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Abstract

We show that a weighted manifold which admits a relative Faber Krahn inequality admits the Fefferman Phong inequality $\langle V\psi, \psi\rangle \leq C_V\|\psi\|^2$, with the constant depending on a Morrey norm of V, and we deduce from it a condition for a L^2 Hardy inequality to holds, as well as conditions for Schrödinger operators to be positive. We also obtain an estimate on the bottom of the spectrum for Schrödinger operators.

Résumé

On montre qu'une variété à poids admettant une inégalité de Faber-Krahn relative admet une inégalité de Fefferman Phong $\langle V\psi, \psi\rangle \leq C_V\|\psi\|^2$, où la constante dépend d'une norme de Morrey de V. On en déduit une condition pour qu'une inégalité de Hardy L^2 soit vérifiée, et des conditions de positivité des opérateurs de Schrödinger sur M. On obtient aussi un estimé du bas du spectre des opérateurs de Schrödinger.

1 Introduction

In [8, 9] Fefferman and Phong established the inequality, for $p > 1$:

$$\int_{\mathbf{R}^n} V(x)\psi(x)^2 \, \mathrm{d}x \leq C_{n,p} N_p(V) \int_{\mathbf{R}^n} |\nabla\psi(x)|^2 \, \mathrm{d}x, \tag{1.1}$$

with ψ a compactly supported smooth function, V non negative and locally integrable, $C_{n,p}$ is a constant depending only on the dimension and p, and N_p is the Morrey norm :

$$N_p(V) = \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \left(r^{2p-n} \int_{B(x,r)} |V(y)|^p \, \mathrm{d}y \right)^{1/p}. \tag{1.2}$$

Such an inequality yields a positivity condition for the Schrödinger operator $H = \Delta - V$ (with $\Delta = -\sum_{i=1}^n \partial_i^2$), namely that if $N_p(V) \leq 1/C_{n,p}$, then H is a positive operator. In fact the following estimates on the lower bound of the spectrum of H, $\lambda_1(H)$ were also given :

$$\sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \left(C_1 r^{-n} \int_{B(x,r)} V \, \mathrm{d}y - r^{-2} \right) \leq -\lambda_1(H) \leq \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \left(C_p r^{-n} \left(\int_{B(x,r)} V^p \, \mathrm{d}y \right)^{1/p} - r^{-2} \right) \tag{1.3}$$

The conditions for such inequalities (though with a constant that doesn't necessarily depends on the Morrey norm) to hold in $\mathbf{R}^n$ has been studied extensively, see for example in [4, 15, 18].

And in [19], Maz'ya and Verbitsky establish necessary and sufficient conditions for (1.1) to hold with complex valued V. That being the case, it seems interesting to study to what extends, and under which geometrical hypotheses those results extend on other spaces, such as Riemannian manifolds.

The first aim of this article is to generalize the initial result of Fefferman and Phong to a weighted Riemannian manifold M. A natural way to do that would be to use the Poincaré inequality : $\int_{B(x,r)} |f - f_{B(x,r)}| \, \mathrm{d}\mu \leq Cr \int_{B(x,\kappa r)} |\nabla f| \, \mathrm{d}\mu$, $f \in \mathcal{C}^\infty(\kappa B)$, for any $x \in M$, $r > 0$, with $\kappa > 1$, $f_B = \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu$. It turns out that the result still holds under some weaker hypothesis. Our proof will follow the general idea used by Schechter in [26], that (1.1) follows from the inequality (which holds in $\mathbf{R}^n$ following a result of Muckenhoupt and Wheeden[21]) :

$$\|I_1 f\|_{L^2} \leq C \|M_1 f\|_{L^2}, \tag{1.4}$$

with $I_1 f(x) = c_n \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}} \, \mathrm{d}\mu(y)$, and $M_1 f(x) = \sup_{r>0} r^{1-n} \int_{B(x,r)} |f(y)| \, \mathrm{d}y$, and that (1.3) is proved using similar estimates, this time on $(\Delta + \lambda^2)^{-1/2}$.

The proof of the generalisation of (1.3) will naturally yields weak versions of (1.1), which holds under weaker hypothesis.

1.1 Definitions and Notations

A weighted Riemannian manifold (M, g, μ), or simply a weighted manifold, is the data of a smooth manifold M, g a smooth Riemannian metric on M, and a Borel measure $\mathrm{d}\mu = \sigma^2 \, \mathrm{d}v_g$ on M, with σ a smooth positive function on M and v_g is the Riemannian volume measure associated with the metric g. We define the (weighted) Dirichlet Laplace operator as the Friedrichs extension of the operator on $\mathcal{C}_0^\infty(M)$ defined by $\Delta_\mu f = -\sigma^{-2} \mathrm{div}(\sigma^2 \nabla f)$, with associated quadratic form $Q(\psi) = \int_M |\nabla \psi|^2 \, \mathrm{d}\mu$. We will usually write the Dirichlet Laplace operator as simply Δ.

On a metric space (X, d), for $x \in X$, $r > 0$, the ball of center x and radius r is the set $B(x,r) = \{y : \ d(x,y) < r\}$. If $B = B(x,r)$ is the ball, $\theta \in R$, then θB refers to the set $B(x, \theta r)$.

For $p \geq 1$, we let $\|\cdot\|_p$ be the L^p norm on (M, μ). We recall $\|f\|_p = \left(\int_M |f|^p \, \mathrm{d}\mu\right)^{1/p}$. For T a bounded operator on L^p, we use $\|T\|_{L^p \to L^p}$ or $\|T\|_p$ to refer to its operator norm : i.e. $\|T\|_p = \sup_{\substack{\psi \in L^2 \\ \psi \neq 0}} \frac{\|T\psi\|_p}{\|\psi\|_p}$.

For an open $U \subset M$, $\lambda_1(U)$ refers to the first Dirichlet eigenvalue of Δ_μ on U :

$$\lambda_1(U) = \inf_{\substack{\psi \in \mathcal{C}_0^\infty(U) \\ \psi \neq 0}} \frac{\|\nabla \psi\|_2^2}{\|\psi\|_2^2}. \tag{1.5}$$

When H is an operator defined on smooth function with compact support, $\lambda_1(H)$ is similarly defined to be :

$$\lambda_1(H) = \inf_{\substack{\psi \in \mathcal{C}_0^\infty(M) \\ \psi \neq 0}} \frac{\langle H\psi, \psi \rangle}{\|\psi\|_2^2} \tag{1.6}$$

On a weighted manifold (M, g, μ), for $p \geq 0$ we define the Morrey norms N_p as follows : if V is a non-negative, locally integrable function, we let $N_p(V)$ be :

$$N_p(V) = \sup_{\substack{x \in M \\ r > 0}} \left(r^{2p} \fint_{B(x,r)} V^p \, \mathrm{d}\mu \right)^{1/p}, \tag{1.7}$$

where $\fint_B f \, \mathrm{d}\mu = \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu$ is the mean of f over B. We also define the Morrey norm taken on balls of radius less than $R > 0$:

$$N_{p,R}(V) = \sup_{\substack{x \in M \\ 0 < r < R}} \left(r^{2p} \fint_{B(x,r)} V^p \, \mathrm{d}\mu \right)^{1/p} \tag{1.8}$$

For our generalization to hold, it is important that (M, g, μ) must admits a *relative Faber Krahn inequality*, $(\mathbf{RFK})^\eta$, defined as follows :

Definition 1.1. *A weighted Riemannian manifold (M, g, μ) admits a relative Faber-Krahn inequality if there are constants $b, \eta > 0$, such that for all $x \in M$, $r > 0$, and for any U open subset of the open ball $B(x, r)$ relatively compact in $B(x, r)$, then :*

$$\lambda_1(U) \geq \frac{b}{r^2} \left(\frac{\mu(B(x,r))}{\mu(U)} \right)^{\frac{2}{\eta}}. \tag{1.9}$$

It instead admits a relative Faber-Krahn inequality at scale R, $(\mathbf{RFK})^\eta_R$ if (1.9) holds only for $0 \leq r \leq R$.

In what follows, we will call b, η in either $(\mathbf{RFK})^\eta$ or $(\mathbf{RFK})^\eta_R$ the *Faber-Krahn constants* of the manifold.

1.2 Statements of the results

Theorem 1.1. *Let (M, g, μ) be a weighted Riemannian manifold satisfying $(\mathbf{RFK})^\eta$, then for any $p > 1$, there is a constant C_p, which depends only on the Faber-Krahn constants and on p, such that for any $V \in L^1_{loc}(M)$, $V \geq 0$, and any $\psi \in \mathcal{C}^\infty_0(M)$, we have*

$$\int_M V\psi^2 \, \mathrm{d}\mu \leq C_p N_p(V) \int_M |\nabla \psi|^2 \, \mathrm{d}\mu. \tag{1.10}$$

If only $(\mathbf{RFK})^\eta_R$ holds, then we still have the following weaker result :

Theorem 1.2. *Let (M, g, μ) be a weighted Riemannian manifold, such that, for some $R > 0$, $(\mathbf{RFK})^\eta_R$ holds. Then for any $p > 1$ there is a constant $C_p > 0$, which depends only on the Faber-Krahn and on p, such that $V \in L^1_{loc}(M), V \geq 0$, and any $\psi \in \mathcal{C}^\infty_0(M)$,*

$$\int_M V\psi^2 \, \mathrm{d}\mu \leq C_p N_{p,R}(V) \left(\int_M |\nabla \psi|^2 \, \mathrm{d}\mu + \frac{1}{R^2} \int_M \psi^2 \, \mathrm{d}\mu \right). \tag{1.11}$$

From this inequality we can generalize the Fefferman Phong estimate on the lower bound of the spectrum of the operator $H = \Delta - V$. Indeed if $(\mathbf{RFK})^\eta$ holds, then for any $R > 0$, $(\mathbf{RFK})^\eta_R$ is satisfied. Thus (1.11) is true for any R. Then the following theorem follows easily :

Theorem 1.3. *Let (M, g, μ) be a complete non-compact weighted Riemannian manifold satisfying $(\mathbf{RFK})^\eta$. Then for any $p > 1$ we have two constants $C_1, C_p > 0$, which depends only on the Faber-Krahn constants (and for C_p, on p), such that, for any $V \in L^1_{loc}(M)$, $V \geq 0$:*

$$\sup_{\substack{x \in M \\ \delta > 0}} \left(C_1 \fint_{B(x,\delta)} V \, \mathrm{d}\mu - \delta^{-2} \right) \leq -\lambda_1(\Delta_\mu - V) \leq \sup_{\substack{x \in M \\ \delta > 0}} \left(C_p \left(\fint_{B(x,\delta)} V^p \, \mathrm{d}\mu \right)^{1/p} - \delta^2 \right). \tag{1.12}$$

In addition, if $\lambda_1(M) > 0$, then we can strengthen (1.11), and obtain the following result, giving a condition for $\Delta - V$ to be positive :

Theorem 1.4. *Let (M, g, μ) be a complete non-compact weighted Riemannian manifold, such that $(\mathbf{RFK})^\eta_R$ holds for $R > 0$. If in addition, if $\lambda_1(M) > 0$, then for any $p > 1$, there is a constant $C_p > 0$ depending only on the Faber-Krahn constants such that, for $V \in L^1_{loc}(M), V \geq 0$, and any $\psi \in \mathcal{C}_0^\infty(M)$,*

$$\int_M V\psi^2 \, \mathrm{d}\mu \leq C_p N_{p,R}(V) \frac{1 + \lambda_1(M)R^2}{\lambda_1(M)R^2} \left(\int_M |\nabla\psi|^2 \, \mathrm{d}\mu + \frac{\lambda_1(M)}{2} \int_M \psi^2 \, \mathrm{d}\mu \right). \tag{1.13}$$

1.3 L^2 Hardy inequality

Notice that the inequality (1.10) is, for potentials V with $N_p(V) < +\infty$, nothing more than a generalized L^2 Hardy inequality. Thus, on manifolds for which theorem 1.1 holds, the classical Hardy inequality is true whenever $N_p(d(o, \cdot)^{-2})$ is finite for all points $o \in M$. For this, we must make an additional assumption on the measure μ.

Definition 1.2. *A measured metric space (X, d, μ) satisfy the reverse doubling property $(\mathbf{RD})^\nu$ of order ν if, there is some constant $a > 0$ such that for all $x \in M$, $0 < r \leq r'$,*

$$a \left(\frac{r'}{r} \right)^\nu \leq \frac{\mu(B(x, r'))}{\mu(B(x, r))}. \tag{1.14}$$

Theorem 1.5. *Let (M, g, μ) be a weighted Riemannian manifold, for which $(\mathbf{RFK})^\eta$ holds, and where μ admits the reverse doubling property of order ν, $(\mathbf{RD})^\nu$, with $\nu > 2$. For an arbitrary $o \in M$, let $\rho(x) = d(o, x)$. Then there is some constant $C > 0$, which depends only on the Faber-Krahn and reverse doubling constants, such that for any $\psi \in \mathcal{C}_0^\infty(M)$ we have :*

$$\int_M \frac{\psi(x)^2}{\rho(x)^2} \, \mathrm{d}\mu(x) \leq C \int_M |\nabla\psi|^2 \, \mathrm{d}\mu \tag{1.15}$$

We can compare this to the results of V. Minerbe [20] or G. Grillo [13], who proved L^p Hardy inequalities assuming a Poincaré inequalities and a doubling measure. While we only get a L^2 inequality, it holds true under the weaker hypothesis of a relative Faber-Krahn inequality.

Cao, Grigor'yan and Liu [2] proved Hardy inequalities as a consequence of volume doubling, reverse doubling, and certain estimates on either the Green function or the heat kernel. Their results are far more general than this article.

1.4 Examples of manifolds satisfying relative Faber-Krahn inequalities

We give various cases of manifolds which will satisfy a relative Faber-Krahn inequality (at scale R). Then our results will follow.

1.4.1 Complete manifolds with Ricci curvature bounded from below

From Li and Yau[17], the heat kernel of a complete manifold (M, g, μ) of dimension n, with μ here being the Riemannian volume measure, with Ricci curvature bounded from below by $-K$, for a constant $K \geq 0$, admits the following diagonal estimate :

$$p_t(x, x) \leq \frac{C_0}{\mu\left(B\left(x, \sqrt{t}\right)\right)} e^{C_1 K t}$$

Also, as a consequence of the Bishop-Gromov volume comparison theorem, we get that (see [6, 5, 23] for example), for any $0 < r \leq r'$:

$$\frac{\mu\left(B\left(x, r'\right)\right)}{\mu\left(B\left(x, r\right)\right)} \leq \left(\frac{r'}{r}\right)^{n} \exp\left(\sqrt{(n-1)K}R\right)$$

Those two conditions implies, (see for example [23, 14]), that there is some $R > 0$ such that M satisfy $(\mathbf{RFK})^n_R$. If the Ricci curvature is non-negative, then we also have $(\mathbf{RFK})^n$.

1.4.2 Manifolds satisfying Faber Krahn inequalities outside a compact set

We consider a complete weighted manifold M, and remove from it a compact set with smooth boundary K. We let $E_1, \ldots, E_k$ be the connected components of $M \setminus K$, and suppose that each E_i is the exterior of a compact set with smooth boundary in a complete manifold M_i.

A simple example of such manifold is the connected sum of two (or more) copies of $\mathbf{R}^n$. It admits $(\mathbf{RFK})^n$, but it is known that such manifold doesn't satisfy a Poincaré inequality (see for example [1]).

Using [12], we get that if each M_i satisfy $(\mathbf{RFK})^\eta$, then there is some $R > 0$ such that M satisfies $(\mathbf{RFK})^\eta_R$.

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2 Some techniques of harmonic analysis

Remark. *We will often use C or c for generic constants which values might change from line to line. When we need to make it clear on which parameters the constant depends, new constant factors will be written when they appear before being folded into this generic constant.*

2.1 Dydadic cubes

In $\mathbf{R}^n$, the natural decomposition of the space into cubes of length 2^k, $k \in \mathbf{Z}$ is a very powerful tool. It turns out that families of open sets satisfying similar properties to those of the dyadic cubes in the euclidean space can be constructed in a more general setting. See for example the third part of [7].

We will use the construction of such "dyadic cubes" given by E. Sawyer and R. L. Wheeden in [24]. Though it remains true in a more general setting, for our purposes it can be stated as :

Theorem 2.1. *Let (X, d) be a separable metric space, then there is $\rho > 1$ ($\rho = 8$ works), such that for any (large negative) integer m, there are points $\left\{x^k_\alpha\right\}$ and a family $\mathcal{D}_m = \left\{\mathcal{E}^k_\alpha\right\}$ of Borel sets for $k = m, m+1, \ldots, \alpha = 1, 2, \ldots,$ such that*

- $B(x^k_\alpha, \rho^k) \subset \mathcal{E}^k_\alpha \subset B(x^k_\alpha, \rho^{k+1})$.
- *For each $k = m, m+1, \ldots,$ the family $\left\{\mathcal{E}^k_\alpha\right\}_\alpha$ is pairwise disjoint in α and $X = \bigcup_\alpha \mathcal{E}^k_\alpha$.*
- *If $m \leq k < l$, then either $\mathcal{E}^k_\alpha \cap \mathcal{E}^l_\beta = \emptyset$ or $\mathcal{E}^k_\alpha \subset \mathcal{E}^l_\beta$.*

Given such a family $\mathcal{D}_m$, the sets $\mathcal{E}_\alpha^k$ will be called *dyadic cubes* of M, or simply *cubes.* The ball $B(x_\alpha^k, \rho^{k+1})$ is called the containing ball of the cube $\mathcal{E}_\alpha^k$. For any cube Q the containing ball is denoted by $B(Q)$. ρ will be called the sidelength constant of dyadic cubes.

The length of a cube Q is the radius of $\rho^{-1}B(Q)$, written $\ell(Q)$.

2.2 Properties of doubling measures

We start by recalling the definitions and some standard properties of doubling measures, while covering at the same time the R-doubling case.

