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# GEOM-SPIDER-EM: FASTER VARIANCE REDUCED STOCHASTIC EXPECTATION MAXIMIZATION FOR NONCONVEX FINITE-SUM OPTIMIZATION

Gersende Fort<sup>◇</sup>   Eric Moulines<sup>\*</sup>   Hoi-To Wai<sup>†</sup>

<sup>◇</sup> Institut Mathématique de Toulouse, Université de Toulouse; CNRS UPS, F-31062 Toulouse Cedex, France

<sup>\*</sup> Centre de Mathématiques Appliquées; Ecole Polytechnique; 91128 Palaiseau Cedex, France

<sup>†</sup> Department of SEEM; The Chinese University of Hong Kong; Shatin, Hong Kong

## ABSTRACT

The Expectation Maximization (EM) algorithm is a key reference for inference in latent variable models; unfortunately, its computational cost is prohibitive in the large scale learning setting. In this paper, we propose an extension of the Stochastic Path-Integrated Differential Estimator EM (SPIDER-EM) and derive complexity bounds for this novel algorithm, designed to solve smooth nonconvex finite-sum optimization problems. We show that it reaches the same state of the art complexity bounds as SPIDER-EM; and provide conditions for a linear rate of convergence. Numerical results support our findings.

**Index Terms**— Large scale learning, Latent variable analysis, Expectation Maximization, Stochastic nonconvex optimization, Variance reduction.

## 1. INTRODUCTION

Intelligent processing of large data set and efficient learning of high-dimensional models require new optimization algorithms designed to be robust to big data and complex models era (see e.g. [1, 2, 3]). This paper is concerned with stochastic optimization of a nonconvex finite-sum smooth objective function

$$\operatorname{Argmin}_{\theta \in \Theta} F(\theta), \quad F(\theta) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\theta) + R(\theta), \quad (1)$$

when  $\Theta \subseteq \mathbb{R}^d$  and  $F$  cannot be explicitly evaluated (nor its gradient). Many statistical learning problems can be cast into this framework, where  $n$  is the number of observations or examples,  $\mathcal{L}_i$  is a loss function associated to example  $\#i$  (most of often a negated log-likelihood), and  $R$  is a penalty term promoting sparsity, regularity, etc.. Intractability of  $F(\theta)$  might come from two sources. The first, referred to as *large scale learning* setting, is that the number  $n$  is very large so that the computations involving a sum over  $n$  terms should be either simply avoided or sparingly used during the run of the optimization algorithm (see e.g. [4] for an introduction to the bridge between large scale learning and stochastic approximation; see [5, 6] for applications to training of deep neural networks for signal and image processing; and more generally, empirical risk minimization in machine learning is a matter for (1)). The second is due to the presence of latent variables: for any  $i$ , the function  $\mathcal{L}_i$  as a (high-dimensional) integral over latent variables. Such a latent variable context is a classical statistical modeling; for example as a tool for solving inference in mixture models [7], for the definition of mixed models capturing variability among examples [8] or for

modeling hidden and/or missing variables (see e.g. applications in text modeling through latent Dirichlet allocation [9], in audio source separation [10, 11], in hyper-spectral imaging [12]).

In this contribution, we address the two levels of intractability in the case  $\mathcal{L}_i$  is of the form

$$\mathcal{L}_i(\theta) \stackrel{\text{def}}{=} -\log \int_{\mathcal{Z}} h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta)) \mu(dz). \quad (2)$$

This setting in particular covers the case when  $\sum_{i=1}^n \mathcal{L}_i(\theta)$  is the negated log-likelihood of the observations  $(Y_1, \dots, Y_n)$ , the pairs observation/latent variable  $\{(Y_i, Z_i), i \leq n\}$  are independent, and the distribution of the complete data  $(Y_i, Z_i)$  given by  $(y_i, z) \mapsto h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta)) \mu(dz)$  is from the curved exponential family. Gaussian mixture models are typical examples, as well as mixtures of distributions from the curved exponential family.

In the framework (1)-(2), a Majorize-Minimization approach through the Expectation-Maximization (EM) algorithm [13] is standard; unfortunately, the computational cost of the batch EM can be prohibitive in the large scale learning setting. Different strategies were proposed to address this issue [14, 15, 16, 17, 18]: they combine mini-batches processing, Stochastic Approximation (SA) techniques (see e.g. [19, 20]) and variance reduction methods.

The first contribution of this paper is to provide a novel algorithm, the generalized Stochastic Path-Integrated Differential Estimator EM ( $\sigma$ -SPIDER-EM), which is among the variance reduced stochastic EM methods for nonconvex finite-sum optimization of the form (1)-(2); the generalizations allow a reduced computational cost without altering the convergence properties. The second contribution is the proof of complexity bounds, that is the number of parameter updates (M-step) and the number of conditional expectations evaluations (E-step), in order to reach  $\epsilon$ -approximate stationary points; these bounds are derived for a specific form of  $\sigma$ -SPIDER-EM: we show that its complexity bounds are the same as those of SPIDER-EM, bounds which are state of the art ones and overpass all the previous ones. Linear convergence rates are proved under a Polyak-Łojasiewicz condition. Finally, numerical results support our findings and provide insights on how to implement  $\sigma$ -SPIDER-EM in order to inherit the properties of SPIDER-EM while reducing the computational cost.

**Notations** For two vectors  $a, b \in \mathbb{R}^q$ ,  $\langle a, b \rangle$  is the scalar product, and  $\|\cdot\|$  the associated norm. For a matrix  $A$ ,  $A^T$  is its transpose. For a positive integer  $n$ , set  $[n]^* \stackrel{\text{def}}{=} \{1, \dots, n\}$  and  $[n] \stackrel{\text{def}}{=} \{0, \dots, n\}$ .  $\nabla f$  denotes the gradient of a differentiable function  $f$ . The minimum of  $a$  and  $b$  is denoted by  $a \wedge b$ . Finally, we use standard big O notation to leave out constants.

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## 2. EM-BASED METHODS IN THE EXPECTATION SPACE

We begin by formulating the model assumptions:

**A1.**  $\Theta \subseteq \mathbb{R}^d$  is a convex set.  $(Z, \mathcal{Z})$  is a measurable space and  $\mu$  is a  $\sigma$ -finite positive measure on  $Z$ . The functions  $R : \Theta \rightarrow \mathbb{R}$ ,  $\phi : \Theta \rightarrow \mathbb{R}^q$ ,  $\psi : \Theta \rightarrow \mathbb{R}$ ,  $s_i : Z \rightarrow \mathbb{R}^q$ ,  $h_i : Z \rightarrow \mathbb{R}_+$  for all  $i \in [n]^*$  are measurable. For any  $\theta \in \Theta$  and  $i \in [n]^*$ ,  $|\mathcal{L}_i(\theta)| < \infty$ .

For any  $\theta \in \Theta$  and  $i \in [n]^*$ , define the posterior density of the latent variable  $Z_i$  given the observation  $Y_i$ :

$$p_i(z; \theta) \stackrel{\text{def}}{=} h_i(z) \exp(\langle s_i(z), \phi(\theta) \rangle - \psi(\theta) + \mathcal{L}_i(\theta)), \quad (3)$$

note that the dependence upon  $y_i$  follows through the index  $i$  in the above. Set

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} \int_Z s_i(z) p_i(z; \theta) \mu(dz), \quad \bar{s}(\theta) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n \bar{s}_i(\theta). \quad (4)$$

**A2.** The expectations  $\bar{s}_i(\theta)$  are well defined for all  $\theta \in \Theta$  and  $i \in [n]^*$ . For any  $s \in \mathbb{R}^q$ ,  $\text{Argmin}_{\theta \in \Theta} (\psi(\theta) - \langle s, \phi(\theta) \rangle + R(\theta))$  is a (non empty) singleton denoted by  $\{\mathbb{T}(s)\}$ .

