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AUGMENTED LAGRANGIAN METHODS FOR DEGENERATE HAMILTON–JACOBI EQUATIONS

HAMZA ENNAJI[†], NOUREDDINE IGBIDA[†], AND VAN THANH NGUYEN[‡]

ABSTRACT. We suggest a new approach to solve a class of degenerate Hamilton–Jacobi equations without any assumptions on the emptiness of the Aubry set. It is based on the characterization of the maximal subsolution by means of the Fenchel–Rockafellar duality. This approach enables us to use augmented Lagrangian methods as alternatives to the commonly used methods for numerical approximation of the solution, based on finite difference approximation or on optimal control interpretation of the solution.

1. INTRODUCTION

Our main interest lies in the study from theoretical and numerical point of view the Hamilton–Jacobi (HJ) equation

$$H(x, \nabla u) = 0$$

in an open bounded domain $\Omega \subseteq \mathbb{R}^N$ coupled, for instance, with a Dirichlet boundary condition on $\partial\Omega$. Here, H is continuous in both arguments, coercive and convex in the second, but not enjoying any property of existence of a smooth strict subsolution. Our study here takes place in the framework of viscosity solution theory, so that the term (sub, super) solution must be understood in this sense.

Recall that the nonexistence of smooth strict subsolution is connected to the fact that the so-called Aubry set, noted by \mathcal{A} , is not empty. It is known that in this case the problem has many solutions. Nevertheless, one needs to select the pertinent one from the physical point of view as well as the optimal control one, such as, the maximal subsolution. Such kind of equations are relevant, for instance, in the study of Shape from Shading [16, 29, 31, 32, 36], granular material [24] and many other problems ranging from mechanics, geometry, seismic analysis, optics to image processing, etc.

Typical situation appears in the study of the so-called Eikonal equation subject to Dirichlet boundary condition, that is

$$\begin{cases} |\nabla u| = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

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where $f \geq 0$ is a continuous function. In this case the Aubry set corresponds to the zero set $[f = 0]$. In the case where this set is empty, i.e. $f > 0$ in $\overline{\Omega}$, the equation (1) falls into the scope of Eikonal equation of the type $k|\nabla u| = 1$, where k is a continuous positive function on $\overline{\Omega}$. Among famous numerical methods (especially for Eikonal equations), remain the Fast Sweeping Method (FSM) and the Fast Marching Method (FMM) (see e.g. [30, 34]).

The case where \mathcal{A} is nonempty is different. Recall that the main role of this set concerns the uniqueness of the solution. Roughly speaking a solution is characterized in a unique way by its values on the set of degeneracy, i.e. on \mathcal{A} . So, some additional condition is imposed to single out numerically the reasonable solution among infinitely many ones: for example, it is assumed that the solution is given on the Aubry set, which becomes then a part of the boundary of the domain where the problem is discretized (see e.g. [32]). Nevertheless, in practice we do not, in general, have this knowledge. Another method suggests to regularize the problem by cutting from below f at some level $\epsilon > 0$, and then let $\epsilon \rightarrow 0$. This approach was exploited in [7] using deeply control theoretic interpretation for the approximation of the maximal subsolution. Their scheme was based on a two-step discretization of the control problem associated with the regularized problem: first in the time variable, and then in the space variable. Under some appropriate condition, the approximated solution is shown to be convergent to the maximal solution using stability results. For comprehensive exposition of Hamilton–Jacobi equations and optimal control one can see, for example, [28, 9].

In this paper, we suggest a new approach of variational type based on the characterization of the maximal subsolution of the degenerate Hamilton–Jacobi equations. We use duality method without requiring any additional assumption on the Aubry set. This enables us to give a direct algorithm to approximate the solution. More precisely, we use augmented Lagrangian methods as alternatives to the commonly used methods we presented previously. Actually, it is known that problem (1) aims to characterize the maximal subsolution; i.e. the maximal continuous functions subject to a Dirichlet condition and the constraint on the gradient

$$|\nabla u| \leq f,$$

which can be rewritten as $\nabla u \in B_{|\cdot|}(0, f)$ where $|\cdot|$ stands for the Euclidean norm. This enables us to rewrite the problem into the following sub-gradient constrained optimization

$$\max \left\{ \int_{\Omega} u dx : |\nabla u| \leq f \text{ and } u = 0 \text{ on } \partial\Omega \right\}. \quad (1.2)$$

Similar problems as (1) will be addressed for general Hamilton–Jacobi equations $H(x, \nabla u) = 0$ exploiting more precisely the metric character of these equations. This will allow us to establish the equivalence between being a subsolution of HJ equations, i.e. $H(x, \nabla u) \leq 0$, and $\nabla u \in B_{\sigma^*}$ where σ^* is the dual of the support function σ of the sublevel sets of H , which will be given later. So, in order to solve the HJ equations we will focus on the problem of the form

$$\max \left\{ \int_{\Omega} u dx : \nabla u \in B_{\sigma^*} \text{ and } u = g \text{ on } \partial\Omega \right\}. \quad (1.3)$$

One sees that exploiting the intrinsic metric associated to the Hamiltonian, d_σ , we end up in some sense with the problem of maximizing $\int_\Omega u d\rho$ for a given measure ρ ($\rho \equiv 1$ in (1)), among 1-Lipschitz functions with respect to d_σ .

Dealing with the degenerate case, the main difficulty remains in the strong duality since the qualification conditions are not fulfilled even for the Eikonal equation (with $f \geq 0$). Moreover, the dual problem involves a normal trace of the vector-valued dual variable. To handle rigorously these difficulties, we make use of a trace-like operator for the so called divergence-measure field (cf. [10, 11, 12]). Then, using the perturbation method, we prove a duality result which provides the equivalence with a minimal flux type problem under some divergence constraint. In particular, we will see that this kind of problems can be seen as Finsler variants of Beckmann's model for transportation (cf. [1, 33]). With such a duality result, the problem then consists in determining saddle points of an augmented Lagrangian. This strategy works for very general (even degenerate) quasiconvex Hamiltonians. Up to our knowledge, most of the existing methods to approximate solutions of HJ equations of the form $H(x, p) = F(p) - k(x)$ assuming that $k > 0$. We will see in the examples that with the augmented Lagrangian we can deal with source terms $k \geq 0$, which is meaningful from modeling viewpoint.

At last, let us notice that though the approach proposed in [7] is more concerned with the discretization of the intrinsic metric d_σ , the FMM and FSM repose on a direct discretization of the equation, our strategy is more in the spirit of "optimize then discretize". As we will see, the analysis in the continuous setting leads, via augmented Lagrangian methods, to the discretization of a Laplace equation and to perform projections onto convex sets, which are easy to implement in general. Let us mention also that our approach is close to the one in [6] as they are considering a maximization problem. However, they deal with nonconvex constraint contrary to the convex problem we are considering.

This paper is organized as follows. In section 2, we recall the metric character of HJ equations. More precisely, we present a distance d_σ of Finsler type, constructed from the sublevel sets of the Hamiltonian, which is essential to characterize the subsolutions of HJ equations. We also show that the unique maximal subsolution is characterized by the maximization problem of the form

$$\max \left\{ \int_\Omega u dx : \nabla u \in B_{\sigma^*} \text{ and } u = g \text{ on } D \subset \overline{\Omega} \right\}. \quad (1.4)$$

Dealing with the problem (1), it is natural to derive its dual problem. This will be the main goal of section 3 where we discuss the duality issue for a general Hamiltonian. This will be done via perturbation techniques. Finally we present the augmented Lagrangian, and we see that thanks to the duality result, solving (1) amounts to finding a saddle point of a Lagrangian \mathcal{L} . We give several numerical tests to illustrate our results.

2. METRIC CHARACTER OF HAMILTON–JACOBI EQUATIONS

2.1. Preliminaries. In this paper $\Omega \subset \mathbb{R}^N$ is a regular connected open domain. We consider the following HJ equation

$$H(x, \nabla u) = 0, \quad x \in \Omega, \quad (2.5)$$

where $H: \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous Hamiltonian satisfying, for each $x \in \bar{\Omega}$, $Z(x) := \{p \in \mathbb{R}^N, H(x, p) \leq 0\}$,

- (H1) coercivity: $Z(x)$ is compact;
- (H2) convexity: $Z(x)$ is convex for any $x \in \bar{\Omega}$;
- (H3) $H(x, 0) \leq 0$, i.e. $0 \in Z(x)$ for any $x \in \bar{\Omega}$.

A typical example is the Eikonal equation with $H(x, p) = |p| - f(x)$ for a nonnegative continuous function f .

The properties and notions introduced in this subsection are more or less known. Proofs and more details can be found in [13, 28, 35, 14]. We recall briefly some definitions related to viscosity theory. Given two continuous functions ϕ and ψ , one says that ψ is a *strict* supertangent (respectively sub-tangent) to ϕ at some point $x \in \Omega$ if x is a *strict* local maximizer (respectively minimizer) of $\phi - \psi$.

Definition 2.1. A continuous function $u: \Omega \rightarrow \mathbb{R}$ is said to be a viscosity subsolution of (2.1) if $H(x, \nabla\phi(x)) \leq 0$ for any $x \in \Omega$ and any C^1 supertangent function ϕ to u at x . It is said to be a viscosity supersolution of (2.1) if $H(x, \nabla\psi(x)) \geq 0$ for any $x \in \Omega$ and any C^1 sub-tangent function to u at x . Finally, u is a viscosity solution of (2.1) if it is both a subsolution and a supersolution.

We denote by $\mathcal{S}_H^-(\Omega)$ the family of subsolutions of (2.1). One pertinent property of this family is its stability with respect to uniform convergence, which is not the case for a.e. solutions as we can see by considering a sawtooth function. Moreover, whenever we consider a family $\mathcal{C} \subset \mathcal{S}_H^-(\Omega)$ of locally equibounded functions, then $\inf_{u \in \mathcal{C}} u(x)$ and $\sup_{u \in \mathcal{C}} u(x)$ are still subsolutions to (2.1). We

similarly denote by $\mathcal{S}_H^+(\Omega)$ (resp. $\mathcal{S}_H(\Omega)$) the family of supersolutions (resp. solutions) of (2.1).

