{2} \mathbb{E} \left[\|\mathcal{E}_{t+1}\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \mathbb{E} \left[\|S_{t+1,0} - \hat{S}_{t+1,-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

The previous inequality remains true when $\mathbb{E} \left[W(\hat{S}_{t,\xi_t}) | \mathcal{F}_{t,0} \right]$ is replaced with $\mathbb{E}_t \left[W(\hat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right]$; and $\mathbb{E} \left[\|h(\hat{S}_{t,\xi_t})\|^2 | \mathcal{F}_{t,0} \right] = \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0} \right]$. The proof follows from Theorem 11, and (see Lemma 2)

$$\mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \geq \rho_t \mathbb{E}_t \left[\|h(\hat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0} \right].$$

\square