Definition 2.1. *A measured metric space (X, d, μ) satisfy the doubling property $(\mathbf{D})^\eta$ of order η if, there is some constant $A > 0$ such that for all $x \in M$, $0 < r \leq r'$,*

$$\frac{\mu\left(B\left(x, r'\right)\right)}{\mu\left(B\left(x, r\right)\right)} \leq A\left(\frac{r'}{r}\right)^\eta. \tag{2.1}$$

We call A the doubling constant, and η the doubling order. We will also say "the doubling constants" to refer to both A and η at the same time. The property $(\mathbf{D})^\eta$ is equivalent to the fact that for some constant $A > 0$, for any ball $B \subset M$:

$$\mu(2B) \leq A\mu(B) \tag{2.2}$$

The proof of the equivalence is the same as that of the R-doubling case given after definition 2.3, (with $R = \infty$).

A note on the constants : (2.2) implies (2.1) with $\eta = \log_2 A$ (and A the same in both inequalities), while conversely, (2.1) implies that the constant in (2.2) be $2^\eta A$.

We state again the reverse doubling property :

Definition 2.2. *A measured metric space (X, d, μ) satisfy the reverse doubling property $(\mathbf{RD})^\nu$ of order ν if, there is some constant $a > 0$ such that for all $x \in M$, $0 < r \leq r'$,*

$$a\left(\frac{r'}{r}\right)^\nu \leq \frac{\mu\left(B\left(x, r'\right)\right)}{\mu\left(B\left(x, r\right)\right)}. \tag{2.3}$$

We call a the reverse doubling constant, and ν the reverse doubling order. The property $(\mathbf{RD})^\nu$ is equivalent to the fact that for some constant $a \in (0, 1)$, for any ball $B \subset M$:

$$\mu(B) \leq a\mu(2B) \tag{2.4}$$

Proof of (2.4) *implies* (2.3). We can assume that $a \leq 1$. Let $x \in X$, $0 < r \leq r'$. Writing $\lfloor t \rfloor$ for the integer part of $t \in \mathbf{R}$, let $k = \left\lfloor \log_2 \frac{r'}{r} \right\rfloor$. Then :

$$\begin{aligned}
\mu\left(B\left(x, r\right)\right) &\leq a^k \mu\left(B\left(x, 2^k r\right)\right) \\
&\leq a^k \mu\left(B\left(x, r'\right)\right) \\
&\leq a^{-1+\log_2 \frac{r'}{r}} \mu\left(B\left(x, r'\right)\right) \quad (a \leq 1) \\
&\leq \frac{1}{a}\left(\frac{r'}{r}\right)^{-\nu} \mu\left(B\left(x, r'\right)\right)
\end{aligned}$$

With $\nu = -\log_2 a$. Thus :

$$a\left(\frac{r'}{r}\right)^{\nu} \leq \frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(x,r\right)\right)}.$$

□

Proposition 2.1. *Let (X,d,μ) satisfies $(\mathbf{D})^{\eta}$, then for any $x,y \in M$, $r,r' > 0$ such that $B(y,r) \subset B(x,r')$, we have :*

$$\frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,r\right)\right)} \leq A^2\left(\frac{r'}{r}\right)^{\eta}. \tag{2.5}$$

This is a classical result. The proof is similar to what we will do to prove proposition 2.2.

Definition 2.3. *A measured metric space (X,d,μ) satisfy the R-doubling property $(\mathbf{D})^{\eta}_{R}$ if there is some constant $A > 0$ such that* (2.1) *holds for all $x \in M$, and $0 < r \leq r' \leq 2R$. This is equivalent to* (2.2) *being true for all ball B with radius less than R.*

We define in the same way the R-reverse doubling property $(\mathbf{RD})^{\nu}_{R}$.

We will write A_R for the doubling constant when it's important to precise which R the constant is associated with.

Some care is needed to get precisely those maximal radius. That (2.2) follows from (2.1) is immediate.

Proof of (2.2) *implies* (2.1). Suppose that there is some constant A such that for all ball B of radius less than R, then $\mu(2B) \leq A\mu(B)$. Let $r \leq r' \leq 2R$. $k = \left\lfloor \log_2 \frac{r'}{r} \right\rfloor$. We have $2^{-k-1}r' < r \leq 2^{-k}r'$, and, using repeatedly $\mu\left(B\left(x,\rho\right)\right) \leq A\mu\left(B\left(x,\rho/2\right)\right)$, valid for all $\rho \leq 2R$, we have :

$$\begin{aligned}
\mu\left(B\left(x,r'\right)\right) &\leq A^{k+1}\mu\left(B\left(x,2^{-k-1}r'\right)\right) \\
&\leq A^{k+1}\mu\left(B\left(x,r\right)\right) \\
&\leq Ae^{\left(\log A \log \frac{r'}{r}\right)/\log 2}\mu\left(B\left(x,r\right)\right) \\
&\leq A\left(\frac{r'}{r}\right)^{\eta}\mu\left(B\left(x,r\right)\right),
\end{aligned}$$

with $\eta = \log_2 A$. □

Proposition 2.2. *Let X satisfies $(\mathbf{D})^{\eta}_{R}$, then for all $x,y \in X$, $r,r' > 0$ such that $B(y,r) \subset B(x,r')$ and with $r' < R$, then for $\eta = \log_2 A$:*

$$\frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,r\right)\right)} \leq A^2\left(\frac{r'}{r}\right)^{\eta}. \tag{2.6}$$

If in addition X satisfies $(\mathbf{RD})^{\nu}_{R}$, then we also have for some constant $c > 0$, that for all $0 < r,r' < R$ and $B(y,r) \subset B(x,r')$,

$$c\left(\frac{r'}{r}\right)^{\nu} \leq \frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,r\right)\right)}. \tag{2.7}$$

Proof. For the first part, we simply use $B(x,r) \subset B(y,2r)$ then applies (2.1).

For the second part, since $B(x,r') \subset B(y,2r')$, we can use (2.5) and we get :

$$\begin{aligned}\frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,r\right)\right)} &= \frac{\mu\left(B\left(y,r'\right)\right)}{\mu\left(B\left(y,r\right)\right)}\frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,r'\right)\right)} \\ &\geq a\left(\frac{r'}{r}\right)^{\nu}\frac{\mu\left(B\left(x,r'\right)\right)}{\mu\left(B\left(y,2r'\right)\right)} \\ &\geq aA^{-2}2^{-\eta}\left(\frac{r'}{r}\right)^{\nu}\end{aligned}$$

□

We now suppose that (X,d) is a *path metric space*, i.e. that the distance $d(x,y)$ is realised as the infimum of the length of continuous path with end points x and y. We will keep making this assumption in everything that follows (though some results are still true in a more general setting).

Proposition 2.3. *Let X satisfy $(\mathbf{D})_R^\eta$, and suppose that $X \setminus B(x,3R/4)$ is non empty for all $x \in X$. Then there is some $\nu > 0$ such that X satisfy $(\mathbf{RD})_{R/2}^\nu$.*

Proof. Let $x \in X$, $r < R/2$. We take $y \in X$ such that $d(x,y) = \frac{3}{2}r$. Then $B(y,r/2) \subset B(x,2r) \setminus B(x,r)$. Thus $\mu\left(B\left(x,2r\right)\right) \leq A^2 4^\eta \mu\left(B\left(y,r/2\right)\right) = A^4 \mu\left(B\left(y,r/2\right)\right)$.

Thus $(1+A^{-4})\mu(B(x,r)) \leq \mu(B(x,2r))$. From this we show $(\mathbf{RD})_{R/2}^\nu$ in a similar way as in what follows definition 2.3. □

The R-doubling also implies some upper bound on the volume of balls of large radius. The two following propositions, and their proof, come from [14].

Proposition 2.4. *If (X,d,μ) satisfy $(\mathbf{D})_R^\eta$, then there is some $C > 0$ that depends only on the doubling constant and order, such that we have, for any $r > 0$, $R' \leq R$:*

$$\mu\left(B\left(x,r+R'/4\right)\right) \leq C\mu\left(B\left(x,r\right)\right) \tag{2.8}$$

Proof. The case $r \leq R$ is obvious by the doubling property. For $r > R$, then let $\{x_i\}_i$ be a maximal family in $B(x,r-R/4)$ such that for any $i \neq j$, $d(x_i,x_j) > R'/2$. Then the balls $B(x_i,R'/4) \subset B(x,r)$ are disjoints, and the balls $B(x_i,R')$ cover $B(x,r+R'/4)$, since a point of $B(x,r+R'/4)$ is at distance at most $R'/2$ of $B(x,r-R'/4)$ (because (X,d) is a path-metric space). Thus $\mu\left(B\left(x,r+R'/4\right)\right) \leq \sum_i \mu\left(B\left(x_i,R'\right)\right) \leq A^2 \sum_i \mu\left(B\left(x_i,R'/4\right)\right) \leq A^2\mu\left(B\left(x,r\right)\right)$. □

Proposition 2.5. *If (X,d,μ) satisfy $(\mathbf{D})_R^\eta$ then, there is a $D > 0$, that depends only on the doubling constant and doubling order, such that for any $r > 0$, we have :*

$$\mu\left(B\left(x,r\right)\right) \leq e^{D\frac{r}{R}}\mu(B(x,R)) \tag{2.9}$$

Proof. Let $r > R$, $k = \left\lfloor 4\frac{r-R}{R}\right\rfloor$, then we have $\mu\left(B\left(x,r\right)\right) \leq \mu\left(B\left(x,R+(k+1)R/4\right)\right)$. Thus by proposition 2.4, $\mu\left(B\left(x,r\right)\right) \leq C^{k+1}\mu\left(B\left(x,R\right)\right)$. Moreover, $k+1 \leq 4\frac{r}{R} - 3 \leq 4\frac{r}{R}$, and so :

$$\mu\left(B\left(x,r\right)\right) \leq \exp\left(4\ln\left(C\right)\frac{r}{R}\right)\mu\left(B\left(x,R\right)\right)$$

And thus we get (2.9) with $D = 4\ln(C)$.

If $r \leq R$, then $\mu(B(x,r)) \leq \mu(B(x,R)) \leq e^{D\frac{r}{R}}\mu(B(x,R))$ and thus (2.9) still holds. □

Similarly to how we always use A for the doubling constant, D will always be used for this constant $D = 8\log A$.

Proposition 2.6. *Let X satisfies $(\mathbf{D})_R^\eta$, let $r \leq R$, then there exists a constant $C > 0$, that depends only on the doubling constant and order, such that for any $x, y \in X$, $\mu(B(x,r)) \leq Ce^{D\frac{d(x,y)}{r}}\mu(B(y,r))$.*

Proof. $B(x,r) \subset B(y, r + d(x,y))$. Since $r \leq R$, by proposition 2.4, we have $\mu(B(x,r)) \leq A^8\mu(B(y,d(x,y)))$. Then using proposition 2.5 :

$$\mu(B(x,r)) \leq Ce^{D\frac{d(x,y)}{R}}\mu(B(y,r)) \leq Ce^{D\frac{d(x,y)}{r}}\mu(B(y,r))$$

□

Proposition 2.7. *If (X,d,μ) satisfy $(\mathbf{D})_R^\eta$, then it also satisfy $(\mathbf{D})_{R'}^\eta$ for any $R' > 0$, with a doubling constant $A_{R'} = A_R$ if $R' \leq R$, and $A_{R'} = e^{2D\frac{R'}{R}}$ if $R' > R$.*

Proof. The case $R' \leq R$ is obvious. Thus assume $R > R'$, let $r \leq R'$. If $r \leq R$ then the result is trivial since $A_R \leq A_{R'}$. If $r > R$, then by proposition 2.5, $\mu(B(x,2r)) \leq e^{D\frac{2r}{R}}\mu(B(x,r))$, and $e^{2D\frac{r}{R}} \leq e^{2D\frac{R'}{R}}$. Thus μ is R'-doubling, with a doubling constant $A_{R'} = e^{2D\frac{R'}{R}}$. □

With this we can generalise proposition 2.6 for any $r > 0$: if $r > R$, we can use the r-doubling and apply proposition 2.6 for it. The constants are $A_r = e^{2D\frac{r}{R}}$, $D_r = 4\log\left(A_r^2\right) = 16D\frac{r}{R}$, $A_r^8 = e^{16D\frac{r}{R}}$. Then we have, for any $x, y \in X$, $r > 0$:

$$\mu(B(x,r)) \leq e^{16D\frac{r+d(x,y)}{R}}\mu(B(y,r)). \tag{2.10}$$

Proposition 2.8. *Let (X,d,μ) be a measured metric space that satisfy $(\mathbf{D})_R^\eta$. If it also satisfy $(\mathbf{RD})_R^\nu$, then for any $\kappa > 1$, it satisfy $(\mathbf{RD})_{\kappa R}^\nu$ with a different reverse doubling constant, that depends only on the doubling, reverse doubling constant and orders, and on κ.*

The notable part of this proposition is that the reverse doubling order is conserved.

Proof. By proposition 2.7, μ is κR-doubling for all κ, with some doubling order $\eta = \eta(\kappa)$. We take a point $x \in M$, and r, r' with $0 < r \leq r' \leq \kappa R$. We want to prove that there's some constant a_κ such that, for any such x, r, r' :

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \geq a_\kappa\left(\frac{r'}{r}\right)^\nu$$

If $0 < r \leq r' \leq R$, then there's nothing to do but apply $(\mathbf{RD})_R^\nu$. If $0 < r \leq R < r' \leq \kappa R$, then :

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \geq \frac{\mu(B(x,R))}{\mu(B(x,r))} \geq a\left(\frac{R}{r}\right)^\nu \geq a\kappa^{-\nu}\left(\frac{r'}{r}\right)^\nu$$

Finally, when $R < r \leq r' < \kappa R$, then :

$$\begin{aligned}\frac{\mu(B(x,r'))}{\mu(B(x,r))} &\geq \frac{\mu(B(x,r'))}{\mu(B(x,R))}\frac{\mu(B(x,R))}{\mu(B(x,r))}\\ &\geq a\kappa^{-\nu}\left(\frac{r'}{R}\right)^\nu A^{-1}\left(\frac{R}{r}\right)^\eta\end{aligned}$$

$$
\begin{aligned}
&\geq aA^{-1}\kappa^{-\nu}\left(\frac{R}{r}\right)^{\eta-\nu}\left(\frac{r'}{r}\right)^{\nu} \\
&\geq aA^{-1}\kappa^{-\eta}\left(\frac{r'}{r}\right)^{\nu}
\end{aligned}
$$

Thus (2.2) holds for $a_\kappa = \min\left(a, a\kappa^{-\nu}, aA^{-1}\kappa^{-\eta}\right) = aA^{-1}\kappa^{-\eta}$. $\square$

Proposition 2.9. *Let (X, d, μ) satisfy $(\mathbf{D})_R^\eta$. Take $x \in X$, $r > 0$, and let $B = B(x, r)$. Let δ be such that $0 < \delta \leq \min(r, R)$, and $\{x_i\}_i \subset B$ be a family of points such that the balls $B_i = B(x_i, \delta)$ form a covering of B and that for any $i \neq j$, $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$.*

Then there are constants C, c, depending only on the doubling constant such that

$$
\operatorname{card}(I) \leq Ce^{c\frac{r}{\delta}} \tag{2.11}
$$

Proof. For any i, $B_i \subset B(x, r+\delta)$, and since $\delta \leq R$, then by proposition 2.4, $\mu\left(B\left(x, r+\delta\right)\right) \leq C\mu\left(B\left(x, r\right)\right)$. Now, if $r > R$, then by proposition 2.5, $\mu\left(B\left(x, r\right)\right) \leq e^{D\frac{r}{\delta}}\mu\left(B\left(x, \delta\right)\right)$ ($\delta \leq R$ and so we use that μ is δ-doubling with the same doubling constant as that of the R-doubling).

Moreover by proposition 2.6, $\mu\left(B\left(x, \delta\right)\right) \leq Ce^{D\frac{d(x,x_i)}{\delta}}\mu\left(B\left(x_i, \delta\right)\right) \leq Ce^{D\frac{r}{\delta}}\mu(B_i)$, since $x_i \in B$, thus $d(x, x_i) \leq r$.

Thus we have $\mu(B(x, r)) \leq Ce^{2D\frac{r}{\delta}}\mu(B_i)$. Up to this point the constant C depends only on the doubling constants.

$$
\begin{aligned}
\left(\operatorname{card} I\right)\mu\left(B\left(x, r+\delta\right)\right) &\leq Ce^{2D\frac{r}{\delta}}\sum_{i \in I}\mu(B_i) \\
&\leq ACe^{2D\frac{r}{\delta}}\sum_{i}\mu\left(\frac{1}{2}B_i\right) \\
&\leq Ce^{2D\frac{r}{\delta}}\mu\left(B\left(x, r+\delta\right)\right)
\end{aligned}
$$

Thus $\operatorname{card}(I) \leq Ce^{2D\frac{r}{\delta}}$ and the constant C depends only on the doubling constants. $\square$

Remark. *For any ball B, such a covering always exists : take for $\{x_i\}_i \subset B$ a maximal family with $d(x_i, x_j) \geq \delta$ for any $i \neq j$.*

Proposition 2.10. *Let M_R be the centered maximal function defined by :*

$$
\forall f \in L^1_{loc}(M),\ M_R f(x) = \sup_{r<R} \fint_{B(x,r)} |f| \,\mathrm{d}\mu \tag{2.12}
$$

Then, if μ satisfies $(\mathbf{D})_R^\eta$, $M_{R/2}$ is bounded on L^p for all $p \in (1, +\infty]$, and the operator norm is bounded by a constant that only depends on the doubling constant A and on p.

