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EXISTENCE, UNIQUENESS AND REGULARITY FOR THE STOCHASTIC ERICKSEN-LESLIE EQUATION

ANNE DE BOUARD, ANTOINE HOCQUET, AND ANDREAS PROHL

ABSTRACT. We investigate existence and uniqueness for the liquid crystal flow driven by colored noise on the two-dimensional torus. After giving a natural uniqueness criterion, we prove local solvability in L^p -based spaces, for every $p > 2$. Thanks to a bootstrap principle together with a Gyöngy-Krylov-type compactness argument, this will ultimately lead us to prove the existence of a particular class of global solutions which are partially regular, strong in the probabilistic sense, and taking values in the “critical space” $L^2 \times H^1$.

Keywords— stochastic partial differential equations, liquid crystals, non-linear parabolic equations, harmonic maps

Mathematics Subject Classification — 60H15, 76A15, 35K55, 58E20

1. INTRODUCTION

1.1. **Motivations.** Let $T > 0$ and $\mathbb{T}^2 = (0, 1)^2$. The simplified incompressible Ericksen-Leslie equations (EL) model the combined dynamics of a director field $u : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{S}^2$ (also referred to as “molecular orientation”) and a velocity field $v : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ of a thermotropic nematic liquid crystal which is contained in the domain \mathbb{T}^2 . The director field gives an averaged orientation of the constituent molecules (“mesogens”) of the liquid crystal phase to e.g. predict the evolution of a texture that exhibits several defects, and which could possibly annihilate or nucleate due to a nontrivial fluid flow dynamics [28]. In this work, we include a stochastic forcing term to the simplified Ericksen-Leslie equations to account for thermal fluctuation effects [22, 5, 3]. The noise acts on the molecular orientation so that the system is written as

$$\left\{ \begin{array}{l} dv - (\Delta v - v \cdot \nabla v - \operatorname{div}(\nabla u \odot \nabla u) - \nabla \pi) dt = 0 \\ \operatorname{div} v = 0 \\ du - (\Delta u - v \cdot \nabla u + |\nabla u|^2 u) dt = \nu u \times \circ dW_t \\ |u|_{\mathbb{R}^3} = 1 \text{ a.e.} \end{array} \right. \quad \text{in } (0, T] \times \mathbb{T}^2, \quad (\text{SEL})$$

where π is the hydrostatic pressure while $\nu \geq 0$ denotes a constant, and we assume that $(v(0), u(0))$ is given with finite energy. In the above, $W(\omega, t, x)$ stands for an \mathbb{R}^3 -valued, spatially correlated Wiener process which models random forces on the mesogens in the liquid crystal. The noise term appears inside a Stratonovitch integral, which allows the process $u \equiv u(\omega, t, x)$ to be \mathbb{S}^2 -valued. Throughout the paper, we will denote by $(\nabla u \odot \nabla u)$ the 2×2 matrix whose entries are given by

$$(\nabla u \odot \nabla u)_{i,j} = \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle_{\mathbb{R}^3}, \quad \text{for } 1 \leq i, j \leq 2.$$

The velocity process $v \equiv v(\omega, t, x)$ is divergence-free, and is driven by a non-trivial director field in the random Navier-Stokes equation. The system (SEL) is supposed to model (small-scale) fluctuating rotations of elongated molecules which are embedded into a liquid; a major motivation here is to study the stability of texture in the nematic phase, and mechanisms which trigger changes in the orientation of the director field.

The system (EL) of nonlinear PDEs (i.e., $\nu = 0$ in (SEL)) couples the convected harmonic map flow with the incompressible Navier-Stokes equation through the term $-\operatorname{div}(\nabla u \odot \nabla u)$. Its non-trivial interplay is essentially due to the latter quadratic term, since weak solutions of the harmonic map heat equation (i.e., $v \equiv 0$ and $\nu = 0$ in (SEL)₃) might exhibit a singular behavior at finite time $T_1 > 0$. According to [31, 32], such a singularity manifests as concentration of the “local energy” $\sup_{x \in \mathbb{T}^2} \frac{1}{2} \int_{B_\rho(x)} (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy$ of the regular local solution as t gets close to T_1 . This local solution may however be continued to a weak solution which is global in time, by choosing the weak limit $w\text{-}\lim_{t \uparrow T_1} u(t, \cdot)$ in $H^1(\mathbb{T}^2, \mathbb{S}^2)$ as new initial value for times $t \geq T_1$, and then proceeding inductively. This construction was used by Lin, Lin and Wang [20], Lin and Wang [25], or Hong [16]. A partially regular weak solution of (EL) was constructed, for which there exist at most finitely many times $0 < T_1 < \dots < T_L < \infty$ such that the local energy concentrates, i.e.,

$$\liminf_{t \uparrow T_j} \sup_{x \in \mathbb{T}^2} \frac{1}{2} \int_{B_\rho(x)} [v(t, y)|^2 + |\nabla u(t, y)|^2] dy \geq \varepsilon_1, \quad \forall \rho > 0, \quad (1.1)$$

for some geometric quantity $\varepsilon_1 > 0$ (depending only on the domain). We also mention the works of Lin and Liu [18, 19] (see also [34]) where (EL) is approximated via a Ginzburg-Landau penalization term, allowing to relax the constraint $|u| = 1$. In this setting, the third equation in (SEL) changes, for some positive ϵ , to

$$du^\epsilon = \left(\Delta u^\epsilon - [v^\epsilon \cdot \nabla] u^\epsilon - \frac{1}{\epsilon} f(u^\epsilon) \right) dt + \nu u^\epsilon \times \circ dW, \quad \text{in } \mathbb{T}^2 \times (0, \infty) \quad (1.2)$$

where $f(x) = \nabla F(x)$ and $F(x) = \frac{1}{4} \int_{\mathbb{T}^2} (|x|^2 - 1)^2 dx$. For $\nu = 0$, the director field u^ϵ is more regular, which allows to construct a weak solution by a Galerkin method through uniform bounds for the related energy [18, 19]; however, passing to the limit $\epsilon \downarrow 0$ is an open problem, mostly due to the quadratic term in the first equation in (SEL) that was discussed above; see also [19, Thm. 7.1] for the deterministic case.

Solvability of the SPDE given by a stochastic perturbation of the penalized (EL) equation is established in [3]; the authors use the Lyapunov structure of the problem to construct a global strong solution by continuation of a local in time mild solution, and Itô’s formula to verify \mathbb{P} -a.s. non-negativity of $1 - |u|$ in space and time; a weak martingale solution to the 3D system is constructed by a Galerkin method in [2], and it is shown to be the unique strong solution in the two-dimensional setting. All these results were obtained for $\epsilon > 0$ fixed.

In this work, we construct a partially regular, global solution for (SEL), which is strong in the usual probabilistic sense, and which for any time lies in the critical space $L^2(\mathbb{T}^2; \mathbb{R}^2) \times H^1(\mathbb{T}^2; \mathbb{R}^3)$, see Definition 1.1 below. In addition, we will see that the solutions of (SEL) are unique under some integrability property which, to the best of our knowledge, is new even in the deterministic context (it suffices to let

$\nu = 0$ in (SEL)). Thanks to a classical interpolation inequality (see (2.1)), the partially regular solutions constructed above fulfill (correspondingly) the integrability condition (1.16), which implies in turn that these solutions are unique in the class described in Theorem 1.2.

From the point of view of stochastic analysis, the strategy employed to prove our main *existence* theorem (i.e. Theorem 1.2 below) benefits from the arguments of the second author's previous contribution [15]. While some of the key ideas (e.g. local energy estimates and bootstrap) may appear to be directly adapted from the proof of Theorem 2 in [15], we point out that the presence of a velocity component requires to face a significant amount of hurdles, due to two basic facts. First, the local energy step requires to test (SEL) formally against a localization function of the form $\mathbf{1}_{B(x,\rho)}$, which fails to be divergence-free in general. This in turn requires a careful analysis of the “pressure” term, which will be carried out in Section 4.2. This particular pitfall is also responsible for increasing the complexity of the bootstrap argument done in Section 5.3, in comparison with that of the stochastic harmonic map flow. Second, taking the limit in the equation as the mollification of the data $(v(0), u(0), \psi)$ is removed requires a calculation trick which unveils additional cancellations between the nonlinearities associated with two different solutions. To the best of our knowledge, this observation (summarized succinctly in the formula (6.17)) is new in this context. The resulting convergence which is stated in Corollary 6.1 is here particularly emphasized, which also has the merit to fill a gap in [15] concerning the uniform integrability property (6.24) (the counterpart for the stochastic harmonic map flow is obtained by ignoring the velocity).

An interesting extension of the model (SEL) could include an additive noise term dV_t in the right hand side of the velocity equation. In fact if V is a divergence-free, $L^2(\mathbb{T}^2; \mathbb{R}^2)$ -Wiener process, it is easy to convince oneself that, modulo some notational difficulties, the proofs below could be carried out *mutatis mutandis*. Since the paper is already rather technical, we here stuck to (SEL) and leave the proof of such an extension to the reader. We nevertheless point out that, due to the important scale differences between the velocity and the molecular orientation, our model (SEL) is physically relevant, as for instance discussed in [22]. Similarly, it should be possible to deal with more general domains and boundary conditions, but we chose to restrict ourselves to the torus for simplicity.

After completing the manuscript, we were told about the existence of [4], which deals with local existence and uniqueness of strong solutions to equations (SEL) with in addition a (multiplicative) correlated noise in the Navier-Stokes equation, using arguments similar to those of Section 3. However, although the equation in [4] is more general than (SEL), our results go further in the analysis, since we have a better control on the bubbling time (see Section 6.3), and we are able to construct a unique global weak solution.

Organization of the paper. Section 1.2 introduces the used notations, while in Section 1.3 we define two different notions of solution and provide our main solvability results (theorems 1.1 and 1.2). For the reader's convenience, in Section 1.4 we will explain the main ideas in the proof of Theorem 1.2 and provide a “sketch of proof”. The proof of Theorem 1.1 will be given in Section 2. Section 3 shows local solvability for (SEL), using a fixed point argument in the case where the data are

“subcritical”. In Section 4, we derive a priori estimates, while a bootstrap principle will be shown in Section 5. The proof of Theorem 1.2 will be addressed in Section 6. Finally, computational details related to Itô-Stratonovitch corrections of the form (1.10) will be given in Appendix A.

1.2. Notation. Throughout the paper the symbol $T \in (0, \infty)$ refers to a fixed, deterministic time horizon. If u, v are measurable maps, the notation “ $u \times w$ ” refers to the pointwise vector product, namely for each $i \in \{1, 2, 3\} \simeq \mathbb{Z}/3\mathbb{Z}$ we let $(u \times w)^i := u^{i+1}w^{i+2} - u^{i+2}w^{i+1}$.

For $p \in [1, \infty]$, we will denote by L^p the usual Lebesgue spaces. We will make use of the Sobolev-Slobodeckij spaces $W^{\alpha,p}$ defined as usual for $\alpha \geq 0$ and by duality as $W^{\alpha,p} = (W^{-\alpha, \frac{p}{p-1}})^*$ whenever $p \in (1, \infty]$ and $\alpha < 0$. For notational simplicity, we also denote $H^\alpha := W^{\alpha,2}$, for $\alpha \in \mathbb{R}$. Throughout the paper, we denote by $\mathbb{T}^2 \equiv (0, 1)^2$ the two-dimensional torus, and for $d \geq 1$ the notations

$$L^p(\mathbb{T}^2; \mathbb{R}^d), \quad W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^d), \quad H^\alpha(\mathbb{T}^2; \mathbb{R}^d),$$

will be used as shorthands for the spaces $L^p_{\text{per}}(\mathbb{R}^2; \mathbb{R}^d)$, $W^{\alpha,p}_{\text{per}}(\mathbb{R}^2; \mathbb{R}^d)$, $H^\alpha_{\text{per}}(\mathbb{R}^2; \mathbb{R}^d)$, consisting of 1-periodic elements $f : \mathbb{R}^2 \rightarrow \mathbb{R}^d$ that belong to the corresponding local Sobolev space (endowed with the appropriate topology). For any square-integrable $f, g : \mathbb{T}^2 \rightarrow \mathbb{R}^d$, we denote

$$\langle f, g \rangle := \int_{\mathbb{T}^2} f(x) \cdot g(x) dx \equiv \sum_{i=1}^d \int_{(0,1)^2} f^i(x) g^i(x) dx,$$

and we denote by the same bracket the bilinear mapping

$$\langle \cdot, \cdot \rangle : (W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^d))^* \times W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^d) \longrightarrow \mathbb{R}, \quad (f, g) \longmapsto \langle f, g \rangle := f(g).$$

Given a Banach space E , the space of continuous paths $u : [0, T] \rightarrow E$, endowed with the supremum norm will be denoted by $C(0, T; E)$. If $p \in [1, \infty)$ we similarly denote $L^p(0, T; E)$ the space of Bochner p -integrable functions with values in E .

Following [33], we introduce the linear space

$$\mathcal{V} := \{f \in C^\infty(\mathbb{T}^2; \mathbb{R}^2), \text{ div } f = 0\}.$$

For $p \in (1, \infty]$ and $\alpha \in \mathbb{R}$, we denote by $\mathbb{W}^{\alpha,p}$ the completion of \mathcal{V} with respect to the norm of $W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^2)$, and we further let $\mathbb{H}^\alpha := \mathbb{W}^{\alpha,2}$. For $\alpha = 0$ we will also use the notation $\mathbb{L}^p := \mathbb{W}^{0,p}$. Similarly, and in order to distinguish between the velocity component v and the molecular orientation u , we adopt the notations $\mathbf{W}^{\alpha,p} := W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^3)$, $\mathbf{H}^\alpha := W^{\alpha,2}(\mathbb{T}^2; \mathbb{R}^3)$, and $\mathbf{L}^p := L^p(\mathbb{T}^2; \mathbb{R}^3)$.

Given a two-dimensional vector field $f \in L^2(\mathbb{T}^2; \mathbb{R}^2)$, we denote by $\mathcal{P}f$ its linear projection onto the space \mathbb{L}^2 and the same notation is used for its unique extension as a linear map $\mathcal{P} : W^{\alpha,p}(\mathbb{T}^2; \mathbb{R}^2) \rightarrow \mathbb{W}^{\alpha,p}$ for every $\alpha \in \mathbb{R}$ and $p \in (1, \infty]$. For convenience, note that $\mathcal{P}f$ is explicitly given on the torus by the formula

$$\widehat{\mathcal{P}f}(k) = \left(\text{id} - \frac{kk^T}{k_1^2 + k_2^2} \right) \hat{f}(k), \quad k \equiv (k_1, k_2) \in \mathbb{Z}^2, \quad (1.3)$$

where $\hat{f}(k) := \int_{\mathbb{T}^2} f(x) \exp(2i\pi k \cdot x) dx$ denotes the spatial Fourier Transform, and we further recall that \mathcal{P} is continuous (see for instance [23]).

When $p \in (1, \infty]$ we further denote by $A : D(A) \subset \mathbb{W}^{-1,p} \rightarrow \mathbb{W}^{-1,p}$ the Stokes operator, that is the operator defined as

$$Av = -\mathcal{P}\Delta v, \quad \text{for } v \in D(A) := \mathbb{W}^{1,p}.$$

If in addition I denotes a fixed time-interval we define the Banach spaces

$$\mathcal{V}_I^\alpha := C(I; \mathbb{H}^\alpha) \cap L^2(I; \mathbb{H}^{\alpha+1}), \quad \text{and} \quad \mathcal{U}_I^\alpha := C(I; \mathbf{H}^\alpha) \cap L^2(I; \mathbf{H}^{\alpha+1}). \quad (1.4)$$

The notation “ $v \in \mathcal{V}_{\text{loc};I}^\alpha$ ” means that v belongs to \mathcal{V}_J^α for every compact interval $J \subset I$. Whenever $I = [0, T]$ for some deterministic $T > 0$, we also use the abbreviation $\mathcal{V}_T^\alpha := \mathcal{V}_{[0,T]}^\alpha$. We will use similar notations for \mathcal{U} .

Given a Banach space B , the set of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (i.e. measurable maps from $\Omega \rightarrow B$ w.r.t. \mathcal{A}) will be denoted by $L^0(\Omega; B)$. If $0 \leq \tau_1 < \tau_2 \leq T$ are stopping times with respect to some filtration on Ω , we use the notation

$$v \in L^0(\Omega; \mathcal{V}_{\text{loc};[\tau_1, \tau_2]}^\alpha)$$

if and only if $v(\cdot \wedge \sigma) \in L^0(\Omega; \mathcal{V}_{[\tau_1, T]}^\alpha)$ for every stopping time $\tau_1 \leq \sigma < \tau_2$, and we define the space $\mathcal{U}_{\text{loc};[\tau_1, \tau_2]}^\alpha$ in a similar fashion. Similarly, if $z : \Omega \times [0, T] \rightarrow X$ is a stochastic process with values in some Banach space X , $\tau > 0$ is a stopping time, and $q \in [1, \infty]$, we write $z \in L^q(\Omega; C([0, \tau]; X))$ to indicate that the stopped process $z(\cdot \wedge \sigma)$ belongs to $L^q(\Omega; C(0, T; X))$ for any stopping time $\sigma < \tau$.

Given a Hilbert space K , we denote by $\mathcal{L}_2(L^2, K)$ the class of Hilbert-Schmidt linear maps $\Theta : L^2 \rightarrow K$, i.e. such that

$$|\Theta|_{\mathcal{L}_2(L^2, K)}^2 := \sum_{l \in \mathbb{N}} |\Theta f_l|_K^2 < \infty,$$

where in the sequel $(f_l)_{l \in \mathbb{N}}$ is a given orthonormal basis of the Hilbert space $L^2(\mathbb{T}^2; \mathbb{R})$. Similarly, given a Banach space Y , we denote by $\gamma(Y)$ the space of γ -radonifying operators from L^2 to Y , namely $\Theta \in \gamma(Y)$ if and only if

$$|\Theta|_{\gamma(Y)}^2 := \int_{\tilde{\Omega}} \left| \sum_{l \in \mathbb{N}} \gamma_l(\tilde{\omega}) \Theta f_l \right|_Y^2 \hat{\mathbb{P}}(d\tilde{\omega}) < \infty,$$

for every independent, identically distributed normal family $(\gamma_l)_{l \in \mathbb{N}}$ on some probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$.

1.3. Assumption on the noise and main results. Before we introduce an appropriate notion of solution for (SEL), we need to make some assumptions on the noise term. In what follows, we will denote by $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ a filtered probability space satisfying the usual assumptions.

Assumption 1.1. *Assume that we are given $\psi \in \mathcal{L}_2(L^2, H^1)$, and let $W : \Omega \times [0, T] \rightarrow \mathbf{H}^1$ be the (\mathcal{F}_t) -adapted Wiener process given by the infinite series*

$$W(\omega, t, x) = \sum_{l \in \mathbb{N}} \mathbf{B}_l(\omega, t) \psi f_l(x), \quad (1.5)$$

where $(f_l)_{l \in \mathbb{N}}$ denotes a fixed complete orthonormal system for the Hilbert space $L^2(\mathbb{T}^2; \mathbb{R})$, while $(\mathbf{B}_l)_{l \in \mathbb{N}}$ is a given family of independent and identically distributed Brownian motions in \mathbb{R}^3 .

In the sequel, we will make use of the notation

$$\psi_l := \psi f_l \in L^2(\mathbb{T}^2; \mathbb{R}), \quad (1.6)$$

while for $j = 1, 2, 3$, we set

$$\boldsymbol{\psi}_l^j := \psi_l \mathbf{e}_j \in \mathbf{L}^2, \quad (1.7)$$

where

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \text{ is the canonical basis of } \mathbb{R}^3. \quad (1.8)$$

Note that Assumption 1.1 leads to the following expression for the covariance of W :

$$\mathbb{E} [\langle W(t), a \rangle \langle W(s), b \rangle] = \min(t, s) \sum_{i,j=1}^3 \langle \psi^* a^i, \psi^* b^j \rangle_{L^2(\mathbb{T}^2; \mathbb{R})}, \quad (1.9)$$

for all $a, b \in \mathbf{L}^2$, and every $(s, t) \in [0, T]^2$.