EM is an iterative algorithm: given a current value  $\tau_k \in \Theta$ , the next value is  $\tau_{k+1} \leftarrow \mathbb{T} \circ \bar{s}(\tau_k)$ . It combines an expectation step which boils down to the computation of  $\bar{s}(\tau_k)$ , the conditional expectation of  $s(z)$  under  $p(\cdot; \tau_k)$ ; and a maximization step which corresponds to the computation of the map  $\mathbb{T}$ . Equivalently, by using  $\mathbb{T}$  which maps  $\mathbb{R}^q$  to  $\Theta$ , it can be described in the *expectation space* (see [21]): given the current value  $\bar{s}^k \in \bar{s}(\Theta)$ , the next value is  $\bar{s}^{k+1} \leftarrow \bar{s} \circ \mathbb{T}(\bar{s}^k)$ .

In this paper, we see EM as an iterative algorithm operating in the expectation space. In that case, the fixed points of the EM operator  $\bar{s} \circ \mathbb{T}$  are the roots of the function  $h$

$$h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathbb{T}(s) - s. \quad (5)$$

EM possesses a Lyapunov function: in the parameter space, it is the objective function  $F$  where by definition of the EM sequence, it holds  $F(\tau_{k+1}) \leq F(\tau_k)$ ; in the expectation space, it is  $W \stackrel{\text{def}}{=} F \circ \mathbb{T}$ , and  $W(\bar{s}^{k+1}) \leq W(\bar{s}^k)$  holds. In order to derive complexity bounds, regularity assumptions are required on  $W$ :

**A3.** The functions  $\phi$ ,  $\psi$  and  $R$  are continuously differentiable on  $\Theta^v$ , where  $\Theta^v$  is a neighborhood of  $\Theta$ .  $\mathbb{T}$  is continuously differentiable on  $\mathbb{R}^q$ . The function  $F$  is continuously differentiable on  $\Theta^v$  and for any  $\theta \in \Theta$ ,  $\nabla F(\theta) = -\nabla \phi(\theta)^T \bar{s}(\theta) + \nabla \psi(\theta) + \nabla R(\theta)$ . For any  $s \in \mathbb{R}^q$ ,  $B(s) \stackrel{\text{def}}{=} \nabla(\phi \circ \mathbb{T})(s)$  is a symmetric  $q \times q$  matrix and there exist  $0 < v_{\min} \leq v_{\max} < \infty$  such that for all  $s \in \mathbb{R}^q$ , the spectrum of  $B(s)$  is in  $[v_{\min}, v_{\max}]$ . For any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathbb{T}$  is globally Lipschitz on  $\mathbb{R}^q$  with constant  $L_i$ . The function  $s \mapsto \nabla(F \circ \mathbb{T})(s) = B(s) (\bar{s} \circ \mathbb{T}(s) - s)$  is globally Lipschitz on  $\mathbb{R}^q$  with constant  $L_{\bar{W}}$ .

A6 implies that  $W$  has globally Lipschitz gradient and  $\nabla W(s) = -B(s)h(s)$  for some positive definite matrix  $B(s)$  (see e.g. [21, Lemma 2]; see also [22, Propositions 1 and 2]). Note that this implies that  $\nabla W(s^*) = 0$  iff  $h(s^*) = 0$ .

Unfortunately, in the large scale learning setting (when  $n \gg 1$ ), EM can not be easily applied since each iteration involves  $n$  conditional expectations (CE) evaluations through  $\bar{s} = n^{-1} \sum_{i=1}^n \bar{s}_i$ . Incremental EM techniques have been proposed to address this issue: the most straightforward approach amounts to use a SA scheme with mean field  $h$  since. Upon noting that  $h(s) = \mathbb{E}[\bar{s}_I \circ \mathbb{T}(s)] - s$

where  $I$  is a uniform random variable (r.v.) on  $[n]^*$ , the fixed points of the EM operator  $\bar{s} \circ \mathbb{T}$  are those of the SA scheme

$$\hat{S}_{k+1} = \hat{S}_k + \gamma_{k+1} \left( \mathbf{b}^{-1} \sum_{i \in \mathcal{B}_{k+1}} \bar{s}_i \circ \mathbb{T}(\hat{S}_k) - \hat{S}_k \right) \quad (6)$$

where  $\{\gamma_k, k \geq 0\}$  is a deterministic positive step size sequence, and  $\mathcal{B}_{k+1}$  is sampled in  $[n]^*$  independently from the past of the algorithm. This forms the basis of `Online-EM` proposed by [15] (see also [23]). Variance reduced versions were also proposed and studied: Incremental EM (i-EM) [14, 24], Stochastic EM with variance reduction (sEM-vr) [16], Fast Incremental EM [17, 22] (FIEM) and more recently, Stochastic Path-Integrated Differential Estimator (SPIDER-EM) [18].

As shown in [22, section 2.3], these algorithms can be seen as a combination of SA with *control variate*: upon noting that  $h(s) = h(s) + \mathbb{E}[U]$  for any r.v.  $U$  such that  $\mathbb{E}[U] = 0$ , *control variates within SA* procedures replace (6) with

$$\hat{S}_{k+1} = \hat{S}_k + \gamma_{k+1} \left( \mathbf{b}^{-1} \sum_{i \in \mathcal{B}_{k+1}} \bar{s}_i \circ \mathbb{T}(\hat{S}_k) + U_{k+1} - \hat{S}_k \right)$$

for a choice of  $U_{k+1}$  such that the new algorithm has better properties (for example, in terms of complexity - see the end of Section 3).

Lastly, we remark that A4–A6 are common assumptions satisfied by many statistical models such as the Gaussian Mixture Model; see [18] for a rigorous justification of these assumptions.

## 3. THE GEOM-SPIDER-EM ALGORITHM

**Data:**  $k_{\text{out}} \in \mathbb{N}^*$ ;  $\hat{S}_{\text{init}} \in \mathbb{R}^q$ ;  $\xi_t \in \mathbb{N}^*$  for  $t \in [k_{\text{out}}]^*$ ;  $\gamma_{t,0} \geq 0, \gamma_{t,k} > 0$  for  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t]^*$ .

**Result:** The SPIDER-EM sequence:  $\{\hat{S}_{t,k}\}$

```

1  $\hat{S}_{1,0} = \hat{S}_{1,-1} = \hat{S}_{\text{init}}$ ;
2  $S_{1,0} = \bar{s} \circ \mathbb{T}(\hat{S}_{1,-1}) + \mathcal{E}_1$ ;
3 for  $t = 1, \dots, k_{\text{out}}$  do
4   for  $k = 0, \dots, \xi_t - 1$  do
5     Sample a mini batch  $\mathcal{B}_{t,k+1}$  of size  $\mathbf{b}$  in  $[n]^*$ ;
6      $S_{t,k+1} = S_{t,k} +$ 
7        $\mathbf{b}^{-1} \sum_{i \in \mathcal{B}_{t,k+1}} (\bar{s}_i \circ \mathbb{T}(\hat{S}_{t,k}) - \bar{s}_i \circ \mathbb{T}(\hat{S}_{t,k-1}))$ 
8      $\hat{S}_{t,k+1} = \hat{S}_{t,k} + \gamma_{t,k+1} (S_{t,k+1} - \hat{S}_{t,k})$ 
9    $\hat{S}_{t+1,-1} = \hat{S}_{t,\xi_t}$ ;
10   $S_{t+1,0} = \bar{s} \circ \mathbb{T}(\hat{S}_{t+1,-1}) + \mathcal{E}_{t+1}$ ;
11   $\hat{S}_{t+1,0} = \hat{S}_{t+1,-1} + \gamma_{t+1,0} (S_{t+1,0} - \hat{S}_{t+1,-1})$ 

```

**Algorithm 1:** The  $\mathfrak{g}$ -SPIDER-EM algorithm. The  $\mathcal{E}_t$ 's are introduced as a perturbation to the computation of  $\bar{s} \circ \mathbb{T}(\hat{S}_{t,-1})$ ; they can be null.