For $x \in \bar{\Omega}$, we define the support function of the 0-sublevel set $Z(x)$ by

$$\sigma(x, q) := \sup q \cdot Z(x) = \sup\{q \cdot p, p \in Z(x)\} \quad \text{for } q \in \mathbb{R}^N.$$

The assumption (H1) – (H2) ensures that σ is a continuous nonnegative function in $\Omega \times \mathbb{R}^N$, convex and positively homogeneous with respect to q . Due to the assumption (H3), $\sigma(x, q)$ is possible to equal to 0 for $q \neq 0$, which leads to the degeneracy and its dual σ^* , as given below, may take the value $+\infty$. Here, the dual σ^* (also called polar) is defined by

$$\sigma^*(x, p) := \sup_q \{p \cdot q : \sigma(x, q) \leq 1\}.$$

For example, in the case of Eikonal equation $|\nabla u(x)| = f(x)$, one has $\sigma(x, q) = f(x)|q|$ and $\sigma^*(x, p) = \frac{1}{f(x)}|p|$. We see that $\sigma^*(x, p)$ take the value $+\infty$ for $p \neq 0$ on the zero set of f .

We denote by $\Gamma(x, y)$ the set of Lipschitz curves defined on $[0, 1]$ joining x, y in $\bar{\Omega}$. We then define the intrinsic distance by

$$d_\sigma(x, y) := \inf_{\zeta \in \Gamma(x, y)} \int_0^1 \sigma(\zeta(t), \dot{\zeta}(t)) dt,$$

which is a quasi-distance, *i.e.* satisfying $d_\sigma(x, x) = 0$ and the triangular inequality, but not necessarily symmetric. We summarize the basic properties of d_σ in the following.

Proposition 2.2. ([19])

- 1) d_σ is a quasi-metric, in the sense that for any $x, y \in \bar{\Omega}$ $d_\sigma(x, y) \geq 0$ and $d_\sigma(x, x) = 0$. Moreover, for all $x, y, z \in \bar{\Omega}$ one has $d_\sigma(x, y) \leq d_\sigma(x, z) + d_\sigma(z, y)$.
- 2) For any $x \in \Omega$ $d_\sigma(x, \cdot) \in \mathcal{S}_H^-(\Omega) \cup \mathcal{S}_H^+(\Omega \setminus \{x\})$.
- 3) Compatibility condition: $v \in \mathcal{S}_H^-(\Omega)$ if and only if $v(x) - v(y) \leq d_\sigma(y, x)$ for any $x, y \in \bar{\Omega}$.

The so called Aubry set is defined as the set where the quasi-metric d_σ degenerates. Prescribing a boundary value on $\partial\Omega$ does not guarantee the uniqueness of viscosity solutions to (2.1) unless $\mathcal{A} = \emptyset$, which is not the case in our situation due to the assumption (H3). The Aubry set \mathcal{A} appears then to be a uniqueness set for (2.1).

Definition 2.3. We define the Aubry set \mathcal{A} as the set of points $x \in \Omega$ such that there exists $(\zeta_n)_n \in \Gamma(x, x)$ with $l(\zeta_n) \geq \delta > 0$ and

$$\inf_n \left\{ \int_0^1 \sigma(\zeta_n(t), \dot{\zeta}_n(t)) dt \right\} = 0,$$

where $l(\zeta_n)$ is the Euclidean length of the curve ζ_n .

Proposition 2.4. ([19],[8])

- 1) The Aubry set \mathcal{A} is a closed subset of Ω .
- 2) If $x \in \mathcal{A}$ then $d_\sigma(x, \cdot) \in \mathcal{S}_H(\Omega)$. Moreover, $x \notin \mathcal{A}$ if and only if (2.1) admits a strict subsolution around x .
- 3) If $g : \mathcal{A} \cup \partial\Omega \rightarrow \mathbb{R}$ is a continuous function satisfying the compatibility condition $g(x) - g(y) \leq d_\sigma(y, x)$ on $\mathcal{A} \cup \partial\Omega$, then

$$u(x) = \min_{y \in \mathcal{A} \cup \partial\Omega} \{d_\sigma(y, x) + g(y)\}$$

is the unique viscosity solution of the equation (2.1) such that $u = g$ on $\mathcal{A} \cup \partial\Omega$.

2.2. Equivalence between HJ and maximization problem. Given a closed subset $D \subset \bar{\Omega}$ (typically $D = \partial\Omega$ or $D = \{x\}$ for some $x \in \bar{\Omega}$), we consider the following HJ equation

$$\begin{cases} H(x, \nabla u) = 0 & \Omega \setminus D \\ u = g & D \end{cases} \quad (2.6)$$

where $g : D \rightarrow \mathbb{R}$ is continuous function satisfying compatibility condition. Then the unique maximal viscosity subsolution of the equation (2.2) such that $u(x) = g(x)$ for any $x \in D$, is given by

$$u(x) = \min_{y \in D} d_\sigma(y, x) + g(y) \quad (2.7)$$

with $d_\sigma(\cdot, \cdot)$ being the intrinsic metric associated to the Hamiltonian H .

The considerations given in the introduction lead us to look for the maximal subsolutions of the HJ equation, *i.e.*, $H(x, \nabla u) \leq 0$ or equivalently $u(x) - u(y) \leq d_\sigma(y, x)$. We will show that such functions are precisely the ones with gradient in the unit ball of σ^* . More precisely, we prove the following

Proposition 2.5. $\mathcal{S}_H^-(\Omega) = \{u \in W^{1,\infty}(\Omega) \text{ and } \sigma^*(x, \nabla u(x)) \leq 1 \text{ for a.e. } x \in \Omega\} := \mathcal{B}_{\sigma^*}$.

Then, we transform the problem into a question of maximization of the volume $\int_{\Omega} u dx$ among the subsolutions u . This leads to the following theorem which is an important step to treat the equation (2.2) via augmented Lagrangian methods.

Theorem 2.6. *The maximal viscosity subsolution of (2.2), given by (2.2), is the unique solution of the problem*

$$\max \left\{ \int_{\Omega} z(x) dx, \sigma^*(x, \nabla z(x)) \leq 1 \text{ and } z = g \text{ on } D \right\}. \quad (2.8)$$

For the proof of Proposition 2.5, we recall that the result is more or less known in the case where $H(x, 0) < 0$ which corresponds to $\mathcal{A} = \emptyset$ (see [26, 25] for example). Here, under the general condition (H3), we need the following Cauchy–Schwartz-like lemma.

Lemma 2.7. *For any $q \in \mathbb{R}^N$, and for any $x \in \overline{\Omega}$ we have*

$$\sigma^*(x, q) \leq 1 \Leftrightarrow \langle p, q \rangle \leq \sigma(x, p) \text{ for any } p \in \mathbb{R}^N.$$

Proof. Assume first that $\sigma^*(x, q) \leq 1$. If $\langle p, q \rangle = 0$, it is obvious. For the case $\langle p, q \rangle > 0$, if $\sigma(x, p) = 0$ then by homogeneity of σ , $\sigma(x, \lambda p) = 0$ for every $\lambda \geq 0$. Consequently, $\sigma^*(x, q) \geq \lambda \langle p, q \rangle \rightarrow \infty$ as $\lambda \rightarrow \infty$, this contradicts $\sigma^*(x, q) \leq 1$. This implies that $\sigma(x, p) > 0$ and $\left\langle \frac{p}{\sigma(x, p)}, q \right\rangle \leq \sigma^*(x, q) \leq 1$, as desired. Conversely, if $\langle p, q \rangle \leq \sigma(x, p)$, by definition of σ^* , we take the *sup* over all p such that $\sigma(x, p) \leq 1$ in the previous inequality to obtain that $\sigma^*(x, q) \leq 1$. \square

Proof of Proposition 2.5. We divide the proof into two parts. Firstly, $\mathcal{S}_H^-(\Omega) \subset \mathcal{B}_{\sigma^*}$. Take a point $x \in \Omega$ such that u is differentiable at x . For every vector $v \in \mathbb{S}^{N-1}$ on the unit sphere, we take $\zeta_h(t) = tx + (1-t)(x - hv)$ for every $h > 0$ and $t \in [0, 1]$ and we notice that $\zeta_h \in \Gamma(x - hv, x)$ and for h small enough, ζ_h is close to x . We then have

$$\begin{aligned} \langle \nabla u(x), v \rangle &\leq \lim_{h \rightarrow 0^+} h^{-1} (u(x) - u(x - hv)) \\ &\leq \liminf_{h \rightarrow 0^+} h^{-1} d_{\sigma}(x - hv, x) \\ &\leq \liminf_{h \rightarrow 0^+} h^{-1} \int_0^1 \sigma(\zeta_h, \dot{\zeta}_h) dt = \liminf_{h \rightarrow 0^+} \int_0^1 \sigma(\zeta_h, v) dt \leq \sigma(x, v), \end{aligned}$$

where we have used the continuity of $x \mapsto \sigma(\cdot, v)$ and Lebesgue theorem. Using the definition of σ^* , we deduce that $\sigma^*(x, \nabla u(x)) \leq 1$ as desired.

Secondly, $\mathcal{B}_{\sigma^*} \subset \mathcal{S}_H^-(\Omega)$. Assume now that $\sigma^*(x, \nabla u(x)) \leq 1$ at any differentiable point x of u , i.e., $\langle \nabla u(x), p \rangle \leq \sigma(x, p)$ for all $p \in \mathbb{R}^N$. For smooth function u , the argument is simply given

by

$$\begin{aligned} u(y) - u(x) &= \int_0^1 \nabla u(\xi(s)) \cdot \dot{\xi}(s) ds \\ &\leq \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds, \end{aligned}$$

for Lipschitz curves ξ joining x to y in $\bar{\Omega}$. This implies that $u(y) - u(x) \leq d_\sigma(x, y)$. For general Lipschitz function u , one can make use of smooth approximation (see [26, Proposition 5]). \square

Proof of Theorem 2.6. Let us see that for any subsolution v (2.2), we have $v \leq u$. Pick any subsolution v of (2.2) satisfying the condition $v = g$ on D . Let $x \in \Omega$, we have for any $y \in D$

$$\begin{aligned} v(x) &= v(x) - v(y) + g(y) \\ &\leq d_\sigma(y, x) + g(y). \end{aligned}$$

This gives that $v \leq u$ in Ω and then $\int_\Omega v(x) dx \leq \int_\Omega u(x) dx$. Clearly u is 1-Lipschitz with respect to d_σ , hence by Proposition (2.5) we have $\sigma^*(x, \nabla u) \leq 1$. Consequently u solves (2.6). If w is another solution, then $\int_\Omega u(x) dx = \int_\Omega w(x) dx$ and $w \leq u$ in Ω by the first step. Consequently $u = w$, as desired. \square

3. HJ AND DUALITY RESULTS

3.1. Generalities. As we said in the introduction among our main interests in this paper is to use augmented Lagrangian methods to give a direct algorithm to approximate the solution of the Hamilton–Jacobi equations.