In order to define a solution (v, u) of (SEL), it is convenient to switch to an Itô equation. For $\Phi \in C^1(L^2; \mathcal{L}_2(L^2, L^2))$, we have the Itô-Stratonovitch conversion formula:

$$\int \Phi(u) \circ dW = \int \Phi(u) dW + \frac{1}{2} \int \sum_{l \in \mathbb{N}} \sum_{j=1}^3 [\Phi'(u) \cdot (u \times \boldsymbol{\psi}_l^j)] (\boldsymbol{\psi}_l^j) dt, \quad (1.10)$$

where $\boldsymbol{\psi}_l^j$ is as in (1.7) and here “ \circ ” denotes the usual Stratonovitch product. In particular, an immediate computation using (A.1) gives the formula $\int u \times \circ dW = \int u \times dW + \int F_\psi u dt$ where we denote

$$F_\psi(x) := - \sum_{l \in \mathbb{N}} \psi_l(x)^2, \quad x \in \mathbb{T}^2. \quad (1.11)$$

Definition 1.1. Let \mathfrak{P} be a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$, and denote by $W(\omega, t, x)$ an \mathbf{L}^2 -Wiener process whose covariance verifies (1.9) for some $\psi \in \mathcal{L}_2(L^2, L^2)$.

We will call (v, u) a *solution* with respect to (\mathfrak{P}, W) if the following properties are fulfilled:

- (i) $(v, u) : \Omega \times [0, T] \rightarrow \mathbb{H}^{-1} \times \mathbf{L}^2$ is a progressively measurable process with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$, such that in addition

$$v \in L^0(\Omega; \mathcal{V}_{\text{loc}; [\tau_1, \tau_2]}^{-1}), \quad u \in L^0(\Omega; \mathcal{U}_{\text{loc}; [\tau_1, \tau_2]}^0),$$

(in particular we have $\text{div} v = 0$ by definition of the spaces \mathcal{V}^{-1} and \mathbb{H}^{-1});

- (ii) the following constraint holds on u :

$$|u(\omega, t, x)| = 1 \quad \text{for almost every } (\omega, t, x) \in \Omega \times [0, T] \times \mathbb{T}^2; \quad (1.12)$$

- (iii) we have

$$\mathbb{E} \left[\int_0^T (|v \cdot \nabla v|_{H^{-2}}^2 + |\nabla u \odot \nabla u|_{H^{-1}}^2 + |\Delta u + u|\nabla u|^2|_{H^{-1}}^2) ds \right] < \infty ; \quad (1.13)$$

- (iv) for every $t \in [0, T]$, \mathbb{P} -a.s.:

$$v(t) - v(0) + \int_0^t [Av + \mathcal{P}(v \cdot \nabla v + \text{div} \nabla u \odot \nabla u)] ds = 0 \quad (1.14)$$

and

$$u(t) - u(0) - \int_0^t [\Delta u + |\nabla u|^2 u - v \cdot \nabla u + F_\psi u] ds = \int_0^t u \times dW(s), \quad (1.15)$$

where (1.14) is to be understood in the Bochner sense in H^{-2} , while the integrals in (1.15) are taken in the Bochner sense (respectively Itô sense), in H^{-1} .

Furthermore, let τ_1 and τ_2 be two stopping times with respect to $(\mathcal{F}_t)_{t \in [0, T]}$ such that \mathbb{P} -as $0 \leq \tau_1 < \tau_2 \leq T$. We will say that (v, u) is a *local solution* on $[\tau_1, \tau_2)$, provided (i), (ii) and (iii) hold, but (iv) is replaced by the relations (1.14)-(1.15) for $t \in [\tau_1(\omega), \tau_2(\omega))$, for \mathbb{P} -almost every $\omega \in \Omega$.

Having this notion at hand, we are now in position to state our first main theorem. The following result is concerned with uniqueness of weak solutions, conditionally to appropriate moment bounds. As already alluded to, the corresponding uniqueness criterion will be crucially used in the existence part. It is however of interest in its own right, whether or not stochasticity.

Theorem 1.1 (Uniqueness). *Fix a stochastic basis (\mathfrak{P}, W) as in Definition 1.1. Let (v_j, u_j) , $j = 1, 2$, be solutions with respect to (\mathfrak{P}, W) , starting from the same initial datum $(v(0), u(0)) \in \mathbb{H}^{-1} \times \mathbf{L}^2$. Assume that $\psi \in \mathcal{L}_2(L^2, L^2)$ and suppose furthermore that for $j = 1, 2$, it holds*

$$v_j \in L^4(\Omega; L^4(0, T; \mathbb{L}^4)), \quad u_j \in L^4(\Omega; L^4(0, T; \mathbf{W}^{1,4})). \quad (1.16)$$

Then, we have $(v_1, u_1) = (v_2, u_2)$.

Before we state our second result, we need another (stronger) notion of solution.

Definition 1.2 (strong solution). Fix a stochastic basis (\mathfrak{P}, W) , $\mathfrak{P} = (\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in [0, T]})$, and let τ_1 and τ_2 be two stopping times with respect to $(\mathcal{F}_t)_{t \in [0, T]}$ such that $0 \leq \tau_1 < \tau_2 \leq T$, \mathbb{P} -a.s. We say that (v, u) is a *strong solution* of (SEL) on $[\tau_1, \tau_2)$ if the following holds

- (i) (v, u) is a local solution on $[\tau_1, \tau_2)$ with respect to (\mathfrak{P}, W) , in the sense of Definition 1.1;
- (ii) it has the additional regularity

$$v \in L^0(\Omega; \mathcal{V}_{\text{loc}; [\tau_1, \tau_2)}^0), \quad u \in L^0(\Omega; \mathcal{U}_{\text{loc}; [\tau_1, \tau_2)}^1).$$

We are interested in a special class of local strong solutions, which is uniquely characterized by the “forward bubbling” property. In [15], these were referred to as “Struwe solutions”, in analogy with the deterministic theory in [31].

Theorem 1.2 (Existence and uniqueness of strong solutions). *Let $(v_0, u_0) \in \mathbb{L}^2 \times \mathbf{H}^1$, $\psi \in \mathcal{L}_2(L^2, H^1)$, and assume that $|u_0|_{\mathbb{R}^3} = 1$ almost everywhere. Fix a stochastic basis (\mathfrak{P}, W) , where $\mathfrak{P} = (\Omega, \mathcal{A}, \mathbb{P})$ and (W, ψ) satisfies Assumption 1.1.*

There exists a solution (v, u) of (SEL), starting from the initial datum (v_0, u_0) , and a random variable $J \in L^0(\Omega; \mathbb{N})$ such that the following properties hold:

- (P1) *There is a sequence of stopping times $0 \equiv \tau_0 < \tau_1 < \dots < \tau_J \equiv T$ with the property that for each $j \in \{0, \dots, J-1\}$,*

$$(v, u)|_{[\tau_j, \tau_{j+1})} \text{ is supported in } \mathcal{V}_{\text{loc}; [\tau_j, \tau_{j+1})}^0 \times \mathcal{U}_{\text{loc}; [\tau_j, \tau_{j+1})}^1,$$

and has moments of all order in this space. In particular the solution is strong on each subinterval $[\tau_j, \tau_{j+1})$ for $j \in \{0, \dots, J-1\}$.

(P2) For each $j \in \{0, \dots, J-1\}$, the random variable $(v(\tau_{j+1}), u(\tau_{j+1}))$ belongs to the space $L^2(\Omega, \mathbb{L}^2 \times \mathbf{H}^1)$. Moreover, for every sequence of stopping times $\{\sigma_j^k, k \in \mathbb{N}\}$ such that $\mathbb{P}(\sigma_j^k \nearrow \tau_{j+1}) = 1$, we have

$$(v(\sigma_j^k), u(\sigma_j^k)) \xrightarrow[k \rightarrow \infty]{} (v(\tau_{j+1}), u(\tau_{j+1})) \text{ weakly in } L^2(\Omega; \mathbb{L}^2 \times \mathbf{H}^1).$$

(P3) The solution (v, u) may become singular at $t = \tau_{j+1} - 0$, for each $j \in \{0, \dots, J-1\}$. Namely, we have the alternative that either $\mathbb{P}(J=1) = 1$, or $\mathbb{P}(J > 1) > 0$ in which case the solution “bubbles forward” in the following sense:

$$\inf_{\rho > 0} \sup_{t \nearrow \tau_{j+1}} \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho)} (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy > 0, \quad \mathbb{P}\text{-a.s. on } \{\tau_{j+1} < T\}. \quad (1.17)$$

(P4) The above solution is unique in the class described by (P1), (P2) and (P3).

1.4. Strategy of the proof of Theorem 1.2. The existence part generalizes arguments for the deterministic case from [20, 16, 25] to the SPDE (SEL), and benefits from [15]. As it turns out, solutions are arbitrarily regular in the space-like variable (as permitted by the data), as could be easily seen by a higher order generalization of Theorem 5.1. As this fact does not play any specific role in the proof of Theorem 1.2, we will restrict ourselves to show that, under the assumptions that $(v(0), u(0)) \in \mathbb{H}^2 \times \mathbf{H}^3$ and $\psi \in \mathcal{L}_2(L^2, H^3)$, the full trajectory takes values in $\mathbb{H}^2 \times \mathbf{H}^3$, up to $t = \zeta(\rho)$, where $\zeta(\rho)$ denotes the concentration time defined in (5.1) for any fixed $\rho > 0$. This property will be obtained as a consequence of the bootstrap argument shown in Section 5.

Before proving Theorem 1.2, we will first need local solvability results in the “subcritical case” i.e. when

$$(v(0), u(0)) \in \mathbb{L}^p \times \mathbf{W}^{1,p}, \quad \text{and} \quad \psi \in \gamma(W^{\alpha,p}) \text{ with } \alpha > 2/p,$$

for some $p > 2$. For $\psi \in \mathcal{L}_2(L^2, H^\alpha)$, α being sufficiently large, $(v(0), u(0)) \in \mathbb{W}^{s-1,p} \times \mathbf{W}^{s,p}$ with $s \in [1, 3]$ and $p > 2$, we show that the Cauchy problem for (SEL) is locally well-posed, i.e. up to some stopping time $\tau^{s,p}$, and that the trajectories of (v, u) belong to $C([0, \tau^{s,p}); \mathbb{W}^{s-1,p} \times \mathbf{W}^{s,p})$. The proof of these facts will be done in Section 3 via a fixed point argument and for an appropriate mild formulation for (SEL).

The proof of Theorem 1.2 will be addressed progressively in sections 4, 5 and 6. These steps are summarized below.

Step 1: A priori estimates. In a first step, we shall collect a priori estimates on the “energy” $E(t) := \frac{1}{2}(|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2)$ associated to a strong solution (v, u) . To prove this, we use the fact that the quantity

$$\frac{1}{2} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2) + \int_0^t (|\nabla v(s)|_{L^2}^2 + |\Delta u + u|\nabla u|_{L^2}^2(s)) ds, \quad (1.18)$$

modulo correction by a suitable semi-martingale term, is preserved along the flow. In particular, the energy estimate comes with an $L_t^2(L_x^2)$ estimate on the “tension”

$$\mathcal{T} := \Delta u + u|\nabla u|^2. \quad (1.19)$$

We will also derive “local estimates” for this energy, where the locality is to be understood in the spatial sense, i.e. on small balls $B(x, \rho)$ for $x \in \mathbb{T}^2$ and $\rho > 0$. These estimate will also prove useful in Section 6.3, when we will show that the concentration time is non-trivial for a limit of approximate solutions, i.e. $\mathbb{P}(\zeta(\lim v_n, \lim u_n; \rho) = 0) = 0$.

Step 2: Bootstrap. In a second step, we will prove a priori estimates that are local *in time*, i.e. up to some energy-concentration time $\zeta(v, u; \rho) > 0$. Given a local solution (v, u, τ) and $\rho > 0$, the energy-concentration time $\zeta(v, u; \rho)$ is defined as the first time $t \in [0, \tau)$ such that there exists $x \in \mathbb{T}^2$ for which the local energy

$$\varepsilon(t, \rho, x) := \frac{1}{2} \int_{B(x, \rho)} (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy$$

attains a threshold $\varepsilon_1 > 0$ (the exact value of ε_1 is related to the optimal constant in (1.20)). The core of the argument is then to show that, for this value of ε_1 and for every $t \leq \zeta(v, u; \rho)$, the $L^2(0, T; \mathbf{H}^2)$ -norm of $u(\cdot \wedge \zeta)$ has moments of arbitrary order, estimated above by a constant that only depends on $E(0)$, $|\psi|_{\mathcal{L}_2(L^2, H^1)}$ and ρ . This fact will follow from the very definition of $\zeta(v, u; \rho)$, and the following inequality due to Struwe [31]: if $u \in L^2(0, T; \mathbf{H}^2)$, it holds

$$\iint_{[0, T] \times \mathbb{T}^2} |\nabla u|^4 dy dt \leq \mu_1 \left(\sup_{\substack{t \in [0, T] \\ x \in \mathbb{T}^2}} \int_{y \in B(x, \rho)} |\nabla u(t, y)|^2 dy \right) \iint_{[0, T] \times \mathbb{T}^2} \left(|\Delta u|^2 + \frac{|\nabla u|^2}{\rho^2} \right) dy dt \quad (1.20)$$

The $L_t^2(\mathbf{H}_x^2)$ -estimate then follows rather easily from (1.18), (1.20) and the fact that, since $\mathcal{F} \perp u$ (as can be seen from the norm constraint $|u|_{\mathbb{R}^3} = 1$ a.e.), it holds

$$|\mathcal{F}|_{\mathbb{R}^3}^2 = \mathcal{F} \cdot \Delta u = |\Delta u|_{\mathbb{R}^3}^2 - |\nabla u|_{\mathbb{R}^{3 \times 2}}^4.$$

Together with the estimate on the energy E , this will allow us to obtain suitable bounds in the “critical space” $\mathcal{V}_T^0 \times \mathcal{U}_T^1$, locally in time.

Step 3: Construction of the local strong solution by approximation. We will approximate the equation (SEL) by considering the local solution (v_n, u_n, τ_n) to a problem with smooth data $(v_n(0), u_n(0); \psi_n)$. The existence and uniqueness of such local solutions for each $n \in \mathbb{N}$ is ensured by the previous local solvability results. By the above steps, for any $\delta < 1$, the image laws of the sequence $\{(v_n, u_n), n \in \mathbb{N}\}$ forms a weakly compact set in the space $\mathcal{V}_{\text{loc}}^{\delta-1} \times \mathcal{U}_{\text{loc}}^\delta$ and therefore there exists – up to a change of probability space in the use of the Skorohod embedding theorem – a limiting process (\hat{v}, \hat{u}) which will be shown to provide a local strong solution. This solution is regular enough to satisfy the assumptions of the uniqueness theorem, and therefore, any jointly converging subsequence $((v_{n_\ell}, u_{n_\ell}), (v_{m_\ell}, u_{m_\ell}))$ should, in the limit, be supported on the diagonal of $\mathcal{V}_{\text{loc}}^0 \times \mathcal{U}_{\text{loc}}^1$. By Gyöngy-Krylov Theorem, this will show that the solution is in fact probabilistically strong, up to the singular time $\zeta(v, u; \rho) = \lim_{n \rightarrow \infty} \zeta(v_n, u_n; \rho)$. The local solution $(v, u; \tau)$ will be then constructed by letting $\tau_1 := \lim_{\rho \rightarrow 0} \zeta(v, u; \rho)$, and then by induction on each $[\tau_j, \tau_{j+1})$, by taking as initial datum the weak limit of (v, u) as $t \nearrow \tau_j$ (see the details in Section 6.4).

2. PROOF OF UNIQUENESS

2.1. Main interpolation inequality. The following well-known interpolation inequality will be crucial in the sequel: there exists a constant $\mu_0 > 0$, such that for every $\phi \in H^1$,

$$\int_{\mathbb{T}^2} |\phi|^4 dx \leq \mu_0 \left(\int_{\mathbb{T}^2} |\phi|^2 dx \right) \left(\int_{\mathbb{T}^2} [|\phi|^2 + |\nabla \phi|^2] dx \right). \quad (2.1)$$

For a proof, we refer e.g. to [24, II Thm. 2.2].

2.2. Proof of Theorem 1.1. Let $(v_1, u_1), (v_2, u_2)$ be two solutions with respect to (\mathfrak{B}, W) , starting from the same initial datum. Denote further by $g = v_1 - v_2$, $f = u_1 - u_2$, and $U = u_1 + u_2$. We have the system

$$\begin{cases} dg + Agdt = \mathcal{P}[-g \cdot \nabla v_2 - v_1 \cdot \nabla g - \operatorname{div}(\nabla f \odot \nabla u_2 + \nabla u_1 \odot \nabla f)] dt \\ df - \Delta f dt = [f|\nabla u_1|^2 + u_2 \nabla f \cdot \nabla U - g \cdot \nabla u_1 - v_2 \cdot \nabla f] dt + f \times \circ dW \\ g(0) = f(0) = 0. \end{cases}$$

Our strategy is to apply the Itô Formula [27, Theorem 4.2.5], for a suitable ‘‘Gelfand triple’’ $V \subset H \subset V^*$.

By Definition 1.1, the stochastic processes

$$\begin{aligned} Y &:= \mathcal{P}[-g \cdot \nabla v_2 - v_1 \cdot \nabla g - \operatorname{div}(\nabla f \odot \nabla u_2 + \nabla u_1 \odot \nabla f)] \\ \tilde{Y} &:= f|\nabla u_1|^2 + u_2 \nabla f \cdot \nabla U - g \cdot \nabla u_1 - v_2 \cdot \nabla f \\ Z &:= u \times (\psi(\cdot)), \end{aligned}$$

are progressively measurable. Furthermore, using (1.13) we see that $Z \in L^2(\Omega \times [0, T]; \mathcal{L}_2(L^2, L^2))$, and thus

$$(0, Z) \in L^2(\Omega \times [0, T]; \mathcal{L}_2(U; \mathbf{L}^2)), \quad \text{where } U = \{0\} \times \mathbf{L}^2.$$

Next, observe that the shifted operator

$$\Lambda := I + A, \quad (2.2)$$

defines a continuous isomorphism between \mathbb{H}^1 and \mathbb{H}^{-1} (as is classical, see e.g. [12]) and introduce the spaces $H := \mathbb{H}^{-1} \times \mathbf{L}^2$, $V := \mathbb{L}^2 \times \mathbf{H}^1$, where H is endowed with the norm

$$|(g, f)|_H := |\Lambda^{-1/2} g|_{L^2}^2 + |f|_{L^2}^2. \quad (2.3)$$

With this definition, the Riesz isomorphism $i : H \rightarrow H^*$ allows for the identification $V^* \simeq \mathbb{H}^{-2} \times \mathbf{H}^{-1}$. As a consequence, using (1.16), it is easily seen that $(Y, \tilde{Y}) \in L^2(\Omega \times [0, T]; V^*)$ and therefore we have all in hand to apply [27, Theorem 4.2.5].