We will use the following classical results :

Lemma 2.1 (Vitali's covering lemma)**.** *Let (X, d) be a separable metric space, and $\{B_j\}_{j\in J}$ a collection of balls, such that $\sup_j r(B_j) < \infty$. Then for any $c > 3$ there exists a subcollection $\{B_{j_n}\}_{n\in\mathbf{N}} \subset \{B_j\}_{j\in J}$ such that the B_{j_n} are pairwise disjoint and $\bigcup_{j\in J} B_j \subset \bigcup_{n\in\mathbf{N}} cB_{j_n}$.*

Theorem 2.2 (Marcinkiewicz interpolation theorem)**.** *Let (X, μ) be a measure space, T a sub-linear operator acting on functions, i.e. there is a $\kappa > 0$ such that for any f, g measurable, then Tf, Tg are measurable and $T(f+g)(x) \leq \kappa\left(Tf(x) + Tg(x)\right)$ for almost every $x \in X$.*

Assume that for $1 \leq p < r < \infty$:

$$\forall f \in L^p,\ \mu\{x \in X :\ Tf(x) > \lambda\} \leq \frac{A}{\lambda^p}\|f\|_p^p,$$
$$\forall f \in L^r,\ \mu\{x \in X :\ Tf(x) > \lambda\} \leq \frac{B}{\lambda^r}\|f\|_r^r,$$

or that, for $1 \leq p < r = \infty$, we replace the second line by : $\forall f \in L^\infty, |Tf(x)| \leq B|f(x)|$ for almost every $x \in X$.

Then, for every $s \in (p, r)$, for all $f \in L^s$, $Tf \in L^s$ and :

$$\|Tf\|_s \leq C(A, B, p, r, s, \kappa)\|f\|_s \tag{2.13}$$

Proof of the proposition. We have, for any $f \in L^\infty(M)$, $\|M_R f\|_\infty \leq \|f\|_\infty$.

If $f \in L^1(M)$, then for any $\lambda > 0$, define $E_\lambda = \{x \in M :\ M_{R/2}Rf(x) > \lambda\}$. If $x \in E_\lambda$, then there is some $r_x > 0$ such that $\lambda < \fint_{B(x,r_x)} |f|\ \mathrm{d}\mu$, and $2r_x \leq R$. Then $\mu(B(x, r_x)) \leq \lambda^{-1} \int_{B(x,r)} |f|\ \mathrm{d}\mu$.

We have $E_\lambda \subset \bigcup_x B(x, r_x)$, thus by Vitali's covering lemma, there is a subcollection $\{x_n\}$ such that the $B(x_n, r_n)$ are pairwise disjoint and $E_\lambda \subset \bigcup_n B(x_n, 4r_n)$.

Also, since $r_n < R/2$, and μ is R-doubling, we have $\mu\left(B\left(x_n, 4r_n\right)\right) \leq A^2\mu\left(B\left(x_n, r_n\right)\right)$. Then :

$$\begin{aligned}
\mu(E_\lambda) &\leq \sum_n \mu(B(x_n, 4r_n)) \\
&\leq A^2 \sum_n \mu(B(x_n, r_n)) \\
&\leq A^2\lambda^{-1} \sum_n \int_{B(x_n, r_n)} |f|\ \mathrm{d}\mu \\
&\leq A^2 \frac{\|f\|_1}{\lambda}.
\end{aligned}$$

And so by the Marcinkiewicz interpolation theorem, for any $p \in (1, +\infty)$, $M_{R/2}$ is bounded on L^p with an operator norm $\|M_{R/2}\|_{p\to p} \leq C_p$, with C_p depending only on A and p. □

Remark. *Of course, $(\mathbf{D})_R^\eta$ implies $(\mathbf{D})_{R'}^\eta$ for all $R' > R$, then M_R itself is also bounded, but with the constant C_p depending on the constant for $(\mathbf{D})_{2R}^\eta$. And so are all the $M_{R'}$ with $R' > R$, with the constant C_p depending on p, the R-doubling constant, and the ratio R'/R.*

Proposition 2.11. *Let $\tilde{M}_R$ the uncentered maximal function defined by : for all $f \in L^1_{loc}(M)$,*

$$\tilde{M}_R f(x) = \sup_{\substack{x\in B,\\ r(B)\leq R}} \fint_B |f|\ \mathrm{d}\mu \tag{2.14}$$

With this supremum to be interpretated as being over all balls B satisfying the given condition, and $r(B)$ being the radius of B.

Then, if μ is R-doubling, there exist some constant $C > 0$ such that $M_R \leq \tilde{M}_R \leq CM_{2R}$.

Proof. Since a ball centered at x is a ball containing x, $M_R \leq \tilde{M}_R$ is obvious. Now, for some balls $B = B(y,r)$ containing x, with radius less than R, we have $B \subset B(x,2r)$ and :

$$\fint_B |f| \, \mathrm{d}\mu \leq \frac{\mu\left(B\left(x,2r\right)\right)}{\mu(B)} \fint_{B(x,2r)} |f| \, \mathrm{d}\mu \leq C M_{2R} f(x)$$

□

Proposition 2.12. *Let (X,d,μ) be a separable, measured metric space, and $\mathcal{D}_m$ be a chosen construction of dyadic cubes on X. Define the associated dyadic maximal function $M_{d,m}$ by :*

$$M_{d,m} f(x) = \sup_{\substack{Q \in \mathcal{D}_m \\ x \in Q}} \fint_Q |f| \, \mathrm{d}\mu \tag{2.15}$$

Then there is a constant C_p such that for any $p > 1$, for any $f \in L^p$, $\|M_{d,m} f\|_p \leq C_p \|f\|_p$.

As a consequence, $M_{d,m,l}$, the maximal function defined the same way, but with the cubes in the supremum being only those of length less than l, is also bounded on L^p for all $p > 1$.

Proof. Let $f \in L^1(X)$, $\lambda > 0$, we define $E_\lambda = \{x \in X : \ M_{d,m} f(x) > \lambda\}$. If $x \in E_\lambda$, then there is a cube $Q \in \mathcal{D}_m$ such that $\fint_Q |f| \, \mathrm{d}\mu > \lambda$, and so $Q \subset E_\lambda$. Then there is two possibilities : First, there is a maximal dyadic cube P containing x such that $\fint_P |f| \, \mathrm{d}\mu > \lambda$, then $P \subset E_\lambda$. Second, there is no such cube, then $\Omega = \bigcup_{\substack{Q \in \mathcal{D}_m \\ x \in Q}} Q \subset E_\lambda$, and we have $\mu(\Omega) \leq \lambda^{-1} \int_\Omega |f| \, \mathrm{d}\mu < \infty$.

Then take $\{Q_i\}_i$ to be the family of all the maximal dyadic cubes such that $\fint_{Q_i} |f| \, \mathrm{d}\mu > \lambda$, and $\{\Omega_j\}_j$ be the family of all the the regions $\Omega_j = \bigcup_k Q_k^j$, where $\{Q_k^j\}$ is an infinite increasing sequence of cubes with $\fint_{Q_j^k} |f| \, \mathrm{d}\mu > \lambda$. The Q_i, Ω_j are pairwise disjoints : it is clear that the Q_i are. Now, if for a cube Q, we have $Q \cap \Omega_j \neq \emptyset$, then there is a cube $P \subset \Omega_j$ such that $P \cap Q \neq \emptyset$, thus we have either $P \subset Q$ or $Q \subset P$. In both case, $Q \subset \Omega_j$ since Ω_j is the union of all cubes containing P. This mean both that $Q_i \cap \Omega_j = \emptyset$ for all i, j, and that $\Omega_j \cap \Omega_l = \emptyset$ for $j \neq l$.

Thus, we have the disjoint union :

$$E_\lambda = \bigcup_i Q_i \cup \bigcup_j \Omega_j,$$

Then $\mu(Q_i) < \lambda^{-1} \int_{Q_i} |f| \, \mathrm{d}\mu$, and $\mu(\Omega_j) \leq \lambda^{-1} \int_{\Omega_j} |f| \, \mathrm{d}\mu$. Summing on all cubes and all regions, $\mu(E_\lambda) \leq \lambda^{-1} \int_{E_\lambda} |f| \, \mathrm{d}\mu \leq \lambda^{-1} \|f\|_1$. Thus :

$$\mu\left(\{x \in X : \ M_{d,m} f(x) > \lambda\}\right) \leq \frac{\|f\|_1}{\lambda} \tag{2.16}$$

Moreover, for $f \in L^\infty(X)$, we clearly have $M_{d,m} f(x) \leq \|f\|_\infty$. Then by Marcienkiewicz interpolation theorem, there is a constant $C_p > 1$ such that $\|M_{d,m} f\|_p \leq C_p \|f\|_p$. □

2.3 Estimates of operator norms by that of a maximal function

We refers to the works of C. Pérez and R.L. Wheeden [22] for a more general approach.

In what follows, we let (X,d) be a separable R-doubling metric space. We take T an operator given by a kernel $K : X \times X \setminus \mathrm{Diag} \to \mathbf{R}$, i.e.

$$Tf(x) = \int_X f(y) K(x,y) \, \mathrm{d}\mu(y) \tag{2.17}$$

We say that T, or K, satisfy the condtion $(\mathbf{K})$ if K is non negative and if there are constants $C_1, C_2 > 1$ such that :

$$\begin{aligned} d(x',y) \leq C_2 d(x,y) &\Rightarrow K(x,y) \leq C_1 K(x',y), \\ d(x,y') \leq C_2 d(x,y) &\Rightarrow K(x,y) \leq C_1 K(x,y'). \end{aligned} \tag{2.18}$$

For each $m \in \mathbf{Z}$, X admits a decomposition in dyadic cube. We take $\rho > 1$ to be as in theorem 2.1. We define φ as the following functional on balls

$$\varphi(B) = \sup_{\substack{x,y\in B \\ d(x,y)\geq \frac{1}{2\rho} r(B)}} K(x,y), \tag{2.19}$$

and M_φ to be the following maximal functions :

$$M_\varphi f(x) = \sup_{x\in B} \varphi(B) \int_B |f| \, \mathrm{d}\mu \tag{2.20}$$

For T satisfying $(\mathbf{K})$, it is shown in (4.3) of [25] that φ is decreasing in the following sense :

Proposition 2.13. *There is a constant α, which depends only on C_1, C_2, ρ such that for any balls $B \subset B'$, $\varphi(B') \leq \alpha\varphi(B)$*

Proof. First we want to prove that if (2.18) holds, then for any integer $k \geq 1$, $d(x',y) \leq C_2^k d(x,y)$ implies $K(x,y) \leq C_1^k K(x',y)$ (and the same with x, y'.

We proceed by induction. The case $k = 1$ is obvious. Let $k > 2$, $x, x', y \in X$ such that $d(x',y) \leq C_2^k d(x,y)$, and suppose $d(x',y) \leq C_2^{k-1} d(x,y) \Rightarrow K(x,y) \leq C_1^{k-1} K(x'y)$. Then, if $d(x',y) \leq C_2^{k-1} d(x,y)$, the results holds and there is nothing to prove. If $d(x',y) > C_2^{k-1} d(x,y)$, then X is a path metric space, so there is a path from y to x' of length $d(x',y)$, and on this path is a point z such that $d(y,z) = C_2^{k-1} d(x,y)$. But then :

$$d(x',y) \leq C_2^k d(x,y) = C_2 d(z,y)$$

And thus $K(z,y) \leq C_1 K(x',y)$. Then by induction we get that $K(x,y) \leq C_1^k K(x',y)$ for all x, x', y with $d(x',y) \leq C_2^k d(x,y)$. We can generalize sligthly, and we have that for any $C_2 > 1$ there exist a $C_1 > 1$ such that (2.18) holds.

Now we can prove the proposition proper. For $x', y' \in B'$, $x, y \in B$ such that $d(x',y') \geq cr(B')$ and $d(x,y) \geq cr(B)$, with $c = \frac{1}{2\rho}$. We can suppose that $d(x,y') \geq d(x,x')$ (if not, we can exchange x' and y').

Then $cr(B') \leq d(x',y') \leq d(x',x) + d(x,y') \leq 2d(x,y')$. Moreover, since $B \subset B'$, $d(x,y') \leq 2r(B')$, and thus :

$$d(x,y') \leq \frac{2}{c} d(x',y')$$

Thus by (2.18) there is a constant $c_1 > 1$ such that $K(x',y') \leq c_1 K(x,y')$.

Moreover $d(x,y) \leq d(x,y') + d(y',y) \leq d(x,y') + 2r(B') \leq (1 + 4/c)d(x,y')$. Thus by (2.18) there is a constant $c_2 > 1$ such that $K(x,y') \leq c_2 K(x,y)$. Thus

$$K(x',y') \leq c_1 c_2 K(x,y)$$

And thus $\varphi(B') \leq c_1 c_2 \varphi(B)$. □

We further assume that φ satisfy the following condition : there is some $\varepsilon > 0$ and some constant $L > 0$ such that for any balls B_1, B_2, with $B_1 \subset B_2$, we have :

$$\varphi(B_1)\mu(B_1) \leq L \left(\frac{r(B_1)}{r(B_2)}\right)^{\varepsilon} \varphi(B_2)\mu(B_2) \tag{2.21}$$

Theorem 2.3 (C. Pérez and R.L. Wheeden [22]). *Let (X, d, μ) be a metric space with a doubling measure μ. For T an operator defined by (2.17) satisfying $(\mathbf{K})$, and with φ satisfying (2.21), then there is some constant C, depending only on the doubling constant and p, such that, for any $f : X \to \mathbf{R}$:*

$$\|Tf\|_p \leq C\|M_\varphi f\|_p \tag{2.22}$$

In addition for the operator $Tf(x) = \int_M \frac{d(x,y)^s}{\mu(B(x,d(x,y)))} f(y)\ \mathrm{d}\mu(y)$, we can replace M_φ by the maximal function defined by $M_s f(x) = \sup_{r>0} r^s ⨍_{B(x,r)} |f|\ \mathrm{d}\mu$. See corollary 2.1.

We will also show a variant on this theorem. We consider the operator T_δ, $\delta < R$, with kernel $K_\delta(x, y) = K(x, y)\chi_{\{d(x,y)<\delta\}}$, and we want to compare its L^p norm to that of the maximal function $M_{\varphi,\delta}$ defined by :

$$M_{\varphi,\delta} f(x) = \sup_{\substack{x\in B \\ r(B)<\delta}} \varphi(B) \int_B |f|\ \mathrm{d}\mu. \tag{2.23}$$

The idea of the proof of this result will be essentially the same as that of theorem 2.3 given in [22], but some care must be taken to account for the different hypotheses properly, and thus we will give the details in what follows.

The hypothesis to prove $\|Tf\|_p \leq C\|M_{\varphi,\delta}\|_p$ can be weakened compared to those of theorem 2.3. A key point is that proposition 2.13 has to hold at least for balls of radius at most 2δ. Looking at the proof of the proposition, this is true as long as (2.18) holds for $C_2 \leq (1 + 8\rho)$ and $d(x, y) \leq 4\delta$.

Then we take (X, d, μ) a R-doubling space. T an operator defined by a kernel K. We say that T, or K verify the condition $(\mathbf{K})_\delta$, if there exist constants $C_1 > 1$, $C_2 \geq 1 + 8\rho$, such that for any x, y such that $d(x, y) \leq 4\delta$, we have :

$$\begin{aligned} &\forall x' \in X,\ d(x', y) \leq C_2 d(x, y), \quad K(x, y) \leq C_1 K(x', y) \\ &\forall x' \in X,\ d(x, y') \leq C_2 d(x, y), \quad K(x, y) \leq C_1 K(x, y'). \end{aligned} \tag{2.24}$$

Property $(\mathbf{K})_\delta$ ensure that 2.13 holds for balls of radius less than 2δ.

Since we will end up considering balls of a radius slightly larger than δ, the following proposition will be useful.

Proposition 2.14. *Let (X, d, μ) satisfy $(\mathbf{D})^\eta_{2(2\kappa+1)\delta}$ for $\delta > 0$, $\kappa > 1$, T an operator satisfying $(\mathbf{K})_{4(2\kappa+1)\delta}$, and such that the associated functional φ satisfies (2.21) when $r(B_1), r(B_2) \leq 2(2\kappa+1)\delta$. Then for any $p \in (1, \infty]$, there is some constant C which depends only on p, κ, the doubling constants, and the constants α, L, ε, in proposition 2.13 and in (2.21) such that for any non negative f, $\|M_{\varphi,\kappa\delta} f\|_p \leq C\|M_{\varphi,\delta} f\|_p$.*

Proof. We have :

$$M_{\varphi,\kappa\delta} f(x) = M_{\varphi,\delta} f(x) + \sup_{\substack{x\in B, \\ \delta<r(B)\leq\kappa\delta}} \varphi(B) \int_B |f|\ \mathrm{d}\mu$$

$$\leq M_{\varphi,\delta}f(x) + C \sup_{\substack{x\in B,\\ r(B)=\kappa\delta}} \varphi(B)\int_{B(x,2\kappa\delta)} |f| \, \mathrm{d}\mu$$

Using that for $x \in B$, $B \subset B(x, 2r(B)) \subset B(x, 2\kappa\delta)$ and that for any ball B with radius greater than δ, by (2.21) (on balls with radius at most $\kappa\delta$), we have $\varphi(B) \leq AL\kappa^\eta\varphi\left(\frac{\kappa\delta}{r(B)}B\right)$.

Now, for any ball B containing x with radius equal to $\kappa\delta$. For $y \in B(x, 2\kappa\delta)$, consider the ball $Q(y) = B(y, \delta)$. We have $Q(y) \subset B(x,(2\kappa + 1)\delta)$, thus using $(\mathbf{D})^\eta_{(2\kappa+1)\delta}$, we have that $\mu(B(x, 2\kappa\delta)) \leq A^2(2\kappa + 1)^\eta\mu(Q(y))$.