Actually, the problem (2.6) falls into the scope of the following class of optimization problem

$$(\mathcal{P}) \quad \inf_{u \in \mathcal{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

where \mathcal{X} and \mathcal{Y} are two Banach spaces with the topological dual spaces \mathcal{X}^* and \mathcal{Y}^* , $\mathcal{F} : \mathcal{X} \rightarrow (-\infty, +\infty]$, $\mathcal{G} : \mathcal{Y} \rightarrow (-\infty, +\infty]$ are proper, l.s.c., convex functions and $\Lambda : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear operator. The main future for this kind of formulation is to look at its dual problem, which is given by

$$(\mathcal{D}) \quad \sup_{v \in \mathcal{Y}^*} \left(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) \right).$$

Here Λ^* is the adjoint operator of Λ , while \mathcal{F}^* , \mathcal{G}^* are the Legendre–Fenchel transformations of \mathcal{F} and \mathcal{G} given by

$$\begin{aligned} \mathcal{F}^*(f) &= \sup_{u \in \mathcal{X}} (\langle f, u \rangle - \mathcal{F}(u)) \quad \text{for any } f \in \mathcal{X}^* \\ \mathcal{G}^*(g) &= \sup_{q \in \mathcal{Y}} (\langle g, q \rangle - \mathcal{G}(q)) \quad \text{for any } g \in \mathcal{Y}^*. \end{aligned}$$

Using the definitions of \mathcal{F}^* , \mathcal{G}^* and Λ^* , it is not difficult to get the following inequality

$$\sup_{v \in \mathcal{Y}^*} \left(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) \right) \leq \inf_{u \in \mathcal{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u),$$

usually called *weak duality*. The reverse inequality, the so-called *strong duality*, plays an important role since the point is that the solutions of (\mathcal{P}) and (\mathcal{D}) are given by the saddle-points of the problem

$$\min_{u,q} \max_{\phi} L(u, q; \phi)$$

where $L(u, q; \phi) = \mathcal{F}(u) + \mathcal{G}(q) + \langle \phi, \Lambda u - q \rangle$. Yet, the strong duality is not true in general. Some qualification conditions are needed. For instance, one has

Theorem 3.8. (*Strong duality* [18, Chap. III]) *Assume moreover that there exists $u_0 \in \mathcal{X}$ such that $\mathcal{F}(u_0) < +\infty$, $\mathcal{G}(\Lambda u_0) < +\infty$ and \mathcal{G} is continuous at Λu_0 . Then the dual problem admits at least a solution $v \in \mathcal{Y}^*$ and the strong duality holds, i.e.*

$$\max_{v \in \mathcal{Y}^*} \left(-\mathcal{F}^*(-\Lambda^*v) - \mathcal{G}^*(v) \right) = \inf_{u \in \mathcal{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Moreover, u solves (\mathcal{P}) if and only if

$$-\Lambda^*u \in \partial\mathcal{F}(u) \text{ and } v \in \partial\mathcal{G}(\Lambda u).$$

As we pointed out in the introduction, because of the degeneracy of the Hamiltonian, we can not directly use Theorem 3.8 to show duality between the maximization problem and its dual problem. The main goal of this section is to show rigorously that the duality still holds true.

3.2. Duality for HJ equation. To the duality, we consider a more general problem by considering, for a nonnegative Radon measure $\rho \in \mathcal{M}_b^+(\Omega)$ and a closed subset $D \subset \bar{\Omega}$, the problem

$$\max \left\{ \int_{\Omega} u d\rho : \sigma^*(x, \nabla u(x)) \leq 1 \text{ and } u = g \text{ on } D \right\}.$$

Since the values of u are prescribed on D and the solution of (2.2) is given by the distance to D , we can consider the following problem

$$(\mathcal{M}_D) : \max \left\{ \int_{\Omega_D} u d\rho : \sigma^*(x, \nabla u(x)) \leq 1 \text{ and } u = g \text{ on } \Gamma_D \right\},$$

where $\Omega_D := \Omega \setminus D$, $\Gamma_D := \partial\Omega_D \cap \partial D$ and $\tilde{\Gamma} = \partial\Omega_D \setminus \Gamma_D$. In particular, if $D = \partial\Omega$ then $\Omega_D = \Omega$ and $\Gamma_D = \partial\Omega$, $\tilde{\Gamma} = \emptyset$.

As we will see, our dual formulation challenges some kind of trace-like operator for the so called divergence-measure field. To begin with let us sort out formally and briefly our approach, at least in the case where $D = \partial\Omega$. Taking \mathcal{X} to be the Banach space $W^{1,\infty}(\Omega)$ and \mathcal{Y} to be the space $L^\infty(\Omega)^N$, we consider simply

$$\mathcal{F}(u) = \begin{cases} - \int_{\Omega} u d\rho & \text{if } u \in W^{1,\infty}(\Omega) \text{ and } u = g \text{ on } \partial\Omega \\ +\infty & \text{otherwise,} \end{cases}, \quad \Lambda u = \nabla u,$$

and

$$\mathcal{G}(\eta) = \begin{cases} 0 & \text{if } \sigma^*(x, \eta) \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

so that the problem (\mathcal{M}_D) reads as (\mathcal{P}) . For a formal computation of the dual problem, let us notice that $\mathcal{G}^* : \mathcal{Y}^* \rightarrow (-\infty, +\infty]$,

$$\mathcal{G}^*(\Phi) = \int_{\Omega} \sigma(x, \Phi) \quad \text{for any } \Phi \in \mathcal{Y}^*.$$

The operator $\Lambda^* : \mathcal{Y}^* \rightarrow \mathcal{X}^*$ is given by

$$\langle \Lambda^* \Phi, \xi \rangle = \langle \Phi, \nabla \xi \rangle \quad \text{for any } \xi \in \mathcal{X}.$$

Also,

$$\begin{aligned} \mathcal{F}^*(-\Lambda^* \Phi) &= \sup_{u \in \mathcal{X}, u=g \text{ on } \partial\Omega} \int_{\Omega} -\Phi \cdot \nabla u \, dx - \mathcal{F}(u) \\ &= \sup_{u \in \mathcal{X}, u=g \text{ on } \partial\Omega} \int_{\Omega} -\Phi \cdot \nabla u \, dx + \int_{\Omega} u \, d\rho \\ &= \sup_{u \in \mathcal{X}, u=g \text{ on } \partial\Omega} \int_{\Omega} u \operatorname{div}(\Phi) - \int_{\partial\Omega} \Phi \cdot \nu u + \int_{\Omega} u \, d\rho \\ &= \begin{cases} - \int_{\partial\Omega} \Phi \cdot \nu g & \text{if } -\operatorname{div}(\Phi) = \rho \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, the dual problem reads

$$\inf \left\{ \int_{\Omega} \sigma(x, \Phi) - \int_{\partial\Omega} \Phi \cdot \nu g : \Phi \in \mathcal{Y}^*, -\operatorname{div}(\Phi) = \rho \text{ in } \mathcal{D}'(\Omega) \right\}. \quad (3.9)$$

Note that the above computation is very formal by the appearance of the trace-like term $\Phi \cdot \nu$ which is not well defined for all Φ .

To handle rigorously the normal trace of the vector-valued dual variable Φ in the dual problem of the type (3.2), we recall the trace-like operator for the so called divergence-measure field (cf. [10, 11, 12]). To this aim, we assume in this section that

$$\Omega_D = \Omega \setminus D \text{ is a regular domain with a deformable Lipschitz boundary } \partial\Omega_D.$$

This is achieved for instance in the case where Ω is a regular domain and $D = \partial\Omega$ or $D = \bar{\omega}$ with a regular domain $\omega \subset\subset \Omega$.

For any $1 \leq p \leq \infty$, we define the set

$$\mathcal{DM}^p(\Omega_D) := \{F \in L^p(\Omega_D)^N : \operatorname{div} F =: \mu \in \mathcal{M}_b(\Omega_D)\},$$

where $\mu = \operatorname{div} F$ is taken in $\mathcal{D}'(\Omega_D)$. See here, that for any $F \in \mathcal{DM}^p(\Omega_D)$, the total variation of $\operatorname{div} F$ is given by

$$\begin{aligned} |\operatorname{div} F|(\Omega_D) &:= \sup \left\{ \int \varphi \, d\mu : \varphi \in \mathcal{C}_0(\Omega_D), |\varphi(x)| \leq 1 \text{ for any } x \in \Omega_D \right\} \\ &= \sup \left\{ \int \varphi \, d\mu : \varphi \in \mathcal{C}_0^1(\Omega_D), |\varphi(x)| \leq 1 \text{ for any } x \in \Omega_D \right\} \end{aligned}$$

$$= \sup \left\{ \int F \cdot \nabla \varphi : \varphi \in \mathcal{C}_0^1(\Omega_D), |\varphi(x)| \leq 1 \text{ for any } x \in \Omega_D \right\}.$$

In particular, the space $\mathcal{DM}^p(\Omega_D)$ endowed with the norm

$$\|F\|_{\mathcal{DM}^p(\Omega_D)} := \|F\|_{L^p(\Omega_D)} + |\operatorname{div} F|(\Omega_D)$$

is a Banach space.

Thanks to [10, 11, 12], for any $1 < p \leq \infty$, it is possible to define a trace-like operator on the set $\mathcal{DM}^p(\Omega_D)$. Actually, for any $F \in \mathcal{DM}^p(\Omega_D)$, we define $F \cdot \nu$ the normal trace of F on $\partial\Omega_D$, given by $F \cdot \nu : Lip(\partial\Omega_D) \rightarrow \mathbb{R}$ the continuous linear functional such that

$$\langle F \cdot \nu, \xi_{/\partial\Omega_D} \rangle = \int_{\Omega_D} \xi \operatorname{div} F + \int_{\Omega_D} \nabla \xi \cdot F, \quad \text{for any } \xi \in \mathcal{C}^1(\overline{\Omega_D}).$$

Moreover, since Γ_D is a deformable Lipschitz boundary, then the restriction of $F \cdot \nu$ to Γ_D is well defined, this will be denoted by $F \cdot \nu_D$ (cf. [10, 11, 12], see also Remark 3.9).