This yields the relation:

$$\begin{aligned}
& \frac{1}{2}(|\Lambda^{-1/2}g(t)|_{L^2}^2 + |f(t)|_{L^2}^2) + \int_0^t (|g|_{L^2}^2 + |\nabla f|_{L^2}^2) ds \\
&= \iint_{[0,t] \times \mathbb{T}^2} \left[\Lambda^{-1/2}(-g \cdot \nabla v_2) \cdot \Lambda^{-1/2}g - \Lambda^{-1/2}(v_1 \cdot \nabla g) \cdot \Lambda^{-1/2}g \right. \\
&\quad \left. + (\nabla f \odot \nabla u_2) \cdot \nabla \Lambda^{-1}g + (\nabla u_1 \odot \nabla f) \cdot \nabla \Lambda^{-1}g \right. \\
&\quad \left. + |f|^2 |\nabla u_1|^2 + u_2 \cdot f(\nabla f \cdot \nabla U) - f \cdot (g \cdot \nabla u_1) - f \cdot (v_2 \cdot \nabla f) \right] dx ds \\
&= \sum_{\gamma=1}^8 I_\gamma,
\end{aligned}$$

and we now evaluate each term separately. Concerning the first term, we have using that $\operatorname{div} g = 0$

$$\begin{aligned}
I_1 &= \iint_{[0,t] \times \mathbb{T}^2} (\Lambda^{-1/2} \partial_i (g^i v_2^j)) (\Lambda^{-1/2} g^j) dx ds \\
&\leq C \int_0^t |\partial_i (g^i v_2)|_{W^{-1,4/3}} |g|_{W^{-1,4}} ds \\
&\leq C \int_0^t |g|_{L^2} |v_2|_{L^4} |g|_{W^{-1,4}} ds.
\end{aligned} \tag{2.4}$$

Now, observe that an immediate generalization of (2.1) yields the existence of $\tilde{\mu}_0 > 0$ such that for any $\phi \in W^{-1,4}$:

$$|\phi|_{W^{-1,4}}^2 \leq \tilde{\mu}_0 |\phi|_{H^{-1}} |\phi|_{L^2}. \tag{2.5}$$

Whence, making use of Young inequality in (2.4) together with (2.5), one obtains

$$\begin{aligned}
I_1 &\leq \epsilon \int_0^t |g|_{L^2}^2 ds + C(\epsilon, \tilde{\mu}_0) \int_0^t |v_2|_{L^4}^2 |g|_{H^{-1}} |g|_{L^2} ds \\
&\leq 2\epsilon \int_0^t |g|_{L^2}^2 ds + \tilde{C}(\epsilon, \tilde{\mu}_0) \int_0^t |v_2|_{L^4}^4 |g|_{H^{-1}}^2 ds,
\end{aligned}$$

for any $\epsilon > 0$. Similar computations give for the second term

$$\begin{aligned}
I_2 &= - \iint_{[0,t] \times \mathbb{T}^2} (\Lambda^{-1/2} \partial_i (v_1^i g)) (\Lambda^{-1/2} g) dx ds \\
&\leq C \int_0^t |v_1|_{L^4} |g|_{L^2} |g|_{W^{-1,4}} ds \\
&\leq \epsilon \int_0^t |g|_{L^2}^2 ds + C(\epsilon, \tilde{\mu}_0) \int_0^t |v_1|_{L^4}^2 |g|_{H^{-1}} |g|_{L^2} ds \\
&\leq 2\epsilon \int_0^t |g|_{L^2}^2 ds + \tilde{C}(\epsilon, \tilde{\mu}_0) \int_0^t |v_1|_{L^4}^4 |g|_{H^{-1}}^2 ds,
\end{aligned}$$

$\epsilon > 0$ being arbitrary. For the third term, we have using (2.5):

$$\begin{aligned}
I_3 &\leq \int_0^t |\nabla f|_{L^2} |\nabla u_2|_{L^4} |\nabla \Lambda^{-1} g|_{L^4} ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon) \int_0^t |\nabla u_2|_{L^4}^2 |g|_{W^{-1,4}}^2 ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon, \tilde{\mu}_0) \int_0^t |\nabla u_2|_{L^4}^2 |g|_{H^{-1}} |g|_{L^2} ds \\
&\leq \epsilon \int_0^t (|\nabla f|_{L^2}^2 + |g|^2) ds + \tilde{C}(\epsilon, \tilde{\mu}_0) \int_0^t |\nabla u_2|_{L^4}^4 |g|_{H^{-1}}^2 ds,
\end{aligned}$$

for any $\epsilon > 0$, and the same computations as above yield for the fourth term:

$$\begin{aligned}
I_4 &\leq C \int_0^t |\nabla u_1|_{L^4} |\nabla f|_{L^2} |g|_{W^{-1,4}} ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon, \tilde{\mu}_0) \int_0^t |\nabla u_1|_{L^4}^2 |g|_{H^{-1}} |g|_{L^2} ds \\
&\leq \epsilon \int_0^t (|\nabla f|_{L^2}^2 + |g|^2) ds + C(\epsilon, \tilde{\mu}_0) \int_0^t |\nabla u_1|_{L^4}^4 |g|_{H^{-1}}^2 ds.
\end{aligned}$$

Next, using this time (2.1), we have for any $\epsilon > 0$:

$$\begin{aligned}
I_5 &\leq \int_0^t |f|_{L^4}^2 |\nabla u_1|_{L^4}^2 ds \\
&\leq \mu_0 \int_0^t |f|_{L^2} (|f|_{L^2} + |\nabla f|_{L^2}) |\nabla u_1|_{L^4}^2 ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon, \mu_0) \int_0^t (|\nabla u_1|_{L^4}^2 + |\nabla u_1|_{L^4}^4) |f|_{L^2}^2 ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + \tilde{C}(\epsilon, \mu_0) \int_0^t (1 + |\nabla u_1|_{L^4}^4) |f|_{L^2}^2 ds
\end{aligned}$$

Similarly, using $|u_2|_{\mathbb{R}^3} = 1$ and (2.1), we have

$$\begin{aligned}
I_6 &\leq \int_0^t |f|_{L^4} |\nabla f|_{L^2} |\nabla U|_{L^4} ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon) \int_0^t |f|_{L^4}^2 |\nabla U|_{L^4}^2 ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon, \mu_0) \int_0^t |f|_{L^2} (|f|_{L^2} + |\nabla f|_{L^2}) |\nabla U|_{L^4}^2 ds \\
&\leq 2\epsilon \int_0^t |\nabla f|_{L^2}^2 ds + \tilde{C}(\epsilon, \mu_0) \int_0^t (1 + |\nabla U|_{L^4}^4) |f|_{L^2}^2 ds
\end{aligned}$$

while

$$\begin{aligned}
I_7 &\leq \int_0^t |f|_{L^4} |g|_{L^2} |\nabla u_1|_{L^4} ds \\
&\leq \epsilon \int_0^t |g|_{L^2}^2 ds + C(\epsilon, \mu_0) \int_0^t |f|_{L^2} (|f|_{L^2} + |\nabla f|_{L^2}) |\nabla u_1|_{L^4}^2 ds \\
&\leq \epsilon \int_0^t (|g|_{L^2}^2 + |\nabla f|_{L^2}^2) ds + \tilde{C}(\epsilon, \mu_0) \int_0^t (1 + |\nabla u_1|_{L^4}^4) |f|_{L^2}^2 ds
\end{aligned}$$

Using again (2.1) with Hölder and Young Inequalities, we have finally

$$\begin{aligned}
I_8 &\leq \int_0^t |f|_{L^4} |v_2|_{L^4} |\nabla f|_{L^2} ds \\
&\leq \epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon) \mu_0 \int_0^t |f|_{L^2} (|f|_{L^2} + |\nabla f|_{L^2}) |v_2|_{L^4}^2 ds \\
&\leq 2\epsilon \int_0^t |\nabla f|_{L^2}^2 ds + C(\epsilon, \mu_0) \int_0^t (1 + |v_2|_{L^4}^4) |f|_{L^2}^2 ds.
\end{aligned}$$

Conclusion. Since $|\Lambda^{-1/2}g|_{L^2}$ and $|g|_{H^{-1}}$ are equivalent quantities, we see that provided $\epsilon > 0$ is chosen sufficiently small, the summation of all the above contributions leads to the relation

$$\Psi(t) \leq C \int_0^t \sum_{j=1,2} (1 + |v_j(s)|_{L^4}^4 + |\nabla u_j(s)|_{L^4}^4) \Psi(s) ds$$

where $\Psi(t) := \sup_{s \in [0,t]} \frac{1}{2} (|\Lambda^{-1/2}g(s)|_{L^2}^2 + |f(s)|_{H^1}^2)$ and $C > 0$ is a universal constant. Applying Gronwall Lemma for \mathbb{P} -a.e. $\omega \in \Omega$, we find that $f = g = 0$. This ends the proof of Theorem 1.1. \blacksquare

3. LOCAL SOLVABILITY

This section is devoted to the proof of existence and uniqueness of local solutions. In order to express this as a fixed point problem, it will be convenient to switch to a mild form for (SEL). As seen for instance in [10, Section 6], mild solutions that are sufficiently regular (in a sense made precise below) are also strong solutions in the usual sense.

We shall say that a triplet $(v, u; \tau)$ is a local mild solution to (SEL) if $\tau > 0$ denotes a stopping time, such that for some $p > 2$, the following holds:

- (M1) for any stopping time $0 < \zeta < \tau$ the couple $(v(\cdot \wedge \zeta), u(\cdot \wedge \zeta))$ is progressively measurable as a process with values in $\mathbb{L}^p \times \mathbf{W}^{1,p}$;
- (M2) \mathbb{P} -almost surely on $\{t < \tau\}$:

$$v(t) = e^{-tA} v_0 + \int_0^t e^{A(s-t)} \mathcal{P}(-v \cdot \nabla v - \operatorname{div}(\nabla u \odot \nabla u)) ds \quad (3.1)$$

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{\Delta(t-s)} (u |\nabla u|^2 - v \cdot \nabla u + F_\psi u) ds + \int_0^t e^{\Delta(t-s)} [u \times dW], \quad (3.2)$$

where the above correspond to Bochner integral in \mathbb{L}^p , respectively in $\mathbf{W}^{1,p}$, and Itô integral in $\mathbb{W}^{1,p}$.

As will be seen below, the condition (M1) ensures the summability of the integrals in (M2).

Our main result in this section is the following.

Theorem 3.1. *Let $p \in (2, \infty)$, $s \in [1, 3]$, fix $\alpha > 2/p$, and $q > 2/\alpha$. For every $\psi \in \gamma(W^{s+\alpha,p})$ and $(v_0, u_0) \in \mathbb{W}^{s-1,p} \times \mathbf{W}^{s,p}$ with $|u_0|_{\mathbb{R}^3} = 1$ almost everywhere, there exists a local mild solution $(v, u; \tau^{s,p})$ to (SEL), unique in the space*

$$L^q(\Omega; C([0, \tau^{s,p}); \mathbb{W}^{s-1,p})) \times L^q(\Omega; C([0, \tau^{s,p}); \mathbf{W}^{s,p})).$$

The existence time $\tau^{s,p} > 0$ is maximal in the sense that

$$\text{on } \{\tau^{s,p} < T\}: \quad \limsup_{t \rightarrow \tau^{s,p}} \max(|v(t)|_{W^{s-1,p}}, |u(t)|_{W^{s,p}}) = \infty.$$

Moreover, if $p > 2$ and $s \geq 2$, then u satisfies the spherical constraint (1.12) for a.e. $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{T}^2$ such that $0 \leq t < \tau^{p,s}(\omega)$.

Remark 3.1. In particular, by Theorem 3.1 if $p > 2$ and $s \geq 2$, then (v, u) takes values in the domain of the linear part, and thus (v, u) is a strong solution in the usual analytic (and probabilistic) sense, see [10, Chapter 6]. It is also a solution in the sense of Definition 1.1 (the properties (i), (iii) and (iv) are immediate in this case).

Prior to proving Theorem 3.1, we need some preparatory steps.

3.1. Hypercontractivity bounds. We will make use of the following well-known inequality. Let $k \in \mathbb{Z}$ and assume that $\Lambda : D(\Lambda) \subset H^k \rightarrow H^k$, is a negative self-adjoint operator such that for any $1 < p < \infty$, the semigroup $e^{t\Lambda}$ extends canonically to a strongly continuous, analytic semigroup on $W^{k,p}$, $1 < p < \infty$. Then, for any $t \in (0, 1]$, $k - 2 \leq \alpha, \beta \leq k + 2$ and $1 < p, q < \infty$ it holds the estimate

$$|e^{t\Lambda}|_{\mathcal{L}(W^{\alpha,p}, W^{\beta,q})} \leq \frac{C}{t^{\frac{\beta-\alpha}{2} + \frac{1}{p} - \frac{1}{q}}}, \quad (3.3)$$

where the constant C depends only on the largest spectral value of Λ . Note that such semigroups are in some references referred to as ‘‘hypercontractive’’ or ‘‘ L^p -contractive’’ – see, e.g., [30, Theorem X.55].

Proof of (3.3). The inequality (3.3) is well-known in principle, hence we only provide references. The case $p = q = 2$ is treated for instance in [26, Theorem 5.2]. The general case follows by interpolation and duality, as can be seen for instance in [29, p. 25]. \blacksquare

Note that $-A$ (resp. Δ) with $k = 1$ (resp. $k = 2$) satisfies the above assumptions (this is standard for the Laplacian; for the Stokes operator, we refer to [12]).

3.2. Stochastic parabolic estimates. We now recall some well-known stochastic parabolic estimates: consider the solution, of the equation

$$\begin{cases} dZ - \Delta Z dt = \Psi(t) d\xi \\ Z(0) = 0, \end{cases} \quad (3.4)$$

(Itô sense) where $\xi \equiv \sum_{k \in \mathbb{N}} \mathbf{B}_k(t) f_k$ is a cylindrical Wiener process and the unknown is a continuous process $Z : \Omega \times [0, T] \rightarrow H$. Under suitable assumptions on Ψ (see the proposition below) the solution of (3.4) is written as the stochastic convolution process:

$$Z(t) := \int_0^t e^{(t-s)\Delta} \Psi(s) d\xi(s), \quad t \in [0, T]. \quad (3.5)$$

The following result is proven in [6], see also [17].

Proposition 3.1. *Let $\alpha \geq 0, p \in [2, \infty), r \geq 1$, and assume we are given a progressively measurable process $\Psi : \Omega \times [0, T] \rightarrow \mathcal{L}_2(L^2, L^2)$ such that Ψ belongs to $L^r(\Omega; L^r(0, T; \gamma(W^{\alpha, p})))$. The following holds:*

- (i) *for each $p > 2$, every $\delta \in [0, 1 - 2/r)$ and $\lambda \in [0, 1 - 1/r - \delta/2)$, the stochastic convolution (3.5) is well-defined and belongs to $L^r(\Omega; C^\lambda(0, T; W^{\alpha+\delta, p}))$. Moreover, it holds*

$$\mathbb{E} \left[\|Z\|_{C^\lambda(0, T; W^{\alpha+\delta, p})}^r \right] \leq C \mathbb{E} \left[\|\Psi\|_{L^r(0, T; \gamma(W^{\alpha, p}))}^r \right].$$

- (ii) *For each $p \geq 2, \delta \in (0, 1)$, Z is well-defined and belongs to $L^r(\Omega; L^r(0, T; W^{\alpha+\delta, p}))$. It holds as well:*

$$\mathbb{E} \left[\|Z\|_{L^r(0, T; W^{\alpha+\delta, p})}^r \right] \leq C \mathbb{E} \left[\|\Psi\|_{L^r(0, T; \gamma(W^{\alpha, p}))}^r \right].$$

The above constants do not depend on Ψ in the indicated classes.

We can now proceed to the proof of the main theorem.

3.3. Proof of Theorem 3.1. Let $p > 2$. We shall first let $s = 1$, and show that the conclusions of the above theorem hold in this particular case. The proof is based on a contraction mapping principle for a truncated version of (SEL), into the Banach space

$$\mathfrak{X}_{q, T} := L^q(\Omega; C(0, T; \mathbb{L}^p)) \times L^q(\Omega; C(0, T; \mathbf{W}^{1, p})), \quad (3.6)$$

where $T > 0$ and $q \geq 2$ are parameters to be fixed later. It is endowed with the product norm $\| (X_1, X_2) \|_{\mathfrak{X}_{q, T}} := \| X_1 \|_{q, T, \mathbb{L}^p} + \| X_2 \|_{q, T, \mathbf{W}^{1, p}}$ where for convenience, whenever E is a Banach space and $X : \Omega \times [0, T] \rightarrow E$ is a stochastic process, we shall denote

$$\| X \|_{q, T, E} := \mathbb{E} \left[\sup_{t \in [0, T]} |X(t)|_E^q \right]^{1/q}.$$

Because the noise term cannot be estimated pathwise, we define a cut-off function $\theta \in C_c^\infty((0, \infty), \mathbb{R})$, such that

$$\text{Supp} \theta \subset (0, 2), \quad 0 \leq \theta \leq 1 \quad \text{and} \quad \theta(x) = 1 \quad \text{for all } 0 \leq x \leq 1, \quad (3.7)$$

and for $R > 0, x \in \mathbb{R}^+$, we denote

$$\theta_R(x) = \theta\left(\frac{x}{R}\right).$$

Next, we fix $X_0 \equiv (v_0, u_0) \in \mathbb{L}^p \times \mathbf{W}^{1, p}$ and solve a problem where the non-linearity is truncated, that is: given $R > 0$ for any $(w, y) \in \mathfrak{X}_{q, T}$, we define the map

$$\Gamma_{X_0, R} : \mathfrak{X}_{q, T} \rightarrow \mathfrak{X}_{q, T}, \quad (w, y) \mapsto \Gamma_{X_0, R}(w, y) := (v, u),$$

where for every $0 \leq t \leq T$, a.s.:

$$\begin{aligned} v(t) &= e^{-tA} v_0 - \int_0^t e^{(s-t)A} \mathcal{P} [\Theta_R(w \cdot \nabla w + \text{div}(\nabla y \odot \nabla y))] ds \\ u(t) &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} [\Theta_R(y |\nabla y|^2 + w \cdot \nabla y) + F_\psi y] ds + \int_0^t e^{(t-s)\Delta} (y \times dW), \end{aligned} \quad (3.8)$$

and we make use of the following abbreviation

$$\Theta_R(t) := \theta_R(\max(|w(t)|_{L^p}, |y(t)|_{W^{1, p}})).$$

In the sequel, we shall consider $R > 0$ as fixed, and show that provided T is sufficiently small, *depending only on R* , then:

- (P1) for $q > 2/\alpha$, $\Gamma_{X_0, R}$ maps $\mathfrak{X}_{q, T}$ into itself;
- (P2) $\Gamma_{X_0, R}$ is a contraction in $\mathfrak{X}_{q, T}$.

Then, Picard Theorem yields existence and uniqueness of a fixed point (v_R, u_R) , solution to (3.1)-(3.2) up to the stopping time $\tau_R := \inf\{t \in [0, T], |w(t)|_{L^p} = R \text{ or } |y(t)|_{W^{1,p}} = R\}$.