The algorithm *generalized Stochastic Path-Integrated Differential Estimator Expectation Maximization* ( $\mathfrak{g}$ -SPIDER-EM) described by Algorithm 1 uses a new strategy when defining the approximation of  $\bar{s} \circ \mathbb{T}(s)$  at each iteration. It is composed of nested loops:  $k_{\text{out}}$  outer loops, each of them formed with a possibly random number of inner loops. Within the  $t$ th outer loop,  $\mathfrak{g}$ -SPIDER-EM mimics the identity  $\bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) = \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1}) + \{\bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1})\}$ . More precisely, at iteration  $k+1$ , the approximation

$S_{t,k+1}$  of the full sum  $\bar{s} \circ \mathbb{T}(\widehat{S}_{t,k})$  is the sum of the current approximation  $S_{t,k}$  and of a Monte Carlo approximation of the difference (see Lines 5, 6, in Algorithm 1); the examples  $i$  in  $\mathcal{B}_{t,k+1}$  used in the approximation of  $\bar{s} \circ \mathbb{T}(\widehat{S}_{t,k})$  and those used for the approximation of  $\bar{s} \circ \mathbb{T}(\widehat{S}_{t,k-1})$  are the same - which make the approximations correlated and favor a variance reduction when plugged in the SA update (Line 7).  $\mathcal{B}_{t,k+1}$  is sampled with or without replacement; even when  $\mathcal{B}_{t,k+1}$  collects independent examples sampled uniformly in  $[n]^*$ , we have  $\mathbb{E}[S_{t,k+1} | \mathcal{F}_{t,k}] - \bar{s} \circ \mathbb{T}(\widehat{S}_{t,k}) = S_{t,k} - \bar{s} \circ \mathbb{T}(\widehat{S}_{t,k-1})$  where  $\mathcal{F}_{t,k}$  is the sigma-field collecting the randomness up to the end of the outer loop  $\#t$  and inner loop  $\#k$ : the approximation  $S_{t,k+1}$  of  $\bar{s} \circ \mathbb{T}(\widehat{S}_{t,k})$  is biased - a property which makes the theoretical analysis of the algorithm challenging. This approximation is reset (see Lines 2,9) at the end of an outer loop: in the "standard" SPIDER-EM,  $S_{t,0} = \bar{s} \circ \mathbb{T}(\widehat{S}_{t,-1})$  is computed, but this "refresh" can be only partial, by computing an update on a (large) batch  $\tilde{\mathcal{B}}_{t,0}$  (size  $\tilde{b}_t$ ) of observations:  $S_{t,0} = \tilde{b}_t^{-1} \sum_{i \in \tilde{\mathcal{B}}_{t,0}} \bar{s}_i \circ \mathbb{T}(\widehat{S}_{t,-1})$ . Such a reset starts a so-called *epoch* (see Line 3). The number of inner loops  $\xi_t$  at epoch  $\#t$  can be deterministic  $\xi_t$ ; or random, such as a uniform distribution on  $[k_{\text{in}}]^*$  or a geometric distribution, and drawn prior the run of the algorithm.

Comparing  $\mathfrak{g}$ -SPIDER-EM with SPIDER-EM [18], we notice that the former allows a perturbation  $\mathcal{E}_t$  when initializing  $S_{t,0}$ . This is important for computational cost reduction. Moreover,  $\mathfrak{g}$ -SPIDER-EM considers epochs with time-varying length  $\xi_t$  which covers situations when it is random and chosen independently of the other sources of randomness (the errors  $\mathcal{E}_t$ , the batches  $\mathcal{B}_{t,k+1}$ ). Hereafter, we provide an original analysis of an  $\mathfrak{g}$ -SPIDER-EM, namely Geom-SPIDER-EM which corresponds to the case  $\xi_t \leftarrow \Xi_t$ ,  $\Xi_t$  being a geometric r.v. on  $\mathbb{N}^*$  with success probability  $1 - \rho_t \in (0, 1)$ :  $\mathbb{P}(\Xi_t = k) = (1 - \rho_t)\rho_t^{k-1}$  for  $k \geq 1$  (hereafter, we will write  $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$ ). Since  $\Xi_t$  is also the first success distribution in a sequence of independent Bernoulli trials, the geometric length could be replaced with: (i) at each iteration  $k$  of epoch  $t$ , sample a Bernoulli r.v. with a probability of success  $(1 - \rho_t)$ ; (ii) when the coin comes up head, start a new epoch (see [25, 26] for similar ideas on stochastic gradient algorithms).

Let us establish complexity bounds for Geom-SPIDER-EM. We analyze a randomized terminating iteration  $\Xi^*$  [27] and discuss how to choose  $k_{\text{out}}$ ,  $\mathfrak{b}$  and  $\xi_1, \dots, \xi_{k_{\text{out}}}$  as a function of the batch size  $n$  and an accuracy  $\epsilon > 0$  to reach  $\epsilon$ -approximate stationarity i.e.  $\mathbb{E}[\|h(\widehat{S}_{\Xi^*})\|^2] \leq \epsilon$ . To this end, we endow the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  with the sigma-fields  $\mathcal{F}_{1,0} \stackrel{\text{def}}{=} \sigma(\mathcal{E}_1)$ ,  $\mathcal{F}_{t,0} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t-1,\xi_t}, \mathcal{E}_t)$  for  $t \geq 2$ , and  $\mathcal{F}_{t,k+1} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$  for  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ . For a r.v.  $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$ , set  $\overline{\mathbb{E}}_t[\phi(\Xi_t) | \mathcal{F}_{t,0}] \stackrel{\text{def}}{=} (1 - \rho_t) \sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k) | \mathcal{F}_{t,0}]$  for any bounded measurable function  $\phi$ .

**Theorem 1.** *Assume A4 to A6. For any  $t \in [k_{\text{out}}]^*$ , let  $\rho_t \in (0, 1)$  and  $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$ . Run Algorithm 1 with  $\gamma_{t,k+1} = \gamma_t > 0$  and  $\xi_t \leftarrow \Xi_t$  for any  $t \in [k_{\text{out}}]^*$ ,  $k \geq 0$ . Then, for any  $t \in [k_{\text{out}}]^*$ ,*

$$\begin{aligned} & \frac{v_{\min} \gamma_t}{2(1 - \rho_t)} \overline{\mathbb{E}}_t \left[ \|h(\widehat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq W(\widehat{S}_{t,0}) - \overline{\mathbb{E}}_t \left[ W(\widehat{S}_{t,\Xi_t}) | \mathcal{F}_{t,0} \right] + \frac{v_{\max} \gamma_t}{2(1 - \rho_t)} \|\mathcal{E}_t\|^2 \\ & + \frac{v_{\max} \gamma_t \gamma_{t,0}^2 L^2}{2(1 - \rho_t) \mathfrak{b}} \|\Delta \widehat{S}_{t,0}\|^2 + \mathcal{N}_t \overline{\mathbb{E}}_t \left[ \|\Delta \widehat{S}_{t,\Xi_t}\|^2 | \mathcal{F}_{t,0} \right]; \end{aligned}$$

where  $\Delta \widehat{S}_{t,\xi} \stackrel{\text{def}}{=} S_{t,\xi} - \widehat{S}_{t,\xi-1}$ ,  $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i$ , and

$$\mathcal{N}_t \stackrel{\text{def}}{=} \frac{\gamma_t}{2(1 - \rho_t)} \left( v_{\min} - \gamma_t L_{\mathbb{W}} - \frac{v_{\max} L^2 \rho_t}{(1 - \rho_t) \mathfrak{b}} \gamma_t^2 \right).$$

Theorem 1 is the key result from which our conclusions are drawn; its proof is adapted from [18, section 8] (also see [28]).