Remark 3.9. (1) Thanks to [10, 11, 12], for any $F \in \mathcal{DM}^p(\Omega_D)$, with $1 < p \leq \infty$, it is possible to define the normal trace $F \cdot \nu$ locally by using Lipschitz deformation of the boundary. This formulation is very useful in the case where the boundary is partitioned into disjoint deformable Lipschitz patches.

(2) In the case where $p = 1$ as well as the case where $L^p(\Omega_D)^N$ is replaced by the space $\mathcal{M}_b(\Omega_D)^N$, the trace may be defined as well, but only as continuous linear form on a subset of $Lip(\partial\Omega_D)$ (cf. [10, 11, 12]).

Now, combining this consideration with the formal computation for (3.2), we introduce the following optimization problem

$$(\mathcal{OF}_D) : \inf_{\phi \in \mathcal{DM}^p(\Omega_D)} \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) dx - \langle \phi \cdot \nu_D, g \rangle : -\operatorname{div}(\phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega_D} \setminus \Gamma_D) \right\},$$

where the divergence constraint is understood as follows:

$$\int_{\Omega_D} \nabla \xi \cdot \phi \, dx - \langle \phi \cdot \nu_D, \xi \rangle = \int_{\overline{\Omega_D} \setminus \Gamma_D} \xi \, d\rho, \quad \text{for any } \xi \in Lip(\overline{\Omega_D}),$$

with $\phi \cdot \nu_D$ being trace-like term on Γ_D as defined above. In other words, we impose that $\phi \cdot \nu_{\tilde{\Gamma}} = 0$ on $\tilde{\Gamma}$.

Our main result in this section is the following duality result.

Theorem 3.10. *Let $\rho \in \mathcal{M}_b^+(\Omega)$, then (\mathcal{M}_D) and (\mathcal{OF}_D) coincide.*

Proof. Consider on $\mathcal{M}_b(\overline{\Omega_D})$ the following functional $F : \mathcal{M}_b(\overline{\Omega_D}) \mapsto]-\infty, \infty]$ defined by

$$F(h) = \inf_{\phi \in \mathcal{DM}^p(\Omega_D)} \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) dx + \int_{\Gamma_D} g dh - \langle \phi \cdot \nu_D, g \rangle : -\operatorname{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega_D} \setminus \Gamma_D) \right\},$$

for any $h \in \mathcal{M}_b(\overline{\Omega}_D)$. Then F is convex and l.s.c. Indeed, take $h_1, h_2 \in \mathcal{M}_b(\overline{\Omega}_D)$ and set $h := th_1 + (1-t)h_2$ for $t \in [0, 1]$. Let $\phi_{1,n}, \phi_{2,n} \in \mathcal{DM}^p(\Omega_D)$ be two minimizing sequences of fluxes corresponding to h_1 and h_2 respectively, *i.e.* $-\operatorname{div}(\phi_{i,n}) = \rho + h_i$, in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$ and

$$F(h_i) = \lim_n \int_{\Omega_D} \sigma(x, \phi_{i,n}(x)) dx + \int_{\Gamma_D} g dh_i - \langle \phi_{i,n} \cdot \nu_D, g \rangle \quad \text{for } i = 1, 2.$$

Set $\phi_n = t\phi_{1,n} + (1-t)\phi_{2,n}$. We clearly see that ϕ_n are admissible for h and

$$\begin{aligned} F(h) &\leq \lim_n \int_{\Omega_D} \sigma(x, \phi_n(x)) dx + \int_{\Gamma_D} g dh - \langle \phi_n \cdot \nu_D, g \rangle \\ &= \lim_n \int_{\Omega_D} \sigma(x, t\phi_{1,n} + (1-t)\phi_{2,n}) dx + \int_{\Gamma_D} g dh - \langle (t\phi_{1,n} + (1-t)\phi_{2,n}) \cdot \nu_D, g \rangle \\ &\leq \lim_n t \left(\int_{\Omega_D} \sigma(x, \phi_{1,n}) dx + \int_{\Gamma_D} g dh - \langle \phi_{1,n} \cdot \nu_D, g \rangle \right) \\ &\quad + (1-t) \left(\int_{\Omega_D} \sigma(x, \phi_{2,n}) dx + \int_{\Gamma_D} g dh - \langle \phi_{2,n} \cdot \nu_D, g \rangle \right) \\ &\leq tF(h_1) + (1-t)F(h_2) \end{aligned}$$

and this proves convexity. For the lower semicontinuity, take a sequence $h_n \rightarrow h$ in $\mathcal{M}_b(\overline{\Omega}_D)$. For every $n \in \mathbb{N}$, we consider a sequence $(\phi_n^k)_{k \in \mathbb{N}}$ of $L^1(\Omega_D, \mathbb{R}^N)$ such that

$$F(h_n) = \lim_{k \rightarrow \infty} \int_{\Omega_D} \sigma(x, \phi_n^k(x)) dx + \int_{\Gamma_D} g dh_n - \langle \phi_n^k \cdot \nu_D, g \rangle.$$

We may find some $\psi_n \in L^1(\Omega_D, \mathbb{R}^N)$ such that $-\operatorname{div}(\psi_n) = h - h_n$ in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$, $\|\psi_n\|_{L^1} \rightarrow 0$ and $\langle \psi_n \cdot \nu_D, g \rangle \rightarrow 0$. In fact, we have $h - h_n \in \mathcal{M}_b(\overline{\Omega}_D) \hookrightarrow W^{-1,p'}(\Omega_D)$ for $p > N$ and $p' := \frac{p}{p-1}$. We consider the following p -Laplace equation

$$\begin{cases} -\Delta_p u_n = h - h_n & \text{in } \Omega_D \\ u_n = 0 & \text{on } \Gamma_D \\ |\nabla u_n|^{p-2} \nabla u_n \cdot \nu_{\tilde{\Gamma}} = 0 & \text{on } \tilde{\Gamma}. \end{cases} \quad (3.10)$$

The system (3.2) admits a unique solution $u_n \in W^{1,p}(\Omega_D)$ such that $u_n = 0$ on Γ_D . Hence, if we set $\psi_n = |\nabla u_n|^{p-2} \nabla u_n$, we see that $\psi_n \in L^{p'}(\Omega_D)$, and then in $L^1(\Omega_D)$. Moreover, we have $-\operatorname{div}(\psi_n) = h - h_n$ in $\mathcal{D}'(\Omega_D)$. Since $h - h_n$ is bounded in $W^{-1,p'}(\Omega_D)$, it is not difficult to prove that u_n is bounded in $W^{1,p}(\Omega_D)$. So, by taking a subsequence if necessary, we have $u_n \rightarrow u$ in $W^{1,p}(\Omega_D)$, and uniformly in Ω_D . On the other hand, we have

$$\int_{\Omega_D} |\psi_n|^{p'} dx = \int_{\Omega_D} |\nabla u_n|^p dx = \langle h - h_n, u_n \rangle_{W^{-1,p'}(\Omega_D), W^{1,p}(\Omega_D)} \xrightarrow{n \rightarrow \infty} 0.$$

In particular, this implies that $|\psi_n| \xrightarrow[n \rightarrow \infty]{} 0$ in $L^1(\Omega_D)$. Moreover, taking $\tilde{g} \in Lip(\Omega_D)$ be such that $\tilde{g} = g$ on Γ_D , we have

$$\langle \psi_n \cdot \nu_D, g \rangle = \int_{\Omega_D} \psi_n \cdot \nabla \tilde{g} \, dx - \int_{\Omega_D} \tilde{g} \, d(h - h_n) \xrightarrow[n \rightarrow \infty]{} 0.$$

This being said, we clearly have $-\operatorname{div}(\phi_n^k + \psi_n) = \rho + h$, i.e. $\phi_n^k + \psi_n$ are admissible fluxes for h . By semicontinuity of the integral, we have

$$\begin{aligned} F(h) &\leq \int_{\Omega_D} \sigma(x, (\phi_n^k + \psi_n)(x)) \, dx + \int_{\Gamma_D} g \, dh - \langle (\phi_n^k + \psi_n) \cdot \nu_D, g \rangle \\ &\leq \int_{\Omega_D} \sigma(x, \phi_n^k(x)) \, dx + \int_{\Gamma_D} g \, dh_n - \langle \phi_n^k \cdot \nu_D, g \rangle \\ &\quad + \int_{\Omega_D} \sigma(x, \psi_n(x)) \, dx + \int_{\Gamma_D} g \, d(h - h_n) - \langle \psi_n \cdot \nu_D, g \rangle. \end{aligned}$$

Letting $k \rightarrow \infty$ we get

$$F(h) \leq F(h_n) + \int_{\Omega_D} \sigma(x, \psi_n(x)) \, dx + \int_{\Gamma_D} g \, d(h - h_n) - \langle \psi_n \cdot \nu_D, g \rangle.$$

Now, letting $n \rightarrow \infty$, and using the fact that $\psi_n \rightarrow 0$ in $L^1(\Omega_D)^N$, and $h_n \rightarrow h$ in $\mathcal{M}_b(\overline{\Omega}_D)$, as $n \rightarrow \infty$, we obtain the lower semicontinuity, i.e.

$$F(h) \leq \liminf_n F(h_n).$$

Next let us compute F^* . For any $u \in C(\overline{\Omega}_D)$, we have

$$\begin{aligned} F^*(u) &= \sup_{h \in \mathcal{M}_b(\overline{\Omega}_D)} \int_{\overline{\Omega}_D} u \, dh - F(h) \\ &= \sup_{\substack{h \in \mathcal{M}_b(\overline{\Omega}_D) \\ \phi \in \mathcal{DM}^p(\Omega_D)}} \left\{ \int_{\overline{\Omega}_D} u \, dh - \int_{\Omega_D} \sigma(x, \phi(x)) \, dx - \int_{\Gamma_D} g \, dh + \langle \phi \cdot \nu_D, g \rangle : \right. \\ &\quad \left. -\operatorname{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\} \\ &= I_1(u) + I_2(u), \end{aligned}$$

where $I_1(u) := - \int_{\Omega_D} u \, d\rho$ and

$$\begin{aligned} I_2(u) &:= \sup_{\substack{h \in \mathcal{M}_b(\overline{\Omega}_D) \\ \phi \in \mathcal{DM}^p(\Omega_D)}} \left\{ \int_{\overline{\Omega}_D \setminus \Gamma_D} u \, d(\rho + h) - \int_{\Omega_D} \sigma(x, \phi(x)) \, dx + \int_{\Gamma_D} (u - g) \, dh + \langle \phi \cdot \nu_D, g \rangle \right. \\ &\quad \left. : -\operatorname{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\}. \end{aligned}$$