For $y \in B(x,(2\kappa + 1)\delta)$, we also have that $B \subset B(z, 2(2\kappa + 1)\delta)$, thus using (2.21) (for balls with radius at most $2(2\kappa + 1)\delta)$), $(\mathbf{D})^\eta_{2(2\kappa+1)}$ and $(\mathbf{K})_{4(2\kappa+1)\delta)}$, we get that $\varphi(B) \leq A^2\left(\frac{2(2\kappa+1)}{2\kappa}\right)^\eta\alpha\varphi(Q(y))$. Putting all this together, we get :

$$\begin{aligned}
\varphi(B)\int_{B(x,2\kappa\delta)} |f| \, \mathrm{d}\mu &= \varphi(B) ⨍_{B(x,2\kappa\delta)} \mu(B(x, 2\kappa\delta))|f| \, \mathrm{d}\mu \\
&\leq C\varphi(B) ⨍_{B(x,2\kappa\delta)} \mu(Q(y))|f(y)| \, \mathrm{d}\mu(y) \\
&\leq C ⨍_{B(x,2\kappa\delta)} \varphi(B) \int_{Q(y)} \mathrm{d}\mu(z)|f(y)| \, \mathrm{d}\mu(y) \\
&\leq C\frac{1}{\mu\left(B\left(x, 2\kappa\delta\right)\right)} \int_{B(x,(2\kappa+1)\delta)} \varphi(B) \int_{B(x,2\kappa\delta)\cap B(z,\delta)} |f(y)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\leq CA\left(\frac{2\kappa + 1}{2\kappa}\right)^\eta ⨍_{B(x,(2\kappa+1)\delta)} \varphi(B(z, \delta)) \int_{B(z,\delta)} |f(y)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\leq C ⨍_{B(x,(2\kappa+1)\delta)} M_{\varphi,\delta}f \, \mathrm{d}\mu
\end{aligned}$$

And the constant C depends only on the doubling constants, L, α and κ. Then we have :

$$M_{\varphi,\kappa\delta}f(x) \leq M_{\varphi,\delta}f(x) + CM_{(2\kappa+1)\delta}\left(M_{\varphi,\delta}f\right)(x) \tag{2.25}$$

The theorem follows from the boundedness of the classical maximal function $M_{(2\kappa+1)\delta}$ on any L^p, $p > 1$, under $(\mathbf{D})^\eta_{2(2\kappa+1)\delta}$. □

Theorem 2.4. *Let $\delta > 0$. Let $\rho > 0$ be the sidelength constant of dyadic cubes. Suppose that (X, d, μ) satisfy $(\mathbf{D})^\eta_{2(6\rho+1)\delta}$. Assume that K satisfies $(\mathbf{K})_{4(6\rho+1)\delta}$, and that φ satisfies* (2.21) *for balls with radius at most $2(6\rho + 1)\delta$. Let $p \geq 1$. Then there is a constant $C > 0$ (which depends only on the doubling constants, ρ, p and of the constants in* (2.21), (2.18)*) such that we have :*

$$\int_X |T_\delta f|^p \, \mathrm{d}\mu \leq C \int_X (M_{\varphi,\delta}f)^p \, \mathrm{d}\mu \tag{2.26}$$

Proof. We will show that there exist some constant $C > 0$ such that for any non negative function f, we have $\int_X |T_\delta f|^p \, \mathrm{d}\mu \leq C \int_X (M_{\varphi,3\rho\delta}f)^p \, \mathrm{d}\mu$. Then the theorem will follows by proposition 2.14.

To prove this, we define, for any $m \in \mathbf{Z}$, the operator T_m by :

$$T_m f(x) = \int_{d(x,y)>\rho^m} K_\delta(x, y)f(y) \, \mathrm{d}\mu(y)$$

Then, if for any $m \in \mathbf{Z}$, and for any non negative measurable functions f, g, we have :

$$\int_X T_m f g \, \mathrm{d}\mu = \int_{d(x,y)>\rho^m} K_\delta(x,y) f(y) g(x) \, \mathrm{d}\mu(x,y) \le C \|M_{\varphi,3\delta} f\|_p \|g\|_{p'} \tag{2.27}$$

Then by the monotone convergence theorem, taking $m \to -\infty$, the same inequality holds but with T_m replaced by T, and by duality, (2.26) is true.

Take $m \in \mathbf{Z}$, and let f, g be non negative measurable functions. Let $\mathcal{D}_m = \left\{\mathcal{E}_\alpha^k\right\}_{\alpha \in \mathbf{N}^*}^{k \ge m}$ be a decomposition of X in dyadic cubes given by theorem 2.1 with sidelengths ρ^k. If $(x,y) \in X$ are such that $d(x,y) > \rho^m$, we take the integer $l \ge m$ such that $\rho^l < d(x,y) \le \rho^{l+1}$. Let Q be the cube of length ρ^l containing x, $B(Q) = B\left(c_Q, \rho^{l+1}\right)$ the containing ball. We recall that we have $\rho^{-1} B(Q) \subset Q \subset B(Q)$.

$d(c_Q, y) \le d(c_Q, x) + d(x,y) \le 2\rho^{l+1}$, thus $y \in 2B(Q)$. Since $d(x,y) > \rho^l = \frac{1}{2\rho} r\left(2B(Q)\right)$, we have by definition of φ, $K(x,y) \le \varphi(2B(Q)) \le \alpha \varphi(B(Q))$ by proposition 2.13 (which needs to hold for balls of radius $2\rho\delta$, thus we need $(\mathbf{K})_{4\rho\delta}$. And if we suppose that $\delta \le \rho^l = \ell(Q)$, then $d(x,y) \ge \delta$ and $K_\delta(x,y) = 0$.

We have proved that if Q is the cube of length comparable with $d(x,y)$, containing x, we have $y \in 2B(Q)$ and :

$$K_\delta(x,y) \le C \varphi(B(Q)) \chi_{\{R \in \mathcal{D}_m,\, \ell(R) < \delta\}}(Q) \chi_Q(x) \chi_{2B(Q)}(y)$$

If r is the largest integer such that $\rho^r < \delta$, define $\mathcal{D}_m^r = \left\{\mathcal{E}_\alpha^k;\ m \le k \le r\right\}$. For any $x, y \in X$ with $d(x,y) > \rho^m$, there is at least one cube $Q \in \mathcal{D}_m$ such that the previous inequation holds, and since both sides of it are zero if $\ell(Q) \ge \delta$, we have, for any $x, y \in X$:

$$K_\delta(x,y) \le \sum_{Q \in \mathcal{D}_m^r} C \varphi(B(Q)) \chi_Q(x) \chi_{2B(Q)}(y)$$

And so, for any $f, g \ge 0$:

$$\int_X T_m f g \, \mathrm{d}\mu \le C \sum_{Q \in \mathcal{D}_m^r} \varphi(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu \int_Q g \, \mathrm{d}\mu$$

But for any fixed integer $k \ge m$, the cubes of length of length ρ^k, $\left\{\mathcal{E}_\alpha^k\right\}$ are pairwise disjoints, and $X = \bigcup_\alpha \mathcal{E}_\alpha^k$. Then using this decomposition for $k = r$,

$$\int_X T_m f g \, \mathrm{d}\mu \le C \sum_{\alpha \ge 1} \sum_{\substack{Q \in \mathcal{D}_m^r \\ Q \subset \mathcal{E}_\alpha^r}} \varphi(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu \int_Q g \, \mathrm{d}\mu$$

Then for a constant $\gamma \ge 1$ to be determined, for any $\alpha \ge 1$, and $n \in \mathbf{Z}$, define :

$$\mathcal{C}_\alpha^n = \left\{Q \in \mathcal{D}_m^r, Q \subset \mathcal{E}_\alpha^r;\ \gamma^n < \frac{1}{\mu(B(Q))} \int_Q g \, \mathrm{d}\mu \le \gamma^{n+1}\right\} \tag{2.28}$$

We let n_α be the unique integer such that $\mathcal{E}_\alpha^r \in \mathcal{C}_\alpha^{n_\alpha}$. Notice that $\{\mathcal{C}_\alpha^n\}_{n \in \mathbf{Z}}$ is a partition of $\{Q \in \mathcal{D}_m^r;\ Q \subset \mathcal{E}_\alpha^r\}$. Then we have :

$$\int_X T_m f g \, \mathrm{d}\mu \le C \sum_{\alpha \ge 1} \sum_{n \in \mathbf{Z}} \gamma^{n+1} \sum_{Q \in \mathcal{C}_\alpha^n} \varphi(B(Q)) \mu(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu$$

For any $\alpha \geq 1$, we let $\left\{Q^n_{j,\alpha}\right\}_{j\in J_n}$, for some index set J_n, be the collection of the maximal dyadic cubes subset of $\mathcal{E}^r_\alpha$ such that $\gamma^n < \frac{1}{\mu\left(B\left(Q^n_{j,\alpha}\right)\right)}\int_{Q^n_{j,\alpha}} g \,\mathrm{d}\mu$. If $n \leq n_\alpha$, then there is exactly one such maximal cube : $\mathcal{E}^r_\alpha$. Also, we have an injection from the set of the couples (n, Q) with $n \leq n_\alpha$, $Q \in \mathcal{C}^n_\alpha$ to $\{Q \in D^r_m : Q \subset \mathcal{E}^r_\alpha\}$, thus :

$$\sum_{n\leq n_\alpha} \gamma^{n+1} \sum_{Q\in\mathcal{C}^n_\alpha} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \,\mathrm{d}\mu \leq \gamma^{n_\alpha+1} \sum_{\substack{Q\in\mathcal{D}^r_m\\ Q\subset\mathcal{E}^r_\alpha}} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q))} f \,\mathrm{d}\mu$$

If $n > n_\alpha$, then any $Q^n_{j,\alpha}$ is a strict subset of $\mathcal{E}^r_\alpha$. For such a maximal cube $\mathcal{F}$, we let P be his dyadic parent i.e. the only cube of length $\rho\ell(R)$ containing P. We have $P \subset \mathcal{E}^r_\alpha$, and by using the maximality of $\mathcal{F}$, and that $B(R) \subset 2B(P)$, and using the $\rho\delta$-doubling ($B(P)$ has radius less than $\rho\delta$) :

$$\gamma^n < \frac{1}{\mu(B(\mathcal{F}))}\int_{\mathcal{F}} g \,\mathrm{d}\mu \leq \frac{\mu(B(P))}{\mu(B(\mathcal{F}))}\frac{1}{\mu(B(P))}\int_P g \,\mathrm{d}\mu \leq C\rho^\eta\gamma^n = \kappa\gamma^n, \tag{2.29}$$

the constant κ depending only on ρ and on the doubling constant. Then choosing $\gamma > \kappa$, we have $\frac{1}{\mu(B(\mathcal{F}))}\int_{\mathcal{F}} g \,\mathrm{d}\mu \leq \gamma^{n+1}$, thus $\mathcal{F} \in \mathcal{C}^n_\alpha$. Thus for a fixed $n > n_\alpha$, every cube in $\mathcal{C}^n_\alpha$ is in a (unique) $Q^n_{j,\alpha}$, which are disjoint in j by maximality. Thus, writing $Q^{n_\alpha}_{j,\alpha}$ for $\mathcal{E}^r_\alpha$ we have :

$$\int_X (T_m f)\, g \,\mathrm{d}\mu \leq C\sum_{\alpha\geq1}\sum_{n\geq n_\alpha}\gamma^{n+1}\sum_{j\in J_n}\sum_{\substack{Q\in\mathcal{D}^m_\alpha\\ Q\subset Q^n_{j,\alpha}}} \varphi(B(Q))\mu(B(Q))\int_{2B(Q)} f \,\mathrm{d}\mu$$

Now we use the following lemma (see lemma 6.1 of [22]) :

Lemma 2.2. *Let (X, d, μ) satisfy $(\mathbf{D})^\eta_\delta$. Let φ be a functional on balls that satisfy* (2.21) *for balls of radius at most $\rho\delta$. Then there is a constant C which depends only on the constant L of* (2.21) *and on the doubling constant such that for any $f \geq 0$ and any dyadic cube $Q_0 \in \mathcal{D}^r_m$, with $\rho^r \leq \delta$,*

$$\sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0}} \varphi(B(Q))\mu(B(Q))\int_{2B(Q))} f \,\mathrm{d}\mu \leq C\varphi(B(Q_0))\mu(B(Q_0))\int_{3B(Q_0)} f \,\mathrm{d}\mu \tag{2.30}$$

Proof. By (2.21), we have :

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0}} \varphi(B(Q))\mu(B(Q))\int_{2B(Q))} f \,\mathrm{d}\mu &\leq L\varphi(B(Q_0))\mu(B(Q_0))\sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0}}\left(\frac{\ell(Q)}{\ell(Q_0)}\right)^\varepsilon\int_{2B(Q))} f \,\mathrm{d}\mu\\ &\leq L\varphi(B(Q_0))\mu(B(Q_0))\sum_{l=0}^{+\infty}\rho^{-\varepsilon l}\sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0\\ \ell(Q)=\rho^{-l}\ell(Q_0)}}\int_{2B(Q))} f \,\mathrm{d}\mu.\end{aligned} \tag{2.31}$$

Then for $Q \in \mathcal{D}_m, Q \subset Q_0$, and $\ell(Q) \leq \ell(Q_0)$ we have $2B(Q) \subset 3B(Q_0)$. Indeed, if $y \in 2B(Q)$, then :

$$
\begin{aligned}
d(y, x_{Q_0}) &\le d(y, x_Q) + d(x_Q, x_{Q_0}) \\
&\le 2r(B(Q)) + r(B(Q_0)) \\
&\le 3r(B(Q_0)).
\end{aligned}
$$

Thus, the left hand side of (2.31) is less than :

$$
L\varphi(B(Q_0))\mu(B(Q_0)) \int_{3B(Q_0))} f(x) \sum_{l=0}^{\infty} \rho^{-\varepsilon l} \sum_{\substack{Q\in\mathcal{D}_m \\ Q\subset Q_0 \\ \ell(Q)=\rho^{-l}\ell(Q_0)}} \chi_{2B(Q)}(x)\ \mathrm{d}\mu(x).
$$

Then it suffices to show that for each l, any x of $3B(Q_0)$ is in at most N of the $2B(Q)$, with $\ell(Q) = \rho^{-l}\ell(Q_0)$, with N independant of the choices of x and Q_0. For $l = 0$, there is only one Q : Q_0 itself, and thus it is true.

Now fix $l > 1$, let $x \in M$, and Q be a cube of sidelength $\rho^{-l}\ell(Q_0)$ such that $x \in 2B(Q)$. We write $\ell = \ell(Q) \le \rho^{-1}\delta$. Then for $y \in Q$, $d(x, y) \le d(x, x_Q) + d(y, x_Q) \le 3\rho\ell \le 3\delta$. Then we have $B(x_Q, \ell) \subset Q \subset B(x, 3\rho\ell)$. By the proposition 2.9, then there can be at most N disjoint balls of radius $\ell \le \delta$ with center in a ball of radius $3\rho\ell$, with the constant N depending only on ρ and on the δ-doubling constant.

Thus

$$
\sum_{l=0}^{\infty} \rho^{-\varepsilon l} \sum_{\substack{Q\in\mathcal{D}_m \\ Q\subset Q_0 \\ \ell(Q)=\rho^{-l}\ell(Q_0)}} 1 \le N \frac{1}{1-\rho^{-\varepsilon}},
$$

and the lemma follows. $\square$

Then applying the lemma :

$$
\int_X (T_m f)\, g\ \mathrm{d}\mu \le C \sum_{\alpha\ge 1} \sum_{n\ge n_\alpha} \gamma^{n+1} \sum_{j\in J_n} \varphi\left(B\left(Q^n_{j,\alpha}\right)\right) \mu\left(B\left(Q^n_{j,\alpha}\right)\right) \int_{3B\left(Q^n_{j,\alpha}\right)} f\ \mathrm{d}\mu.
$$

And thus since $Q^n_{j,\alpha} \in \mathcal{C}^n_\alpha$, $\gamma^n \le \frac{1}{\mu\left(B(Q^n_{j,\alpha})\right)} \int_{Q^\alpha_{j,n}} g\ \mathrm{d}\mu$, and so,

$$
\int_X (T_m f)\, g\ \mathrm{d}\mu \le C\gamma \sum_{\alpha\ge 1} \sum_{n\ge n_\alpha} \sum_{j\in J_n} \varphi\left(B((Q^n_{j,\alpha})\right) \int_{3B\left(Q^n_{j,\alpha}\right)} f\ \mathrm{d}\mu \int_{Q^n_{j,\alpha}} g\ \mathrm{d}\mu,
$$

and we have :

$$
\int_X (T_m f)\, g\ \mathrm{d}\mu \le c \sum_{\alpha,n,j} \varphi\left(B\left(Q^n_{j,\alpha}\right)\right) \mu\left(Q^n_{j,\alpha}\right) \int_{3B(Q^n_{j,\alpha})} f\ \mathrm{d}\mu \frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g\ \mathrm{d}\mu. \tag{2.32}
$$

Then using Hölder's inequality, and that by (2.21) there is some constant c depending only on $\alpha, A, L, \varepsilon$ such that $\varphi(B) \le c\varphi(3B)$ (ball of radius $3\rho\delta$), we get :

$$\int_X (T_m f)\, g \, \mathrm{d}\mu \leq C \left(\sum_{\alpha,n,j} \mu\left(Q^n_{j,\alpha}\right) \left(\varphi\left(B\left(3Q^n_{j,\alpha}\right)\right) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu \right)^p \right)^{\frac{1}{p}}$$

$$\left(\sum_{\alpha,n,j} \mu\left(Q^n_{j,\alpha}\right) \left(\frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu \right)^{p'} \right)^{\frac{1}{p'}}$$

Now we just need to establish a majoration of $\mu(Q^n_{j,\alpha})$ by a constant time the measure of a set $E^n_{j,\alpha}$, with the $E^n_{j,\alpha}$ being pairwise disjoint in j, n, α. For this, define Ω^n_α by

$$\Omega^n_\alpha = \left\{ x \in \mathcal{E}^r_\alpha;\ \sup_{\substack{Q \in \mathcal{D}^r_m \\ x \in Q}} \frac{1}{\mu(B(Q))} \int_Q g \, \mathrm{d}\mu > \gamma^n \right\} = \bigcup_{j \in J_n} Q^n_{j,\alpha} \tag{2.33}$$

and define the set $E^n_{j,\alpha} = Q^n_{j,\alpha} \setminus \Omega^{n+1}_\alpha$. We have that $E^n_{j,\alpha} \subset \Omega^n_\alpha \setminus \Omega^{n+1}_\alpha$, and the $E^n_{j,\alpha}$ are pairwise disjoints in j, n, α.