Let us discuss the rate of convergence and the complexity of Geom-SPIDER-SA in the case: for any  $t \in [k_{\text{out}}]^*$ , the mean number of inner loops is  $(1 - \rho_t)^{-1} = k_{\text{in}}$ ,  $\gamma_{t,0} = 0$  and  $\gamma_t = \alpha/L$  for  $\alpha > 0$  satisfying

$$v_{\min} - \alpha \frac{L_{\mathbb{W}}}{L} - \alpha^2 v_{\max} \frac{k_{\text{in}}}{\mathfrak{b}} \left( 1 - \frac{1}{k_{\text{in}}} \right) > 0.$$

**Linear rate.** When  $\Xi \sim \mathcal{G}^*(1 - \rho)$ , we have

$$\rho \mathbb{E}[D_{\Xi}] \leq \rho \mathbb{E}[D_{\Xi}] + (1 - \rho) D_0 = \mathbb{E}[D_{\Xi-1}] \quad (7)$$

for any positive sequence  $\{D_k, k \geq 0\}$ , Theorem 1 implies

$$\begin{aligned} \overline{\mathbb{E}}_t \left[ \|h(\widehat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0} \right] & \leq \frac{2L}{v_{\min} \alpha (k_{\text{in}} - 1)} \left( W(\widehat{S}_{t,0}) - \min W \right) \\ & + \frac{v_{\max}}{v_{\min}} \frac{k_{\text{in}}}{k_{\text{in}} - 1} \|\mathcal{E}_t\|^2. \end{aligned} \quad (8)$$

Hence, when  $\|\mathcal{E}_t\| = 0$  and  $W$  satisfies a Polyak-Łojasiewicz condition [29], i.e.

$$\exists \tau > 0, \forall s \in \mathbb{R}^q, \quad W(s) - \min W \leq \tau \|\nabla W(s)\|^2 \quad (9)$$

then (8) yields

$$\mathcal{H}_t \stackrel{\text{def}}{=} \overline{\mathbb{E}}_t \left[ \|h(\widehat{S}_{t,\Xi_t})\|^2 | \mathcal{F}_{t,0} \right] \leq \frac{2L\tau v_{\max}^2}{v_{\min} \alpha (k_{\text{in}} - 1)} \|h(\widehat{S}_{t-1,\xi_{t-1}})\|^2,$$

thus establishing a linear rate of the algorithm along the path  $\{\widehat{S}_{t,\Xi_t}, t \in [k_{\text{out}}]^*\}$  as soon as  $k_{\text{in}}$  is large enough:

$$\mathbb{E}[\mathcal{H}_t] \leq \left( \frac{2L\tau v_{\max}^2}{v_{\min} \alpha (k_{\text{in}} - 1)} \right)^t \|h(\widehat{S}_{\text{init}})\|^2.$$

Even if the Polyak-Łojasiewicz condition (9) is quite restrictive, the above discussion gives the intuition of the *lock-in* phenomenon which often happens at convergence: a linear rate of convergence is observed when the path is trapped in a neighborhood of its limiting point, which may be the consequence that locally, the Polyak-Łojasiewicz condition holds (see figure 1 in Section 4).

**Complexity for  $\epsilon$ -approximate stationarity.** From Theorem 1, Eq. (7) and  $\widehat{S}_{t,\Xi_t} = \widehat{S}_{t+1,0}$  (here  $\gamma_{t,0} = 0$ ), it holds

$$\frac{v_{\min} \alpha (k_{\text{in}} - 1)}{2L} \mathbb{E}[\mathcal{H}_t] \leq \mathbb{E} \left[ W(\widehat{S}_{t,0}) - W(\widehat{S}_{t+1,0}) \right].$$

Therefore,

$$\frac{1}{k_{\text{out}}} \sum_{t=1}^{k_{\text{out}}} \mathbb{E}[\mathcal{H}_t] \leq \frac{2L \left( W(\widehat{S}_{\text{init}}) - \min W \right)}{v_{\min} \alpha (k_{\text{in}} - 1) k_{\text{out}}}. \quad (10)$$

Eq. (10) establishes that in order to obtain an  $\epsilon$ -approximate stationary point, it is sufficient to stop the algorithm at the end of the epoch  $\#T$ , where  $T$  is sampled uniformly in  $[k_{\text{out}}]^*$  with  $k_{\text{out}} = O(\epsilon^{-1} L/k_{\text{in}})$  - and return  $\widehat{S}_{T,\xi_T}$ . To do such, the mean number of conditional expectations evaluations is  $\mathcal{K}_{\text{CE}} \stackrel{\text{def}}{=} n + nk_{\text{out}} +$

$2bk_{\text{in}}k_{\text{out}}$ ; and the mean number of optimization steps is  $\mathcal{K}_{\text{Opt}} \stackrel{\text{def}}{=} k_{\text{out}} + k_{\text{in}}k_{\text{out}}$ . By choosing  $k_{\text{in}} = O(\sqrt{n})$  and  $b = O(\sqrt{n})$ , we have  $\mathcal{K}_{\text{CE}} = O(L\sqrt{n}\epsilon^{-1})$  and  $\mathcal{K}_{\text{Opt}} = O(L\epsilon^{-1})$ . Similar randomized terminating strategies were proposed in the literature: their optimal complexity in terms of conditional expectations evaluations is  $O(\epsilon^{-2})$  for Online-EM [15],  $O(\epsilon^{-1}n)$  for i-EM [14],  $O(\epsilon^{-1}n^{2/3})$  for sEM-vr [16, 17],  $O(\{\epsilon^{-1}n^{2/3}\} \wedge \{\epsilon^{-3/2}\sqrt{n}\})$  for FIEM [17, 22] and  $O(\epsilon^{-1}\sqrt{n})$  for SPIDER-EM - see [18, section 6] for a comparison of the complexities  $\mathcal{K}_{\text{CE}}$  and  $\mathcal{K}_{\text{Opt}}$  of these incremental EM algorithms. Hence, Geom-SPIDER-EM has the same complexity bounds as SPIDER-EM, and they are optimal among the class of incremental EM algorithms.

#### 4. NUMERICAL ILLUSTRATION

We perform experiments on the MNIST dataset, which consists of  $n = 6 \times 10^4$  images of handwritten digits, each with 784 pixels. We pre-process the data as detailed in [22, section 5]: 67 uninformative pixels are removed from each image, and then a principal component analysis is applied to further reduce the dimension; we keep the 20 principal components of each observation. The learning problem consists in fitting a Gaussian mixture model with  $g = 12$  components:  $\theta$  collects the weights of the mixture, the expectations of the components (i.e.  $g$  vectors in  $\mathbb{R}^{20}$ ) and a full  $20 \times 20$  covariance matrix; here,  $R = 0$  (no penalty term). All the algorithms start from an initial value  $\hat{S}_{\text{init}} = \bar{s} \circ T(\theta_{\text{init}})$  such that  $-F(\theta_{\text{init}}) = -58.3$ , and their first two epochs are Online-EM. The first epoch with a variance reduction technique is epoch #3; on Fig. 1, the plot starts at epoch #2.