Using Lemma 3.11 below, we deduce that, for any $u \in Lip(\Omega_D)$, we have

$$F^*(u) = \begin{cases} - \int_{\Omega_D} u d\rho & \text{if } \begin{cases} \sigma^*(x, \nabla u) \leq 1 \\ \text{and } u = g \text{ on } \Gamma_D \end{cases} \\ \infty & \text{otherwise.} \end{cases}$$

Finally, using the fact

$$\inf(\mathcal{O}\mathcal{F}_D) = F(0) = F^{**}(0) = \sup_{u \in Lip(\Omega_D)} -F^*(u) = \max(\mathcal{M}_D),$$

we deduce the result. \square

Lemma 3.11. *Let $u \in Lip(\Omega_D)$, we have*

$$\begin{aligned} & \sup_{\substack{h \in \mathcal{M}_b(\overline{\Omega}_D) \\ \phi \in \mathcal{DM}^p(\Omega_D)}} \left\{ \int_{\overline{\Omega}_D \setminus \Gamma_D} u d(\rho + h) - \int_{\Omega_D} \sigma(x, \phi(x)) dx + \int_{\Gamma_D} (u - g) dh + \langle \phi \cdot \nu_D, g \rangle \right. \\ & \qquad \qquad \qquad \left. : -\operatorname{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\} \\ & = \begin{cases} 0 & \text{if } \begin{cases} \sigma^*(x, \nabla u) \leq 1 \\ \text{and } u = g \text{ on } \Gamma_D \end{cases} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Take u as a test function in the divergence constraint $-\operatorname{div}(\phi) = \rho + h$ in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$, we get

$$\begin{aligned} I(h, \phi) & := \int_{\overline{\Omega}_D \setminus \Gamma_D} u d(\rho + h) - \int_{\Omega_D} \sigma(x, \phi(x)) dx + \int_{\Gamma_D} (u - g) dh + \langle \phi \cdot \nu_D, g \rangle \\ & = \int_{\Omega_D} \nabla u \cdot \phi dx - \int_{\Omega_D} \sigma(x, \phi(x)) dx + \int_{\Gamma_D} (u - g) dh + \langle \phi \cdot \nu_D, g - u \rangle. \end{aligned}$$

If $\sigma^*(x, \nabla u) \leq 1$ and $u = g$ on Γ_D , then following Lemma 2.7, we obtain $\sup I(h, \phi) \leq 0$. Actually, $\sup I(h, \phi) = 0$ in this case by taking $h \equiv -\rho$ and $\phi \equiv 0$. If $u(x_0) \neq g(x_0)$ for some $x_0 \in \Gamma_D$, then we consider Dirac mass at x_0 , $h = n \operatorname{sign}(u(x_0) - g(x_0)) \delta_{x_0}$ for $n \in \mathbb{N}$, and fix $\Phi_0 \in \mathcal{DM}^p(\Omega_D)$ such that $-\operatorname{div} \Phi_0 = \rho$ in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$, we have

$$\sup I(h, \phi) \geq \int_{\Omega_D} \Phi_0 \cdot \nabla u - \int_{\Omega_D} \sigma(x, \Phi_0(x)) dx + n|u(x_0) - g(x_0)| + \langle \Phi_0 \cdot \nu_D, g - u \rangle.$$

Letting $n \rightarrow \infty$, we get the result. For the remaining case, i.e. $u = g$ on Γ_D and $\sigma^*(x, \nabla u) > 1$ on a non negligible set, we see first that, for any $u \in Lip(\Omega_D)$, there exists a measurable function that we denote by $q_u : \Omega_D \rightarrow \mathbb{R}^N$, such that

$$q_u(x) \cdot \nabla u(x) = \sigma^*(x, \nabla u(x)), \quad \text{a.e. in } \Omega_D.$$

Indeed, recall that $\sigma^*(x, \nabla u(x)) = \max_p \{\langle p, \nabla u(x) \rangle : \sigma(x, p) = 1\}$ and the function $x \rightarrow \langle p, \nabla u(x) \rangle + \mathbb{I}_{[\sigma(x, \cdot)=1]}$ is measurable. Then, q_u is given by the measurable representation in the set

$$\arg \max_p \{\langle p, \nabla u \rangle : \sigma(x, p) = 1\}.$$

Now, if $u = g$ on Γ_D and $\sigma^*(x, \nabla u) > 1$ in a subset $A \subset \Omega_D$ such that $|A| \neq 0$, we consider $\Phi_{n\epsilon} = n \frac{q_u}{|q_u|} \chi_A * \eta_\epsilon$, where η_ϵ is a sequence of mollifiers. It is clear that there exists $h \in \mathcal{M}_b(\overline{\Omega}_D)$, such that $-\operatorname{div} \Phi_{n\epsilon} = \rho + h$. For any n , we have

$$\sup I(h, \phi) \geq \int_{\Omega_D} \Phi_{n\epsilon} \cdot \nabla u - \int_{\Omega_D} \sigma(x, \Phi_{n\epsilon}(x)) dx.$$

Letting $\epsilon \rightarrow 0$, we get

$$\sup I(h, \phi) \geq n \int_A \frac{1}{|q_u|} (q_u \cdot \nabla u - \sigma(x, q_u)) dx \geq n \int_A \frac{1}{|q_u|} (\sigma^*(x, \nabla u(x)) - 1) dx.$$

Then, letting $n \rightarrow \infty$, we get the result. \square

Remark 3.12. Going over the duality inferred by Theorem 3.10, we have

$$\begin{aligned} & \inf \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) dx - \langle \phi \cdot \nu_D, g \rangle : \phi \in \mathcal{DM}^p(\Omega_D), -\operatorname{div}(\phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\} \\ & = \max \left\{ \int_{\Omega_D} u(x) d\rho, \sigma^*(x, \nabla u(x)) \leq 1 \text{ a.e. } x \text{ in } \Omega_D \text{ and } u = g \text{ on } \Gamma_D \right\}. \end{aligned}$$

It is not clear if the inf is a min. This is closely connected to the regularity of the trace of divergence-measure field. However, one sees that if this is true, i.e. the inf is a min, then the respective extremums u and ϕ satisfy the following PDE:

$$\left\{ \begin{array}{ll} -\operatorname{div}(\phi) = \rho & \text{in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \\ \phi(x) \cdot \nabla u(x) = \sigma(x, \phi(x)) & \text{in } \Omega_D \\ u = g & \text{on } \Gamma_D \\ \sigma^*(x, \nabla u(x)) \leq 1 & \text{in } \Omega_D \\ \phi \cdot \nu_{\tilde{\Gamma}} = 0 & \text{on } \tilde{\Gamma}. \end{array} \right.$$

4. THE AUGMENTED LAGRANGIAN TECHNIQUE

4.1. Formulation of the problem. We set again $\mathcal{X} = W^{1,\infty}(\Omega_D)$ and $\mathcal{Y} = L^\infty(\Omega_D)^N$. For any $u \in \mathcal{X}$ and $\eta \in \mathcal{Y}$, we define

$$\mathcal{F}(u) = \begin{cases} -\int_{\Omega_D} u d\rho & \text{if } u = g \text{ on } \Gamma_D \\ +\infty & \text{otherwise} \end{cases}, \quad \mathcal{G}(\eta) = \begin{cases} 0 & \text{if } \sigma^*(x, \eta) \leq 1 \\ +\infty & \text{otherwise} \end{cases}, \quad \text{and } \Lambda u = \nabla u.$$

Thus, the problem (\mathcal{M}_D) can be rewritten in the form

$$- \inf_{u \in \mathcal{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Thanks to Theorem 3.10, we can write

$$\begin{aligned} & - \min \left\{ \mathcal{F}(u) + \mathcal{G}(\Lambda u) : u \in \mathcal{X} \right\} = - \sup \left\{ -\mathcal{F}^*(-\Lambda^* \phi) - \mathcal{G}^*(\phi) : \phi \in \mathcal{DM}^p(\Omega_D) \right\} \\ & = \inf \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) dx - \langle \phi \cdot \nu_D, g \rangle : \phi \in \mathcal{DM}^p(\Omega_D), -\operatorname{div}(\phi) = \rho \text{ in } \mathcal{D}'(\bar{\Omega}_D \setminus \Gamma_D) \right\}. \end{aligned}$$

Introducing a new primal variable $q \in \mathcal{Y}$ we can write (\mathcal{M}_D) in the following alternative form

$$- \inf_{\substack{(u,q) \in \mathcal{X} \times \mathcal{Y} \\ \Lambda u = q}} \mathcal{F}(u) + \mathcal{G}(q) \quad (4.11)$$

so that (\mathcal{M}_D) and (\mathcal{OF}_D) can be recast in a saddle point form

$$\inf_{(u,q) \in \mathcal{X} \times \mathcal{Y}} \sup_{\phi \in \mathcal{DM}^p(\Omega_D)} L(u, q; \phi)$$

where

$$L(u, q; \phi) = \mathcal{F}(u) + \mathcal{G}(q) + \int_{\Omega_D} \phi \cdot (\Lambda(u) - q) dx, \quad \text{for any } (u, q, \phi) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{DM}^p(\Omega_D).$$

More precisely, we have

Proposition 4.13. *u is a solution of (\mathcal{M}_D) if and only if the couple $(u, q := \Lambda u) \in \mathcal{X} \times \mathcal{Y}$ is a solution of*

$$\sup_{\phi \in \mathcal{DM}^p(\Omega_D)} \min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} L(u, q; \phi)$$

which is equal to

$$\min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} \sup_{\phi \in \mathcal{DM}^p(\Omega_D)} L(u, q; \phi).$$

Proof. The proof of this result is standard. For completeness let us give the main arguments showing how the duality of the previous section takes part of this result. Using (4.1), it is not difficult to see that

$$\min \left\{ \mathcal{F}(u) + \mathcal{G}(\Lambda u) : u \in \mathcal{X} \right\} = \min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} \sup_{\phi \in \mathcal{DM}^p(\Omega_D)} L(u, q; \phi).$$