Step 1: proof of (P1). For the velocity component, we have for every $t \leq T$:

$$\begin{aligned} |v(t)|_{L^p} &\leq C(p, T)|v_0|_{L^p} + \left| \int_0^t \Theta_R(s) e^{(s-t)A} \mathcal{P}[w(s) \cdot \nabla w(s)] ds \right|_{L^p} \\ &\quad + \left| \int_0^t \Theta_R(s) e^{(s-t)A} \mathcal{P} \operatorname{div}(\nabla u \odot \nabla u(s)) ds \right|_{L^p} \\ &=: C(p, T)|v_0|_{L^p} + |I_1|_{L^p} + |I_2|_{L^p}. \end{aligned}$$

The first term is estimated as follows: since w is divergence-free, we have for each $i = 1, 2$, the relation $[w \cdot \nabla]w^i \equiv \sum_{j=1}^2 w^j \partial_j w^i = \sum_{j=1}^2 \partial_j (w^j w^i)$. Using in addition (3.3), we have for the first term

$$\begin{aligned} |I_1|_{L^p} &\leq \int_0^t \Theta_R(s) \frac{|\mathcal{P} \sum_{j=1}^2 \partial_j (w^j(s) w(s))|_{W^{-1,p/2}}}{(t-s)^{1/2+1/p}} ds \\ &\leq C(\mathcal{P}, p) T^{1/2-1/p} \sup_{s \in [0, t]} (\Theta_R(s) |w(s)|_{L^p}^2) \\ &\leq C(\mathcal{P}, p) T^{1/2-1/p} R^2, \end{aligned}$$

by continuity of $\mathcal{P} : W^{-1,p} \rightarrow \mathbb{W}^{-1,p}$ since $p > 1$. Whence, for some universal constant $C > 0$, it holds:

$$\| \| I_1 \| \|_{q, T, \mathbb{L}^p} \leq C T^{1/2-1/p} R^2. \quad (3.9)$$

Similarly, we have for the second term

$$\begin{aligned} |I_2|_{L^p} &\leq \int_0^t \Theta_R(s) \frac{|\mathcal{P} \operatorname{div}(\nabla y(s) \odot \nabla y(s))|_{W^{-1,p/2}}}{(t-s)^{1/2+1/p}} ds \\ &\leq C(\mathcal{P}, p) T^{1/2-1/p} \sup_{s \in [0, t]} (\Theta_R(s) |\nabla y(s) \odot \nabla y(s)|_{L^{p/2}}) \\ &\leq C(\mathcal{P}, p) T^{1/2-1/p} \sup_{s \in [0, t]} (\Theta_R(s) |\nabla y(s)|_{L^p}^2) \\ &\leq C(\mathcal{P}, p) T^{1/2-1/p} R^2, \end{aligned}$$

and therefore:

$$\| \| I_2 \| \|_{q, T, \mathbb{L}^p} \leq C T^{1/2-1/p} R^2. \quad (3.10)$$

Next, for $t \in [0, T]$, almost surely, we write

$$\begin{aligned} |u(t)|_{W^{1,p}} &\leq C_p |u_0|_{W^{1,p}} + \left| \int_0^t \Theta_R(s) e^{(t-s)\Delta} (y(s) |\nabla y(s)|^2) ds \right|_{W^{1,p}} \\ &\quad + \left| \int_0^t \Theta_R(s) e^{(t-s)\Delta} (w \cdot \nabla y(s)) ds \right|_{W^{1,p}} \\ &\quad + \left| \int_0^t e^{(t-s)\Delta} (F_\psi y(s)) ds \right|_{W^{1,p}} + \left| \int_0^t e^{(t-s)\Delta} y(s) \times dW \right|_{W^{1,p}} \\ &=: C_p |u_0|_{W^{1,p}} + \sum_{i=3}^6 |I_i|_{W^{1,p}}. \end{aligned}$$

Making use of (3.3) and the Sobolev inequality $|f|_{L^\infty} \leq C_p |f|_{W^{1,p}}$ for $p > 2$, the first term above is estimated as follows:

$$\begin{aligned} |I_3|_{W^{1,p}} &\leq C \int_0^t \Theta_R(s) \frac{|y(s) |\nabla y(s)|^2|_{L^{p/2}}}{(t-s)^{1/2+1/p}} ds \\ &\leq CT^{1/2-1/p} \sup_{s \leq t} (\Theta_R(s) |y|_{L^\infty} |\nabla y|_{L^p}^2) \\ &\leq CT^{1/2-1/p} \sup_{0 \leq s \leq T} (\Theta_R(s) |y(s)|_{W^{1,p}}^3). \end{aligned}$$

Consequently it holds true that

$$\| \| I_3 \| \|_{q,T,\mathbf{W}^{1,p}} \leq CT^{1/2-1/p} R^3. \quad (3.11)$$

Similarly, we have

$$\begin{aligned} |I_5|_{W^{1,p}} &\leq \int_0^t \frac{C}{(t-s)^{1/2+1/p}} |y(s)|_{L^\infty} ds \left| \sum_{l \geq 1} (\psi f_l)^2 \right|_{L^{p/2}} \\ &\leq C' T^{1/2-1/p} |\psi|_{\gamma(L^p)}^2 \sup_{s \in [0,T]} |y(s)|_{W^{1,p}}, \end{aligned}$$

and hence,

$$\| \| I_5 \| \|_{q,T,\mathbf{W}^{1,p}} \leq CT^{1/2-1/p} |\psi|_{\gamma(L^p)}^2 R. \quad (3.12)$$

Concerning the transport term, we have

$$\begin{aligned} |I_4|_{W^{1,p}} &\leq C \int_0^t \frac{\Theta_R(s) |(w(s) \cdot \nabla) y(s)|_{L^{p/2}}}{(t-s)^{1/2+1/p}} ds \\ &\leq C \int_0^T \frac{\Theta_R(s) |w(s)|_{L^p} |y(s)|_{W^{1,p}}}{(t-s)^{1/2+1/p}} ds, \end{aligned}$$

and thus:

$$\| \| I_4 \| \|_{q,T,\mathbf{W}^{1,p}} \leq CT^{1/2-1/p} R^2. \quad (3.13)$$

The treatment of the stochastic convolution $I_6 \equiv \int_0^\cdot e^{(\cdot-s)\Delta} y(s) \times dW$ works as follows: letting $\epsilon > 0$ such that

$$\min(\alpha, 1) > \epsilon > \frac{2}{p}, \quad (3.14)$$

then the stochastic estimates (Proposition 3.1) with $\lambda = 0, \delta = 1 - \epsilon$ imply

$$\begin{aligned} \mathbb{E} \left[\| I_6 \|_{C(0,T;\mathbf{W}^{1,p})}^q \right] &\leq CT \mathbb{E} \left[\| y \times \psi \|_{C(0,T;\gamma(\mathbf{W}^{\epsilon,p}))}^q \right] \\ &\leq CT |\psi|_{\gamma(W^{\epsilon,p})}^q \mathbb{E} \left[\| y \|_{C(0,T;\mathbf{W}^{\epsilon,p})}^q \right], \end{aligned}$$

since the space $W^{\epsilon,p}$, where ϵ, p are subject to conditions (3.14), is an algebra. Hence, we end up with the estimate:

$$\| \| I_6 \| \|_{q,T,\mathbf{W}^{1,p}} \leq C^{1/q} T^{1/q} |\psi|_{\gamma(W^{\alpha,p})} \| \| y \| \|_{q,T,\mathbf{W}^{1,p}} . \quad (3.15)$$

Summing the estimates (3.9), (3.10), (3.11), (3.13), (3.12) and (3.15), we obtain (P1).

Step 2: proof of (P2). For $j = 1, 2$, take $(w_j, y_j) \in \mathfrak{X}_{q,T}$, let $(v_j, u_j) := \Gamma_{X_0,R}(w_j, y_j)$, and denote by

$$\Theta_R^j(t) := \theta_R(\max(|w_j(t)|_{L^p}, |y_j(t)|_{W^{1,p}}).$$

We also define

$$\tau_j := \inf\{s \in [0, T] : \max(|w_j(s)|_{L^p}, |y_j(s)|_{W^{1,p}}) = R\}. \quad (3.16)$$

For the velocity component, we write

$$\begin{aligned} v_1(t) - v_2(t) &= \int_0^t e^{(s-t)A} \mathcal{P} \left(\Theta_R^1 w_1 \cdot \nabla w_1 - \Theta_R^2 w_2 \cdot \nabla w_2 \right) ds \\ &\quad + \int_0^t e^{(s-t)A} \mathcal{P} \operatorname{div} \left(\Theta_R^1 \nabla y_1 \odot \nabla y_1 - \Theta_R^2 \nabla y_2 \odot \nabla y_2 \right) ds \\ &=: \Delta_1 + \Delta_2. \end{aligned}$$

To estimate the first term, we can assume without loss of generality that $\tau_1 \leq \tau_2$. This gives

$$\begin{aligned} |\Delta_1|_{L^p} &\leq C \int_0^{\tau_2} (\Theta_R^1 - \Theta_R^2) \frac{\left| \mathcal{P} \sum_{j=1}^2 \partial_j (w_2^j w_2) \right|_{W^{-1,p/2}}}{(t-s)^{\frac{1}{2} + \frac{1}{p}}} ds \\ &\quad + C \int_0^{\tau_2} \Theta_R^1 \frac{\left| \mathcal{P} \sum_{j=1}^2 \partial_j (w_1^j w_1 - w_2^j w_2) \right|_{W^{-1,p/2}}}{(t-s)^{\frac{1}{2} + \frac{1}{p}}} ds \\ &= \text{I} + \text{II}. \end{aligned}$$

Thanks to the continuity of \mathcal{P} , and the fact that $x, y \mapsto \max(x, y)$ is Lipschitz, we have

$$\begin{aligned} |\text{I}|_{L^q(\Omega)} &\leq C(\mathcal{P}, p) T^{1/2-1/p} \sup_{s \in [0, T]} (\Theta_R(s) |w_2(s)|_{L^p}^2) \\ &\quad \times \frac{|\theta'|_{L^\infty}}{R} (\| \| w_1 - w_2 \| \|_{q,T,\mathbb{L}^p} + \| \| y_1 - y_2 \| \|_{q,T,\mathbf{W}^{1,p}}) \\ &\leq CRT^{1/2-1/p} |\theta'|_{L^\infty} \| \| (w_1 - w_2, y_1 - y_2) \| \|_{\mathfrak{X}_{q,T}} . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \text{II} &\leq C(\mathcal{P}, p) \int_0^{\tau_2} \Theta_R^1 \frac{\left| \sum_{j=1}^2 \partial_j [(w_1^j(s) - w_2^j(s)) w_1(s)] \right|_{W^{-1,p/2}}}{(t-s)^{\frac{1}{2} + \frac{1}{p}}} ds \\ &\quad + C(\mathcal{P}, p) \int_0^{\tau_2} \Theta_R^1 \frac{\left| \sum_{j=1}^2 \partial_j [w_2^j (w_1(s) - w_2(s))] \right|_{W^{-1,p/2}}}{(t-s)^{\frac{1}{2} + \frac{1}{p}}} ds , \end{aligned}$$

which yields the estimate $\|\text{II}\|_{L^q(\Omega)} \leq CT^{1/2-1/p}R \|\| w_1 - w_2 \|\|_{q,T,\mathbb{L}^p}$. Hence, we obtain that

$$\|\| \Delta_1 \|\|_{q,T,\mathbb{L}^p} \leq CT^{1/2-1/p}R \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}} \quad (3.17)$$

The second term is similar: assuming without loss of generality that $\tau_1 \leq \tau_2$, we have

$$\begin{aligned} |\Delta_2|_{L^p} &\leq \int_0^{\tau_2} |\Theta_R^1 - \Theta_R^2| e^{(s-t)A} \mathcal{P} \operatorname{div}(\nabla y_2 \odot \nabla y_2)|_{L^p} ds \\ &\quad + \int_0^{\tau_1} \Theta_R^1 e^{(s-t)A} \mathcal{P} \operatorname{div}(\nabla(y_1 - y_2) \odot \nabla y_1)|_{L^p} ds \\ &\quad + \int_0^{\tau_1} \Theta_R^1 e^{(s-t)A} \mathcal{P} \operatorname{div}(\nabla y_2 \odot \nabla(y_1 - y_2))|_{L^p} ds \\ &=: \text{I} + \text{II} + \text{III}. \end{aligned}$$

By a similar argument as before, we have for the first term:

$$\|\text{I}\|_{L^q(\Omega)} \leq C(\mathcal{P}, p) |\theta'|_{L^\infty} T^{1/2-1/p} R \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}}.$$

Next, proceeding as for Δ_1 , it is easy to see that:

$$\mathbb{E} \left[\sup_{t \in [0, T]} (\text{II}(t) + \text{III}(t))^q \right]^{1/q} \leq CRT^{1/2-1/p} \|\| y_1 - y_2 \|\|_{q,T, \mathbf{W}^{1,p}}.$$

Hence, we end up with the estimate:

$$\|\| \Delta_2 \|\|_{q,T,\mathbb{L}^p} \leq CRT^{1/2-1/p} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}} \quad (3.18)$$

We now proceed to the evaluation of the second component. We write

$$\begin{aligned} (u_1 - u_2)(t) &= \int_0^t e^{(t-s)\Delta} (\Theta_R^1 y_1 |\nabla y_1|^2 - \Theta_R^2 y_2 |\nabla y_2|^2) ds \\ &\quad - \int_0^t e^{(t-s)\Delta} (\Theta_R^1 w_1 \cdot \nabla y_1 - \Theta_R^2 w_2 \cdot \nabla y_2) ds \\ &\quad + \int_0^t e^{(t-s)\Delta} (F_\psi(y_1 - y_2)) ds + \int_0^t e^{(t-s)\Delta} (y_1 - y_2) \times dW \\ &=: \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6. \end{aligned}$$

Now, linearity and Step 1 provide the bound

$$\|\| \Delta_5 + \Delta_6 \|\|_{q,T, \mathbf{W}^{1,p}} \leq \|\| y_1 - y_2 \|\|_{q,T, \mathbf{W}^{1,p}} C \left(T^{1/2-1/p} |\psi|_{\gamma(L^p)}^2 + T |\psi|_{\gamma(W^{\alpha,p})} \right). \quad (3.19)$$

Next, recalling (3.16) and assuming without loss of generality that $\tau_1 \leq \tau_2$, we have for the third term:

$$\begin{aligned} |\Delta_3|_{W^{1,p}} &\leq \int_0^{\tau_2} \left| e^{(t-s)\Delta} (\Theta_R^1 y_1 |\nabla y_1|^2 - \Theta_R^2 y_2 |\nabla y_2|^2) \right|_{W^{1,p}} ds \\ &\leq \int_0^{\tau_2} (\Theta_R^1 - \Theta_R^2) \left| e^{(t-s)\Delta} (y_2 |\nabla y_2|^2) \right|_{W^{1,p}} ds \\ &\quad + \int_0^{\tau_2} \Theta_R^1 \left| e^{(t-s)\Delta} (y_1 |\nabla y_1|^2 - y_2 |\nabla y_2|^2) \right|_{W^{1,p}} ds \\ &=: \text{I} + \text{II}. \end{aligned}$$

Proceeding as for the velocity component, we can estimate the first term as

$$\|\text{I}\|_{L^q(\Omega)} \leq C' R^2 T^{1/2-1/p} |\theta'|_{L^\infty} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}}.$$

For the second term, using again (3.3), we have

$$\begin{aligned} \text{II} &\leq C \int_0^{\tau_2} \Theta_R^1 \frac{\left| (y_1 - y_2) |\nabla y_1|^2 \right|_{L^{p/2}}}{(t-s)^{1/2+1/p}} ds \\ &\quad + C \int_0^{\tau_2} \Theta_R^1 \frac{\left| y_2 (|\nabla y_1|^2 - |\nabla y_2|^2) \right|_{L^{p/2}}}{(t-s)^{1/2+1/p}} ds \\ &= \text{II}' + \text{III}. \end{aligned}$$

Since $\mathbf{W}^{1,p} \hookrightarrow \mathbf{L}^\infty$, it holds immediately

$$\|\text{II}'\|_{L^q(\Omega)} \leq CR^2 T^{1/2-1/p} \|\| y_1 - y_2 \|\|_{q,T,\mathbf{W}^{1,p}}.$$

Similarly, we have

$$\begin{aligned} \|\text{III}\|_{L^q(\Omega)} &\leq \mathbb{E} \left[\left(\int_0^{\tau_2} \Theta_R^1 \frac{\left| y_2 \nabla(y_1 - y_2) \cdot \nabla(y_1 + y_2) \right|_{L^{p/2}}}{(t-s)^{1/2+1/p}} ds \right)^q \right]^{1/q} \\ &\leq CR^2 T^{1/2-1/p} \|\| y_1 - y_2 \|\|_{q,T,\mathbf{W}^{1,p}}. \end{aligned}$$

Finally, we see that there is a universal constant $C > 0$ such that:

$$\|\| \Delta_3 \|\|_{q,T,\mathbf{W}^{1,p}} \leq CR^2 T^{1/2-1/p} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}}. \quad (3.20)$$

For the fourth term, it is again sufficient to assume that $\tau_1 \leq \tau_2$. But in this case it holds

$$\begin{aligned} |\Delta_4|_{W^{1,p}} &\leq \int_0^{\tau_2} \left| e^{(t-s)\Delta} (\Theta_R^1 w_1 \cdot \nabla y_1 - \Theta_R^2 w_2 \cdot \nabla y_2) \right|_{W^{1,p}} ds \\ &\leq \int_0^{\tau_2} |\Theta_R^1 - \Theta_R^2| \left| e^{(t-s)\Delta} w_2 \cdot \nabla y_2 \right|_{W^{1,p}} ds \\ &\quad + \int_0^{\tau_2} \Theta_R^1 \left| e^{(t-s)\Delta} (w_1 \cdot \nabla y_1 - w_2 \cdot \nabla y_2) \right|_{W^{1,p}} ds. \end{aligned}$$

Whence, we obtain as before:

$$\begin{aligned} \|\| \Delta_4 \|\|_{q,T,W^{1,p}} &\leq CR^2 |\theta'|_{L^\infty} T^{1/2-1/p} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}} \\ &\quad + \mathbb{E} \left[\left(\int_0^T \Theta_R^1 \left| e^{(t-s)\Delta} [(w_1 - w_2) \cdot \nabla y_2] \right|_{W^{1,p}} ds \right)^q \right]^{1/q} \\ &\quad + \mathbb{E} \left[\left(\int_0^T \Theta_R^1 \left| e^{(t-s)\Delta} [w_1 \cdot \nabla (y_1 - y_2)] \right|_{W^{1,p}} ds \right)^q \right]^{1/q} \\ &\leq C' R^2 T^{1/2-1/p} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}}. \end{aligned}$$

Hence, it follows that for some universal constant $C > 0$:

$$\|\| \Delta_4 \|\|_{q,T,\mathbf{W}^{1,p}} \leq CR^2 T^{1/2-1/p} \|\| (w_1 - w_2, y_1 - y_2) \|\|_{\mathfrak{X}_{q,T}}. \quad (3.21)$$

Summing the estimates (3.17), (3.18), (3.20), (3.21) and (3.19), we see that there exists a constant $C(p, |\psi|_{\gamma(W^{\alpha,p})}) > 0$ such that

$$\begin{aligned} &\|\| \Gamma(w_1, y_1) - \Gamma(w_2, y_2) \|\|_{\mathfrak{X}_{q,T}} \\ &\leq C(p, |\psi|_{\gamma(W^{\alpha,p})}) (1 + R^2) (T^{1/2-1/p} + T) \|\| (w_1, y_1) - (w_2, y_2) \|\|_{\mathfrak{X}_{q,T}}. \end{aligned}$$

If we choose T such that

$$0 < T \leq T^*(R) := \min \left(1, 4C(p, |\psi|_{\gamma(W^{\alpha,p})}) (1 + R^2) \right)^{-\frac{2p}{p-2}},$$

then $\Gamma_{X_0,R} : \mathfrak{X}_{q,T} \rightarrow \mathfrak{X}_{q,T}$ is a contraction. This proves (P2).

Step 3: definition of the maximal solution. Using a localization procedure (see [7, Theorem 4.1]) we can build a maximal solution as follows: we define the stopping times

$$\tau_m = \inf \{t \in [0, T], \max(|v_m(t)|_{L^p}, |u_m(t)|_{W^{1,p}}) \geq m\}, \quad (3.22)$$

and show that the sequence $\{\tau_m\}_{m \in \mathbb{N}}$ is non-decreasing and that

$$(v_{m+1}(t), u_{m+1}(t)) = (v_m(t), u_m(t)) \quad \text{for } t \in [0, \tau_m], \text{ a.s.}$$

Since the proof of these properties does not involve any new idea compared to steps 1 and 2, we leave the details to the reader. The maximal solution $(v; u, \tau \equiv \sup_{m \geq 0} \tau_m)$ is then defined by “gluing together” (v_m, u_m) for each $m \geq 1$.

Step 4: General case. It is sufficient to let $s = 3$ (the general case follows by interpolation).

Let $(v, u) \in L^q(\Omega; C([0, \tau^{p,1}]; \mathbb{L}^p \times \mathbf{W}^{1,p}))$ be the solution provided by the above fixed point, and assume that $X_0 = (v_0, u_0)$ belongs to $\mathbb{W}^{2,p} \times \mathbf{W}^{3,p}$. For $(\tilde{w}, \tilde{y}) \in \mathfrak{X}_{q,T}$, define $(\tilde{v}, \tilde{u}) := \Gamma_{X_0,R}^\Delta(\tilde{w}, \tilde{y})$ via the relation

$$\begin{aligned} \tilde{v}(t) &= e^{-tA} \Delta v_0 + \int_0^t \theta_R(\max(|\tilde{w}(s)|_{L^p}, |\tilde{y}(s)|_{W^{1,p}})) \\ &\quad \times e^{A(s-t)} \mathcal{P}[-\tilde{w} \cdot \nabla v - 2\nabla v \cdot \nabla^2 v - v \cdot \nabla \tilde{w} \\ &\quad - \operatorname{div}(\nabla \tilde{y} \odot \nabla u + 2\nabla^2 u \odot \nabla^2 u + \nabla u \odot \nabla \tilde{y})] ds \\ \tilde{u}(t) &= e^{t\Delta} \Delta u_0 + \int_0^t \theta_R(\max(|\tilde{w}(s)|_{L^p}, |\tilde{y}(s)|_{W^{1,p}})) \\ &\quad \times e^{\Delta(t-s)} (\tilde{y} |\nabla u|^2 + 4\nabla u \cdot \nabla^2 u \nabla u + 2u(\nabla \tilde{y} + \nabla^2 u \cdot \nabla^2 u) \\ &\quad - \tilde{w} \cdot \nabla u - 2\nabla v \cdot \nabla^2 u - v \cdot \nabla \tilde{y} + \Delta F_\psi u + 2\nabla F_\psi \cdot \nabla u + F_\psi \tilde{y}) ds \\ &\quad + \int_0^t e^{\Delta(t-s)} [\tilde{y} \times dW + 2\nabla u \times d\nabla W + u \times d\Delta W], \end{aligned}$$

Using similar arguments as that of Step 1 and Step 2, it can be shown that the map $\Gamma_{X_0,R}^\Delta$ admits a unique fixed point (\tilde{v}, \tilde{u}) , for every $R > 0$. Moreover, observing that Δ commutes with e^A and with \mathcal{P} (see (1.3)), it is immediately checked that on $\{\max(|\tilde{v}(t)|_{L^p}, |\tilde{u}(t)|_{W^{1,p}}) \leq R\}$, we have $\tilde{v}(t) = \Delta v(t)$ while $\tilde{u}(t) = \Delta u(t)$, which shows the claimed regularity on (v, u) .