The proposed Geom-SPIDER-EM is run with a constant step size  $\gamma_{t,k} = 0.01$  (and  $\gamma_{t,0} = 0$ );  $k_{\text{out}} = 148$  epochs (which are preceded with 2 epochs of Online-EM); a mini batch size  $b = \sqrt{n}$ . Different strategies are considered for the initialization  $S_{t,0}$  and the parameter of the geometric r.v.  $\Xi_t$ . In full-geom,  $k_{\text{in}} = \sqrt{n}/2$  so that the mean total number of conditional expectations evaluations per outer loop is  $2bk_{\text{in}} = n$ ; and  $\mathcal{E}_t = 0$  which means that  $S_{t,0}$  requires the computation of the full sum  $\bar{s}$  over  $n$  terms. In half-geom,  $k_{\text{in}}$  is defined as in full-geom, but for all  $t \in [k_{\text{out}}]^*$ ,  $S_{t,0} = (2/n) \sum_{i \in \mathcal{B}_{t,0}} \bar{s}_i \circ T(\hat{S}_{t,-1})$  where  $\mathcal{B}_{t,0}$  is of cardinality  $n/2$ ; therefore  $\mathcal{E}_t \neq 0$ . In quad-geom, a quadratic growth is considered both for the mean of the geometric random variables:  $\mathbb{E}[\xi_t] = \min(n, \max(20t^2, n/50))/(2b)$ ; and for the size of the mini batch when computing  $S_{t,0}$ :  $S_{t,0} = \tilde{b}_t^{-1} \sum_{i \in \mathcal{B}_t} \bar{s}_i \circ T(\hat{S}_{t,-1})$  with  $\tilde{b}_t = \min(n, \max(20t^2, n/50))$ . The g-SPIDER-EM with a constant number of inner loops  $\xi_t = k_{\text{in}} = n/(2b)$  is also run for comparison: different strategies for  $S_{t,0}$  are considered, the same as above (it corresponds to full-ctt, half-ctt and quad-ctt on the plots). Finally, in order to illustrate the benefit of the variance reduction, a pure Online-EM is run for 150 epochs, one epoch corresponding to  $\sqrt{n}$  updates of the statistics  $\hat{S}$ , each of them requiring a mini batch  $\mathcal{B}_{k+1}$  of size  $\sqrt{n}$  (see Eq.(6)).

The algorithms are compared through an estimation of the quantile of order 0.5 of  $\|h(\hat{S}_{t,\Xi_t})\|^2$  over 30 independent realizations. It is plotted versus the number of epochs  $t$  in Fig. 1 and the number of conditional expectations (CE) evaluations in Fig. 2. They are also compared through the objective function  $F$  along the path; the mean value over 30 independent paths is displayed versus the number of CE, see Fig. 3.

We first observe that Online-EM has a poor convergence rate, thus justifying the interest of variance reduction techniques as shown in Fig. 1. Having a persistent bias along iterations when defining  $S_{t,0}$

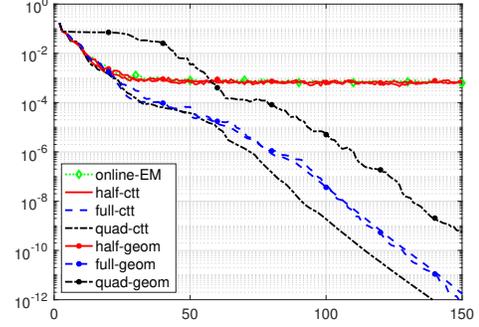


Fig. 1. Quantile 0.5 of  $\|h(\hat{S}_{t,\Xi_t})\|^2$  vs the number of epochs

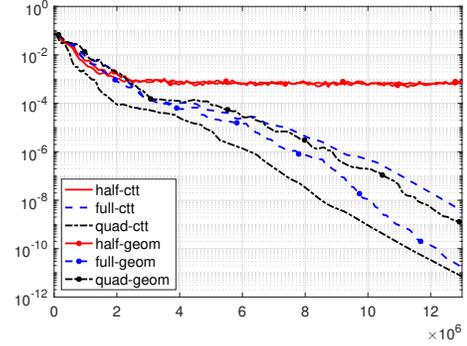


Fig. 2. Quantile 0.5 of  $\|h(\hat{S}_{t,\Xi_t})\|^2$  vs the number of CE evaluations

i.e. considering  $\tilde{b}_t \neq n$  and therefore  $\mathcal{E}_t \neq 0$ , is also a bad strategy as seen in Fig. 1, 2 for half-ctt and half-geom. For the four other SPIDER-EM strategies, we observe a linear convergence rate in Fig. 1, 2. The best strategy, both in terms of CE evaluations and in terms of efficiency given a number of epochs, is quad-ctt: a constant and deterministic number of inner loops  $\xi_t$  combined with an increasing accuracy when computing  $S_{t,0}$ ; therefore, during the first iterations, it is better to reduce the computational cost of the algorithm by considering  $\tilde{b}_t \ll n$ . When  $\mathcal{E}_t = 0$  (i.e.  $\tilde{b}_t = n$  so the computational cost of  $S_{t,0}$  is maximal), it is possible to reduce the total CE computational cost of the algorithm by considering a random number of inner loops (see full-geom and full-ctt on Fig. 1, 2). Finally, the strategy which consists in increasing both  $\tilde{b}_t$  and the number of inner loops, does not look the best one (see quad-ctt and quad-geom on Fig. 1 to Fig. 3).

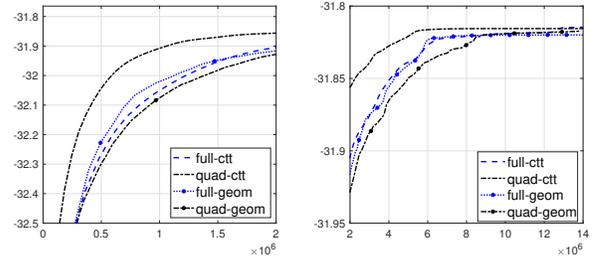


Fig. 3. (Left)  $-F$  vs CE, until  $2e6$ . (Right)  $-F$  vs CE, after  $2e6$ .

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**Supplementary material,  
paper “GEOM-SPIDER-EM: faster variance  
reduced stochastic Expectation Maximization  
for nonconvex finite-sum optimization”**

**6. PROOF OF THEOREM 1**

Let  $\{\mathcal{E}_t, t \in [k_{\text{out}}]^*\}$  and  $\{\mathcal{B}_{t,k+1}, t \in [k_{\text{out}}]^*, k \in [\xi_t - 1]\}$  be random variables defined on the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Define the filtrations  $\mathcal{F}_{t,0} \stackrel{\text{def}}{=} \sigma(\mathcal{E}_1)$ ,  $\mathcal{F}_{t,0} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t-1,\xi_t}, \mathcal{E}_t)$  for  $t \geq 2$ , and  $\mathcal{F}_{t,k+1} \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{t,k}, \mathcal{B}_{t,k+1})$  for  $t \in [k_{\text{out}}]^*, k \in [\xi_t - 1]$ .

For  $\rho_t \in (0, 1)$ , set

$$\bar{\mathbb{E}}_t[\phi(\Xi_t)|\mathcal{F}_{t,0}] \stackrel{\text{def}}{=} (1 - \rho_t) \sum_{k \geq 1} \rho_t^{k-1} \mathbb{E}[\phi(k)|\mathcal{F}_{t,0}],$$

for any measurable positive function  $\phi$ .

**A4.**  $\Theta \subseteq \mathbb{R}^d$  is a convex set.  $(\mathcal{Z}, \mathcal{Z})$  is a measurable space and  $\mu$  is a  $\sigma$ -finite positive measure on  $\mathcal{Z}$ . The functions  $\mathbb{R} : \Theta \rightarrow \mathbb{R}$ ,  $\phi : \Theta \rightarrow \mathbb{R}^q$ ,  $\psi : \Theta \rightarrow \mathbb{R}$ ,  $s_i : \mathcal{Z} \rightarrow \mathbb{R}^q$ ,  $h_i : \mathcal{Z} \rightarrow \mathbb{R}_+$  for all  $i \in [n]^*$  are measurable. For any  $\theta \in \Theta$  and  $i \in [n]^*$ ,  $|\mathcal{L}_i(\theta)| < \infty$ .