Now we want to show that for γ chosen large enough, $\mu(Q^n_{j,\alpha}) \leq 2\mu(E^n_{j,\alpha})$.

First, $Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha = \bigcup_i \left(Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha}\right)$. But we have $\frac{1}{\mu\left(B\left(Q^{n+1}_{i,\alpha}\right)\right)} \int_{Q^{n+1}_{i,\alpha}} g \, \mathrm{d}\mu > \gamma^{n+1} > \gamma^n$, thus by maximality of $Q^n_{j,\alpha}$, and by the properties of dyadic cubes, etiher $Q^{n+1}_{i,\alpha} \subset Q^n_{j,\alpha}$ or $Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha} = \emptyset$. Hence :

$$\mu\left(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha\right) = \sum_{i: Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha} = \emptyset} \mu\left(Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha}\right) = \sum_{i: Q^{n+1}_{i,\alpha} \subset Q^n_{j,\alpha}} \mu\left(Q^{n+1}_{i,\alpha}\right)$$

But :

$$\mu\left(Q^{n+1}_{i,\alpha}\right) \leq \mu\left(B\left(Q^{n+1}_{i,\alpha}\right)\right) \leq \gamma^{-n-1} \int_{Q^{n+1}_{i,\alpha}} g \, \mathrm{d}\mu.$$

And since the $Q^{n+1}_{i,\alpha}$ considered are disjoints and subsets of $Q^n_{j,\alpha}$:

$$\mu(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha) \leq \gamma^{-n-1} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu \leq \kappa \gamma^{-1} \mu(B(Q^n_{j,\alpha})),$$

where κ is the constant in (2.29). But we have :

$$\mu(Q^n_{j,\alpha}) = \mu(E^n_{j,\alpha}) + \mu(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha),$$

and so choosing $\gamma = 2\kappa$, it follows that :

$$\mu\left(Q^n_{j,\alpha}\right) \leq \frac{\gamma}{\gamma - \kappa} \mu\left(E^n_{j,\alpha}\right) = 2\mu\left(E^n_{j,\alpha}\right)$$

Thus we have :

$$\int_X (T_m f)\, g \,\mathrm{d}\mu \leq 2C \left(\sum_{\alpha,n,j} \mu\left(E^n_{j,\alpha}\right) \left(\varphi\left(B\left(3Q^n_{j,\alpha}\right)\right) \int_{3B(Q^n_{j,\alpha})} f \,\mathrm{d}\mu \right)^p \right)^{\frac{1}{p}}$$

$$\left(\sum_{\alpha,n,j} \mu\left(E^n_{j,\alpha}\right) \left(\frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g \,\mathrm{d}\mu \right)^{p'} \right)^{\frac{1}{p'}}.$$

But since $E^n_{j,\alpha} \subset Q^n_{j,\alpha}$, $\mu\left(E^n_{j,\alpha}\right) \left(\varphi\left(B\left(3Q^n_{j,\alpha}\right)\right) \fint_{3B\left(Q^n_{j,\alpha}\right)} f \,\mathrm{d}\mu \right)^p \leq \int_{E^n_{j,\alpha}} \left(M_{\varphi,3\rho^{r+1}} f\right)^p \mathrm{d}\mu$, and a similar inequality for the integral on g. In addition using that the $E^n_{j,\alpha}$ are pairwise disjoint, and that $\rho^r < \delta$, we get :

$$\int_X (T_m f)\, g \,\mathrm{d}\mu \leq 2C \left(\int_X (M_{\varphi,3\rho\delta} f)^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_X (M_{d,\delta} g)^{p'} \,\mathrm{d}\mu \right)^{\frac{1}{p'}}; \tag{2.34}$$

then using proposition 2.12, for all $f, g \geq 0$, there is a constant C which depends only on $p, A, \alpha, \varepsilon$ (specifically it depends on the constants for the $\rho\delta$-doubling) such that :

$$\int_X (T_m f)\, g \,\mathrm{d}\mu \leq C \left\| M_{\varphi,3\rho\delta} f \right\|_p \|g\|_{p'}.$$

This holds under $(\mathbf{D})^\eta_{r\delta}$, $(\mathbf{K})_{2\rho\delta}$ and the fact that (2.21) holds for balls of radius at most $3\rho\delta$. The stronger hypotheses are what we need to apply proposition 2.14 which gives us :

$$\int_X (T_m f)\, g \,\mathrm{d}\mu \leq C \left\| M_{\varphi,\delta} f \right\|_p \|g\|_{p'}. \tag{2.35}$$

Which proves the theorem. □

Finally we have :

Corollary 2.1. *Let μ be a measure satisfy $(\mathbf{D})^\eta_R$ and $(\mathbf{RD})^\nu_R$, for $R > 0$, $\eta \geq \nu > 0$ ($\eta \geq \nu$ is automatic). Let $s \leq \nu$. Let $\delta \leq R$. If $K(x,y) = \frac{d(x,y)^s}{\mu(B(x,d(x,y)))}$, then the associated operator T_δ satisfy the hypotheses of theorem 2.4. Moreover, the theorem still holds with $M_{\varphi,\delta} f$ replaced by the following maximal function :*

$$M_{s,\delta} f(x) = \sup_{0<r<\delta} r^s \fint_{B(x,r)} |f| \,\mathrm{d}\mu. \tag{2.36}$$

Proof. First, take some $b > 1$, by proposition 2.8, μ is bR-reverse doubling of order ν. Then, we must verify that K satisfy the hypotheses of theorem 2.4. Let $d(x,y) \leq R$ and $d(x,y') \leq bd(x,y)$, then we have by doubling and reverse doubling, :

$$\begin{aligned} \frac{1}{\mu\left(B\left(x,d(x,y)\right)\right)} &\leq \frac{1}{\mu\left(B\left(x,d(x,y')\right)\right)} \frac{\mu\left(B\left(x,bd(x,y)\right)\right)}{\mu\left(B\left(x,d(x,y)\right)\right)} \frac{\mu\left(B\left(x,d(x,y')\right)\right)}{\mu\left(B\left(x,bd(x,y)\right)\right)}, \\ &\leq Cb^{\eta-\nu} \left(\frac{d(x,y')}{d(x,y)} \right)^\nu \frac{1}{\mu\left(B\left(x,d(x,y')\right)\right)}. \end{aligned}$$

Thus, provided that $s \leq \nu$:

$$K(x,y) \leq Cb^{\eta-\nu}\left(\frac{d(x,y')}{d(x,y)}\right)^{\nu-s} K(x,y') \leq Cb^{\eta-s}K(x,y').$$

Furthermore, if $d(x',y) \leq \alpha d(x,y)$, by doubling there are c, C such that $c\mu\left(B\left(y, d(x',y)\right)\right) \leq \mu\left(B\left(x', d(x',y)\right)\right) \leq C\mu\left(B\left(y, d(x',y)\right)\right)$, and so doing the same calcuations we have :

$$K(x,y) \leq Cb^{\nu-s}K(x',y).$$

And there are $C_1, C_2 > 1$ such that (2.18) is satisfied.

Then, using the definition of φ and doubling, $c\frac{r(B)^s}{\mu(B)} \leq \varphi(B) \leq C\frac{r(B)^s}{\mu(B)}$ for some constants that depends only on s, ρ and the doubling constant. Then since we have, for $B_1 \subset B_2$, $r(B_1)^s \leq 2^s r(B_2)^s$, we easily verify that φ satisfy (2.21) with $\varepsilon = s$.

Then it is enough to prove that the centered and uncentered version of the maximal function $M_{s,\delta}$ are equivalent in L^p norms. This follow from the same argument as that of proposition 2.11. $\square$

3 Relative Faber-Krahn inequality and estimates on the heat kernel and the Riesz and Bessels potentials

3.1 Faber-Krahn and doubling

The results from this subsection are due to A.A. Grigor'yan [10, 11], or are slight adaptation of his results to the R-doubling case.

Theorem 3.1. *[11] Let (M, g, μ) be a weighted manifold, and let $\{B(x_i, r_i)\}_{i\in I}$ be a family of relatively comacpt balls in M, where I is an arbitrary index set. Assume that, for any $i \in I$, the Faber-Krahn inequality holds :*

$$\lambda_1(U) \geq a_i \mu(U)^{-2/\eta}, \tag{3.1}$$

for any open set $U \subset B(x_i, r_i)$, where $a_i > 0$. Let $\Omega = \bigcup_{i\in I} B\left(x_i, \frac{r_i}{2}\right)$. Then for all $x, y \in \Omega$ and $t \geq t_0 > 0$:

$$p_t(x,y) \leq \frac{C(\eta)\left(1+\frac{d(x,y)^2}{t}\right)^{\eta/2}\exp\left(-\frac{d(x,y)^2}{4t} - \lambda_1(M)(t-t_0)\right)}{\left(a_i a_j \min\left(t_0, r_i^2\right)\min\left(t_0, r_j^2\right)\right)^{\eta/4}}, \tag{3.2}$$

where i, j are the indices such that $x \in B\left(x_i, \frac{r_i}{2}\right)$ and $y \in B\left(x_j, \frac{r_j}{2}\right)$.

On a manifold which admits $(\mathbf{RFK})^{\eta}_{R}$, applying this theorem with the family $\{B(x,r)\}_{\substack{x\in M,\\ 0<r\leq R}}$, $a_{x,r} = \frac{b}{r^2}\mu\left(B\left(x,r\right)\right)^{2/\eta}$, $t_0 = t$, and $r = \sqrt{t}$ when $t \leq R^2$ we get :

$$\begin{aligned} p_t(x,y) &\leq C(\eta)\frac{\left(1+\frac{d(x,y)^2}{t}\right)^{\eta/2} e^{-\frac{d(x,y)^2}{4t}}}{\left(a_{x,\sqrt{t}}b_{y,\sqrt{t}}t^2\right)^{\eta/4}}, \\ &\leq \frac{C(\eta)}{b^{\eta/2}}\frac{e^{-\frac{d(x,y)^2}{ct}}}{\mu\left(B\left(x,\sqrt{t}\right)\right)^{1/2}\mu\left(B\left(y,\sqrt{t}\right)\right)^{1/2}}. \end{aligned}$$

If $t > R^2$, then we do the same thing, but with $r = R$. Thus we obtain the following :

Theorem 3.2. *Let (M, g, μ) be a weighted Riemannian manifold, suppose that there is $R > 0$ such that M satisfy $(\mathbf{RFK})_R^\eta$. Then μ satisfy $(\mathbf{D})_R^\eta$, and for any $c > 4$ there is some constant $C > 0$ such that the heat kernel satisfies the upper bound :*

$$p_t(x,y) \le \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)^{1/2}\mu\left(B\left(y,\sqrt{t}\right)\right)^{1/2}} e^{-\frac{d(x,y)^2}{ct}}, \quad t \le R^2 \tag{3.3}$$

$$p_t(x,y) \le \frac{C}{\mu\left(B\left(x,R\right)\right)^{1/2}\mu\left(B\left(y,R\right)\right)^{1/2}} e^{-\frac{d(x,y)^2}{ct}}, \quad t > R^2. \tag{3.4}$$

The constant C depends only on b and η in the Faber-Krahn inequality and on the $c > 4$ chosen.

The estimate on the heat kernel follows from Theorem 5.2 of [10]. The R-doubling follow from the proof of Proposition 5.2 of the same article.

Conversely, we have :

Proposition 3.1. *[10] Let (M, g, μ) be a complete, weighted Riemannian manifold. If μ satisfies $(\mathbf{D})_R^\eta$, if for any $x \in M$, $X \setminus B\left(x, \frac{3}{4}R\right) \neq \emptyset$ and if there is some constant B such the heat kernel satisfies :*

$$p_t(x,x) \le \frac{B}{\mu\left(B\left(x,\sqrt{t}\right)\right)}, \tag{3.5}$$

for all $x \in M$, and for all $0 < t \le R^2$, then there is some constant $\kappa \in (0,1)$, which depends only on the doubling and reverse doubling constants, such that M admits a relative Faber-Krahn inequality at scale κR, with η being the doubling order and b depending only on A, B, and κ depends only on the doubling constants and on B.

3.2 An estimate on the heat kernel

Proposition 3.2. *Let (M, g, μ) be a weighted manifold satisfying $(\mathbf{RFK})_R^\eta$ for $R > 0$, then for any $c > 4$ and $\gamma \in (0,1)$ there exists constants $C > 0$, $\hat{c} > 1$ such that for any $\lambda > 0$ with $R\lambda > \hat{c}$, we have :*

$$\begin{aligned} p_t(x,y) &\le \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)} e^{-\frac{d(x,y)^2}{ct}}, & \sqrt{t} \le \lambda^{-1} \\ p_t(x,y) &\le \frac{C}{\mu\left(B\left(x,\lambda^{-1}\right)\right)} e^{(1-\gamma)\lambda^2 t} e^{-\frac{d(x,y)^2}{ct}}, & \sqrt{t} > \lambda^{-1}. \end{aligned} \tag{3.6}$$

With C depending only on b, η and c, and $\hat{c}$ depending on b, η, c and γ.

Proof. Let $c > 4$, $\gamma \in (0,1)$. If $t \le \lambda^{-1} < R^2$, then applying theorem 3.2, we have, for any $\kappa > 1$ such that $c/\kappa > 4$:

$$\begin{aligned} p_t(x,y) &\le \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)^{1/2}\mu\left(B\left(y,\sqrt{t}\right)\right)^{1/2}} e^{-\frac{\kappa d(x,y)^2}{ct}} \\ &\le \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)} e^{\frac{D}{2}\frac{d(x,y)}{\sqrt{t}} - \frac{\kappa}{c}\frac{d(x,y)^2}{t}} \end{aligned}$$

$$\leq \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)}e^{-\frac{d(x,y)^2}{ct}},$$

using proposition 2.6. If $t \geq \lambda^{-1}$, then similarly :

$$p_t(x,y) \leq \frac{C}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}e^{\frac{Dd(x,y)}{2R}-\frac{\kappa d(x,y)^2}{ct}}.$$

Then we have $\frac{Dd(x,y)}{2R} - \frac{(\kappa-1)d(x,y)^2}{ct} - (1-\gamma)\lambda^2 t \leq \left(\frac{cD^2}{16(\kappa-1)R^2} - (1-\gamma)\lambda^2\right)t$, for all $t > 0$, $x, y \in M$. Thus for $\hat{c} = \sqrt{\frac{c}{(1-\gamma)(\kappa-1)}}\frac{D}{4}$, the for all λ such that $\lambda R \geq \hat{c}$, we have:

$$p_t(x,y) \leq \frac{C}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}e^{(1-\gamma)\lambda^2 t}e^{-\frac{d(x,y)^2}{ct}} \tag{3.7}$$

□

Remark. *If $d(x,y) \leq R$ then we can actually do better, then $e^{Dd(x,y)/2R} \leq C$ for a constant which doesn't depends on $d(x,y)$. Then we have :*

$$p_t(x,y) \leq \frac{C}{\mu\left(B\left(x,\min(\sqrt{t},\lambda^{-1})\right)\right)}e^{-\frac{d(x,y)^2}{ct}} \tag{3.8}$$

3.3 Estimation of the Riesz potential

Let $s > 0$. Define the Riesz potential to be the operator $I_s = \Delta^{-s/2}$ on $L^2(M,\mu)$. We have by the spectral theorem, for f positive, measurable :

$$\begin{aligned}
I_s f(x) &= \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty t^{s/2-1}e^{-t\Delta}f(x)\ \mathrm{d}t \\
&= \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_M f(y)\int_0^\infty t^{s/2-1}p_t(x,y)\ \mathrm{d}t\ \mathrm{d}\mu(y) \\
&= \int_M i_s(x,y)f(y)\ \mathrm{d}\mu
\end{aligned}$$

With the "kernel" i_s defined by :

$$i_s(x,y) = \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty t^{s/2-1}p_t(x,y)\ \mathrm{d}t. \tag{3.9}$$

Proposition 3.3. *Let (M,g,μ) be a manifold satisfying $(\mathbf{RFK})^\eta$ and $(\mathbf{RD})^\nu$, $\nu > 0$. Then for any $s < \nu$, there is a constant C, depending only on the Faber-Krahn and reverse doubling constants, such that :*

$$i_s(x,y) \leq C\frac{d(x,y)^s}{\mu\left(B\left(x,d(x,y)\right)\right)} \tag{3.10}$$

Proof. If M admits a relative Faber-Krahn inequality, then there are constants $C > 0$, $c > 4$ such that $p_t(x,y) \leq \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)}e^{-\frac{d(x,y)^2}{ct}}$, for all $x, y \in M$, $t > 0$.

Thus :

$$i_s(x,y) \leq C_s \int_0^\infty \frac{t^{s/2-1}}{\mu\left(B\left(x,\sqrt{t}\right)\right)} e^{-\frac{d(x,y)^2}{ct}} \; \mathrm{d}t$$

We integrate separately between 0 and d^2 and d^2 and $+\infty$, and using the doubling and reverse doubling properties of the measure we get :

$$i_s(x,y) \leq C \frac{d^\eta}{\mu\left(B\left(x,d(x,y)\right)\right)} \int_0^{d^2} t^{s/2-\eta/2-1} e^{-\frac{d(x,y)^2}{ct}} \; \mathrm{d}t + \\ C \frac{d^\nu}{\mu\left(B\left(x,d(x,y)\right)\right)} \int_{d^2}^\infty t^{s/2-\nu/2-1} e^{-\frac{d(x,y)^2}{ct}} \; \mathrm{d}t$$

□

When $\nu > s$, the second integral is convergent and less than $\frac{2}{\nu - s} d^{s-\nu}$.