The details of the proof being similar to step 1 and step 2 (the difficulties are merely notational), we leave them to the reader.

Step 5: spherical constraint. For $p > 2$ and $s \geq 2$, if (v, u) is defined as above, we can see it as a continuous path with values in the Hilbert space $H := \mathbb{H}^{-1} \times \mathbf{L}^2$. Because (v, u) takes values in the domain of $L := (A, -\Delta)$, it is in fact a *strong solution* in the sense of [10, p. 160] (up to the stopping time $\tau^{p,s}$). Hence, it is possible to apply Itô Formula to the functional $F(v, u) := \frac{1}{2}|1 - |u|^2|_{L^2}^2$ (which is clearly of class C^2 from $D(L) \equiv \mathbb{H}^1 \times \mathbf{H}^2$ to \mathbb{R}). Using the identity $\Delta(|u|^2 - 1) = 2u \cdot \Delta u + 2|\nabla u|^2$,

this gives

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{T}^2} (|u(t)|^2 - 1)^2 dx \\
&= \iint_{[0,t] \times \mathbb{T}^2} [2|u|^2(|u|^2 - 1)|\nabla u|^2 + 2u \cdot \Delta u(|u|^2 - 1) - 2u \cdot \partial_i u v^i (|u|^2 - 1)] dx ds \\
&= \iint_{[0,t] \times \mathbb{T}^2} [2(|u|^2 - 1)^2 |\nabla u|^2 - 2|\nabla(|u|^2 - 1)|^2 - 2u \cdot \partial_i u v^i (|u|^2 - 1)] dx ds \\
&\leq 4 \int_0^t \|\nabla u(s)\|_{L^\infty}^2 F(v(s), u(s)) ds - 2 \iint_{[0,t] \times \mathbb{T}^2} u \cdot \partial_i u v^i (|u|^2 - 1) dx ds \\
&= \text{I} + \text{II}.
\end{aligned} \tag{3.23}$$

As seen by integration by parts, using that $\operatorname{div} v = 0$ we have

$$\text{II} := - \iint_{[0,t] \times \mathbb{T}^2} 2u \cdot \partial_i u v^i (|u|^2 - 1) dx ds = - \iint_{[0,t] \times \mathbb{T}^2} \partial_i (|u|^2 - 1) v^i (|u|^2 - 1) dx ds = -\text{II},$$

and therefore $\text{II} = 0$. Next, by Gronwall Lemma applied to (3.23), we see that provided $0 \leq t < \tau^{s,p}$ is such that

$$\int_0^t \|\nabla u(s)\|_{L^\infty}^2 ds < \infty, \tag{3.24}$$

(which is always true by the Sobolev embedding $W^{p,s} \hookrightarrow W^{1,\infty}$ for $p > 2$ and $s \geq 2$) then one has $F(v(t), u(t)) = 0$. This proves the norm constraint (1.12). \blacksquare

4. A PRIORI ESTIMATES

Our purpose in this section is to obtain suitable a priori estimates associated to (SEL), assuming that the solution at hand is regular enough. We divide them into two categories, namely the “global estimates”, i.e. estimates on the whole time interval, and the “local estimates”, i.e. estimates on small balls $B(x, \rho) \subset \mathbb{T}^2$. Estimates that are local in the time-like variable will be obtained in Section 5 below.

We start by stating an important byproduct of (2.1). Note that in [31], the left hand side below is a gradient, which is unnecessary (as can be immediately seen in the details of the proof).

Proposition 4.1 (Struwe [31], Lemma 3.1). *There exists a constant $\mu_1 > 0$, such that for all $v \in C(0, T; H^1(\mathbb{T}^2))$, for all $\rho > 0$:*

$$\iint_{[0,T] \times \mathbb{T}^2} |v|^4 dy dt \leq \mu_1 \left(\sup_{t \in [0,T]} \int_{x \in \mathbb{T}^2} \int_{y \in B(x, \rho)} |v(t, y)|^2 dy \right) \iint_{[0,T] \times \mathbb{T}^2} \left(|\nabla v|^2 + \frac{|v|^2}{\rho^2} \right) dy dt. \tag{4.1}$$

4.1. Global estimates. Let us first introduce some notation.

Notation 4.1. Since it plays a specific role along the proofs, we will denote the “tension” by

$$\mathcal{T}_u := \Delta u + u |\nabla u|^2. \tag{4.2}$$

The latter terminology is borrowed from [11], in the context of harmonic map flows. Roughly speaking, the L^2 -norm of \mathcal{T}_u measures “how far” u is from being harmonic. Note indeed that $\mathcal{T}_u = 0$ if and only if $u : \mathbb{T}^2 \rightarrow \mathbb{S}^2$ is weakly harmonic.

An important observation is that for any $u \in \mathbf{H}^1$ with $|u|_{\mathbb{R}^3} = 1$, then \mathcal{T}_u identifies with the orthogonal projection of Δu onto u^\perp , that is:

$$\mathcal{T}_u = \Delta u - (\Delta u \cdot u)u, \quad (4.3)$$

where $(\Delta u \cdot u)u$ is defined in the sense of distributions.

Proposition 4.2. *Let (W, ψ) be as in Assumption 1.1. Assume that (v, u) is a strong solution to (SEL) on $[0, T]$, with respect to (\mathfrak{P}, W) , in the sense of Definition 1.2. Denoting by*

$$E(t) := \frac{1}{2} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2),$$

then it holds true that for any $0 \leq t \leq T$, a.s.:

$$E(t) - E(0) + \int_0^t (|\nabla v(s)|_{L^2}^2 + |\mathcal{T}_u(s)|_{L^2}^2) ds = t|\nabla \psi|_{\mathcal{L}_2(L^2, L^2)}^2 + X(t), \quad (4.4)$$

where $(X(t))$ is the martingale defined by the Itô integral $X(t) := \int_0^t \langle u \times d\nabla W, \nabla u \rangle$, $t \in [0, T]$.

Moreover, for any $m \geq 2$, the following estimate holds:

$$\mathbb{E} \left[\sup_{t \in [0, T]} E(t)^m \right] + \mathbb{E} \left[\left(\int_0^T |\nabla v(t)|_{L^2}^2 + |\mathcal{T}_u(t)|_{L^2}^2 dt \right)^m \right] \leq C(T, m, |\psi|_{\mathcal{L}_2(L^2, H^1)}, E(0)) \quad (4.5)$$

where the above constant depends on the indicated quantities, but not on the individual element (v, u) .

Proof. *Proof of (4.4).* We can apply Itô Formula in the form given e.g. in [27, Chapter 4] to the functional $(v, u) \in \mathbb{L}^2 \times \mathbf{H}^1 \mapsto \frac{1}{2}(|v|_{L^2}^2 + |\nabla u|_{L^2}^2)$. It yields:

$$\begin{aligned} E(t) - E(0) &= \int_0^t \langle dv, v \rangle + \int_0^t \langle \nabla du, \circ \nabla u \rangle \\ &= \int_0^t \left\langle -Av - (v \cdot \nabla)v - \operatorname{div}(\nabla u \odot \nabla u), v \right\rangle dt \\ &\quad + \int_0^t \left\langle \nabla(-v \cdot \nabla u + \Delta u + |\nabla u|^2 u), \nabla u \right\rangle dt + \int_0^t \langle \nabla(u \times \circ dW), \nabla u \rangle, \end{aligned} \quad (4.6)$$

where the above Stratonovitch integral makes sense as

$$\begin{aligned} \int_0^t \langle \nabla(u \times \circ dW), \nabla u \rangle &= \int_0^t \langle u \times d\nabla W, \nabla u \rangle \\ &\quad + \frac{1}{2} \sum_{l \in \mathbb{N}} \sum_{j=1}^3 \int_0^t \left[\langle (u \times \psi_l^j) \times \nabla \psi_l^j, \nabla u \rangle + \langle u \times \nabla \psi_l^j, \nabla(u \times \psi_l^j) \rangle \right] ds \\ &=: X(t) + \int_0^t (A^1(s) + A^2(s)) ds. \end{aligned} \quad (4.7)$$

As easily seen thanks to Assumption 1.1, the above trace terms are finite \mathbb{P} -a.s. Moreover, the computation of A^1, A^2 , which is detailed in Appendix A leads to

$$A^1(t) = 0 \quad \text{and} \quad A^2(t) = 2|\nabla\psi|_{\mathcal{L}_2(L^2, L^2)}^2, \quad t \in [0, T]. \quad (4.8)$$

Therefore, we can rewrite (4.6) as:

$$\begin{aligned} E(t) - E_0 - X(t) - t|\nabla\psi|_{\mathcal{L}_2(L^2, L^2)}^2 &+ \int_0^t |\nabla v(s)|_{L^2}^2 ds \\ &= \int_0^t \left[-\langle v \cdot \nabla v, v \rangle - \langle \operatorname{div}(\nabla u \odot \nabla u), v \rangle \right. \\ &\quad \left. - \langle \nabla(v \cdot \nabla u), \nabla u \rangle - \langle \Delta u, \Delta u + |\nabla u|^2 u \rangle \right] ds \\ &=: \int_0^t [I_1 + I_2 + I_3 + I_4] ds, \end{aligned}$$

and we now treat each term separately.

For the first term, integration by parts gives

$$I_1 \equiv \int_{\mathbb{T}^2} v^i \partial_i v^j v^j dx = - \int_{\mathbb{T}^2} \partial_i (v^i v^j) v^j dx = - \int_{\mathbb{T}^2} |v|^2 \operatorname{div} v dx - \int_{\mathbb{T}^2} v^i \partial_i v^j v^j dx,$$

where we use a summation convention over repeated indices. Hence:

$$I_1 = -I_1 = 0.$$

Concerning the second and third terms, we can proceed as in [18]: we have

$$I_2 \equiv - \int_{\mathbb{T}^2} v^i \partial_j (\partial_j u) \cdot \partial_i u dx = - \int_{\mathbb{T}^2} \Delta u \cdot (v \cdot \nabla u) dx = \int_{\mathbb{T}^2} \nabla u \cdot \nabla (v \cdot \nabla u) dx,$$

so that:

$$I_2 + I_3 = 0.$$

By the fact that $\mathcal{T}_u \perp u$ (see (4.2)), we have the pointwise identity:

$$(\Delta u + u|\nabla u|^2) \cdot \Delta u \equiv \mathcal{T}_u \cdot \Delta u = |\mathcal{T}_u|_{\mathbb{R}^3}^2. \quad (4.9)$$

Therefore,

$$I_4 = - \int_{\mathbb{T}^2} |\mathcal{T}_u|^2.$$

Summing every term above eventually yields (4.4).

Proof of (4.5). Let $m \geq 2$. From the Burkholder-Davis-Gundy inequality, one can bound X as follows:

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X(s)|^m \right] \leq C(m) \mathbb{E} \left[\left(\int_0^t |\psi^* \operatorname{div}(u \times \nabla u)|_{L^2}^2 ds \right)^{m/2} \right].$$

Using the continuous embedding $\mathcal{L}_2(L^2, H^1) \hookrightarrow \mathcal{L}(L^2, H^1)$, Hölder and Young inequalities, we infer

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} |X(s)|^m \right] &\leq C(|\psi|_{\mathcal{L}_2(L^2, H^1)}) \mathbb{E} \left[\left(\int_0^t |u \times \nabla u|_{L^2}^2 ds \right)^{m/2} \right] \\ &\leq C(|\psi|_{\mathcal{L}_2(L^2, H^1)}, T) \mathbb{E} \left[1 + \int_0^t |\nabla u|_{L^2}^{2m} ds \right]. \quad (4.10) \end{aligned}$$

Hence, going back to (4.4), we can use (4.10), yielding for any $t \in [0, T]$:

$$\Phi(t) \leq C \left(1 + \int_0^t \Phi(s) ds \right),$$

where we denote by $\Phi(t) := \mathbb{E}[\sup_{s \in [0, t]} E(s)^m]$, and where the above constant depends only on E_0, m, T and $|\psi|_{\mathcal{L}_2(L^2, H^1)}$. The first part of the conclusion hence follows by Gronwall Lemma.

Using again (4.4), together with the bound obtained on Φ , we can now bound the term $\mathbb{E}[(\int_0^T [|\nabla v|_{L^2}^2 + |\mathcal{F}_u|^2] dt)^m]$ in the same way. \blacksquare

4.2. Local estimates. In this paragraph, we estimate the local energy $\frac{1}{2} \int_{B(x, \rho)} |v(t, y)|^2 + |\nabla u(t, y)|^2 dy$, where $\rho > 0$ and $x \in \mathbb{T}^2$. This step requires to test (SEL) against the localization function $\mathbf{1}_{B(x, \rho)}$ (more exactly: a suitable regularization thereof), which is definitely not divergence-free in general. To proceed further, we therefore need to introduce the pressure term associated to a solution.

Let (v, u) be a strong solution in the sense of Definition 1.2 on $[\tau_1, \tau_2]$. Using (1.14) against test functions of the form $-\nabla \theta$ where $\theta \in L^2(\mathbb{T}^2; \mathbb{R})$, and denoting by $\tilde{\pi}(\omega, t) := \pi - \int_{\mathbb{T}^2} \pi(\omega, t, x) dx$, we see from the original equation (SEL) that $\tilde{\pi}$ must be given by the equation

$$\Delta \tilde{\pi} = -\partial_{ij}(v^i v^j) - \partial_{ij}(\partial_i u \cdot \partial_j u), \quad \langle \tilde{\pi}, 1 \rangle = 0, \quad (4.11)$$

for a.e. $(\omega, t) \in \Omega \times [0, T]$ such that $\tau_1(\omega) \leq t < \tau_2(\omega)$. From (2.1) we see that

$$v \in L^0(\Omega; L^4([\tau_1, \tau_2]; \mathbb{L}^4)),$$

while

$$\nabla u \in L^0(\Omega; L^4([\tau_1, \tau_2]; \mathbb{L}^4)).$$

Therefore, using the fact that the Laplace operator is an isomorphism, from $L^2(\mathbb{T}^2)$ to $H^{-2}(\mathbb{T}^2)$, when restricted to the orthogonal of constant functions, we infer from Hölder Inequality that

$$\tilde{\pi} \in L^0(\Omega; L^2([\tau_1, \tau_2]; L^2)),$$

together with the pathwise estimate:

$$\|\tilde{\pi}(\omega)\|_{L^2(\tau_1; \tau_2'; L^2)} \leq C \left(\|v(\omega)\|_{L^4(\tau_1; \tau_2'; L^4)}^2 + \|\nabla u(\omega)\|_{L^4(\tau_1; \tau_2'; L^4)}^2 \right) \quad (4.12)$$

for \mathbb{P} -a.e. $\omega \in \Omega$ and for every stopping time τ_2' such that $\tau_1 \leq \tau_2' < \tau_2$.

On the other hand, choosing any constant $c \in \mathbb{R}$, one can let $\pi := \tilde{\pi} + c$, and with this definition it is immediately checked that the equation

$$dv + (-\Delta v + \nabla \pi + v \cdot \nabla v + \operatorname{div} \nabla u \odot \nabla u) dt = 0 \quad (4.13)$$

holds for almost every $\omega \in \Omega$, in the sense of Bochner in $H^{-1}(\mathbb{T}^2; \mathbb{R}^2)$.

Proposition 4.3. *Let (W, ψ) be as in Assumption 1.1. Consider $\varphi \in C^\infty(\mathbb{T}^2)$, $0 \leq \varphi \leq 1$, supported in the ball $B(x, \rho)$ for some $x \in \mathbb{T}^2$ and some $\rho > 0$, and assume that there exists $K > 0$ with*

$$\sup_{y \in \mathbb{T}^2} |\nabla \varphi(y)| \leq \frac{K}{\rho}.$$

Let (v, u) be a strong solution to (SEL) on $[0, T]$, with respect to (\mathfrak{B}, W) , and for any $t \in [0, T]$, define $E^\varphi(t) := \frac{1}{2} \int_{\mathbb{T}^2} \varphi(y)^2 (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy$.

Then, there is a universal constant $C_0 > 0$ such that for every $t \in [0, T]$, it holds

$$\begin{aligned} E^\varphi(t) - E^\varphi(0) &+ \frac{1}{2} \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 (|\nabla v|^2 + |\mathcal{T}_u|^2) \, dx ds \\ &\leq C_0 \left(1 + \frac{1}{\rho^2}\right) \iint_{[0,t] \times \mathbb{T}^2} \left(|v|^3 + (|v| + 1)|\nabla u|^2 + |v||\tilde{\pi}|\right) \, dx ds + X^\varphi(t) + C_\psi^\varphi t, \end{aligned} \quad (4.14)$$

where $X^\varphi(t) := \int_0^t \langle \varphi \nabla u, \varphi u \times d\nabla W \rangle$, $C_\psi^\varphi := \sum_{l \in \mathbb{N}} \int_{\mathbb{T}^2} \varphi^2(x) |\nabla \psi f_l(x)|^2 \, dx$, and $\tilde{\pi} \equiv \pi - \langle \pi, 1 \rangle$ is the pressure term corrected by its mean value (see the previous discussion).

Proof. Apply Itô Formula to E^φ . It holds

$$\begin{aligned} E^\varphi(t) - E^\varphi(s) &= \int_0^t \langle \varphi^2 v, dv \rangle + \int_0^t \langle \varphi^2 \nabla u, \circ \nabla du \rangle \\ &= \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 v \cdot (\Delta v - v \cdot \nabla v - \operatorname{div}(\nabla u \odot \nabla u) - \nabla \pi) \, dx ds \\ &\quad + \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 \nabla u \cdot \nabla (\mathcal{T}_u - v \cdot \nabla u) \, dx ds + \int_0^t \langle \varphi^2 \nabla u, \nabla (u \times \circ dW) \rangle. \end{aligned}$$

Integrating by parts and using that a.e. $\Delta u \cdot \mathcal{T}_u = |\mathcal{T}_u|_{\mathbb{R}^3}^2$, we get:

$$\begin{aligned} E^\varphi(t) - E^\varphi(0) &+ \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 (|\nabla v|^2 + |\mathcal{T}_u|^2) \, dx ds \\ &= - \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 v^j v^i \partial_i v^j \, dx ds - \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 v \operatorname{div}(\nabla u \odot \nabla u) \, dx ds \\ &\quad - \iint_{[0,t] \times \mathbb{T}^2} \varphi^2 v \cdot \nabla \pi \, dx ds - 2 \iint_{[0,t] \times \mathbb{T}^2} \varphi (\partial_j \varphi) \partial_j u^i \mathcal{T}_u^i \, dx ds \\ &\quad + \iint_{[0,t] \times \mathbb{T}^2} \partial_j (\varphi^2 \partial_j u^i) v^\ell \partial_\ell u^i \, dx ds + \int_0^t \langle \varphi^2 \nabla u, u \times \nabla \circ dW \rangle \\ &=: \int_0^t \sum_{\gamma=1}^6 I_\gamma \, ds, \end{aligned}$$

where we make use of a summation convention whenever it is convenient. We can now fix the time variable $s \in [0, t]$ and evaluate each term separately.

First, using $\operatorname{div} v = 0$, we have

$$I_1 = \int_{\mathbb{T}^2} \varphi (\partial_i \varphi) (v^j)^2 v^i \, dx.$$

Therefore, from the assumptions on φ , and since $1/\rho \leq 1 + 1/\rho^2$, it holds

$$I_1 \leq \int_{\mathbb{T}^2} \varphi |\nabla \varphi| |v|^3 \, dx \leq K \left(1 + \frac{1}{\rho^2}\right) \int_{\mathbb{T}^2} |v|^3 \, dx.$$

Similarly for the fourth term, we have by Young Inequality:

$$I_4 \leq \frac{K^2}{\rho^2} \int_{\mathbb{T}^2} |\nabla u|^2 \, dx ds + \frac{1}{2} \int_{\mathbb{T}^2} \varphi^2 |\mathcal{T}_u|^2 \, dx,$$

the second term of which will be eventually absorbed to the left.