**A5.** The expectations  $\bar{s}_i(\theta)$  are well defined for all  $\theta \in \Theta$  and  $i \in [n]^*$ . For any  $s \in \mathbb{R}^q$ ,  $\text{Argmin}_{\theta \in \Theta} (\psi(\theta) - \langle s, \phi(\theta) \rangle + \mathbb{R}(\theta))$  is a (non empty) singleton denoted by  $\{\mathbb{T}(s)\}$ .

**A6.** The functions  $\phi$ ,  $\psi$  and  $\mathbb{R}$  are continuously differentiable on  $\Theta^v$ , where  $\Theta^v$  is a neighborhood of  $\Theta$ .  $\mathbb{T}$  is continuously differentiable on  $\mathbb{R}^q$ . The function  $F$  is continuously differentiable on  $\Theta^v$  and for any  $\theta \in \Theta$ ,  $\nabla F(\theta) = -\nabla \phi(\theta)^T \bar{s}(\theta) + \nabla \psi(\theta) + \nabla \mathbb{R}(\theta)$ . For any  $s \in \mathbb{R}^q$ ,  $B(s) \stackrel{\text{def}}{=} \nabla(\phi \circ \mathbb{T})(s)$  is a symmetric  $q \times q$  matrix and there exist  $0 < v_{\min} \leq v_{\max} < \infty$  such that for all  $s \in \mathbb{R}^q$ , the spectrum of  $B(s)$  is in  $[v_{\min}, v_{\max}]$ . For any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathbb{T}$  is globally Lipschitz on  $\mathbb{R}^q$  with constant  $L_i$ . The function  $s \mapsto \nabla(F \circ \mathbb{T})(s) = B(s)(\bar{s} \circ \mathbb{T}(s) - s)$  is globally Lipschitz on  $\mathbb{R}^q$  with constant  $L_{\hat{W}}$ .

**Lemma 2.** Let  $\rho \in (0, 1)$  and  $\{D_k, k \geq 0\}$  be real numbers such that  $\sum_{k \geq 0} \rho^k |D_k| < \infty$ . Let  $\xi \sim \mathcal{G}^*(1 - \rho)$ . Then  $\mathbb{E}[D_{\xi-1}] = \rho \mathbb{E}[D_{\xi}] + (1 - \rho)D_0 = \mathbb{E}[D_{\xi}] + (1 - \rho)(D_0 - \mathbb{E}[D_{\xi}])$ .

*Proof.* By definition of  $\xi$ ,

$$\begin{aligned} \mathbb{E}[D_{\xi}] &= (1 - \rho) \sum_{k \geq 1} \rho^{k-1} D_k \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 1} \rho^k D_k \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 0} \rho^k D_k - \rho^{-1}(1 - \rho)D_0 \\ &= \rho^{-1}(1 - \rho) \sum_{k \geq 1} \rho^{k-1} D_{k-1} - \rho^{-1}(1 - \rho)D_0 \\ &= \rho^{-1} \mathbb{E}[D_{\xi-1}] - \rho^{-1}(1 - \rho)D_0. \end{aligned}$$

This yields  $\rho \mathbb{E}[D_{\xi}] = \mathbb{E}[D_{\xi-1}] - (1 - \rho)D_0$  and concludes the proof.  $\square$

**Lemma 3.** For any  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t]^*$ ,  $\mathcal{B}_{t,k}$  and  $\mathcal{F}_{t,k-1}$  are independent. In addition, for any  $s \in \mathbb{R}^q$ ,  $\mathbb{b}^{-1} \mathbb{E} \left[ \sum_{i \in \mathcal{B}_{t,k}} \bar{s}_i \circ \mathbb{T}(s) \right] =$

$\bar{s} \circ \mathbb{T}(s)$ . Finally, assume that for any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathbb{T}$  is globally Lipschitz with constant  $L_i$ . Then for any  $s, s' \in \mathbb{R}^q$ ,

$$\begin{aligned} \mathbb{E} \left[ \left\| \mathbb{b}^{-1} \sum_{i \in \mathcal{B}_{t,k}} \{\bar{s}_i \circ \mathbb{T}(s) - \bar{s}_i \circ \mathbb{T}(s')\} - \bar{s} \circ \mathbb{T}(s) + \bar{s} \circ \mathbb{T}(s') \right\|^2 \right] \\ \leq \frac{1}{\mathbb{b}} (L^2 \|s - s'\|^2 - \|\bar{s} \circ \mathbb{T}(s) - \bar{s} \circ \mathbb{T}(s')\|^2), \end{aligned}$$

where  $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$ .

*Proof.* See [18, Lemma 4]; the proof holds true when  $\mathcal{B}_{t,k}$  is sampled with or without replacement.  $\square$

**Proposition 4.** For any  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ ,

$$\mathbb{E}[S_{t,k+1}|\mathcal{F}_{t,k}] - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) = S_{t,k} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1}),$$

and

$$\mathbb{E} \left[ S_{t,k+1} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) | \mathcal{F}_{t,0} \right] = \mathcal{E}_t.$$

*Proof.* Let  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ . By Lemma 3,

$$\mathbb{E}[S_{t,k+1}|\mathcal{F}_{t,k}] = S_{t,k} + \bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1}).$$

By definition of  $S_{t,0}$  and of the filtrations,  $S_{t,0} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,-1}) = \mathcal{E}_t \in \mathcal{F}_{t,0}$ . The proof follows by induction on  $k$ .  $\square$

**Proposition 5.** Assume that for any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathbb{T}$  is globally Lipschitz with constant  $L_i$ . For any  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ ,

$$\begin{aligned} \mathbb{E} \left[ \|S_{t,k+1} - \mathbb{E}[S_{t,k+1}|\mathcal{F}_{t,k}]\|^2 | \mathcal{F}_{t,k} \right] \\ \leq \frac{1}{\mathbb{b}} \left( L^2 \|\hat{S}_{t,k} - \hat{S}_{t,k-1}\|^2 - \|\bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1})\|^2 \right) \\ \leq \frac{L^2}{\mathbb{b}} \gamma_{t,k}^2 \|S_{t,k} - \hat{S}_{t,k-1}\|^2, \end{aligned}$$

where  $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$ .

*Proof.* Let  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ . By Lemma 3, Proposition 4, the definition of  $S_{t,k+1}$  and of the filtration  $\mathcal{F}_{t,k}$ ,

$$\begin{aligned} S_{t,k+1} - \mathbb{E}[S_{t,k+1}|\mathcal{F}_{t,k}] \\ = S_{t,k+1} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) - S_{t,k} + \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1}) \\ = \mathbb{b}^{-1} \sum_{i \in \mathcal{B}_{t,k+1}} \{\bar{s}_i \circ \mathbb{T}(\hat{S}_{t,k}) - \bar{s}_i \circ \mathbb{T}(\hat{S}_{t,k-1})\} \\ - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k}) + \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1}). \end{aligned}$$

We then conclude by Lemma 3 for the first inequality; and by using the definition of  $\hat{S}_{t,k}$  for the second one.  $\square$

**Proposition 6.** Assume that for any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathbb{T}$  is globally Lipschitz with constant  $L_i$ . For any  $t \in [k_{\text{out}}]^*$ ,  $k \in [\xi_t - 1]$ ,

$$\begin{aligned} \mathbb{E} \left[ \|S_{t,k+1} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k})\|^2 | \mathcal{F}_{t,k} \right] \\ \leq \frac{L^2}{\mathbb{b}} \gamma_{t,k}^2 \|S_{t,k} - \hat{S}_{t,k-1}\|^2 + \|S_{t,k} - \bar{s} \circ \mathbb{T}(\hat{S}_{t,k-1})\|^2, \end{aligned}$$

where  $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$ .