For the first integral, through the change of variables $t = d^2/cu$ there is some constant c_s such that it is equal to $c_s d^{s-\eta} \int_1^\infty u^{\eta/2-s/2-1} e^{-u} \; \mathrm{d}u$, and this new integral is convergent if $\eta > s$, which it is since we must have $\eta \geq \nu$, and equal to a constant depending only on η, s.

Thus putting all of this together we have :

$$i_s(x,y) \leq C \frac{d(x,y)^s}{\mu\left(B\left(x,d(x,y)\right)\right)}$$

With the constant depending only on s, η, ν as well as the constants of the relative Faber Krahn inequality.

3.4 Estimation of the Bessel potential

Define the Bessel potential for $\lambda > 0$, $s > 0$ to be the operator $G_{s,\lambda} = \left(\Delta + \lambda^2\right)^{-s/2}$ on $L^2(M,\mu)$. We have, by the spectral theorem :

$$G_{s,\lambda} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty t^{s/2-1} e^{-\lambda^2 t} e^{-t\Delta} \; \mathrm{d}t \tag{3.11}$$

Similar to the previous section, we have for positive f :

$$G_{s,\lambda} f(x) = \int_M g_{s,\lambda}(x,y) f(y) \; \mathrm{d}\mu(y), \tag{3.12}$$

with $g_{s,\lambda}$ defined by :

$$g_{s,\lambda}(x,y) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty t^{s/2-1} e^{-\lambda^2 t} p_t(x,y) \; \mathrm{d}t \tag{3.13}$$

Proposition 3.4. *There is a constant $\hat{c}$ such that if (M,g,μ) is a weighted manifold that satisfy $(\mathbf{RFK})_R^\eta$ and $(\mathbf{RD})_R^\nu$ for $R > 0$, $\nu > 1$, then for any λ such that $\lambda R > \hat{c}$, then for any $s < \nu$, there are constants $C > 0$, and $\gamma \in (0,1)$, depending only on the Faber Krahn and reverse doubling constants, such that :*

$$g_{s,\lambda}(x,y) \leq C \left(\frac{d(x,y)^s}{\mu\left(B\left(x,d(x,y)\right)\right)} \chi_{\{\lambda d(x,y) \leq 1\}} + \frac{\lambda^{-s}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)} \left(\chi_{\{\lambda d(x,y) > 1\}}\right) \right) e^{-\gamma \lambda d(x,y)} \tag{3.14}$$

Proof. It is enough to show the proposition for $\lambda = 1$, $R > \hat{c}$.

Indeed, $G_{s,\lambda} = \lambda^{-s}\left(\frac{\Delta}{\lambda^2}+1\right)^{-s/2}$. Δ/λ^2 is the Laplacian Δ' for (M, g', μ') with $g' = \lambda^2 g$ and $\mathrm{d}\mu' = \lambda^n\ \mathrm{d}\mu$. The geodesic distance d' associated with the metric g' is simply $d' = \lambda d$, and if (M, g, μ) admits a relative Faber Krahn inequality at scale R, then (M, g', μ') admits a relative Faber Krahn inequality, with the same constants, at scale λR.

Then using that $g_{s,\lambda}(x,y) = \lambda^{-s} g'_{s,1}(x,y)$, with $g'_{s,1}$ the kernel of $(\Delta'+1)^{-s/2}$, it follows that (3.14) being true for $\lambda = 1$ and all $R > \hat{c}$ implies (3.14) for all (λ, R) such that $R\lambda > \hat{c}$.

We have :

$$g_{s,1}(x,y) = \frac{1}{\Gamma\left(\frac{s}{2}\right)}\left(\int_0^1 t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t + \int_1^\infty t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t\right)$$

Let $J_0 = \int_0^1 t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t$ and $J_\infty = \int_1^\infty t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t$.

To simplify the notations we will write $d = d(x,y)$ until the end of this section.

Lemma 3.1. *There is some constant $\hat{c} > 1$ such that for $R \geq \hat{c}$, there is some $\gamma > 0$, such that for any $s < \nu$ we have :*

$$\int_0^1 t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t \leq c\left(\frac{d^s}{\mu\left(B\left(x,d\right)\right)}\chi_{\{d\leq 1\}} + \frac{1}{\mu\left(B\left(x,1\right)\right)}\chi_{\{d>1\}}\right)e^{-\gamma d}. \tag{3.15}$$

Proof. We treat the cases $d \leq 1$ and $d > 1$ separately. When $d(x,y) \leq 1$ we have : $J_0 = J_{0,1} + J_{0,2}$ with

$$J_{0,2} = \int_{d^2}^1 t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t.$$

We have, by proposition 3.2, that , there is a constant $\hat{c}$ such that if $R \geq \hat{c}$, then for all $t \leq 1$ we have $p_t(x,y) \leq \frac{C}{\mu\left(B\left(x,\sqrt{t}\right)\right)}e^{-\frac{d(x,y)^2}{ct}}$. For such $R \geq \hat{c} > 1$, using the R-reverse doubling, we have that for any $t \in (d^2, 1)$, $\sqrt{t} \leq 1 < R$, and thus we have $\mu\left(B\left(x,d\right)\right) \leq a\left(\frac{d}{\sqrt{t}}\right)^\nu \mu\left(B\left(x,\sqrt{t}\right)\right)$. Using all this we get :

$$\begin{aligned}
J_{0,2} &\leq C\int_{d^2}^1 \frac{t^{s/2-1}e^{-t}e^{-\frac{d^2}{ct}}}{\mu\left(B\left(x,\sqrt{t}\right)\right)}\ \mathrm{d}t, \\
&\leq Ca^{-1}\frac{d^\nu}{\mu\left(B\left(x,d\right)\right)}\int_{d^2}^1 t^{s/2-\nu/2-1}e^{-t}e^{-\frac{d^2}{ct}}\ \mathrm{d}t, \\
&\leq Ce^{1/4}\frac{d^\nu e^{-d}}{\mu\left(B\left(x,d\right)\right)}\int_{d^2}^1 t^{s/2-\nu/2-1}\ \mathrm{d}t; \quad \text{ since } e^{-t} \leq e^{-d^2},\ e^{-d^2} \leq e^{1/4}e^{-d}, \\
&\leq C\frac{d^\nu e^{-d}}{\mu\left(B\left(x,d\right)\right)}\frac{2}{\nu-s}\left(d^{s-\nu}-1\right), \\
&\leq C\frac{d^s}{\mu\left(B\left(x,d\right)\right)}e^{-d},
\end{aligned}$$

since we have $\nu > s$. Now we estimate $J_{0,1}$:

$$J_{0,1} \leq C\int_0^{d^2} \frac{t^{s/2-1}e^{-t}e^{-\frac{d^2}{ct}}}{\mu\left(B\left(x,\sqrt{t}\right)\right)}\ \mathrm{d}t,$$

$$
\begin{aligned}
&\leq AC\frac{d^{\eta}}{\mu\left(B\left(x,d\right)\right)}\int_{0}^{d^2} t^{s/2-\eta/2-1}e^{-t}e^{-\frac{d^2}{ct}}\ \mathrm{d}t,\\
&\leq C\frac{d^{s}}{\mu\left(B\left(x,d\right)\right)}\int_{1}^{\infty} u^{\eta/2-s/2-1}e^{-d^2/cu}e^{-u}\ \mathrm{d}u, \quad \text{change of variable } t=d^2/cu,\\
&\leq C\frac{d^{s}}{\mu\left(B\left(x,d\right)\right)}\int_{1}^{\infty} u^{\eta/2-s/2-1}e^{-d^2/cu-u/2}e^{-u/2}\ \mathrm{d}u
\end{aligned}
$$

We use that $e^{-d^2/cu-u/2}\leq Ce^{-\gamma_1 d}$ for some constant γ_1 which depends on c. Then, $\int_1^{\infty} u^{\eta/2-s/2-1}e^{-u/2}\ \mathrm{d}u$ converges to a constant and :

$$
J_{0,1}\leq C\frac{d^{s}}{\mu\left(B\left(x,d\right)\right)}e^{-\gamma_1 d}
$$

Thus, for a constant C depending only on s,c and the doubling and reverse doubling constants, we have :

$$
J_0\chi_{\{d(x,y)\leq 1\}}\leq C\frac{d^{s}}{\mu\left(B\left(x,d\right)\right)}e^{-\gamma_1 d}\chi_{\{d(x,y)\leq 1\}}.
$$

If $d(x,y)>1$, then we have :

$$
\begin{aligned}
J_0 &\leq C\int_0^1 \frac{t^{s/2-1}e^{-t}e^{-\frac{d^2}{ct}}}{\mu\left(B\left(x,\sqrt{t}\right)\right)}\ \mathrm{d}t,\\
&\leq AC\frac{1}{\mu\left(B\left(x,1\right)\right)}\int_0^1 t^{s/2-\eta/2-1}e^{-t}e^{-\frac{d^2}{ct}}\ \mathrm{d}t, \text{ since } d>1 \text{ we have :}\\
&\leq C\frac{1}{\mu\left(B\left(x,1\right)\right)}e^{-d^2/2c}\int_0^1 t^{s/2-\eta/2-1}e^{-\frac{1}{2ct}}\ \mathrm{d}t,\\
&\leq C\frac{1}{\mu\left(B\left(x,1\right)\right)}e^{-\gamma_2 d}.
\end{aligned}
$$

Since the integral converge and is a constant depending on only s,η,c, and using that $e^{-ax^2}\leq Ce^{-ax}$

Then for $\gamma=\min(\gamma_1,\gamma_2)$ and a constant C which depends only on s,c and the doubling and reverse doubling constants, we have :

$$
J_0\leq c\left(\frac{d^s}{\mu\left(B\left(x,d\right)\right)}\chi_{\{d\leq 1\}}+\frac{1}{\mu\left(B\left(x,1\right)\right)}\chi_{\{d>1\}}\right)e^{-\gamma d}
$$

□

Lemma 3.2. *There is some $\hat{c}$ such that if $R\geq\hat{c}$, there is some constant $\gamma>0$, such that for any s we have :*

$$
\int_1^{\infty} t^{s/2-1}e^{-t}p_t(x,y)\ \mathrm{d}t\leq C\left(\frac{d^s}{\mu\left(B\left(x,d\right)\right)}\chi_{\{d\leq 1\}}+\frac{1}{\mu\left(B\left(x,1\right)\right)}\chi_{\{d>1\}}\right)e^{-\gamma d} \tag{3.16}
$$

Proof. From Proposition 3.2, it follows that, since $R\geq\hat{c}$, then :

$$
J_{\infty}\leq\frac{C}{\mu\left(B\left(x,1\right)\right)}\int_1^{\infty} t^{s/2-1}e^{-\frac{d^2}{ct}}e^{-\gamma_0 t}\ \mathrm{d}t
$$

For any $\alpha \in (0,1)$ $e^{-\alpha t - \frac{d^2}{ct}}$ admits a maximum when $\alpha t = \frac{d^2}{ct}$, and so is less than $e^{-2\sqrt{\frac{\alpha}{c}}d}$. Take $\gamma_1 \in (0,1)$ such that : $\alpha = \frac{c}{4}\gamma_1^2 < \gamma_0$, then we have :

$$J_\infty \leq C\left(\int_1^\infty t^{s/2-1}e^{(\alpha-\gamma_0)t}\,\mathrm{d}t\right)\frac{1}{\mu\left(B\left(x,1\right)\right)}e^{-\gamma_1 d}$$

Thus there is $\gamma \in (0,1)$ depending on γ_0, c, and a constant C depending only on s, c, γ_0 and the doubling constants such that

$$J_\infty \leq C\frac{1}{\mu\left(B\left(x,1\right)\right)}e^{-\gamma d}$$

But we also have, when $d \leq 1$, we have: $\frac{1}{\mu(B(x,1))} \leq a^{-1}\frac{d^\nu}{\mu(B(x,d))} \leq a^{-1}\frac{d^s}{\mu(B(x,d))}$ using $(\mathbf{RD})_R^\nu$ and $s < \nu$. Hence, for $d(x,y) \leq 1$ we have :

$$J_\infty \leq C\frac{d^s}{\mu\left(B\left(x,d\right)\right)}e^{-\gamma_1 d}.$$

Thus there is a constant C such that :

$$J_\infty \leq C\left(\frac{d^s}{\mu\left(B\left(x,d\right)\right)}\chi_{\{d\leq 1\}} + \frac{1}{\mu\left(B\left(x,1\right)\right)}\chi_{\{d>1\}}\right)e^{-\gamma d}.$$

□

And so there is $c_0 > 0$ which depends on s, γ_0, c and the doubling and reverse doubling constants, and $\gamma \in (0,1)$ depending on c and γ_0, such that :

$$g_{s,1}(x,y) \leq c_0\left(\frac{d(x,y)^s}{\mu\left(B\left(x,d(x,y)\right)\right)}\chi_{\{d(x,y)\leq 1\}} + \frac{1}{\mu\left(B\left(x,1\right)\right)}\left(\chi_{\{d(x,y)>1\}}\right)\right)e^{-\gamma d} \tag{3.17}$$

□

4 Proof of the main results

Let (M, g, μ) be a weighted Riemannian manifold. Let $V \in L^1_{loc}(M, \mathrm{d}\mu)$, $V \geq 0$, for any $R > 0$ and $p \geq 1$, we define $N_p(V)$ and $N_{p,R}(V)$ as in (1.7) and (1.8). Notice that $N_p(V) = M_{2p}(V^p)^{1/p}$.

Though we can deduce theorem 1.1 as a special case of 1.2, we start by giving a separate, simpler proof of it. The general idea behind the proof of both theorems remains the same, but in the case of theorem 1.2, much more care will be required in establishing the bounds on the norm of certain operators.

4.1 Proof of Theorem 1.1

We first make the technical hypothesis that μ satisfy the reverse doubling property $(\mathbf{RD})^\nu$, with a reverse doubling order $\nu > 1$.

We assume that M admits $(\mathbf{RFK})^\eta$ and $(\mathbf{RD})^\nu$. Let $\psi \in \mathcal{C}_0^\infty(M)$, define $\varphi = \Delta^{1/2}\psi$, or $\psi = \Delta^{-1/2}\varphi$. We have, using that $\Delta^{-1/2}\left(V^{1/2}\cdot\right)$ is the adjoint of $V^{1/2}\Delta^{-1/2}$:

$$\langle V\psi, \psi\rangle = \left\|V^{1/2}\Delta^{-1/2}\varphi\right\|^2$$

$$
\begin{aligned}
&\leq \left\| V^{1/2}\Delta^{-1/2} \right\|^2_{L^2\to L^2} \|\varphi\|^2 \\
&\leq \left\| \Delta^{-1/2}\left(V^{1/2}\cdot\right) \right\|^2_{L^2\to L^2} \left\|\Delta^{1/2}\psi\right\|^2 \\
&\leq \left\| \Delta^{-1/2}\left(V^{1/2}\cdot\right) \right\|^2_{L^2\to L^2} \|\nabla\psi\|^2
\end{aligned}
$$

But, by proposition 3.3 and theorem 2.3, we have that $\|\Delta^{-1/2}f\|_2 \leq C\|M_1 f\|_2$. Moreover for $q = 2p$, we have :

(something wrong here with N_p)

$$
\begin{aligned}
M_1\left(V^{1/2}f\right)(x) &\leq M_q\left(V^{q/2}\right)(x)^{1/q} M_0(|f|^{q'})(x)^{1/q'} \\
&\leq N_p(V)^{1/2} M_0(|f|^{q'})(x)^{1/q'},
\end{aligned}
$$

using that $N_p(V) = M_{2p}\left(V^p\right)^{1/p}$. Then, using the fact that for any $r > 1$, M_0 is bounded on L^r we have :

$$
\begin{aligned}
\left\| M_1\left(V^{1/2}f\right)\right\|_2 &\leq N_p(V)^{1/2}\left\| M_0(|f|^{q'})\right\|^{1/q'}_{2/q'} \\
&\leq CN_p(V)^{1/2}\left\||f|^{q'}\right\|^{1/q'}_{2/q'} \\
&\leq CN_p(V)^{1/2}\|f\|_2
\end{aligned}
$$

Thus we can estimate the operator norm of $\Delta^{-1/2}\left(V^{1/2}\cdot\right)$, and we get :

$$
\int_M V\psi^2 \,\mathrm{d}\mu \leq CN_p(V)\|\nabla\psi\|^2 \tag{4.1}
$$

4.2 Proof of Theorem 1.2

We first prove the following weaker version of theorem 1.2. The more general result will follows by removing the technical hypothesis of $(\mathbf{RD})^\nu_R$, $\nu > 1$.

Theorem 4.1. *Let (M, g, μ) be a weighted Riemannian manifold, satisfying $(\mathbf{RFK})^\eta_R$ for some $R > 0$, and $(\mathbf{RD})^\nu_R$ for some $\nu > 1$.*

Then for any $p > 1$, there are positive constants $\hat{c}, C_p$, with $\hat{c} > 1$, depending only on the Faber-Krahn and doubling constants (and, for C_p, on p) such that for any $\lambda > \hat{c}R^{-1}$, for any $V \in L^1_{loc}(M, \mathrm{d}\mu)$, $V \geq 0$ and $\psi \in \mathcal{C}^\infty_0(M)$:

$$
\int_M V\psi^2 \,\mathrm{d}\mu \leq C_p N_{p,\lambda^{-1}}(V)\left(\int_M |\nabla\psi|^2 \,\mathrm{d}\mu + \lambda^2\int_M \psi^2 \,\mathrm{d}\mu\right) \tag{4.2}
$$

4.2.1 Proof of Theorem 4.1

Let (M, g, μ) be a weighted manifold satisfying $(\mathbf{RFK})^\eta_R$ and $(\mathbf{RD})^\nu_R$, with $R > 0$ and $\nu > 1$. For $\hat{c}$ the constant given in proposition 3.2, take $\lambda > 2\frac{\hat{c}}{R}$. For $s > 1$, $\delta > 0$ we recall that $M_{s,\delta}$ is the maximal function defined by :

$$M_{s,\delta}f(x) = \sup_{r<\delta} r^s \fint_{B(x,r)} f \,\mathrm{d}\mu \tag{4.3}$$

For a given $\lambda > 0$, and $p \geq 1$ we will note $K_p = N_{p,\lambda^{-1}}(V) = \sup_x M_{2p,\lambda^{-1}}\left(V^p\right)(x)^{1/p}$.