For the third term, using that $\operatorname{div} v = 0$, one can write

$$\begin{aligned} I_3 &= \int_{\mathbb{T}^2} 2\varphi(\partial_i \varphi) v^i \tilde{\pi} \, dx \\ &\leq K \left(1 + \frac{1}{\rho^2}\right) \int_{\mathbb{T}^2} |v| |\tilde{\pi}| \, dx. \end{aligned}$$

Taking into account similar compensations as for the global estimates, we estimate the second and fifth terms together, which yields:

$$\begin{aligned} I_2 + I_5 &= - \int_{\mathbb{T}^2} \varphi^2 v^\ell \partial_j (\partial_j u^i \partial_\ell u^i) \, dx + \int_{\mathbb{T}^2} 2\varphi(\partial_j \varphi) \partial_j u^i v^\ell \partial_\ell u^i \, dx + \int_{\mathbb{T}^2} \varphi^2 \partial_{jj} u^i v^\ell \partial_\ell u^i \, dx \\ &= - \int_{\mathbb{T}^2} \varphi^2 v^\ell \partial_j u^i \partial_{j\ell} u^i \, dx + 2 \int_{\mathbb{T}^2} \varphi(\partial_j \varphi) \partial_j u^i v^\ell \partial_\ell u^i \, dx \\ &=: \text{I} + \text{II} \end{aligned}$$

Integrating by parts and using again that $\operatorname{div} v = 0$, we have

$$\text{I} = \int_{\mathbb{T}^2} \varphi(\partial_\ell \varphi) v^\ell (\partial_j u^i)^2 \, dx \leq K \left(1 + \frac{1}{\rho^2}\right) \int_{\mathbb{T}^2} |v| |\nabla u|^2 \, dx,$$

while the evaluation of II yields a similar bound, namely:

$$\text{II} \leq K \left(1 + \frac{1}{\rho^2}\right) \int_{\mathbb{T}^2} |v| |\nabla u|^2 \, dx,$$

for another such constant $C > 0$.

Finally, computing the Stratonovitch integral as in (4.7), we have

$$\begin{aligned} I_6 &= \int_0^t \langle \varphi \nabla u, \varphi u \times \nabla dW \rangle \\ &\quad + \frac{1}{2} \sum_{l \in \mathbb{N}} \sum_{j=1}^3 \int_0^t \left[\langle \varphi^2 (u \times \psi_l^j) \times \nabla \psi_l^j, \nabla u \rangle + \langle \varphi^2 u \times \nabla \psi_l^j, \nabla (u \times \psi_l^j) \rangle \right] ds. \end{aligned}$$

Hence, similar computations as that of (A.3) and (A.4) yield that

$$\begin{aligned} I_6 &= \int_0^t \langle \varphi \nabla u, \varphi u \times d\nabla W \rangle + t \sum_{l \in \mathbb{N}} \int_{\mathbb{T}^2} \varphi(x)^2 |\nabla \psi_l(x)|^2 \, dx \\ &= X^\varphi(t) + C_\psi^\varphi t. \end{aligned}$$

Now, gathering all the above bounds, integrating in time, and absorbing to the left when needed, we end up with (4.14). \blacksquare

5. BOOTSTRAP

In this section, we will see that provided there is no energy concentration at some fixed $t \in [0, T]$, then the solution can be extended continuously after t (in a suitable space), for a positive time.

More precisely, we aim to show the following.

Theorem 5.1. *Consider (W, ψ) as in Assumption 1.1 with $\psi \in \mathcal{L}_2(L^2, H^3)$. Let (v, u) be a strong solution to (SEL) on $[0, \tau)$ with respect to (\mathfrak{B}, W) , such that furthermore (v, u) belongs to $\mathcal{V}_{\text{loc}, [0, \tau)}^2 \times \mathcal{U}_{\text{loc}, [0, \tau)}^3$ and where the stopping time τ is defined*

by the property that

$$\tau < T \quad \text{if and only if} \quad \limsup_{t \nearrow \tau} |v(t)|_{H^2} + |u(t)|_{H^3} = \infty.$$

For arbitrary $\rho > 0$ and $\varepsilon_1 \in (0, \mu_1^{-1})$, denote by

$$\zeta(v, u; \rho, \varepsilon_1) := \inf \left\{ t \in [0, \tau) : \sup_{s \leq t, x \in \mathbb{T}^2} \int_{B(x, \rho)} (|v(s, y)|^2 + |\nabla u(s, y)|^2) dy \geq \varepsilon_1 \right\}. \quad (5.1)$$

Then, for each $m \in \mathbb{N}$, the stopped process

$$(v(\cdot \wedge \zeta(v, u; \rho, \varepsilon_1)), u(\cdot \wedge \zeta(v, u; \rho, \varepsilon_1)))$$

belongs to $L^m(\Omega; \mathcal{V}_{0,T}^2 \times \mathcal{W}_{0,T}^3)$ and its corresponding norm is bounded in terms of the quantities $m, \rho, T, |\psi|_{\mathcal{L}_2(L^2, H^3)}, |u_0|_{H^3}, |v_0|_{H^2}$ only.

In particular, for any ρ, ε_1 , as above, we have $\mathbb{P}(\zeta(v, u; \rho, \varepsilon_1) < \tau) = 1$.

5.1. Estimates of u in $L_{\text{loc}}^2(\mathbf{H}^2)$. The first step is to show an estimate on the quantity $\mathbb{E}[\|u(\cdot \wedge \zeta(v, u; \rho, \varepsilon_1))\|_{L^2(0, T; \mathbf{H}^2)}^m]$, which depends only on $m, T, |\psi|_{\mathcal{L}_2(L^2, H^1)}, |v_0|_{L^2}$ and $|u_0|_{H^1}$.

Lemma 5.1 ($L_{\text{loc}}^2(\mathbf{H}^2)$ estimate for u). *Fix $\varepsilon_1 \in (0, \mu_1^{-1})$ ($\mu_1 > 0$ being the constant in Proposition 4.1), consider v, u, W, ψ as in Theorem 5.1 and for $\rho > 0$ define $\zeta(v, u; \rho, \varepsilon_1)$ as in (5.1).*

For every $m \geq 2$, it holds for $t \in [0, T]$:

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^{t \wedge \zeta(v, u; \rho, \varepsilon_1)} |\Delta u(s)|_{L^2}^2 ds \right)^m \right] \\ \leq C(\rho, m, |\psi|_{\mathcal{L}_2(L^2, H^1)}, T) \mathbb{E} \left[\left(\int_0^{t \wedge \zeta(v, u; \rho, \varepsilon_1)} |\nabla u|_{L^2}^2 ds \right)^m \right]. \end{aligned} \quad (5.2)$$

In particular, combining with Proposition 4.2, we have an estimate

$$\mathbb{E} \left[\left(\int_0^{T \wedge \zeta(v, u; \rho, \varepsilon_1)} |\Delta u(s)|_{L^2}^2 ds \right)^m \right] \leq C(\rho, m, |\psi|_{\mathcal{L}_2(L^2, H^1)}, T, E(0)). \quad (5.3)$$

Proof. Since (v, u) is a solution, using (4.3), we get $u \perp \mathcal{T}_u$ so that proceeding as in (4.9), and using that, in the weak sense,

$$0 \equiv \Delta \left(\frac{|u|^2}{2} \right) = (\Delta u \cdot u) + |\nabla u|^2,$$

then it holds true that $|\mathcal{T}_u|_{\mathbb{R}^3}^2 = |\Delta u|_{\mathbb{R}^3}^2 - |\nabla u|_{\mathbb{R}^{2 \times 3}}^4$. Applying now Proposition 4.2, and expanding the tension, we obtain

$$\begin{aligned} E(t) - E(0) + \iint_{[0, t] \times \mathbb{T}^2} (|\Delta u|^2 + |\nabla v|^2) dx ds - X(t) - tC_\psi \\ = \iint_{[0, t] \times \mathbb{T}^2} |\nabla u|^4 dx ds. \end{aligned}$$

Next, using Proposition 4.1, we obtain for times $t \leq \zeta(v, u, \rho, \varepsilon_1)$:

$$\begin{aligned}
E(t) - E(0) + \iint_{[0,t] \times \mathbb{T}^2} (|\Delta u|^2 + |\nabla v|^2) dx ds - X(t) - tC_\psi \\
\leq \mu_1 \left(\sup_{s \in [0, T], x \in \mathbb{T}^2} \int_{B(x, \rho)} |\nabla u(s, y)|^2 dy \right) \iint_{[0,t] \times \mathbb{T}^2} \left(|\Delta u|^2 + \frac{C}{\rho^2} |\nabla u|^2 \right) dx ds \\
\leq \mu_1 \varepsilon_1 \iint_{[0,t] \times \mathbb{T}^2} |\Delta u|^2 dx ds + \frac{\mu_1 \varepsilon_1 C}{\rho^2} \iint_{[0,t] \times \mathbb{T}^2} |\nabla u|^2 dx ds,
\end{aligned} \tag{5.4}$$

Since $\varepsilon_1 < \mu_1^{-1}$, the first term above can be absorbed to the left, and the conclusion follows by estimating the martingale part as in Proposition 4.2. \blacksquare

Note that, in the proof of Lemma 5.1, we did not need the local estimate for the term in the energy which involves the velocity. The control of this term will however be important in the following estimates.

Lemma 5.2. *Under the assumptions of Lemma 5.1, for any $m \geq 1$, there hold the exponential bounds:*

$$\mathbb{E} \left[\exp \left\{ m \sup_{t \in [0, \zeta(v, u; \rho, \varepsilon_1)]} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2) \right\} \right] \leq C(m, \rho), \tag{5.5}$$

$$\mathbb{E} \left[\exp \left\{ m \int_0^{\zeta(v, u; \rho, \varepsilon_1)} (|\nabla v|_{L^2}^2 + |\Delta u|_{L^2}^2) dt \right\} \right] \leq C(m, \rho, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}) \tag{5.6}$$

and

$$\mathbb{E} \left[\exp \left\{ m \int_0^{\zeta(v, u; \rho, \varepsilon_1)} (|v|_{L^4}^4 + |\nabla u|_{L^4}^4) dt \right\} \right] \leq C(m, \rho, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}). \tag{5.7}$$

Proof. Fix $m \geq 1$, and for simplicity denote $\zeta := \zeta(v, u; \rho, \varepsilon_1)$. In order to prove (5.5), it is sufficient to observe that the random variable $\sup_{t \in [0, \zeta]} |v(t)|_{L^2}^2 + |\nabla u|_{L^2}^2$ belongs to $L^\infty(\Omega)$. Indeed, as a consequence of the definition of ζ in (5.1) we have

$$\sup_{t \in [0, \zeta]} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2) \leq \sum_{1 \leq i \leq N_\rho} \sup_{t \in [0, \zeta]} \int_{B(x_i, \rho)} (|v(t, y)|^2 + |\nabla u(t, y)|_{L^2}^2) dy \leq N_\rho \varepsilon_1,$$

for any finite sequence $\{x_1, \dots, x_{N_\rho}\}$ such that $\cup_{i \leq N_\rho} B(x_i, \rho) = \mathbb{T}^2$ (note that asymptotically $N_\rho \sim C/\rho^2$, but this has no importance here). This proves our claim, and hence (5.5).

Proof of (5.6). Going back to the proof of Lemma 5.1, we see in particular from (5.4) and (4.4) that \mathbb{P} -a.s., for any $t \in [0, T]$:

$$m \int_0^{t \wedge \zeta} (|\Delta u|_{L^2}^2 + |\nabla v|_{L^2}^2) ds \leq C(1 + X(t \wedge \zeta)), \tag{5.8}$$

where the constant $C > 0$ depends only on the quantities $\varepsilon_1, m, \rho, E_0, T$ and $|\psi|_{\mathcal{L}_2(L^2, H^1)}$.

Next, the process $S(t) := \exp \{mX(t)\}$, $t \in [0, T]$, is a submartingale, and Itô Formula yields, for $t \leq \zeta$,

$$\begin{aligned} S(t) - S(0) - m \int_0^t S(s) \langle \nabla u, u \times \nabla dW \rangle &= m^2 \int_0^t S(s) |\psi^* \operatorname{div}(u \times \nabla u)|_{L^2}^2 ds \\ &\leq C(|\psi|_{\mathcal{L}_2(L^2, H^1)}, \rho, \varepsilon_1, m) \int_0^t S(s) ds. \end{aligned}$$

Hence, taking the expectation, and then applying Gronwall Lemma yields the estimate

$$\sup_{0 \leq t \leq T} \mathbb{E}[S(t \wedge \zeta)] \leq C(T, |\psi|_{\mathcal{L}_2(L^2, H^1)}, \rho, \varepsilon_1, m)$$

where C is as above. Applying Doob's inequality for submartingales, we obtain that

$$\mathbb{E} \left[\sup_{t \leq \zeta} \exp \{mX(t)\} \right] \leq C \sup_{t \leq T} \mathbb{E}[S(t \wedge \zeta)]$$

for another such constant. Taking the exponential and expectation in (5.8), we end up with the desired bound.

Proof of (5.7). By Proposition 4.1, one observes that

$$\begin{aligned} &\mathbb{E} \left[\exp m \int_0^\zeta (|v|_{L^4}^4 + |\nabla u|_{L^4}^4) dt \right] \\ &\leq \mathbb{E} \left[\exp \left\{ C\varepsilon_1 m \int_0^\zeta (|\nabla v|_{L^2}^2 + |\Delta u|_{L^2}^2) dt \right\} \exp \left\{ \frac{C\varepsilon_1 m}{\rho^2} \int_0^\zeta (|v|_{L^2}^2 + |\nabla u|_{L^2}^2) dt \right\} \right], \end{aligned}$$

and the conclusion follows by Hölder Inequality, together with (5.5) and (5.6). \blacksquare

5.2. Higher regularity. Fix $\rho > 0$ and denote $\zeta_\rho := \zeta(v, u; \rho, \varepsilon_1)$. The previous paragraph shows that, if (v, u) is a strong solution restricted to the time interval $[0, \zeta_\rho]$, then a uniform bound holds for (v, u) in $L_t^2(\mathbb{H}_x^1) \times L_t^2(\mathbf{H}_x^2)$, and consequently for the pressure $\pi \in L_t^2(L_x^2)$ as well, as may be seen using (4.12). However, this is certainly not enough to conclude that $(v(\zeta_\rho), u(\zeta_\rho))$ is in $\mathbb{H}^2 \times \mathbf{H}^3$, \mathbb{P} -a.s., as would be needed in order to show the moment bounds of Theorem 5.1 (at this stage it is even uncertain that $(v(\zeta_\rho), u(\zeta_\rho))$ defines a random variable in $\mathbb{H}^1 \times \mathbf{H}^2$). As will be shown below, the latter property is true for smooth enough data (v_0, u_0, ψ) but the proof requires a bootstrap argument in the spirit of [31, Section 3], which is made difficult by the presence of noise in the equation.

The first step is to show the following.

Proposition 5.1 (Higher order estimates). *Under the assumptions of Lemma 5.1, assume in addition that there exists $\rho > 0$, such that a.s.:*

$$\sup_{x \in \mathbb{T}^2, t \in [0, T]} \int_{B(x, \rho)} (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy < \varepsilon_1, \quad (5.9)$$

and also that a.s. :

$$v \in \mathcal{V}_T^1, \quad u \in \mathcal{W}_T^2, \quad \text{and} \quad \psi \in \mathcal{L}_2(L^2, H^3). \quad (5.10)$$

Then, for every $m \in [1, \infty)$ we have the estimate:

$$\mathbb{E} \left[\sup_{t \in [0, \zeta(v, u; \rho, \varepsilon_1)]} \left(|\hat{\mathcal{T}}(t)|_{L^2}^2 + |\Delta u(t)|_{L^2}^2 + |\nabla v(t)|_{L^2}^2 \right)^m + \left(\int_0^{\zeta(v, u; \rho, \varepsilon_1)} [|\nabla \Delta u(t)|_{L^2}^2 + |\nabla \hat{\mathcal{T}}(t)|_{L^2}^2 + |\nabla^2 v(t)|_{L^2}^2 + |\partial_t v(t)|_{L^2}^2] dt \right)^m \right] \leq C(m, |\psi|_{\mathcal{L}_2(L^2, H^3)}, |u(0)|_{H^2}, |v(0)|_{H^1}), \quad (5.11)$$

for a.e. $\omega \in \Omega$, where $\hat{\mathcal{T}}$ is the ‘‘corrected tension’’

$$\hat{\mathcal{T}} := \Delta u + u|\nabla u|^2 - v \cdot \nabla u. \quad (5.12)$$

Proof. For simplicity we denote by $\zeta := \zeta(v, u; \rho, \varepsilon_1)$.

Step 1. Pathwise Gronwall. In order to prove (5.11), we are going to apply Itô Formula to the quantity

$$G(t) := |\Delta u(t)|_{L^2}^2 + |\hat{\mathcal{T}}(t)|_{L^2}^2 + |\nabla v(t)|_{L^2}^2, \quad t \in [0, \zeta], \quad (5.13)$$

and then conclude by applying Gronwall Lemma pathwise.

As a first step, we will show the following.

Claim. We have for any $t \in [0, \zeta]$:

$$G(t) - G(0) + \int_0^t \left(|\nabla \Delta u(s)|_{L^2}^2 + |\nabla \hat{\mathcal{T}}(s)|_{L^2}^2 + |\nabla^2 v(s)|_{L^2}^2 + |\partial_t v(s)|_{L^2}^2 \right) ds \leq \chi(t) \left(\sup_{s \in [0, t]} G(s) \right) + Y(t), \quad (5.14)$$

where we let

$$\chi(t) := C \int_0^t (|v(s)|_{L^4}^4 + |\nabla u(s)|_{L^4}^4) ds,$$

for a sufficiently large but universal constant C , while Y denotes the semi-martingale

$$Y(t) = 2 \int_0^t \langle \hat{\mathcal{T}}, \partial_\beta u \times \text{od} \partial_\beta W \rangle + \int_0^t \langle \hat{\mathcal{T}}, u \times \text{od} \Delta W \rangle - \int_0^t \langle \hat{\mathcal{T}}, v^i (u \times \text{od} \partial_i W) \rangle + 2 \int_0^t \langle \Delta u, \partial_\beta u \times \text{od} \partial_\beta W \rangle + \int_0^t \langle \Delta u, u \times \text{od} \Delta W \rangle, \quad (5.15)$$

for $t \in [0, \zeta]$.

To prove the claim, we proceed as in [16, Lemma 6], an essential difference being that here the (ill-defined) time derivative of u must be replaced by the corrected tension $\hat{\mathcal{T}}$ defined in (5.12).

As a first observation note that, similarly as for \mathcal{T} , the corrected tension is a.e. orthogonal to u . Indeed, by definition $\hat{\mathcal{T}} = \mathcal{T}_u - v \cdot \nabla u$, but on the one hand we have $\mathcal{T}_u \perp u$, and on the other hand:

$$u \cdot (-v \cdot \nabla u) = - \sum_{i=1}^3 u^i \sum_{j=1}^2 v^j \partial_j u^i = - \frac{1}{2} \sum_{j=1}^2 v^j \partial_j (|u|^2) \equiv 0, \quad (5.16)$$

hence showing $\hat{\mathcal{F}} \perp u$. We will now apply Itô Formula to each of the quantities appearing in (5.13).