*Proof.* By definition of the conditional expectation, we have for any r.v.  $\phi(V)$

$$\mathbb{E} [\|U - \phi(V)\|^2 | V] = \mathbb{E} [\|U - \mathbb{E}[U|V]\|^2 | V] + \mathbb{E} [\|U|V - \phi(V)\|^2 | V].$$

The proof follows from this equality and Propositions 4 and 5.  $\square$

**Corollary 7.** Assume that for any  $i \in [n]^*$ ,  $\bar{s}_i \circ \mathsf{T}$  is globally Lipschitz with constant  $L_i$ . For any  $t \in [k_{\text{out}}]^*$ , let  $\rho_t \in (0, 1)$  and  $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$ . For any  $t \in [k_{\text{out}}]^*$ ,

$$\begin{aligned} & \mathbb{E}_t \left[ (\gamma_{t,\Xi_t} - \rho_t \gamma_{t,\Xi_t+1}) \|S_{t,\Xi_t} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \frac{L^2 \rho_t}{b} \mathbb{E}_t \left[ \gamma_{t,\Xi_t+1} \gamma_{t,\Xi_t}^2 \|S_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2 (1 - \rho_t)}{b} \gamma_{t,1} \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2 + (1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2, \end{aligned}$$

where  $L^2 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$ .

*Proof.* Let  $t \in [k_{\text{out}}]^*$  and  $k \in [\xi_t - 1]$ . From Proposition 6 and since  $\mathcal{F}_{t,0} \subseteq \mathcal{F}_{t,k}$  for  $k \in [\xi_t - 1]$ , we have

$$\begin{aligned} & \mathbb{E} \left[ \|S_{t,k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[ \|S_{t,k} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k-1})\|^2 + \frac{L^2}{b} \gamma_{t,k}^2 \|S_{t,k} - \widehat{S}_{t,k-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Multiply by  $\gamma_{t,k+1}$  and apply with  $k = \xi_t - 1$ :

$$\begin{aligned} & \mathbb{E} \left[ \gamma_{t,\xi_t} \|S_{t,\xi_t} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[ \gamma_{t,\xi_t} \|S_{t,\xi_t-1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2}{b} \mathbb{E} \left[ \gamma_{t,\xi_t} \gamma_{t,\xi_t-1}^2 \|S_{t,\xi_t-1} - \widehat{S}_{t,\xi_t-2}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

This implies

$$\begin{aligned} & \mathbb{E}_t \left[ \gamma_{t,\Xi_t} \|S_{t,\Xi_t} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[ \gamma_{t,\Xi_t} \|S_{t,\Xi_t-1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\Xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2}{b} \mathbb{E}_t \left[ \gamma_{t,\Xi_t} \gamma_{t,\Xi_t-1}^2 \|S_{t,\Xi_t-1} - \widehat{S}_{t,\Xi_t-2}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} & \mathbb{E}_t \left[ \gamma_{t,\Xi_t} \|S_{t,\Xi_t-1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\Xi_t-2})\|^2 | \mathcal{F}_{t,0} \right] \\ & = \rho_t \mathbb{E}_t \left[ \gamma_{t,\Xi_t+1} \|S_{t,\Xi_t} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,\Xi_t-1})\|^2 | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \gamma_{t,1} \mathbb{E}_t \left[ \|S_{t,0} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,-1})\|^2 | \mathcal{F}_{t,0} \right]; \end{aligned}$$

by definition of  $S_{t,0}$  and  $\mathcal{F}_{t,0}$ , the last term is equal to  $(1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2$ . By Lemma 2, we have

$$\begin{aligned} & \mathbb{E}_t \left[ \gamma_{t,\Xi_t} \gamma_{t,\Xi_t-1}^2 \|S_{t,\Xi_t-1} - \widehat{S}_{t,\Xi_t-2}\|^2 | \mathcal{F}_{t,0} \right] \\ & = \rho_t \mathbb{E}_t \left[ \gamma_{t,\Xi_t+1} \gamma_{t,\Xi_t}^2 \|S_{t,\Xi_t} - \widehat{S}_{t,\Xi_t-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \gamma_{t,1} \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2. \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 8.** For any  $h, s, S \in \mathbb{R}^q$  and any  $q \times q$  symmetric matrix  $B$ , it holds

$$-2 \langle Bh, S \rangle = -\langle BS, S \rangle - \langle Bh, h \rangle + \langle B\{h - S\}, h - S \rangle.$$

**Proposition 9.** Assume A4 to A6. For any  $t \in [k_{\text{out}}]^*$  and  $k \in [\xi_t - 1]$ ,

$$\begin{aligned} & \mathbb{E} \left[ W(\widehat{S}_{t,k+1}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \gamma_{t,k+1} \mathbb{E} \left[ \|h(\widehat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E} \left[ W(\widehat{S}_{t,k}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \gamma_{t,k+1} \mathbb{E} \left[ \|S_{t,k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}_{t,k})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t,k+1}}{2} (v_{\min} - \gamma_{t,k+1} L_{\dot{W}}) \mathbb{E} \left[ \|S_{t,k+1} - \widehat{S}_{t,k}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

*Proof.* Since  $W$  is continuously differentiable with  $L_{\dot{W}}$ -Lipschitz gradient, then for any  $s, s' \in \mathbb{R}^q$ ,

$$W(s') - W(s) \leq \langle \nabla W(s), s' - s \rangle + \frac{L_{\dot{W}}}{2} \|s' - s\|^2.$$

Set  $s' = s + \gamma S$  where  $\gamma > 0$  and  $S \in \mathbb{R}^q$ . Since  $\nabla W(s) = -B(s)h(s)$  and  $B(s)$  is symmetric, apply Lemma 8 with  $h \leftarrow h(s)$ ,  $B \leftarrow B(s)$  and  $S = (s' - s)/\gamma$ ; this yields

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma}{2} \langle B(s)S, S \rangle - \frac{\gamma}{2} \langle B(s)h(s), h(s) \rangle \\ & + \frac{\gamma}{2} \langle B(s)\{h(s) - S\}, h(s) - S \rangle + \frac{L_{\dot{W}}}{2} \gamma^2 \|S\|^2. \end{aligned}$$

Since  $\|a\|^2 v_{\min} \leq \langle B(s)a, a \rangle \leq v_{\max} \|a\|^2$  for any  $a \in \mathbb{R}^q$ , we have

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma v_{\min}}{2} \|S\|^2 - \frac{\gamma v_{\min}}{2} \|h(s)\|^2 \\ & + \frac{\gamma v_{\max}}{2} \|h(s) - S\|^2 + \frac{L_{\dot{W}}}{2} \gamma^2 \|S\|^2. \end{aligned}$$

Applying this inequality with  $s \leftarrow \widehat{S}_{t,k}$ ,  $\gamma \leftarrow \gamma_{t,k+1}$ ,  $S \leftarrow S_{t,k+1} - \widehat{S}_{t,k}$  (which yields  $s + \gamma S = \widehat{S}_{t,k+1}$ ), and then the conditional expectation yield the result.  $\square$

**Proposition 10.** Assume A4 to A6. For any  $t \in [k_{\text{out}}]^*$

$$\begin{aligned} & W(\widehat{S}_{t+1,0}) - W(\widehat{S}_{t+1,-1}) \\ & \leq -\frac{\gamma_{t+1,0} v_{\min}}{2} \|h(\widehat{S}_{t+1,-1})\|^2 + \frac{v_{\max} \gamma_{t+1,0}}{2} \|\mathcal{E}_{t+1}\|^2 \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \|S_{t+1,0} - \widehat{S}_{t+1,-1}\|^2. \end{aligned}$$