If K_p is infinite, then the previous inequality is obviously true. Then, if we suppose that have $K_p < \infty$, we have :

Lemma 4.1. *Let (M, g, μ) be a weighted manifold. Then for $V \geq 0$ locally integrable, $\lambda \geq 0$ we have :*

$$\langle V\psi, \psi\rangle \leq \left\|G_{1,\lambda}\left(V^{1/2}\cdot\right)\right\|^2_{L^2\to L^2}\left(\|\nabla\psi\|^2 + \lambda^2\|\psi\|^2\right) \tag{4.4}$$

Proof. Let $\psi \in \mathcal{C}_0^\infty(M)$, and define $\varphi = \left(\Delta + \lambda^2\right)^{1/2}\psi$. Then $\psi = G_{1,\lambda}\varphi$ and, using the fact that $G_{1,\lambda}\left(V^{1/2}\cdot\right)$ is the adjoint of $V^{1/2}G_{1,\lambda}$:

$$\begin{aligned}
\langle V\psi,\, \psi\rangle &= \left\langle V^{1/2}G_{1,\lambda}\varphi,\, V^{1/2}G_{1,\lambda}\varphi\right\rangle \\
&= \left\|V^{1/2}G_{1,\lambda}\varphi\right\|_2^2 \\
&\leq \left\|V^{1/2}G_{1,\lambda}\right\|^2_{L^2\to L^2}\|\varphi\|_2^2 \\
&\leq \left\|G_{1,\lambda}\left(V^{1/2}\cdot\right)\right\|^2_{L^2\to L^2}\left\|\left(\Delta+\lambda^2\right)^{1/2}\psi\right\|_2^2 \\
&\leq \left\|G_{1,\lambda}\left(V^{1/2}\cdot\right)\right\|^2_{L^2\to L^2}\left(\|\nabla\psi\|_2^2 + \lambda^2\|\psi\|_2^2\right),
\end{aligned}$$

which is what we wanted to show. □

Now, since M satisfy $(\mathbf{RFK})^\eta_R$ and $(\mathbf{RD})^\nu_R$, with $\nu > 1$, we can apply the proposition 3.4, thus for $R\lambda > \hat{c}$, we have $G_{1,\lambda} \leq c_0(T_1 + T_2)$, with

$$\begin{aligned}
T_1 f(x) &= \int_{\lambda d(x,y)\leq 1} \frac{d(x,y)^s}{\mu\left(B\left(x, d(x,y)\right)\right)} e^{-\gamma\lambda d(x,y)} f(y)\,\mathrm{d}\mu(y) \\
T_2 f(x) &= \frac{\lambda^{-s}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda d(x,y)>1} e^{-\gamma\lambda d(x,y)} f(y)\,\mathrm{d}\mu(y)
\end{aligned} \tag{4.5}$$

Thus, we have $\left\|G_{1,\lambda}\left(V^{1/2}\cdot\right)\right\|_2 \leq c_0\left(\left\|T_1\left(V^{1/2}\cdot\right)\right\|_2 + \left\|T_2\left(V^{1/2}\cdot\right)\right\|_2\right)$. Then all we need to do is to evaluate those two operator norms.

Lemma 4.2. *Let (M, g, μ) be a weighted Riemannian manifold, let $\lambda > 0$. Assume $(\mathbf{D})^\eta_R$ and $(\mathbf{RD})^\nu_R$, for $R \geq \lambda^{-1}$. Then for T_1 defined as in (4.5), and $V \geq 0$ locally integrable, there is some constant $C_{1,p}$ which depends only on p, γ and the reverse doubling and doubling constants, such that :*

$$\left\|T_1\left(V^{1/2}\cdot\right)\right\|_2 \leq C_{1,p}K_p^{1/2} \tag{4.6}$$

Proof. We can apply corollary 2.1 : for any $p \geq 1$, and any locally integrable f, we have $\|T_1 f\|_p \leq c_p\|M_{1,\lambda^{-1}}f\|_p$. Then, for any $\psi \in \mathcal{C}_0^\infty(M)$, for $q = 2p$, $q' = q/(q-1)$:

$$\begin{aligned}
M_{1,\lambda}\left(V^{1/2}\psi\right)(x) &\leq \left(M_{2p,\lambda^{-1}}\left(V^{p}\right)(x)\right)^{1/2p}\left(M_{0,\lambda^{-1}}\left(\psi^{q'}\right)(x)\right)^{1/q'}\\
&\leq K_p^{1/2} M_{0,\lambda^{-1}}\left(\psi^{q'}\right)(x)^{1/q'}\\
\left\|T_1\left(V^{1/2}\psi\right)\right\|_2 &\leq c_p K_p^{1/2}\left\|M_{0,\lambda^{-1}}\left(\psi^{q'}\right)\right\|_{2/q'}^{1/q'}\\
&\leq c_p \tilde{c}_{2/q'} K_p^{1/2}\|\psi\|_2
\end{aligned}$$

With $\|M_{0,\lambda^{-1}}f\|_r \leq \tilde{c}_r\|f\|_r$ for any $f\in L^r$, $r\in(1,\infty]$. Thus :

$$\left\|T_1\left(V^{1/2}\cdot\right)\right\|_{L^2\to L^2} \leq C_{0,p}K_p^{1/2} \tag{4.7}$$

□

Lemma 4.3. *Let (M,g,μ) be a weighted Riemannian manifold, let $\lambda>0$. Assume $(\mathbf{D})_R^{\eta}$, $(\mathbf{RD})_R^{\nu}$ for $R\geq\lambda^{-1}$. Then for T_2 defined as in (4.5), and $V\geq 0$ locally integrable, there is some constant $C_{2,p}$ which depends only on p, γ and the doubling and reverse doubling constants, such that :*

$$\left\|T_2\left(V^{1/2}\cdot\right)\right\|_2 \leq C_{2,p}K_p^{1/2} \tag{4.8}$$

Proof. We majorate $T_2(V^{1/2})$ by an operator for which we can use the Schur Test. We have :

$$\begin{aligned}
T_2 f(x) &= \frac{\lambda^{-1}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda d>1} e^{-\gamma\lambda d(x,y)} f(y)\,\mathrm{d}\mu(y)\\
&= \gamma\frac{1}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}}^{\infty} e^{-\gamma\lambda r}\int_{\lambda^{-1}<d<r} f(y)\,\mathrm{d}\mu(y)\,\mathrm{d}r
\end{aligned}$$

Then, for $\psi\in\mathcal{C}_0^{\infty}(M)$, $q=2p$, $q'=q/(q-1)$, by Hölder's inequality :

$$T_2\left(V^{1/2}\psi\right)(x) \leq \frac{\gamma}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}}^{\infty} e^{-\gamma\lambda r}\left(\int_{\lambda^{-1}<d<r} V^p\,\mathrm{d}\mu\right)^{1/2p}\left(\int_{\lambda^{-1}<d<r}\psi^{q'}\,\mathrm{d}\mu\right)^{1/q'}\mathrm{d}r \tag{4.9}$$

Then we cover the annulus $B(x,r)\setminus B(x,\lambda^{-1})$ by balls $B_i=B(x_i,\lambda^{-1})$, $x_i\in B(x,r)$, such that for $i\neq j$, $\frac{1}{2}B_i\cap\frac{1}{2}B_j=\emptyset$. We have :

$$\begin{aligned}
\int_{\lambda^{-1}<d<r} V^p\,\mathrm{d}\mu &\leq \sum_i\int_{B_i} V^p\,\mathrm{d}\mu \leq \sum_i \lambda^{2p}K_p^p\mu(B_i)\\
&\leq A^2\lambda^{2p}K_p^p\sum_i\mu\left(\frac{1}{2}B_i\right)\\
&\leq C\lambda^{2p}K_p^p\mu\left(B\left(x,r+\frac{1}{2\lambda}\right)\right)
\end{aligned}$$

Then :

$$T_2\left(V^{1/2}\psi\right)(x) \leq \gamma K_p^{1/2}\lambda \int_{\lambda^{-1}}^{\infty} e^{-\gamma\lambda r}\frac{\mu\left(B\left(x, r+\frac{1}{2\lambda}\right)\right)^{1/2p}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\left(\int_{\lambda^{-1}<d<r}\psi^{q'}\,\mathrm{d}\mu\right)^{1/q'}\mathrm{d}r \tag{4.10}$$

Since the measure is R-doubling, with $R > 2\hat{c}\lambda^{-1}$, it is also R'-doubling for all $R' \leq R$ and with the same constants. Then for $1 < \rho < 2$, let $R' = \rho\hat{c}\lambda^{-1}$, we have $\lambda^{-1} < R' \leq R$, then by the propositions 2.4 and 2.5 :

$$\begin{aligned}
\frac{\mu\left(B\left(x, r+\frac{1}{2\lambda}\right)\right)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)} &\leq C\frac{\mu\left(B\left(x,r\right)\right)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\\
&\leq C\frac{\mu\left(B\left(x,r\right)\right)}{\mu\left(B\left(x,R'\right)\right)}\frac{\mu\left(B\left(x,R'\right)\right)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\\
&\leq Ce^{D\frac{r}{R'}}
\end{aligned}$$

With C depending only on $\rho, \hat{c}$ and the doubling constant. Thus :

$$T_2\left(V^{1/2}\psi\right)(x) \leq CK_p^{1/2}\lambda \int_{\lambda^{-1}}^{\infty} e^{\left(\left(\frac{D}{2pR'}-\gamma\lambda\right)r\right)}\left(\frac{1}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}<d<r}\psi^{q'}\,\mathrm{d}\mu\right)^{1/q'}\mathrm{d}r \tag{4.11}$$

And the constant C depends on p, b, η and on the chosen arbitrary parameters.

Finally for $\rho = \frac{D}{(1-\theta)2p\gamma\hat{c}}$ with $\theta \in (0,1)$ we get $\left(\frac{D}{2pR'} - \gamma\lambda\right) = \left(\frac{D}{2p\rho\hat{c}} - \gamma\right)\lambda = -\theta\gamma\lambda$, thus we have :

$$T_2\left(V^{1/2}\psi\right)(x) \leq CK_p^{1/2}\lambda \int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}\left(\frac{1}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}<d<r}\psi^{q'}\,\mathrm{d}\mu\right)^{1/q'}\mathrm{d}r \tag{4.12}$$

Note that we can indeed suppose $\rho = \frac{D}{(1-\theta)2p\gamma\hat{c}}$: by the proof of proposition 3.2, we have $\frac{D}{\hat{c}} = 4\sqrt{\frac{(1-\gamma)(\kappa-1)}{c}}$, with $1 < \kappa < \frac{1}{4}c$, and so $\rho = \frac{\sqrt{(1-\gamma)(\kappa-1)}}{\gamma}\frac{2}{(1-\theta)p\sqrt{c}}$. Since the choice of $c > 4$ in the estimate on the heat kernel is arbitrary, and since γ can always be taken arbitrarily small, we can choose them so that $\frac{D}{(1-\theta)2p\gamma\hat{c}}$ is equal to the chosen ρ.

Then we have by Hölder's inequality :

$$\begin{aligned}
T_2\left(V^{1/2}\psi\right) &\leq CK_p^{1/2}\lambda\left(\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}\,\mathrm{d}r\right)^{1/q}\left(\int_{\lambda^{-1}}^{\infty}\frac{e^{-\theta\gamma\lambda r}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}<d<r}\psi^{q'}\,\mathrm{d}\mu\,\mathrm{d}r\right)^{1/q'}\\
&\leq CK_p^{1/2}\lambda\left(\frac{1}{\theta\gamma\lambda}e^{-\theta\gamma}\right)^{1/q}\left(\frac{1}{\theta\gamma\lambda}\int_{\lambda d>1} e^{-\theta\gamma\lambda d(x,y)}\frac{\psi^{q'}(y)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}\mu(y)\right)^{1/q'}\\
&\leq \frac{CK_p^{1/2}}{\theta\gamma}e^{-\theta\gamma/q}\left(\int_{\lambda d>1} e^{-\theta\gamma\lambda d(x,y)}\frac{\psi^{q'}(y)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}\mu(y)\right)^{1/q'}
\end{aligned}$$

We will now show that there is a $\theta \in (0,1)$ such that the operator S defined by :

$$S\psi = \int_{\lambda d>1} e^{-\theta\gamma\lambda d(x,y)}\frac{\psi(y)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}\mu(y), \tag{4.13}$$

is bounded on every L^p for $p \in [1,\infty]$. We use the Schur test : S is given by the kernel $K(x,y) = \frac{e^{-\theta\gamma\lambda d(x,y)}}{\mu(B(x,\lambda^{-1}))}\chi_{\{\lambda d(x,y)>1\}}$, then if for some constant $L > 0$, $\int_M K(x,y)\,\mathrm{d}\mu(y) < L$ for almost every $x \in M$, and $\int_M K(x,y)\,\mathrm{d}\mu(x) < L$ for almost every $y \in M$, S is bounded on all L^p, $1 \leq p \leq +\infty$, with all the operator norms being less than L. We have :

$$\begin{aligned}
\int_{d\lambda>1} \frac{e^{-\theta\gamma\lambda d(x,y)}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}\mu(y) &= \frac{1}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{d\lambda>1} e^{-\theta\gamma\lambda d(x,y)}\,\mathrm{d}\mu(y) \\
&\leq \frac{1}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\theta\gamma\lambda\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}\mu\left(B\left(x,r\right)\right)\mathrm{d}\mu(r) \\
&\leq \theta\gamma\lambda\frac{\mu\left(B\left(x,R'\right)\right)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}\frac{\mu\left(B\left(x,r\right)\right)}{\mu\left(B\left(x,R'\right)\right)}\,\mathrm{d}r \\
&\leq C\lambda A\left(\rho\hat{c}\right)^{\eta}\int_{\lambda^{-1}}^{\infty} e^{\left(\frac{D}{R'}-\theta\gamma\lambda\right)r}\,\mathrm{d}r \\
&\leq C\lambda\int_{\lambda^{-1}}^{\infty} e^{(2(1-\theta)p\gamma-\theta\gamma)\lambda r}\,\mathrm{d}r \\
&\leq \tilde{c}_1,
\end{aligned}$$

this for any θ such that $2(1-\theta)p\gamma - \theta\gamma \leq -\frac{\gamma}{2}$, and the constant $\tilde{c}_1$ depends on $\theta,\gamma,b,\eta,\rho$, but not on λ or R. We also have

$$\begin{aligned}
\int_{d\lambda>1} \frac{e^{-\theta\gamma\lambda d(x,y)}}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}\mu(x) &= \theta\gamma\lambda\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}\int_{B(y,r)}\frac{\mathrm{d}\mu(x)}{\mu\left(B\left(x,\lambda^{-1}\right)\right)}\,\mathrm{d}r \\
&\leq C\lambda\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}Ce^{D\frac{r}{R'}}\frac{\mu\left(B\left(y,r\right)\right)}{\mu\left(B\left(y,\lambda^{-1}\right)\right)}\,\mathrm{d}r \\
&\leq C\lambda\int_{\lambda^{-1}}^{\infty} e^{-\theta\gamma\lambda r}e^{2D\frac{r}{R'}}\,\mathrm{d}r \\
&\leq C\lambda\int_{\lambda^{-1}}^{\infty} e^{(4(1-\theta)p\gamma-\theta\gamma)\lambda r}\,\mathrm{d}r \\
&\leq \tilde{c}_2,
\end{aligned}$$

where we take θ to be such that $4(1-\theta)p\gamma - \theta\gamma = -\frac{\gamma}{2}$, i.e. $\theta = \frac{\frac{1}{2}+4p}{1+4p} \in (0,1)$. And $\tilde{c}_2$ does not depend on λ, R. Then we also have $2(1-\theta)p\gamma - \theta\gamma \leq -\frac{\gamma}{2}$. Thus by the Schur test, S is bounded on L^p for all $1 \leq p \leq \infty$ with an operator norm that does not depend on λ, R.

Since $T_2\left(V^{1/2}\psi\right) \leq C\left(S\left(\psi^{q'}\right)\right)^{1/q'}$, then :

$$\begin{aligned}
\left\|T_2\left(V^{1/2}\psi\right)\right\|_2^2 &\leq C\left\|S\left(\psi^{q'}\right)\right\|_{2/q'}^{1/q'} \\
&\leq C\left\|\psi\right\|_2^2
\end{aligned}$$

Then we can conclude that there is some constant $C_{1,p}$, which depends only on p,b,η and the γ, c that we chose in the estimation of the heat kernel, such that :

$$\left\|T_2\left(V^{1/2}\cdot\right)\right\|_{L^2\to L^2} \leq C_{1,p}K_p^{1/2} \tag{4.14}$$

□

And so, applying all three lemmas, we have $\left\|G_{1,\lambda}\left(V^{1/2}\cdot\right)\right\|^2_{L^2\to L^2} \leq \left(C_{1,p}+C_{2,p}\right)^2 K_p$.

Thus we have $\hat{c}$ and C_p constants depending only on the doubling constants (and for C_p, on p), such that for $R\lambda > \hat{c}$, $V \geq 0$ locally integrable,

$$\int_M V\psi^2 \, \mathrm{d}\mu \leq C_p N_{p,\lambda^{-1}}(V)\left(\int_M |\nabla\psi|^2 \, \mathrm{d}\mu + \lambda^2 \int_M \psi^2 \, \mathrm{d}\mu\right)$$

We will now prove the theorems 1.2, 1.3 and 1.4 when $(\mathbf{RD})^\nu_R$, $\nu > 1$ holds.