On the other hand, using the definition of \mathcal{F}^* , \mathcal{G}^* and Λ^* , one sees that

$$\sup_{\phi \in \mathcal{DM}^p(\Omega_D)} \min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} L(u, q; \phi) = \sup \left\{ -\mathcal{F}^*(-\Lambda^* \phi) - \mathcal{G}^*(\phi) : \phi \in \mathcal{DM}^p(\Omega_D) \right\}.$$

Thus using Theorem 3.10, the result of the proposition follows. \square

For a given $r > 0$, we recall that the augmented Lagrangian (cf. [22]) is given by

$$L_r(u, q; \phi) = \mathcal{F}(u) + \mathcal{G}(q) + \langle \phi, \Lambda u - q \rangle + \frac{r}{2} |\Lambda u - q|^2, \quad \text{for any } (u, q, \phi) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{DM}^p(\Omega_D).$$

In the same way, it can easily be proved that

Proposition 4.14. *Let $r > 0$. Then, u is a solution of (\mathcal{M}_D) if and only if the couple $(u, q := \Lambda u) \in \mathcal{X} \times \mathcal{Y}$ is a solution of*

$$\sup_{\phi \in \mathcal{DM}^p(\Omega_D)} \min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} L_r(u, q; \phi)$$

which is equal to

$$(\mathcal{S}_r) : \min_{(u,q) \in \mathcal{X} \times \mathcal{Y}} \sup_{\phi \in \mathcal{DM}^p(\Omega_D)} L_r(u, q; \phi).$$

Now for the numerical computation concerning the problem (\mathcal{M}_D) , we will focus on the saddle point problem (\mathcal{S}_r) . Recall that the addition of the quadratic term $\frac{r}{2}|\Lambda u - q|^2$ has the advantage of improving the convergence of the dual approach (one can see [18, 21]).

4.2. Application of ALG2. We approximate the domain Ω_D via a triangulation \mathcal{T}_h . For $k \geq 1$, we denote by \mathbb{P}_k the space of polynomials with real coefficients and of degree at most k . We define $\mathcal{X}_h \subset W^{1,\infty}(\Omega_D)$ as the space of continuous functions on $\bar{\Omega}_D$ belonging to \mathbb{P}_k on each triangle. Similarly, \mathcal{Y}_h is the space of vector valued functions belonging to $(\mathbb{P}_{k-1})^d$ on each triangle. Then the problem (\mathcal{M}_D) is discretized by the following finite-dimensional optimization problem:

$$\inf_{u \in \mathcal{X}_h} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Then, for a given $q_0 \in \mathcal{Y}_h, \phi_0 \in \mathcal{Y}_h^*$, using ALG2 algorithm (also called ADMM [4]) we construct a sequence $\{u_i\}_i, \{q_i\}_i, \{\phi_i\}_i$ by optimizing alternatively in u and q , for $i \geq 1$. Convergence of the algorithm in the finite-dimensional setting is well known (cf. [17]). The convergence can be also established in the case of infinite-dimensional Hilbert spaces. For further details in this direction and about the ALG2, we refer the reader to [22, 20, 17].

Algorithm 1 ALG2 iterations

1st step: $u_{i+1} \in \arg \min_{u \in \mathcal{X}_h} \left\{ \mathcal{F}(u) + \langle \phi_i, \Lambda(u) \rangle + \frac{r}{2} |\Lambda(u) - q_i|^2 \right\}$.

2nd step: $q_{i+1} \in \arg \min_{q \in \mathcal{Y}_h} \left\{ \mathcal{G}(q) - \langle \phi_i, q \rangle + \frac{r}{2} |\Lambda(u_{i+1}) - q|^2 \right\}$.

3rd step: We update the multiplier ϕ by

$$\phi_{i+1} = \phi_i + r(\nabla u_{i+1} - q_{i+1}).$$

Remark 4.15. • The first step amounts to solve a Laplace equation with mixed boundary conditions. Indeed, we have for every $z \in \mathcal{X}_h$ with $z = 0$ on Γ_D

$$r \langle \nabla u_{i+1}, \nabla z \rangle = \langle \rho, z \rangle + \langle (r q_i - \phi_i), \nabla z \rangle$$

which is equivalent to solve

$$-r \Delta u = \rho + \operatorname{div}(\phi - r q) \quad \text{in } \Omega_D,$$

together with the following mixed boundary conditions:

$$\begin{cases} \text{Dirichlet boundary condition } u = g \text{ on } \Gamma_D \\ \text{Homogeneous Neumann boundary condition } (r\nabla u + \phi - rq) \cdot \nu_{\tilde{\Gamma}} = 0 \text{ on } \tilde{\Gamma}. \end{cases}$$

- The second step is a pointwise projection. Indeed, we use P_1 finite element for q and ϕ , we have at each vertex x_k

$$0 \in \partial \mathbb{I}_{B^*(x_k, \cdot)}(q_{i+1}(x_k)) - \phi_i(x_k) - r(\nabla u_{i+1}(x_k) - q_{i+1}(x_k)),$$

which is equivalent to perform pointwise projections:

$$q_{i+1}(x_k) = \text{Proj}_{B^*(x_k, \cdot)}\left(\frac{\phi_i(x_k)}{r} + \nabla u_{i+1}(x_k)\right)$$

where $B^*(x, \cdot) = \{p \in \mathbb{R}^N : \sigma^*(x, p) \leq 1\}$.

4.3. Error criterion. Basing on the primal-dual optimality conditions, we use the following stopping criterion

1. $\text{MaxLip} := \sup_x \sigma^*(x, \nabla u(x))$,
2. $\text{Div} := \|\text{div}(\phi) - \rho\|_{L^2}$,
3. $\text{Dual} := \|\sigma(x, \phi(x)) - \phi(x) \cdot \nabla u\|_{L^2}$,
4. $\text{NBD}_\phi := \left(\int_{\tilde{\Gamma}} (\phi \cdot \nu)^2\right)^{1/2}$.

We expect $\text{MaxLip} \leq 1$ and $\text{Div}, \text{Dual}, \text{NBD}_\phi$ to be small. In addition, we compute $\|u - u_{exact}\|$ for different norms where u is the computed solution and u_{exact} is the exact solution, whenever the latter is easily found. Let us mention that NBD_ϕ will concern only the Test 2 where we prescribe data on a closed set D , so that $\tilde{\Gamma} \neq \emptyset$.

To implement the algorithm we use FreeFem++ [23], which is particularly adapted to solve the Laplace equation in the first step of ALG2. We use P_2 finite element for u and P_1 finite element for ϕ and q (see e.g. [2]). All the tests are executed on a macOS Mojave 10.14.4.

5. NUMERICAL EXPERIMENTS

5.1. Test 1: We first examine the case $|\nabla u| = f(x, y) \equiv 1$ in $\Omega = (0, 1)^2$ and $u = 0$ on $\partial\Omega$.

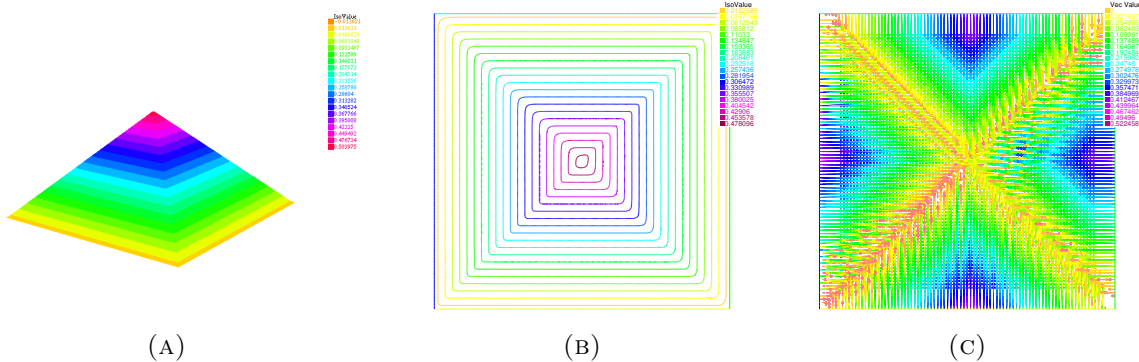
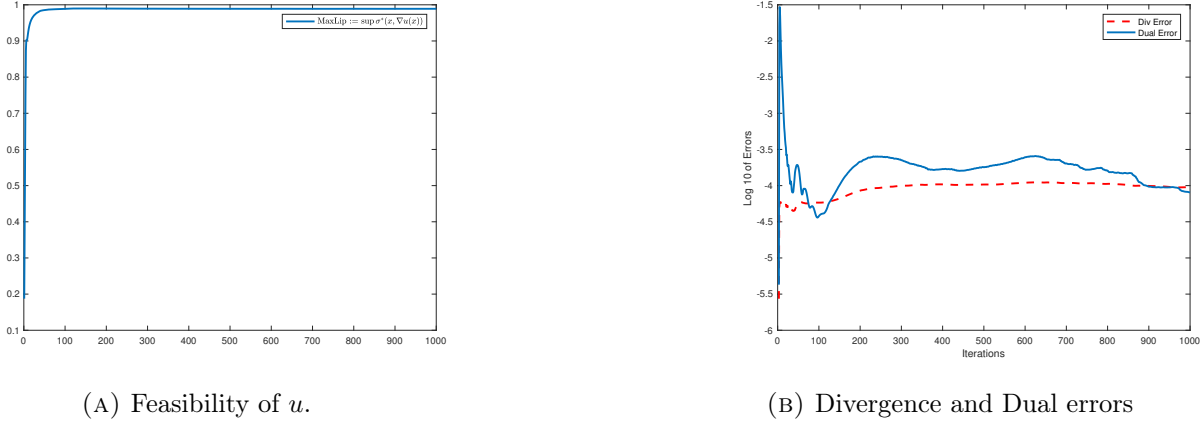