First, using the semi-martingale decomposition of u and the definition (5.12), then Itô Formula applied to the quantity $\frac{1}{2}|\hat{\mathcal{F}}|_{L^2}^2$ gives, using $\operatorname{div} v = 0$:

$$\begin{aligned} \frac{1}{2}|\hat{\mathcal{F}}(t)|_{L^2}^2 - \frac{1}{2}|\hat{\mathcal{F}}(0)|_{L^2}^2 &= \int_0^t \left\langle \hat{\mathcal{F}}, \Delta \hat{\mathcal{F}} + \hat{\mathcal{F}}|\nabla u|^2 + u \left(\nabla u \cdot \nabla \hat{\mathcal{F}} \right) \right\rangle ds \\ &\quad + M(t), \end{aligned} \quad (5.17)$$

for all $0 \leq s \leq t \leq \zeta$, a.s. , where M denotes the semi-martingale

$$\begin{aligned} M(t) &:= \int_0^t \langle \hat{\mathcal{F}}, \Delta(u \times \circ dW) + |\nabla u|^2 u \times \circ dW + 2u \nabla(u \times \circ dW) \cdot \nabla u - v \cdot \nabla(u \times \circ dW) \rangle, \\ &:= \int_0^t \langle \hat{\mathcal{F}}, \Delta(u \times \circ dW) \rangle + \int_0^t \langle \hat{\mathcal{F}}, |\nabla u|^2 u \times \circ dW \rangle - \int_0^t \langle \hat{\mathcal{F}}, v^i \partial_i(u \times \circ dW) \rangle \\ &= 2 \int_0^t \langle \hat{\mathcal{F}}, \partial_\beta u \times \circ d\partial_\beta W \rangle + \int_0^t \langle \hat{\mathcal{F}}, u \times \circ d\Delta W \rangle - \int_0^t \langle \hat{\mathcal{F}}, v^i(u \times \circ d\partial_i W) \rangle, \end{aligned} \quad (5.18)$$

where we have used the fact that the term $\int_0^t \langle \hat{\mathcal{F}}, \hat{\mathcal{F}} \times \circ dW \rangle$ vanishes thanks to orthogonality, together with

$$\int_0^t \langle \hat{\mathcal{F}}, u \nabla(u \times \circ dW) \cdot \nabla u \rangle = 0.$$

Recalling that $\hat{\mathcal{F}} \perp u$, integration by parts yields the relation

$$\frac{1}{2} \left(|\hat{\mathcal{F}}(t)|_{L^2}^2 - |\hat{\mathcal{F}}(0)|_{L^2}^2 \right) - M(t) + \int_0^t |\nabla \hat{\mathcal{F}}|_{L^2}^2 ds = \int_0^t \langle |\hat{\mathcal{F}}|^2, |\nabla u|^2 \rangle ds. \quad (5.19)$$

Using (2.1), the above r.h.s. is estimated as:

$$\begin{aligned} &\iint_{[0,t] \times \mathbb{T}^2} |\hat{\mathcal{F}}|^2 |\nabla u|^2 dx ds \\ &\leq \left(\int_0^t |\hat{\mathcal{F}}|_{L^4}^4 ds \int_0^t |\nabla u|_{L^4}^4 ds \right)^{1/2} \\ &\leq \sqrt{\mu_0} \left(\sup_{s \in [0,t]} |\hat{\mathcal{F}}(s)|_{L^2}^2 \right)^{1/2} \left(\int_0^t |\nabla \hat{\mathcal{F}}|_{L^2}^2 ds \right)^{1/2} \left(\int_0^t |\nabla u|_{L^4}^4 ds \right)^{1/2} \\ &\leq C \left(\sup_{s \in [0,t]} |\hat{\mathcal{F}}(s)|_{L^2}^2 \right) \chi(t) + \frac{1}{2} \int_0^t |\nabla \hat{\mathcal{F}}|_{L^2}^2 ds, \end{aligned}$$

by definition of $\chi(t)$. Hence, from (5.19) we obtain the bound

$$\frac{1}{2} \left(|\hat{\mathcal{F}}(t)|_{L^2}^2 - |\hat{\mathcal{F}}(0)|_{L^2}^2 \right) + \frac{1}{2} \int_0^t |\nabla \hat{\mathcal{F}}|_{L^2}^2 ds - M(t) \leq \chi(t) \left(\sup_{[0,t]} G(s) \right). \quad (5.20)$$

Let us now apply Itô Formula on the second term in (5.13). Using again the semi-martingale decomposition of u , together with the second-order Leibniz Formula for

the laplacian, it holds

$$\begin{aligned}
& \frac{1}{2} (|\Delta u(t)|_{L^2}^2 - |\Delta u(0)|_{L^2}^2) + \int_0^t |\nabla \Delta u|_{L^2}^2 ds \\
&= \int_0^t \langle |\Delta u|^2, |\nabla u|^2 \rangle ds + 4 \int_0^t \langle \Delta u, \nabla u \nabla^2 u \nabla u \rangle ds + 2 \int_0^t \langle \Delta u, u (\nabla \Delta u \cdot \nabla u) \rangle ds \\
&\quad + 2 \int_0^t \langle \Delta u, u |\nabla^2 u|^2 \rangle ds + \int_0^t \langle \nabla \Delta u, \nabla v \nabla u + v \nabla^2 u \rangle ds + N(t), \quad (5.21)
\end{aligned}$$

where $N(t)$ is defined as

$$\begin{aligned}
N(t) &= \int_0^t \langle \Delta u, \Delta [u \times \circ dW] \rangle \\
&:= \int_0^t \langle \Delta u, \Delta u \times \circ dW \rangle + 2 \int_0^t \langle \Delta u, \partial_\beta u \times \circ d\partial_\beta W \rangle + \int_0^t \langle \Delta u, u \times \circ d\Delta W \rangle \\
&= 2 \int_0^t \langle \Delta u, \partial_\beta u \times \circ d\partial_\beta W \rangle + \int_0^t \langle \Delta u, u \times \circ d\Delta W \rangle, \quad (5.22)
\end{aligned}$$

where we have made use of Leibniz Formula and orthogonality.

With the exception of the fourth term in the r.h.s. of (5.21), each term can be evaluated by (2.1) and Hölder Inequality, in order to yield an estimate of the form (5.14). For the fourth term, we write that

$$2 \int_0^t \langle \Delta u, u |\nabla^2 u|^2 \rangle ds \equiv 2 \iint_{[0,t] \times \mathbb{T}^2} (\Delta u \cdot u) |\nabla^2 u|^2 dx ds = -2 \iint_{[0,t] \times \mathbb{T}^2} |\nabla u|^2 |\nabla^2 u|^2 dx ds,$$

using once more the identity $0 = \Delta(\frac{|u|^2}{2}) = u \cdot \Delta u + |\nabla u|^2$.

Hence, using Young Inequality several times, it is easily seen that for any $\delta_1 > 0$ fixed, there is a constant $C_1 > 0$ such that

$$\begin{aligned}
& \frac{1}{2} (|\Delta u(t)|_{L^2}^2 - |\Delta u(0)|_{L^2}^2) + \int_0^t |\nabla \Delta u|_{L^2}^2 ds - N(t) \\
&\leq C_1 \iint_{[0,t] \times \mathbb{T}^2} |\Delta u|^2 |\nabla u|^2 dx ds + C_1 \iint_{[0,t] \times \mathbb{T}^2} |\nabla^2 u|^2 |\nabla u|^2 dx ds \\
&\quad + \delta_1 \int_0^t |\nabla \Delta u|_{L^2}^2 ds + \iint_{[0,t] \times \mathbb{T}^2} |\nabla v|^2 |\nabla u|^2 dx ds + \iint_{[0,t] \times \mathbb{T}^2} |v|^2 |\nabla^2 u|^2 dx ds \\
&\qquad\qquad\qquad =: \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}. \quad (5.23)
\end{aligned}$$

To estimate the first term, we use again (2.1), which gives:

$$\begin{aligned}
\iint_{[0,t] \times \mathbb{T}^2} |\Delta u|^2 |\nabla u|^2 dx ds &\leq \left(\iint_{[0,t] \times \mathbb{T}^2} |\Delta u|^4 dx ds \right)^{1/2} \left(\iint_{[0,t] \times \mathbb{T}^2} |\nabla u|^4 dx ds \right)^{1/2} \\
&\leq \sqrt{\mu_0} \left(\sup_{s \in [0,t]} |\Delta u|_{L^2}^2 \iint_{[0,t] \times \mathbb{T}^2} |\nabla \Delta u|^2 dx ds \right)^{1/2} \chi(t)^{1/2}
\end{aligned}$$

Thus, we have for any $\delta_2 > 0$:

$$\text{I} \leq C_2 \chi(t) \sup_{s \in [0,t]} G(s) + \delta_2 \int_0^t |\nabla \Delta u|_{L^2}^2 ds, \quad (5.24)$$

where $C_2 > 0$ only depends on δ_2 .

Similarly, estimating the H^2 norm by the Laplacian, we have for the second term

$$\text{II} \equiv C_1 \iint_{[0,t] \times \mathbb{T}^2} |\nabla^2 u|^2 |\nabla u|^2 dx ds \leq C_3 \chi(t) \sup_{[0,t]} G(s) + \delta_3 \int_0^t |\nabla \Delta u|_{L^2}^2 ds. \quad (5.25)$$

Concerning the fourth term in (5.23), we proceed similarly to obtain that

$$\text{IV} \leq C_4 \chi(t) \sup_{s \in [0,t]} G(s) + \delta_4 \int_0^t |\nabla^2 v|_{L^2}^2 ds. \quad (5.26)$$

Similarly,

$$\begin{aligned} \iint_{[0,t] \times \mathbb{T}^2} |v|^2 |\nabla^2 u|^2 dx ds &\leq \sqrt{\mu_0} \left(\iint_{[0,t] \times \mathbb{T}^2} |v|^4 dx ds \right)^{1/2} \\ &\quad \times \left(\sup_{s \in [0,t]} |\nabla^2 u|^2 \iint_{[0,t] \times \mathbb{T}^2} |\nabla^3 u|^2 dx ds \right)^{1/2} \end{aligned}$$

and thus:

$$\text{V} \leq C_5 \chi(t) \sup_{s \in [0,t]} G(s) + \delta_5 \int_0^t |\nabla \Delta u|_{L^2}^2 ds, \quad (5.27)$$

It remains to evaluate the velocity terms. First, multiply the equation on v by Δv (which is also divergence free), and integrate on $[0, t] \times \mathbb{T}^2$ to get:

$$\begin{aligned} |\nabla v(t)|_{L^2}^2 - |\nabla v(0)|_{L^2}^2 + \int_0^t |\Delta v|_{L^2}^2 ds \\ = \int_0^t \langle \Delta v, \mathcal{P}[-v \cdot \nabla v - \text{div}(\nabla u \odot \nabla u)] \rangle ds. \end{aligned} \quad (5.28)$$

Note that, integrating by parts

$$- \iint_{[0,t] \times \mathbb{T}^2} \Delta v \cdot (v \cdot \nabla v) dx ds = \iint_{[0,t] \times \mathbb{T}^2} \Delta v \cdot (v \cdot \nabla v) dx ds = 0.$$

Moreover, it holds for $i = 1, 2$:

$$[\text{div}(\nabla u \odot \nabla u)]^i = \partial_{jj} u \cdot \partial_i u + \partial_j u \cdot \partial_{ij} u = [\Delta u \cdot \nabla u + \frac{1}{2} \nabla(|\nabla u|^2)]^i,$$

and therefore

$$\mathcal{P}[\text{div}(\nabla u \odot \nabla u)] = \mathcal{P}[\Delta u \cdot \nabla u].$$

The last term in (5.28) is then estimated as:

$$\int_0^t \langle \Delta v, -\mathcal{P} \text{div}(\nabla u \odot \nabla u) \rangle ds \leq \delta_6 \iint_{[0,t] \times \mathbb{T}^2} |\Delta v|^2 dx ds + C_6 \iint_{[0,t] \times \mathbb{T}^2} |\mathcal{P}[\Delta u \cdot \nabla u]|^2 dx ds.$$

Using (2.1) and the continuity of \mathcal{P} , we have on the other hand

$$\begin{aligned} \iint_{[0,t] \times \mathbb{T}^2} |\mathcal{P}[\Delta u \cdot \nabla u]|^2 dx ds &\leq C(\mathcal{P}) \int_0^t |\Delta u|_{L^4}^2 |\nabla u|_{L^4}^2 ds \\ &\leq C_7 \chi(t) \sup_{s \in [0,t]} G(s) + \delta_7 \int_0^t |\nabla \Delta u|_{L^2}^2 ds, \end{aligned} \quad (5.29)$$

where we have estimated the right hand side as in (5.24).

Now, multiply the equation on v by $\partial_t v$. Integrating over $[0, t] \times \mathbb{T}^2$, it holds

$$\begin{aligned}
& |\nabla v(t)|_{L^2}^2 - |\nabla v(0)|_{L^2}^2 + \int_0^t |\partial_t v|_{L^2}^2 ds \\
& \leq - \iint_{[0,t] \times \mathbb{T}^2} v \cdot \nabla v \cdot \partial_t v dx ds + \iint_{[0,t] \times \mathbb{T}^2} |\partial_t v| |\Delta u| |\nabla u| dx ds \\
& \leq \delta_8 \iint_{[0,t] \times \mathbb{T}^2} |\partial_t v|^2 dx ds \\
& \quad + C_8 \left(\iint_{[0,t] \times \mathbb{T}^2} |\Delta u|^2 |\nabla u|^2 dx ds + \iint_{[0,t] \times \mathbb{T}^2} |\nabla v|^2 |v|^2 dx ds \right). \\
& =: \delta_8 \iint_{[0,t] \times \mathbb{T}^2} |\partial_t v|^2 + C_8(\text{VI} + \text{VII}).
\end{aligned}$$

Using again (5.24), we obtain

$$\text{VI} \leq C_9 \chi(t) \sup_{s \in [0,t]} G(s) + \delta_9 \int_0^t |\nabla \Delta u|_{L^2}^2 ds, \quad (5.30)$$

while

$$\text{VII} \leq C_{10} \chi(t) \sup_{s \in [0,t]} G(s) + \delta_{10} \int_0^t |\nabla^2 v|_{L^2}^2 ds. \quad (5.31)$$

Finally, summing the inequalities (5.18), (5.20) and (5.22)–(5.31), we obtain (5.14) by choosing the constants $\delta_i, i = 1, \dots, 10$ small enough.

Step 2: the main stochastic estimate. Using Burkholder-Davies-Gundy Inequality, we have from Claim A.1:

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \in [0, t \wedge \zeta]} |\hat{Y}(s)|^{2m} \right] \leq C(m) \mathbb{E} [\langle Y \rangle (t \wedge \zeta)^m] \\
& \leq \tilde{C}(m) \mathbb{E} \left[\left(\int_0^{t \wedge \zeta} \left\{ |\psi^* \operatorname{div}(u \times \nabla \hat{\mathcal{F}})|_{L^2}^2 + |\psi^* \Delta(u \times \hat{\mathcal{F}})|_{L^2}^2 + |\psi^* \partial_i(v^i u \times \hat{\mathcal{F}})|_{L^2}^2 \right. \right. \right. \\
& \quad \left. \left. \left. + |\psi^* \Delta(-u \times \Delta u)|_{L^2}^2 + |\psi^* \operatorname{div}(u \times \nabla \Delta u)|_{L^2}^2 \right\} ds \right)^m \right] \quad (5.32)
\end{aligned}$$

Thanks to the fact that $\mathcal{L}_2(L^2, H^k) \hookrightarrow \mathcal{L}(L^2, H^k)$, for each $k = 1, 2, 3$, we deduce from the above estimate the existence of a constant $C(m, |\psi|_{\mathcal{L}_2(L^2, H^3)}) > 0$ so that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \in [0, t \wedge \zeta]} |\hat{Y}(s)|^{2m} \right] \leq C(m, |\psi|_{\mathcal{L}_2(L^2, H^3)}) \mathbb{E} \left[\left\{ \int_0^{t \wedge \zeta} (|\nabla \hat{\mathcal{F}}(s)|_{L^2}^2 + |\hat{\mathcal{F}}(s)|_{L^2}^2 \right. \right. \\
& \quad \left. \left. + |v(s)|_{L^4}^2 |\hat{\mathcal{F}}(t)|_{L^4}^2 + |\Delta u(s)|_{L^2}^2 + |\nabla \Delta u(s)|_{L^2}^2) ds \right\}^m \right]. \quad (5.33)
\end{aligned}$$

Step 3. Conclusion. By Claim A.1, we have $Y(t) = A(t) + \hat{Y}(t)$ where \hat{Y} is a martingale and A has finite variation. Moreover, from (A.6) together with Sobolev embeddings and the fact that $|u|_{\mathbb{R}^3} = 1$, we infer that:

$$\begin{aligned} |A(t)| &\leq C \sum_{l \in \mathbb{N}} |\nabla \psi|_{L^\infty}^2 \int_0^t (|\nabla u(s)|^2 + |v(s)|^2 + |\Delta \psi_l|_{L^2}^2) ds \\ &\leq C(|\psi|_{\mathcal{L}_2(L^2, H^3)}) \int_0^t (1 + |\nabla u(s)|^2 + |v(s)|^2) ds, \end{aligned} \quad (5.34)$$

\mathbb{P} -a.s. for $t \in [0, \zeta]$. Therefore, by (5.14) together with (5.34) and a well-known generalization of Gronwall inequality, we deduce that \mathbb{P} -a.s.:

$$\begin{aligned} \mathcal{G} &:= \sup_{t \in [0, T \wedge \zeta]} G(t) + \int_0^{T \wedge \zeta} (|\nabla \Delta u(t)|_{L^2}^2 + |\nabla \hat{\mathcal{F}}(t)|_{L^2}^2 + |\nabla^2 v(t)|_{L^2}^2 + |\partial_t v(t)|_{L^2}^2) dt \\ &\leq C \left[G(0) + \sup_{t \in [0, T \wedge \zeta]} |\hat{Y}(t)| \right] \exp \left\{ \int_0^{T \wedge \zeta} |v(t)|_{L^4}^4 + |\nabla u(t)|_{L^4}^4 dt \right\}. \end{aligned} \quad (5.35)$$

Next, using (5.33) we have for any $m \geq 1$ in (5.35) and for arbitrary $\delta > 0$:

$$\begin{aligned} \mathbb{E} [\mathcal{G}^m] &\leq C(\delta) \mathbb{E} \left[\exp \left\{ 2m \int_0^{T \wedge \zeta} |v(t)|_{L^4}^4 + |\nabla u(t)|_{L^4}^4 dt \right\} \right] + \delta \mathbb{E} \left[\sup_{t \in [0, T \wedge \zeta]} |\hat{Y}(t)|^{2m} \right] \\ &\leq C(\delta, m, \rho, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}) + \delta \mathbb{E} \left[\left\{ \int_0^{T \wedge \zeta} (|\Delta u|_{L^2}^2 + |\nabla \Delta u|_{L^2}^2 \right. \right. \\ &\quad \left. \left. + |\nabla \hat{\mathcal{F}}|_{L^2}^2 + |\hat{\mathcal{F}}|_{L^2}^2 + |v|_{L^4}^2 |\hat{\mathcal{F}}|_{L^4}^2) dt \right\}^m \right], \end{aligned}$$

and the conclusion follows by (2.1) and absorption to the left, provided $\delta > 0$ is taken sufficiently small. \blacksquare

5.3. Proof of Theorem 5.1. We now have all in hand to prove our main Theorem. The proof is based on an idea of Debussche, De Moor and Hofmanová [8].

For convenience, we again denote $\zeta := \zeta(v, u; \rho, \varepsilon_1)$.

Step 1: increasing the regularity of the stochastic convolution. We define the stochastic convolution

$$Z(t) := \int_0^t e^{t\Delta} u \times dW \equiv \int_0^t e^{t\Delta} u \times \psi d\xi, \quad (5.36)$$

ξ being the cylindrical Wiener process given formally by the infinite sum $\sum_{l \geq 0} \mathbf{B}_l(\cdot) f_l$. Note that the latter series is a well-defined element of $C(H^{-1-\epsilon})$ for any $\epsilon > 0$, where for simplicity we henceforth denote by $C(H^s) := C(0, \zeta; H^s)$, $L^m(H^s) := L^m(0, \zeta; H^s)$ and so on.

By Proposition 5.1, for any $m \in (4, \infty)$ we have

$$\mathbb{E} \left[\|u\|_{L^\infty(H^2)}^m \right] < \infty, \quad (5.37)$$

hence making use of the parabolic estimate (Prop. 3.1) with

$$\delta := \frac{1}{2} \in \left(0, 1 - \frac{2}{m}\right),$$

we obtain the bound

$$\begin{aligned} \mathbb{E} \left[\|Z\|_{C(W^{2,4})}^m \right] &\leq C \mathbb{E} \left[\|Z\|_{C(H^{5/2})}^m \right] \\ &\leq C' \mathbb{E} \left[\|u \times \boldsymbol{\psi}\|_{L^m(\mathcal{L}_2(L^2, H^2))}^m \right] \leq C'' (|\boldsymbol{\psi}|_{\mathcal{L}_2(L^2, H^4)}, T) \mathbb{E} \left[\|u\|_{C(H^2)}^m \right], \end{aligned} \quad (5.38)$$

where we have used the Sobolev embeddings $H^{5/2} \hookrightarrow W^{2,4}$ and $H^4 \hookrightarrow W^{2,\infty}$.