4.2.2 Proof of theorem 1.2

Since theorem 4.1 holds only for $\lambda > \hat{c}R^{-1}$, $\hat{c} > 1$, we immediately get (1.11) only for $R' \leq \frac{1}{\hat{c}}R$. We need just a bit more work to get it for R.

Proof of Theorem 1.2. We have, by theorem 4.1, for any $p > 1$, and $\lambda > 0$ such that $\lambda R > \hat{c}$:

$$\langle V\psi, \psi\rangle \leq C_p N_{p,\lambda^{-1}}(V)\left(\|\nabla\psi\|^2 + \lambda^2\|\psi\|^2\right).$$

In fact, this inequality also holds for $\lambda = \frac{\hat{c}}{R}$. We apply the previous inequality with this lambda, and use that the function $r \mapsto N_{p,r}(V)$ is non-decreasing, and $\lambda^{-1} < R$. Thus :

$$\langle V\psi, \psi\rangle \leq C_p N_{p,R}(V)\left(\|\nabla\psi\|^2 + \frac{\hat{c}^2}{R^2}\|\psi\|^2\right) \leq C_p\hat{c}^2 N_{p,R}(V)\left(\|\nabla\psi\|^2 + \frac{1}{R^2}\|\psi\|^2\right) \tag{4.15}$$

□

4.2.3 Proof of theorem 1.4

We now suppose that $\lambda_1(M) > 0$. Then the previous results can be strenghtened to prove theorem 1.4.

Proof. We apply theorem 1.2, and use that $\lambda_1(M)\int_M \psi^2 \, \mathrm{d}\mu \leq \int_M |\nabla\psi|^2 \, \mathrm{d}\mu$. Then :

$$\langle V\psi, \psi\rangle \leq C_p N_{p,R}(V)\left(1 + \frac{1}{\lambda_1(M)R^2}\right)\int_M |\nabla\psi|^2 \, \mathrm{d}\mu$$

Then :

$$\frac{\lambda_1(M)R^2}{C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}\int_M V\psi^2 \, \mathrm{d}\mu \leq \int_M |\nabla\psi|^2 \, \mathrm{d}\mu$$

And :

$$\frac{\lambda_1(M)R^2}{2C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}\int_M V\psi^2 \, \mathrm{d}\mu + \frac{\lambda_1(M)}{2}\int_M \psi^2 \, \mathrm{d}\mu \leq \int_M |\nabla\psi|^2 \, \mathrm{d}\mu$$

Then, for any V, we have :

$$\langle V\psi, \psi\rangle \leq \frac{C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}{\lambda_1(M)R^2}\left(\|\nabla\psi\|^2 - \frac{\lambda_1(M)}{2}\|\psi\|^2\right), \tag{4.16}$$

which is (1.13). □

4.3 Proof of theorem 1.3

Let C_p be the constant of theorem 1.2. We let

$$L = \sup_{x,\delta} \left(2C_p \left(\fint_{B(x,\delta)} V^p \, \mathrm{d}\mu \right)^{1/p} - \delta^{-2} \right) \tag{4.17}$$

Then we have :

$$\begin{aligned} \left(\fint_{B(x,\delta)} V^p \, \mathrm{d}\mu \right)^{1/p} &\leq \frac{L+\delta^{-2}}{2C_p}, \\ (M_{2p,\delta}(V^p)(x))^{1/p} &\leq \frac{\delta^2 L + 1}{2C_p}. \end{aligned}$$

Take $\delta = L^{-1/2}$, then $N_{p,\delta}(V) \leq \frac{1}{C_p}$. Then by theorem 1.2 we have :

$$\langle V\psi, \psi \rangle - \|\nabla \psi\|_2^2 \leq L \|\psi\|^2, \tag{4.18}$$

thus

$$-\lambda_1(\Delta - V) \leq \sup_{x,\delta} \left(2C_p \left(\fint_{B(x,\delta)} V^p \, \mathrm{d}\mu \right)^{1/p} - \delta^{-2} \right). \tag{4.19}$$

Meanwhile, let $r < \lambda^{-1} \leq R$, and define $f_r : [0,\infty) \to [0,+\infty)$ by $f(t) = r$ if $t \leq r$, $f(t) = 2r - t$ if $t \in (r, 2r]$ and $f_r(t) = 0$ if $t > 2r$. Then for $o \in M$, $\psi = f_r(d(o,x))$. ψ is a Lipschitz function with compact support, and we have, by $(\mathbf{D})_R^\eta$:

$$\begin{aligned} \lambda_1(\Delta - V) &\leq \frac{\|\nabla\psi\|^2 - \int_M V\psi^2 \, \mathrm{d}\mu}{\|\psi\|^2} \\ &\leq \frac{\mu\left(B\left(x,2r\right)\right)}{r^2 \mu\left(B\left(x,r\right)\right)} - \fint_{B(x,r)} V \, \mathrm{d}\mu \\ &\leq Ar^{-2} - \fint_{B(x,r)} V \, \mathrm{d}\mu \\ &\leq (r/\sqrt{A})^{-2} - A^{-1-\eta/2} \fint_{B(x,r/\sqrt{A})} V \, \mathrm{d}\mu, \end{aligned}$$

this for all $r > 0$. Thus :

$$-\lambda_1(\Delta - V) \geq \sup_{x,\delta} \left(A^{-1-\eta/2} \fint_{B(x,\delta)} V \, \mathrm{d}\mu - \delta^{-2} \right). \tag{4.20}$$

4.4 Removing the dependancy on reverse doubling

Let M be a manifold that satisfy $(\mathbf{RFK})^\eta$. We consider $\tilde{M} = \mathbf{R} \times M$, $(\tilde{M}, \tilde{g}, \tilde{\mu})$ the product Riemannian manifold : $\tilde{g} = \mathrm{d}x^2 + g$, $\mathrm{d}\tilde{\mu} = \mathrm{d}x \, \mathrm{d}\mu$. For $V \in L^1_{loc}(M)$ we define $\tilde{V}(x,m) = V(m)$. We write $\tilde{\Delta}$ for the laplacian on $(\tilde{M}, \tilde{g}, \tilde{\mu})$, and Δ for the laplacian on (M, g, μ). The Morrey norm in $\tilde{M}$ is written $\tilde{N}_{p,R}$.

We have :

Proposition 4.1. *$(\tilde{M},\tilde{g},\tilde{\mu})$ satisfies the following properties :*

1. *If μ is R-doubling, then $\tilde{\mu}$ is R-doubling, and R-reverse doubling with order $\nu > 1$.*
2. *The heat kernel of $\tilde{M}$ is $\tilde{p}_t((x,m),(y,n)) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} p_t(m,n)$.*
3. *If M satisfies $(\mathbf{RFK})^\eta_R$, then there is some $\theta \in (0,1)$ such that $\tilde{M}$ satisfies $(\mathbf{RFK})^\eta_{\theta R}$. θ depends only on the Faber Krahn constants.*
4. *$\lambda_1(\tilde{\Delta} - \tilde{V}) = \lambda_1(\Delta - V)$*
5. *If μ is R-doubling, then there are two constants c, C which depends only on the doubling constant, such that $cN_{p,R}(V) \le \tilde{N}_{p,R}(\tilde{V}) \le CN_{p,R}(V)$*

Proof.
1. For $E \subset \mathbf{R}$ measurable, we denote $|E|$ the usual lebesgue measure of E. We have :

$$|(-r/2,r/2)|\mu(B(m,r/2)) \le \tilde{\mu}(\tilde{B}((x,m),r)) \le |(-r,r)|\mu(B(m,r)). \tag{4.21}$$

From this, with $r \le R$ we immediately get $\tilde{\mu}(\tilde{B}((x,m),2r)) \le 4A^2\tilde{\mu}(\tilde{B}((x,m),r))$, with A the R-doubling constant of μ. Moreover, since μ is R-doubling, it is R-reverse doubling, with reverse doubling order $\nu > 0$. Then, we have, for $r < r' < \theta R$:

$$\begin{aligned}
\frac{\tilde{\mu}(\tilde{B}((x,m),r'))}{\tilde{\mu}(\tilde{B}((x,m),r))} &\ge \frac{r'}{2r}\frac{\mu(B(m,r'/2))}{\mu(B(m,r))} \\
&\ge \frac{1}{2A}\frac{r'}{r}\frac{\mu\left(B\left(m,r'\right)\right)}{\mu\left(B\left(m,r\right)\right)} \\
&\ge \frac{a}{2A}\left(\frac{r'}{r}\right)^{1+\nu}
\end{aligned}$$

Thus $\tilde{\mu}$ is reverse doubling of order $\tilde{\nu} = 1+\nu > 1$.

2., 4. We have $\tilde{\Delta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Delta$. Thus $\tilde{p}_t((x,m),(y,n)) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} p_t(m,n)$, and the spectrum of $\tilde{\Delta} - \tilde{V}$ is :

$$Sp(\tilde{\Delta} - \tilde{V}) = \{\lambda + \lambda'; \quad \lambda \in Sp(\Delta - V), \lambda' \ge 0\}.$$

Thus the infimum of the spectrum of $\tilde{\Delta} - \tilde{V}$ is the infimum of the spectrum of $\Delta - V$.

3. We use proposition 3.1.

5. We use (4.21). Using that $\int_{\tilde{B}} \tilde{V}\ \mathrm{d}\tilde{\mu} \le 2r\int_B V\ \mathrm{d}\mu$, we have :

$$\frac{r^{2p}}{\tilde{\mu}(\tilde{B}((x,m),r)}\int_{\tilde{B}} \tilde{V}^p\ \mathrm{d}\tilde{\mu} \le \frac{r^{2p}}{(r/2)\mu\left(B\left(m,r/2\right)\right)}2r\int_B V^p\ \mathrm{d}\mu.$$

Then by R doubling $\tilde{N}_{p,R}(\tilde{V}) \le 4AN_{p,R}(V)$. The other inequality is obtained in a similar same way. □

Proof of theorem 1.2. From the points 1., 3. of the above proposition, if (M,g,μ) is a manifold that satisfy $(\mathbf{RFK})^\eta_R$, then there is some $\theta \in (0,1)$, depending only on the Faber Krahn constants, such that $(\tilde{M},\tilde{g},\tilde{\mu})$ satisfy $(\mathbf{RFK})^\eta_{\theta R}$ and $(\mathbf{RD})^\nu_R$ with $\nu > 1$. Then we can apply 1.2 to $\tilde{M}$: there is a constant $\tilde{C}_p$ such that if $\tilde{V}$ is such that $\tilde{C}_p\tilde{N}_{p,R}(\tilde{V}) \le 1$, then $\lambda_1(\tilde{\Delta} - \tilde{V}) \ge -\frac{1}{\theta^2 R^2}$.

Using 5., then there is a constant $C_p > 0$ such that $C_p N_{p,R}(V) \geq \tilde{C}_p \tilde{N}_{p,R}(\tilde{V})$. Then since $\lambda_1(\Delta - V) = \lambda_1(\tilde{\Delta} - \tilde{V})$, if $\mathbf{C}_p N_{p,R}(V) \leq 1$, then $\lambda_1(\Delta - V) \geq -\frac{1}{\theta^2 R^2}$. For an arbitrary $V \geq 0$, locally integrable, with $N_{p,R}(V) < +\infty$, we can apply the above to $V/C_p N_{p,R}(V)$, then for any $\psi \in \mathcal{C}_0^\infty(M)$:

$$\frac{1}{C_p N_{p,R}(V)} \int_M V\psi^2 \, \mathrm{d}\mu \leq \frac{1}{\theta^2} \int_M \left(|\nabla \psi|^2 + \frac{1}{R^2}\psi^2 \right) \mathrm{d}\mu, \tag{4.22}$$

which is (1.11). □

5 Hardy inequality

For some point $o \in M$, the L^2 Hardy inequality :

$$\forall \psi \in \mathcal{C}_0^\infty(M), \int_M \frac{\psi(x)^2}{d(o,x)^2} \, \mathrm{d}\mu(x) \leq C \int_M |\nabla \psi(x)|^2 \, \mathrm{d}\mu(x) \tag{5.1}$$

is equivalent to $\Delta - V \geq 0$, with $V(x) = \frac{1}{C} d(o,x)^{-2}$. Moreover, we have :

Proposition 5.1. *Let (M, g, μ) be a weighted Riemannian manifold, $R \in (0, \infty]$. If μ satisfy $(\mathbf{D})_R^\eta$ and $(\mathbf{RD})_R^\nu$ with $\nu > 1$, then for any $p \in (1, \nu/2)$, there is a constant $K_p < \infty$ such that for all $r < R$ we have :*

$$r^2 \left(\fint_{B(x,r)} d(o,y)^{-2p} \, \mathrm{d}\mu \right)^{1/p} \leq K_p. \tag{5.2}$$

Proof. We let $\rho(y) = d(o,y)$, $B = B(x,r)$, for $r < R$.

If $r \leq \rho(x)/2$, then for $y \in B(x,r)$, $\rho(y) \geq \rho(x) - r \geq \rho(x)/2 \geq r$. Then :

$$\int_B \rho(y)^{-2p} \, \mathrm{d}\mu \leq r^{-2p} \mu(B).$$

If $r > \rho(x)/2$, then $B(x,r) \subset B(o,3r)$, and :

$$\begin{aligned}
\int_B \rho^{-2p} \, \mathrm{d}\mu \leq & \int_{B(o,3r)} \rho^{-2p} \, \mathrm{d}\mu \\
& \leq \int_0^\infty (2p-1) t^{-2p-1} \mu\left(B\left(o, \min(t,3r)\right)\right) \mathrm{d}t \\
& \leq \int_0^{3r} a^{-1}(2p-1) t^{\nu-2p-1} (3r)^{-\nu} \mu\left(B\left(o,3r\right)\right) \mathrm{d}t + r^{-2p} \mu\left(B\left(o,3r\right)\right) \\
& \leq \left(\frac{1}{3^{3p} a} \frac{2p-1}{\nu - 2p} + 1 \right) r^{-2p} \mu\left(B\left(o,3r\right)\right) \\
& \leq C_p r^{-2p} \mu\left(B\left(x,r\right)\right),
\end{aligned}$$

since $\nu > 2p$, with the constant C_p depending uniquely on p and the doubling and reverse doubling constants. □

Then applying theorems 1.2 and 1.1, we immediately obtain :

Corollary 5.1. *If (M,g,μ) satisfy $(\mathbf{RFK})_R^\eta$ and $(\mathbf{RD})_R^\nu$ with $\nu>2$, then there is a constant C such that for any $\psi \in \mathcal{C}_0^\infty(M)$, $o\in M$,*

$$\int_M \frac{\psi(x)^2}{d(o,x)^2}\,\mathrm{d}\mu(x) \le C\left(\|\nabla\psi\|_2^2 + \frac{1}{R^2}\|\psi\|_2^2\right). \tag{5.3}$$

Corollary 5.2. *If (M,g,μ) satisfy $(\mathbf{RFK})^\eta$, $(\mathbf{RD})^\nu$ with $\nu>2$ then there is a constant C such that :*

$$\forall \psi\in\mathcal{C}_0^\infty(M),\ \int_M \frac{\psi(x)^2}{d(o,x)^2}\,\mathrm{d}\mu(x) \le C\int_M |\nabla\psi|^2\,\mathrm{d}\mu. \tag{5.4}$$

The second corollary being theorem 1.5.

This time the condition on the reverse doubling order is not merely a technical hypothesis. It is, in fact, a necessary condition for the Hardy inequality to holds if we assume the measure μ to be doubling :

Proposition 5.2. *Let (M,g,μ) be a weighted Riemannian manifold, with μ a doubling measure, assume that there is a constant $\nu>2$ such that for any $o\in M$, $\psi\in\mathcal{C}_0^\infty(M)$, M admits the Hardy inequality :*

$$\left(\frac{\nu-2}{2}\right)^2\int_M \frac{\psi(x)^2}{d(o,x)^2}\,\mathrm{d}\mu(x) \le \int_M |\nabla\psi|^2\,\mathrm{d}\mu, \tag{5.5}$$

then μ satisfy $(\mathbf{RD})^\nu$.

Note that that we can always write a Hardy inequality (5.4) in the form (5.5) simply by chosing $\nu = 2+2\sqrt{1/C}$.

Using a method from [3, 16], we have :

Proof. Take $0<r<R$, define $f(t) = r^{-\frac{\nu-2}{2}}$ for $0\le t\le r$, $f(t)=t^{-\frac{\nu-2}{2}}$ for $r\le t\le R$, $f(t) = 2R^{-\frac{\nu-2}{2}} - R^{-\frac{\nu}{2}}t$ for $R\le t\le 2R$ and $f(t)=0$ for $t\ge 2R$.

When $r\le t\le R$, we have $f'(t)^2 = \left(\frac{\nu-2}{2}\right)^2\frac{f(t)^2}{t^2}$. Then for some point $o\in M$ choose $\phi(x) = f(d(o,x))$, the Hardy inequality applied to φ leads to :

$$\left(\frac{\nu-2}{2}\right)^2\int_{B(o,r)} \frac{\phi(x)^2}{d(o,x)^2}\,\mathrm{d}\mu(x) \le \int_{B(o,2R)\setminus B(o,R)} |\nabla\phi|^2\,\mathrm{d}\mu(x), \tag{5.6}$$

then :

$$\left(\frac{\nu-2}{2}\right)^2 r^{-\nu}\mu\left(B\left(o,r\right)\right) \le R^{-\nu}\mu(B(o,2R)\setminus B(o,R)) \le AR^{-\nu}\mu\left(B\left(o,R\right)\right), \tag{5.7}$$

using that μ is doubling. Thus there is some constant $a>0$ such that :

$$a\left(\frac{R}{r}\right)^\nu \le \frac{\mu\left(B\left(o,R\right)\right)}{\mu\left(B\left(o,r\right)\right)}, \tag{5.8}$$

and μ is reverse doubling of order $\nu>2$. □

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