FIGURE 1. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ

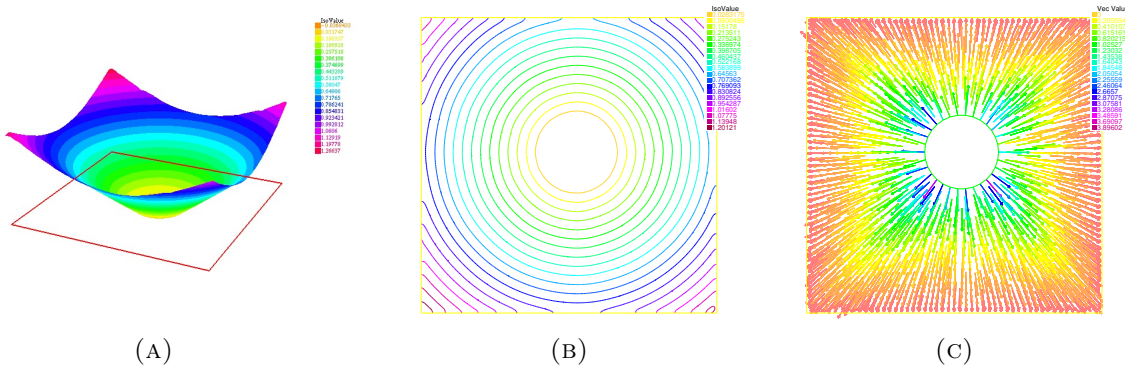
FIGURE 2. Error criterion for 1000 iterations with $N_h = 120$

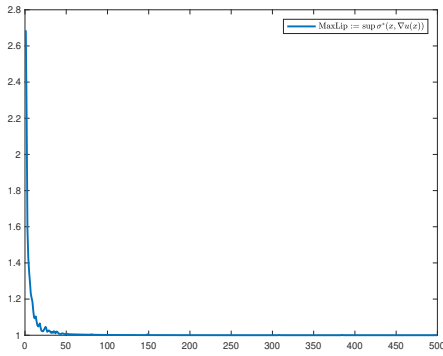
The exact solution is $u_{exact}(x, y) = d((x, y), \partial\Omega) = \min(x, 1 - x, y, 1 - y)$. We measure $\|u - u_{exact}\|$ in different norms, with different mesh sizes N_h and for 1000 iterations.

N_h	Time execution	$\ u - u_{exact}\ _{L^2}$	$\ u - u_{exact}\ _{L^1}$	$\ u - u_{exact}\ _{L^\infty}$
30	55.3254s	1.19311e-3	1.2476e-3	9.71803e-4
60	212.838s	3.68023e-4	3.82991e-4	3.48313e-4
120	855.136s	1.64122e-4	1.7198e-4	1.66483e-4

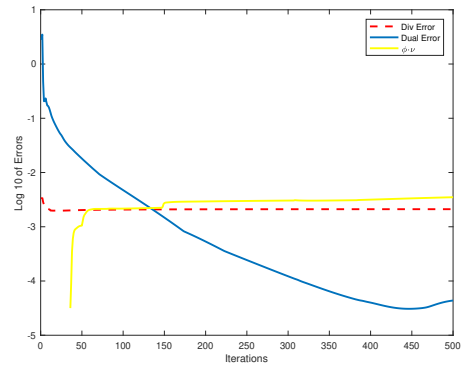
This test shows us that ALG2 iteration converges to the accurate solution u as the mesh is refined.

5.2. Test 2: We consider the same equation $|\nabla u| = 1$ on $\Omega = (-1, 1)^2 \setminus D$ where D is the euclidean ball centered at $(0.05, 0.09)$ and of radius 0.25, and we set u to be equal to 0 on D .

FIGURE 3. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ



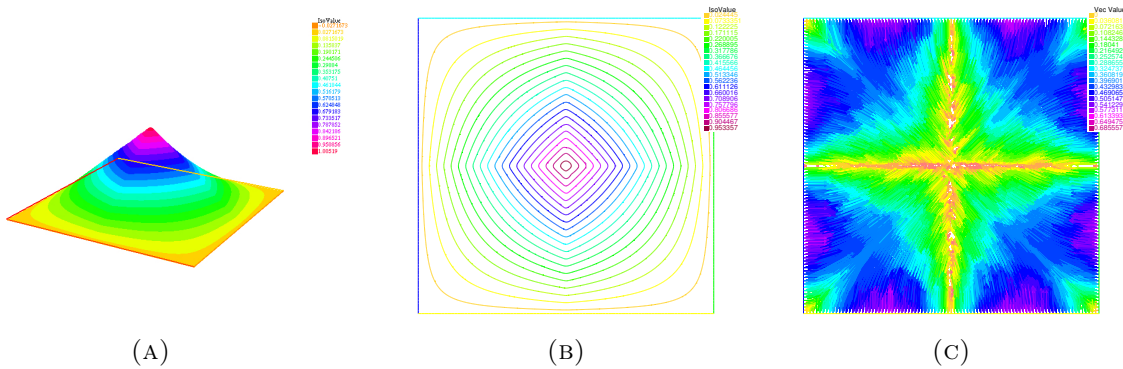
(A) Feasibility of u .



(B) Divergence, Dual and NBD_{ϕ} errors

FIGURE 4. Error criterion for 500 iterations with $N_h = 50$

5.3. **Test 3:** Always with the Eikonal case, take $f(x, y) = \sqrt{(1 - |x|)^2 + (1 - |y|)^2}$. The exact solution in $\Omega = (-1, 1)^2$ is $u_{exact}(x, y) = (1 - |x|) \cdot (1 - |y|)$.

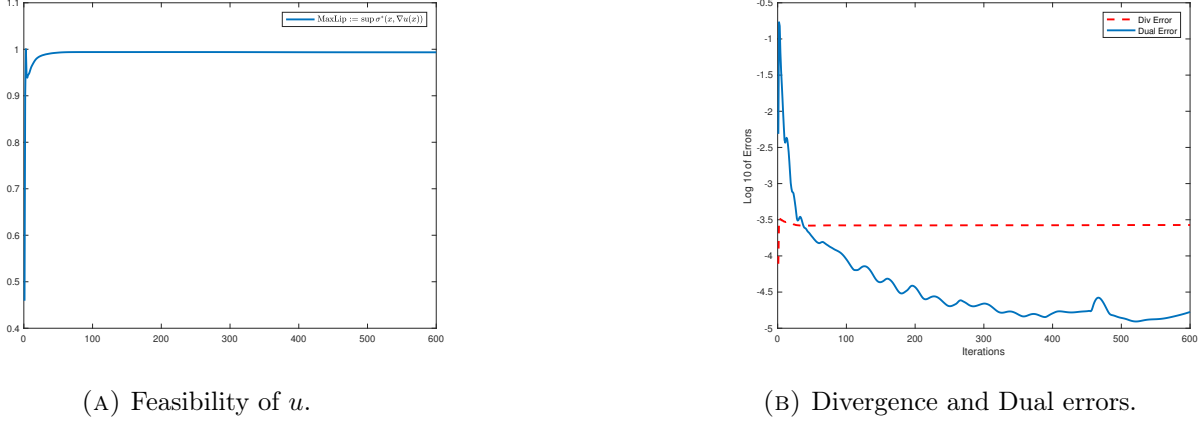


(A)

(B)

(C)

FIGURE 5. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ

FIGURE 6. Error criterion for 600 iterations with $N_h = 120$

N_h	Time execution	$\ u - u_{exact}\ _{L^2}$	$\ u - u_{exact}\ _{L^1}$	$\ u - u_{exact}\ _{L^\infty}$
30	39.1138s	2.27077e-3	2.54665e-3	2.47988e-3
60	158.138s	6.49636e-4	7.41279e-4	1.57558e-3
120	647.024s	2.99273e-4	3.22747e-4	6.54593e-4
240	2610.03s	6.97158e-5	7.83619e-5	2.807e-4

5.4. Riemannian case. We take $\sigma((x, y), v) = \sqrt{\beta_1 v_1^2 + \beta_2 v_2^2}$, with $\beta_1, \beta_2 > 0$. It is not difficult to see that $\sigma^*(q) = \sqrt{\frac{q_1^2}{\beta_1} + \frac{q_2^2}{\beta_2}}$. As in [3], the projection onto the unit ball of σ^* , $B^* = \{q, \sigma^*(q) \leq 1\}$ is given by

$$\text{Proj}_{B^*}(q) = \begin{cases} q & \text{if } q \in B^* \\ \left(\frac{\beta_1 q_1}{\beta_1 + \zeta}, \frac{\beta_2 q_2}{\beta_2 + \zeta} \right) & \text{otherwise,} \end{cases}$$

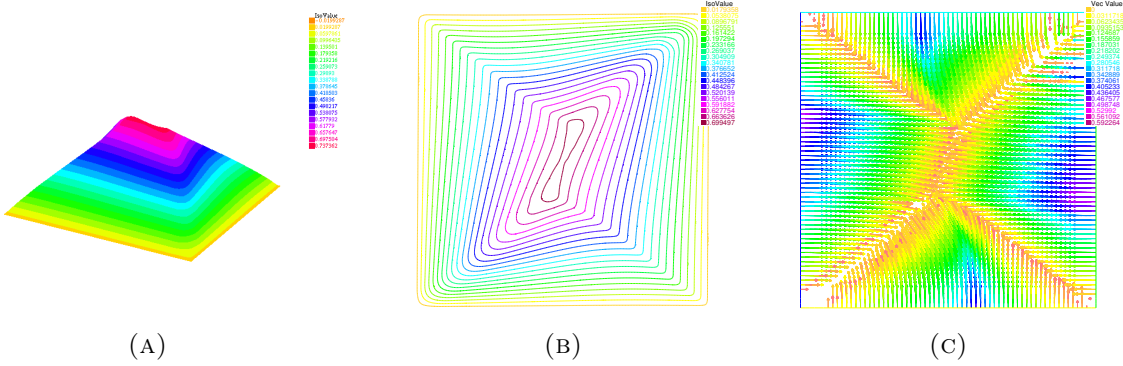
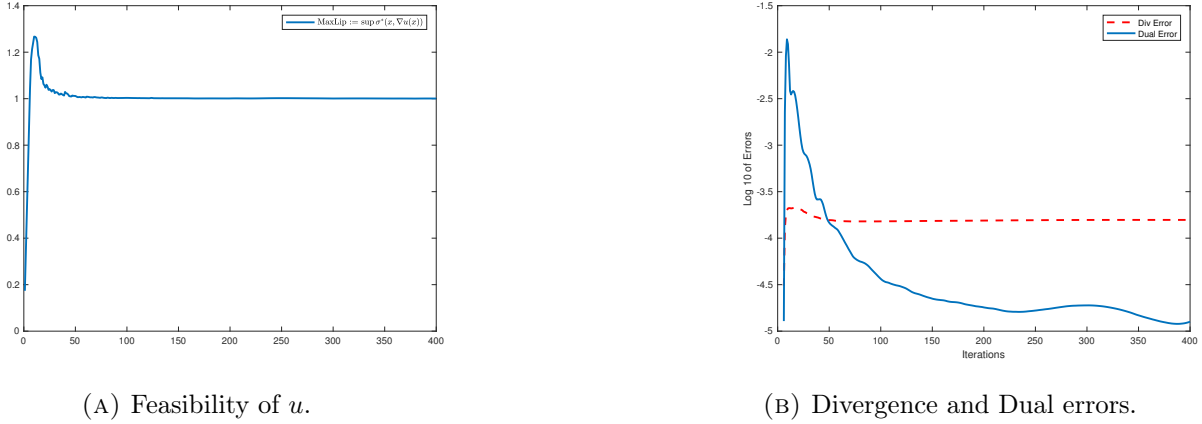
where ζ is the zero of the function

$$F(\zeta) = 1 - \left(\frac{\beta_1 q_1}{(\beta_1 + \zeta)^2} + \frac{\beta_2 q_2}{(\beta_2 + \zeta)^2} \right),$$

which can be computed with a dichotomy algorithm.