*Proof.* As in the proof of Proposition 9, we write for any  $s, s' \in \mathbb{R}^q$ ,

$$W(s') - W(s) \leq \langle \nabla W(s), s' - s \rangle + \frac{L_{\dot{W}}}{2} \|s' - s\|^2.$$

With Lemma 8, this yields when  $s' = s + \gamma S$  for  $\gamma > 0$  and  $S \in \mathbb{R}^q$

$$\begin{aligned} W(s + \gamma S) - W(s) & \leq -\frac{\gamma}{2} (v_{\min} - \gamma L_{\dot{W}}) \|S\|^2 - \frac{\gamma v_{\min}}{2} \|h\|^2 \\ & + \frac{v_{\max} \gamma}{2} \|h - S\|^2. \end{aligned}$$

Apply this inequality with  $\gamma \leftarrow \gamma_{t+1,0}$ ,  $s \leftarrow \widehat{S}_{t+1,-1}$ ,  $S \leftarrow S_{t+1,0} - \widehat{S}_{t+1,-1}$  and  $h \leftarrow h(s)$  (which yields  $s + \gamma S = \widehat{S}_{t+1,0}$ ).  $\square$

**Theorem 11.** Assume A4 to A6. For any  $t \in [k_{\text{out}}]^*$ , let  $\rho_t \in (0, 1)$  and  $\Xi_t \sim \mathcal{G}^*(1 - \rho_t)$ . Finally, choose  $\gamma_{t,k+1} = \gamma_t > 0$  for any  $k \geq 0$ . For any  $t \in [k_{\text{out}}]^*$ ,

$$\begin{aligned} & \frac{v_{\min} \gamma_t}{2(1 - \rho_t)} \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \leq W(\widehat{S}_{t,0}) - \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2(1 - \rho_t)} \frac{L^2}{\mathbf{b}} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2(1 - \rho_t)} \gamma_{t,1} \|\mathcal{E}_t\|^2 \\ & - \frac{\gamma_t}{2(1 - \rho_t)} \left( v_{\min} - \gamma_t L_{\dot{W}} - \frac{v_{\max} L^2 \rho_t}{(1 - \rho_t) \mathbf{b}} \gamma_t^2 \right) \dots \\ & \quad \times \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

*Proof.* Apply Proposition 9 with  $k \leftarrow \xi_t - 1$  and then set  $\xi_t \leftarrow \Xi_t$ ; this yields

$$\begin{aligned} & \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \mathbb{E}_t \left[ \gamma_{t, \Xi_t} \|h(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t - 1}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \mathbb{E}_t \left[ \gamma_{t, \Xi_t} \|S_{t, \Xi_t} - \bar{s} \circ \mathbf{T}(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \mathbb{E}_t \left[ \frac{\gamma_{t, \Xi_t}}{2} (v_{\min} - \gamma_{t, \Xi_t} L_{\dot{W}}) \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

Since  $\Xi_t \geq 1$  and  $\gamma_{t,k} = \gamma_t$  for any  $k \geq 1$ , we have

$$\begin{aligned} & \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] + \frac{v_{\min}}{2} \gamma_t \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t - 1}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \gamma_t \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \bar{s} \circ \mathbf{T}(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_t}{2} (v_{\min} - \gamma_t L_{\dot{W}}) \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

By Lemma 2, it holds

$$\begin{aligned} & \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] = \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t - 1}) | \mathcal{F}_{t,0} \right] \\ & + (1 - \rho_t) \left( \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] - W(\widehat{S}_{t,0}) \right) \end{aligned}$$

Furthermore, by Corollary 7 applied with  $\gamma_{t, \Xi_t} = \gamma_{t, \Xi_t + 1} = \gamma_t$

$$\begin{aligned} & (1 - \rho_t) \gamma_t \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \bar{s} \circ \mathbf{T}(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq \frac{L^2 \rho_t}{\mathbf{b}} \gamma_t^3 \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{L^2 (1 - \rho_t)}{\mathbf{b}} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2 + (1 - \rho_t) \gamma_{t,1} \|\mathcal{E}_t\|^2, \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{v_{\min}}{2} \gamma_t \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq (1 - \rho_t) W(\widehat{S}_{t,0}) - (1 - \rho_t) \mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \frac{L^2 \rho_t}{(1 - \rho_t) \mathbf{b}} \gamma_t^3 \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2} \frac{L^2}{\mathbf{b}} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2 + \frac{v_{\max}}{2} \gamma_{t,1} \|\mathcal{E}_t\|^2 \\ & - \frac{\gamma_t}{2} (v_{\min} - \gamma_t L_{\dot{W}}) \mathbb{E}_t \left[ \|S_{t, \Xi_t} - \widehat{S}_{t, \Xi_t - 1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

This concludes the proof.  $\square$

**Corollary 12** (of Theorem 11). For any  $t \in [k_{\text{out}}]^*$ ,

$$\begin{aligned} & \left( \frac{\gamma_t \rho_t}{(1 - \rho_t)} + \gamma_{t+1,0} \right) \frac{v_{\min}}{2} \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t})\|^2 | \mathcal{F}_{t,0} \right] \\ & \leq W(\widehat{S}_{t,0}) - \mathbb{E}_t \left[ W(\widehat{S}_{t+1,0}) | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \mathbb{E} \left[ \|S_{t+1,0} - \widehat{S}_{t+1,-1}\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max}}{2(1 - \rho_t)} \frac{L^2}{\mathbf{b}} \gamma_t \gamma_{t,0}^2 \|S_{t,0} - \widehat{S}_{t,-1}\|^2 \\ & + \frac{v_{\max}}{2(1 - \rho_t)} \gamma_t \|\mathcal{E}_t\|^2 + \frac{v_{\max} \gamma_{t+1,0}}{2} \mathbb{E} \left[ \|\mathcal{E}_{t+1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

*Proof.* Let  $t \in [k_{\text{out}}]^*$ . By Proposition 10, since  $\widehat{S}_{t, \xi_t} = \widehat{S}_{t+1, -1}$  we have

$$\begin{aligned} & - \mathbb{E} \left[ W(\widehat{S}_{t, \xi_t}) | \mathcal{F}_{t,0} \right] \leq - \mathbb{E} \left[ W(\widehat{S}_{t+1,0}) | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0} v_{\min}}{2} \mathbb{E} \left[ \|h(\widehat{S}_{t, \xi_t})\|^2 | \mathcal{F}_{t,0} \right] \\ & + \frac{v_{\max} \gamma_{t+1,0}}{2} \mathbb{E} \left[ \|\mathcal{E}_{t+1}\|^2 | \mathcal{F}_{t,0} \right] \\ & - \frac{\gamma_{t+1,0}}{2} (v_{\min} - \gamma_{t+1,0} L_{\dot{W}}) \mathbb{E} \left[ \|S_{t+1,0} - \widehat{S}_{t+1,-1}\|^2 | \mathcal{F}_{t,0} \right]. \end{aligned}$$

The previous inequality remains true when  $\mathbb{E} \left[ W(\widehat{S}_{t, \xi_t}) | \mathcal{F}_{t,0} \right]$  is replaced with  $\mathbb{E}_t \left[ W(\widehat{S}_{t, \Xi_t}) | \mathcal{F}_{t,0} \right]$ ; and  $\mathbb{E} \left[ \|h(\widehat{S}_{t, \xi_t})\|^2 | \mathcal{F}_{t,0} \right] = \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t})\|^2 | \mathcal{F}_{t,0} \right]$ . The proof follows from Theorem 11, and (see Lemma 2)

$$\mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t - 1})\|^2 | \mathcal{F}_{t,0} \right] \geq \rho_t \mathbb{E}_t \left[ \|h(\widehat{S}_{t, \Xi_t})\|^2 | \mathcal{F}_{t,0} \right].$$

$\square$