For this test, we take

$$\beta_1 = \frac{1}{e^{-2\left(\sqrt{2(x-0.5)^2+2(x-0.5)(y-0.5)}+(y-0.5)^2\right)}} \text{ and } \beta_2 = \frac{2}{e^{-2\left(\sqrt{2(x-0.5)^2+2(x-0.5)(y-0.5)}+(y-0.5)^2\right)}}.$$


 FIGURE 7. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ

 FIGURE 8. Error criterion for 400 iterations with $N_h = 64$

5.5. Anisotropic Eikonal equation: One interesting case is the so called anisotropic Eikonal equation. Consider a symmetric positive definite matrix M modelling the anisotropy, and define the following equation

$$H(x, \nabla u) = \sqrt{\nabla u^\dagger M \nabla u} - 1 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

In this case B^* is an ellipse and the projection can be computed as in the Riemannian case [27]:

$$\text{Proj}_{B^*}(q) = \begin{cases} q & \text{if } q \in B^* \\ (\zeta M + I_n)^{-1} q & \text{otherwise,} \end{cases}$$

where ζ is the unique positive root of the function

$$F(\zeta) = \bar{q}_\zeta^\dagger M \bar{q}_\zeta - 1 \text{ with } \bar{q}_\zeta = (\zeta M + I_n)^{-1} q,$$

which can be found with a dichotomy method. We perform a test as in [30, Example 2] by taking

$$M = \begin{pmatrix} l_1(x, y) & -l_3(x, y) \\ -l_3(x, y) & l_2(x, y) \end{pmatrix}$$

with

$$l_1(x, y) = \frac{1}{e^{-2(\sqrt{2.(x-0.5)^2+2.(x-0.5).(y-0.5)+(y-0.5)^2})}}$$

$$l_2(x, y) = 2l_1(x, y)$$

$$l_3(x, y) = l_1(x, y).$$

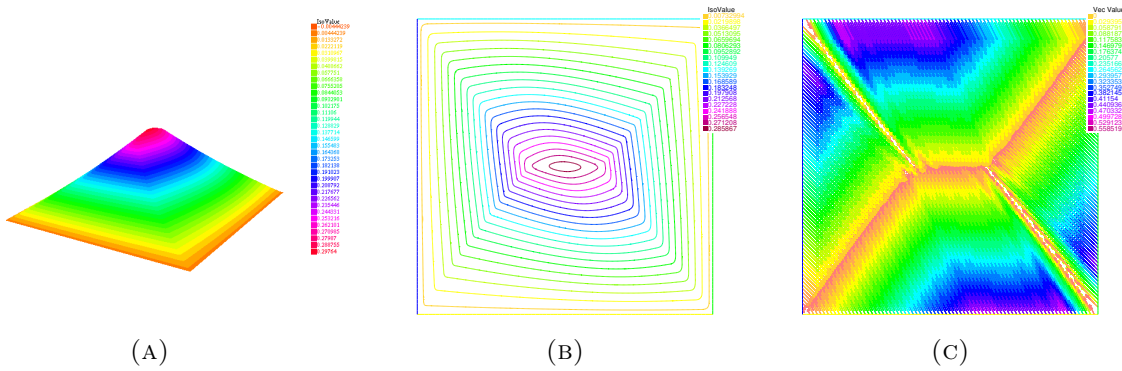


FIGURE 9. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ

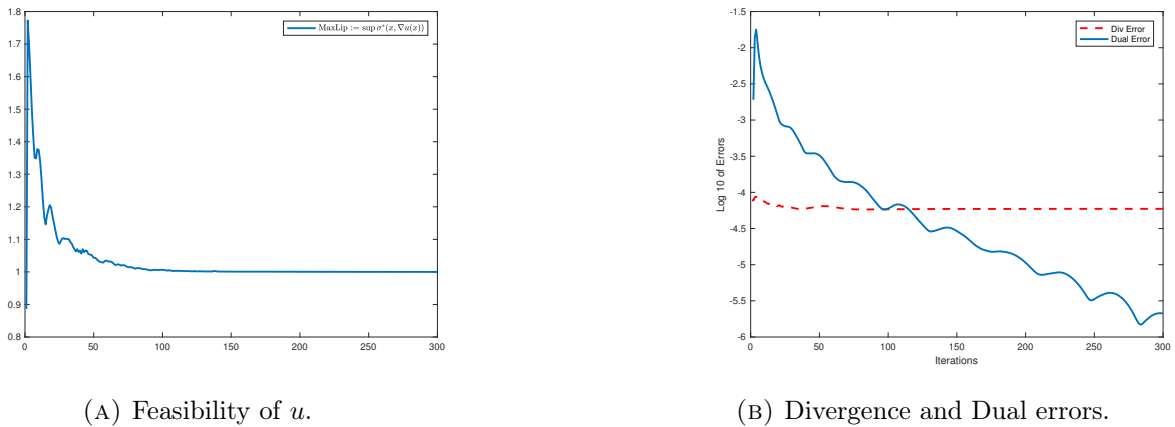


FIGURE 10. Error criterion for 300 iterations with $N_h = 80$.

5.6. **Polyhedral case.** Consider k vectors p_1, \dots, p_k and define for any $v \in \mathbb{R}^N$ the following Finsler metric

$$\sigma(v) = \max_{1 \leq i \leq k} \langle v, p_i \rangle$$

usually called a crystalline norm. We can easily check that the unit ball B^* of σ^* is nothing but the convex hull of the vectors p_1, \dots, p_k :

$$B^* = \text{conv}(p_1, \dots, p_k).$$

The projection onto B^* can be performed easily (see [3, 25]). We start by determining the vertices s_1, \dots, s_k of B^* and the corresponding outward normal vectors ν_i to the edges of B^* . Afterwards, if $v \notin B^*$, we distinguish to cases: either $v \in [s_i, s_{i+1}] + \mathbb{R}^+ \nu_i$, and in this case we project v onto the segment $[s_i, s_{i+1}]$, or it belongs to a sector $s_i + \mathbb{R}^+ \nu_i + \mathbb{R}^+ \nu_{i+1}$ and in this case its projection is s_i .

We perform a test with $p_1 = (1, -1), p_2 = (1, -0.8), p_3 = (-0.8, 1), p_4 = (-1, 1), p_5 = (-1, -1)$. We take $N_h = 64$ and 600 iterations.

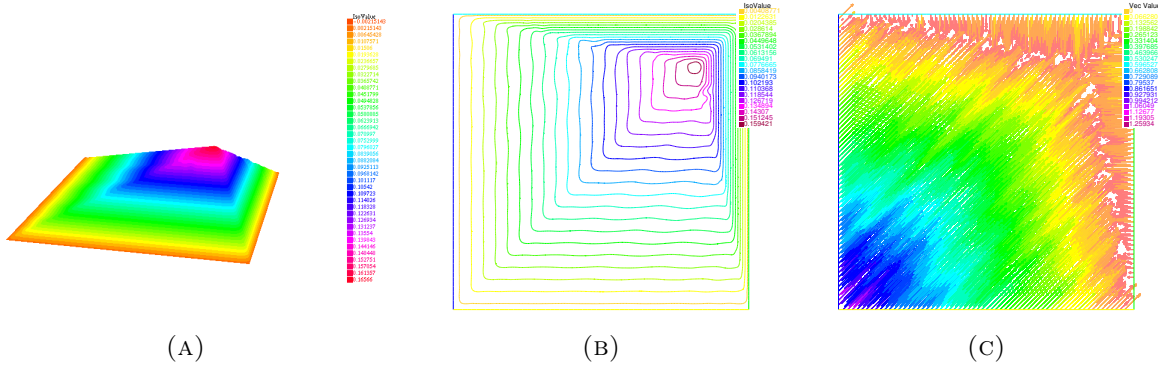
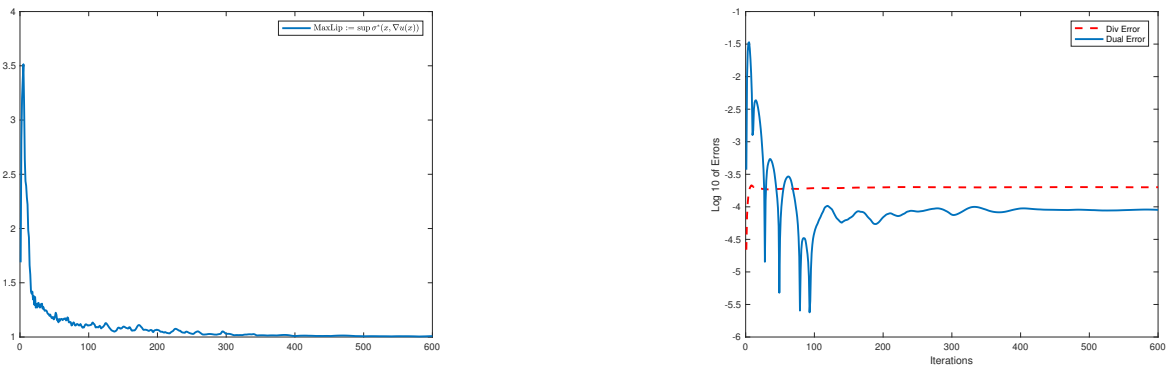


FIGURE 11. Left to right: 3D plot of the solution u , contour plot of u , the flux ϕ



(A) Feasibility of u .

(B) Divergence and Dual errors.

FIGURE 12. Error criterion for 600 iterations with $N_h = 64$.

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