Next, we define $y := u - Z$ which solves the following PDE with random coefficients:

$$\partial_t y - \Delta y = -v \cdot \nabla u + u |\nabla u|^2 := f + g. \quad (5.39)$$

From (5.37) and the Sobolev embedding Theorem, we have

$$g \in L^m(\Omega; C(L^p)), \quad \text{for any } p \in [1, \infty). \quad (5.40)$$

Similarly, from Proposition 5.1, we get

$$\mathbb{E} \left[\|v\|_{C(H^1)}^m \right] < \infty, \quad (5.41)$$

and therefore, it holds as well

$$f \in L^m(\Omega; C(L^p)), \quad \text{for any } p \in [1, \infty). \quad (5.42)$$

Let $p = 4$. As a consequence of (5.39), (5.40), (5.42), one concludes from the standard parabolic estimates (we refer, e.g. to [21], see also [13]) that y belongs to $L^m(\Omega; C(W^{2,4}))$, together with the bound

$$\begin{aligned} \mathbb{E} \left[\|y\|_{C(W^{2,4})}^m \right] &\leq C \mathbb{E} \left[\|v\|_{C(L^8)}^m \|\nabla u\|_{C(L^8)}^m + \|\nabla u\|_{C(L^8)}^{2m} \right] \\ &\leq C' \mathbb{E} \left[\|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right]. \end{aligned} \quad (5.43)$$

Using (5.38), then (5.43) leads to better regularity for $u \equiv y + Z$, namely:

$$\mathbb{E} \left[\|u\|_{C(W^{2,4})}^m \right] \leq C (|\boldsymbol{\psi}|_{\mathcal{L}_2(L^2, H^4)}, T) \mathbb{E} \left[\|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right]. \quad (5.44)$$

Step 2: increasing the regularity of the solution. From the previous step, we also infer better regularity for the velocity term. Indeed, using the equation on v and Sobolev embeddings, we obtain an estimate

$$\begin{aligned} \|v\|_{C(W^{1,4})} &\leq \|\mathcal{P}(v \cdot \nabla v)\|_{C(W^{-1,4})} + \|\mathcal{P} \operatorname{div} \nabla u \odot \nabla u\|_{C(W^{-1,4})} \\ &\leq C (\|v \cdot \nabla v\|_{C(L^{4/3})} + \|\nabla u \odot \nabla u\|_{C(L^4)}) \\ &\leq C \left(\|v\|_{C(L^4)} \|\nabla v\|_{C(L^2)} + \|\nabla u\|_{C(L^8)}^2 \right). \end{aligned}$$

Hence

$$\mathbb{E} \left[\|v\|_{C(W^{1,4})}^m \right] \leq C \mathbb{E} \left[\|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right]. \quad (5.45)$$

Now, because of (5.44) we have $\nabla g \equiv \nabla u |\nabla u|^2 + 2u \nabla^2 u \nabla u \in L^m(\Omega; C(L^2))$, and because of (5.45) it holds as well $\nabla f \equiv -(\nabla v)^T \cdot \nabla u - v \cdot \nabla^2 u \in L^m(\Omega; C(L^2))$. Therefore, we obtain that $y \in L^m(\Omega; C(H^3))$, and there holds

$$\mathbb{E} \|y\|_{C(H^3)}^m \leq C_m \mathbb{E} \left[1 + \|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right]. \quad (5.46)$$

On the other hand, since $u = y + Z$ has gained 1/2 degree of regularity, one can repeat Step 1 to obtain

$$\mathbb{E} \left[\|Z\|_{C(H^3)}^m \right] \leq C(m, |\boldsymbol{\psi}|_{\mathcal{L}_2(L^2, H^5)}, T) \mathbb{E} \left[1 + \|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right].$$

We eventually obtain the needed bound on u , namely

$$\mathbb{E} \left[\|u\|_{C(H^3)}^m \right] \leq C(|\psi|_{\mathcal{L}_2(L^2, H^5)}, T) \mathbb{E} \left[1 + \|v\|_{C(H^1)}^{2m} + \|u\|_{C(H^2)}^{2m} \right].$$

Finally, the bounds on v follow by reusing the parabolic estimates together with appropriate Sobolev embeddings. This finishes the proof of Theorem 5.1. \blacksquare

6. CONVERGENCE OF REGULAR APPROXIMATIONS

6.1. Tightness and passage to the limit. We now define a sequence $\{W_n, n \in \mathbb{N}\}$ of Wiener processes in $L^2(\mathbb{T}^2; \mathbb{R}^3)$ whose covariance belongs to $\mathcal{L}_2(L^2, H^s)$ for every $s \in \mathbb{R}$. More precisely, for each $n \in \mathbb{N}$ we let

$$W_n := \sum_{0 \leq l \leq n} \mathbf{B}_l(\cdot) \psi f_l, \quad (6.1)$$

for $(f_l)_{l \in \mathbb{N}}$ and $(\mathbf{B}_l)_{l \in \mathbb{N}}$ as in (1.5). Next, consider a sequence $(w_n, y_n; \tau_n)$ of strong solutions of (SEL) such that

$$\begin{aligned} w_n(0) \in C^\infty(\mathbb{T}^2; \mathbb{R}^2), \quad \operatorname{div} w_n = 0, \quad y_n(0) \in C^\infty(\mathbb{T}^2; \mathbb{R}^3), \quad |y_n(0)|_{\mathbb{R}^3} = 1 \text{ a.e.} \\ \text{and } (w_n(0), y_n(0)) \rightarrow (v_0, u_0) \text{ in } \mathbb{L}^2 \times \mathbf{H}^1 \text{ strong,} \end{aligned} \quad (6.2)$$

solving the following regularized problem on $[0, \tau_n)$:

$$\begin{cases} dw_n + (Aw_n + \mathcal{P}[w_n \cdot \nabla w_n + \operatorname{div} \nabla y_n \odot \nabla y_n]) dt = 0 \\ dy_n - (\Delta y_n + y_n |\nabla y_n|^2 - w_n \cdot \nabla y_n - F_{\psi_n} y_n) dt = y_n \times dW_n. \end{cases} \quad (6.3)$$

Moreover we define the stopping time τ_n by the property:

$$\mathbb{P}\text{-a.s.}, \quad \tau_n = T \quad \text{or} \quad \limsup_{t \nearrow \tau_n} [|w_n(t)|_{H^2} + |y_n(t)|_{H^3}] = \infty. \quad (6.4)$$

Note that the existence of such a sequence is guaranteed by Theorem 3.1. Now, fix $\epsilon_1 \in (0, 1/\mu_1)$ and choose a positive, non-increasing sequence

$$\rho_k \rightarrow 0, \quad k \rightarrow \infty.$$

For $n, k \in \mathbb{N}$, define the stopping times:

$$\zeta_{n,k} := \inf \left\{ 0 \leq t < \tau_n, \quad \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho_k)} (|w_n(t, y)|^2 + |\nabla y_n(t, y)|^2) dy \geq \epsilon_1 \right\}, \quad (6.5)$$

with the convention that the infimum is equal to τ_n whenever the above set is empty. In the sequel, we will denote by $(v_{n,k}, u_{n,k})$, $k \in \mathbb{N}$, the process:

$$(v_{n,k}(t), u_{n,k}(t)) = \begin{cases} (w_n(t), y_n(t)) & \text{if } 0 \leq t \leq \zeta_{n,k}, \\ \left(e^{(\zeta_{n,k}-t)A^2} w_n(\zeta_{n,k}), e^{-(t-\zeta_{n,k})\Delta^2} y_n(\zeta_{n,k}) \right) & \text{if } \zeta_{n,k} < t \leq T. \end{cases} \quad (6.6)$$

Notation 6.1. For each sequence of random variables $X_k : \Omega \rightarrow E, k \in \mathbb{N}$ where E is some Polish space, we will henceforth denote by \mathbf{X} the corresponding random variable in the product space $\prod_{k \in \mathbb{N}} E$. For instance, we define for each $n \in \mathbb{N}$:

$$\mathbf{v}_n := \{v_{n,k}, k \in \mathbb{N}\}, \quad \mathbf{u}_n := \{u_{n,k}, k \in \mathbb{N}\}, \quad \mathbf{\zeta}_n := \{\zeta_{n,k}, k \in \mathbb{N}\},$$

and so on.

Claim 6.1. There exists $\alpha \in (0, \frac{1}{2})$ such that for every $\delta < 1$ the sequence

$$(\mathbf{v}_n, \mathbf{u}_n, \zeta_n, W_n), \quad n \in \mathbb{N}$$

is tight in the space

$$\mathfrak{U}^\delta := \left(\prod_{k \in \mathbb{N}} \mathcal{V}_T^{\delta-1} \right) \times \left(\prod_{k \in \mathbb{N}} \mathcal{U}_T^\delta \right) \times \left(\prod_{k \in \mathbb{N}} [0, T] \right) \times C^\alpha([0, T]; H^1).$$

Proof. The proof relies in particular on the bound

$$\begin{aligned} \mathbb{E} \left[\|v_{n,k}\|_{\mathcal{V}_T^0}^2 \right] - 1 &\leq \mathbb{E} \left[\sup_{t \in [0, \zeta_{n,k}]} |v_{n,k}(t)|_{L^2}^2 + \int_0^{\zeta_{n,k}} |\nabla v_{n,k}|_{L^2}^2 ds \right] \\ &+ \mathbb{E} \left[\sup_{t \in [\zeta_{n,k}, T]} |e^{A^2(\zeta-t)} v_{n,k}(\zeta_{n,k})|_{L^2}^2 + \int_{\zeta_{n,k}}^T |e^{A^2(\zeta_{n,k}-s)} \nabla v_{n,k}|_{L^2}^2 ds \right] \\ &\leq C(k, E_0, T, |\psi|_{\mathcal{L}_2(L^2, H^1)}) + C \mathbb{E} \left[\int_{\zeta_{n,k}}^T \frac{|v_{n,k}|_{L^2}^2}{(s - \zeta_{n,k})^{1/2}} ds \right] \leq C'''(k, E_0, T, |\psi|_{\mathcal{L}_2(L^2, H^1)}), \end{aligned}$$

by (3.3) particularized for $\Lambda := (A^2, \mathbb{H}^4)$, and $p = q = 2$.

Proceeding similarly for u_n , one obtains as well the estimate

$$\mathbb{E} \left[\|u_n\|_{\mathcal{U}_T^1}^2 \right] \leq C(k, E_0, T, |\psi|_{\mathcal{L}_2(L^2, H^1)}).$$

In order to apply Aubin-Lions' lemma, one needs first higher order estimates with respect to the time-like variable, but in an arbitrary Sobolev space, typically of negative order. Such estimates follow directly from the equation (6.3), and hence the proof is omitted. The conclusion follows from Markov Inequality. \blacksquare

Thanks to 6.1 and Prokhorov Theorem, we infer the existence of an extraction $n_\ell, \ell \in \mathbb{N}$, and a law μ supported in \mathfrak{U}^1 such that

$$\mathbb{P}_{(\mathbf{v}_{n_\ell}, \mathbf{u}_{n_\ell}, \zeta_{n_\ell}, W_{n_\ell})} \rightarrow \mu \quad \text{weakly.}$$

Furthermore, using Skorohod Embedding Theorem (see [35]) we see that there exist

- a stochastic basis $\hat{\mathfrak{P}} = (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t)_{t \in [0, T]}, \hat{W})$, where \hat{W} is a Wiener process in $L^2(\mathbb{T}^2; \mathbb{R}^3)$ such that (1.9) holds;
- a sequence of random variables

$$(\hat{\mathbf{v}}_\ell, \hat{\mathbf{u}}_\ell, \hat{\zeta}_\ell, \hat{W}_\ell) \in L^0(\hat{\Omega}, \mathfrak{U}^3), \quad \ell \in \mathbb{N},$$

such that for each $\ell \in \mathbb{N}$, identifying $\hat{\mathbf{v}}_\ell$ (resp. $\hat{\mathbf{u}}_\ell, \hat{W}_\ell$) as a mapping from $\hat{\Omega} \times [0, T]$ to $\prod_{k \in \mathbb{N}} \mathbb{L}^2$ (resp. to $\prod_{k \in \mathbb{N}} \mathbf{H}^1, \mathbf{H}^1$), then the corresponding stochastic process is predictable with respect to the filtration

$$\hat{\mathcal{F}}_t^\ell := \sigma\{\hat{\mathbf{v}}_\ell(s), \hat{\mathbf{u}}_\ell(s), \hat{W}_\ell(s), s \leq t\}, \quad t \in [0, T];$$

the element \hat{W}_ℓ is an \mathbf{L}^2 -Wiener process with respect to the above filtration, and its covariance is given by (1.9) with ψ_{n_ℓ} instead of ψ ; for each $k \in \mathbb{N}$, the random variable $\hat{\zeta}_{\ell, k}$ is a positive stopping time;

- limits $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\zeta}, \hat{W}) \in \mathfrak{U}^1$, for every $k \in \mathbb{N}$,

such that the following convergences hold for each $k \in \mathbb{N}$:

$$(\hat{v}_{\ell,k}, \hat{u}_{\ell,k}) \rightarrow (\hat{v}_k, \hat{u}_k) \quad \hat{\mathbb{P}} - \text{a.s.} \quad \text{in } \mathcal{V}_T^{\delta-1} \times \mathcal{U}_T^\delta \text{ for every } \delta < 1, \quad (6.7)$$

$$\hat{\zeta}_{\ell,k} \rightarrow \hat{\zeta}_k \quad \hat{\mathbb{P}} - \text{a.s.}, \quad (6.8)$$

$$\hat{W}_\ell \rightarrow \hat{W} \quad \hat{\mathbb{P}} - \text{a.s.} \text{ in } C^\alpha([0, T]; H^1) \text{ for } \alpha \text{ as above.} \quad (6.9)$$

Furthermore, by the Sobolev Embedding Theorem, together with Proposition 4.2 and dominated convergence, we immediately obtain

$$\begin{aligned} (\hat{v}_{\ell,k}, \nabla \hat{u}_{\ell,k}) &\rightarrow (\hat{v}_k, \nabla \hat{u}_k), \quad \text{in } L^2(\Omega \times [0, T]; L^p), \quad \text{for every } p \in [1, \infty) . \\ \hat{u}_{\ell,k} &\rightarrow \hat{u}_k \quad \text{in } L^2(\Omega \times [0, T]; L^\infty), \end{aligned} \quad (6.10)$$

for each $k \in \mathbb{N}$.

In particular, making use of (6.7), (6.8), (6.9), (1.9) and dominated convergence, it is easily seen that for each $k \in \mathbb{N}$,

$$(\hat{\mathfrak{P}}, \hat{W}; \hat{v}_k, \hat{u}_k) \text{ is a martingale solution of (SEL)} \quad (6.11)$$

up to the stopping time $t = \hat{\zeta}_k$, in the sense of Definition 1.1. Details are left to the reader (see, e.g., [1]).

6.2. Improvement of the convergence. We are now going to improve the convergence of $(\hat{v}_{\ell,k}, \hat{u}_{\ell,k})$ in $\mathcal{V}_{\text{loc}}^{\delta-1} \times \mathcal{U}_{\text{loc}}^\delta$ by showing that it remains true for $\delta = 1$.

Claim 6.2. Let $(\hat{v}_{\ell,k}, \hat{u}_{\ell,k})$ be as in the previous paragraph.

The following convergence holds

$$(\hat{v}_n, \hat{u}_n) \rightarrow (\hat{v}, \hat{u}) \quad \text{in } L^2(\Omega; \mathcal{V}^0 \times \mathcal{U}^1). \quad (6.12)$$

Before we proceed to the proof of Claim 6.2, let us collect some inequalities. Consider two martingale solutions $(\hat{\mathfrak{P}}, W_1, v_1, u_1)$, and $(\hat{\mathfrak{P}}, W_2, v_2, u_2)$ defined on the same probability space, and up to a common stopping time $\kappa > 0$. Denote further by $g = v_1 - v_2$ and by $f = u_1 - u_2$.

Computing Itô Formula on $\frac{1}{2}|g(t)|^2$ we find that

$$\frac{1}{2}|g(t)|_{L^2}^2 + \int_0^t |\nabla g|^2 ds = \iint_{[0,t] \times \mathbb{T}^2} (-g^j g^i \partial_i v_1^j - g^j v_2^i \partial_i g^j) dx ds + \mathfrak{t}(v_1, v_2; u_1, u_2), \quad (6.13)$$

where we denote by $\mathfrak{t} : (\mathcal{V}_{[0,\kappa]}^0)^2 \times (\mathcal{U}_{[0,\kappa]}^1)^2 \rightarrow \mathbb{R}$ the operation

$$\mathfrak{t} : (v_1, v_2; u_1, u_2) \mapsto \iint_{[0,t] \times \mathbb{T}^2} (v_1^j - v_2^j)(\Delta u_2 \cdot \partial_j u_2 - \Delta u_1 \cdot \partial_j u_1) dx ds.$$

Proceeding as in the proof of Theorem 1.1, we have on the one's hand:

$$\left| \iint_{[0,t] \times \mathbb{T}^2} -g^j g^i \partial_i v_1^j dx ds \right| \leq C(\epsilon) \int_0^t |g|_{L^2}^2 (1 + |\nabla v_1|_{L^2}^2) ds + \epsilon \int_0^t |\nabla g|_{L^2}^2 ds. \quad (6.14)$$

and similarly

$$\left| \iint_{[0,t] \times \mathbb{T}^2} -g^j v_2^i \partial_i g^j dx ds \right| \leq C'(\epsilon) \int_0^t |g|_{L^2} (1 + |v_2|_{L^4}^4) + \epsilon \int_0^t |\nabla g|_{L^2}^2 ds. \quad (6.15)$$

Next, applying Itô Formula to $\frac{1}{2}|\nabla f|_{L^2}^2$, we obtain

$$\begin{aligned} & \frac{1}{2}|\nabla f(t)|_{L^2}^2 + \int_0^t |\Delta f|_{L^2}^2 ds - X(t) \\ &= \iint_{[0,t] \times \mathbb{T}^2} \left[(-\Delta f \cdot f) |\nabla u_1|^2 - (\Delta f \cdot u_2) (\nabla f \cdot \nabla (u_1 + u_2)) \right] dx ds + \mathbf{q}(v_1, v_2; u_1, u_2), \end{aligned} \quad (6.16)$$

where $\mathbf{q} : (\mathcal{V}_{[0,\kappa]}^0)^2 \times (\mathcal{W}_{[0,\kappa]}^1)^2 \rightarrow \mathbb{R}$ is defined as

$$\mathbf{q} : (v_1, v_2; u_1, u_2) \mapsto \iint_{[0,t] \times \mathbb{T}^2} (\Delta u_2^i - \Delta u_1^i) (v_2^j \partial_j u_2^i - v_1^j \partial_j u_1^i) dx ds.$$

Now, a crucial observation is that

$$\begin{aligned} & \mathbf{t}(v_1, v_2; u_1, u_2) + \mathbf{q}(v_1, v_2; u_1, u_2) \\ &= \iint_{[0,t] \times \mathbb{T}^2} \left[v_1^j \Delta u_2 \cdot \partial_j u_2 - v_2^j \Delta u_2 \cdot \partial_j u_2 - v_1^j \Delta u_1 \cdot \partial_j u_1 + v_2^j \Delta u_1 \cdot \partial_j u_1 \right. \\ & \quad \left. \Delta u_2^i v_2^j \partial_j u_2^i - \Delta u_1^i v_2^j \partial_j u_2^i - \Delta u_2^i v_1^j \partial_j u_1^i + \Delta u_1^i v_1^j \partial_j u_1^i \right] dx ds \\ &= \iint_{[0,t] \times \mathbb{T}^2} \left[v_1^j \Delta u_2 \cdot \partial_j u_2 - v_2^j \Delta u_1 \cdot \partial_j u_2 + v_2^j \Delta u_1 \cdot \partial_j u_1 - v_1^j \Delta u_2 \cdot \partial_j u_1^i \right] dx ds \\ &= \iint_{[0,t] \times \mathbb{T}^2} \left[-g^j \Delta u_1 \cdot \partial_j f + v_1^j \Delta f \cdot \partial_j f \right] dx ds, \end{aligned} \quad (6.17)$$

and hence, similar computations as above show that

$$\begin{aligned} & |\mathbf{t}(v_1, v_2; u_1, u_2) + \mathbf{q}(v_1, v_2; u_1, u_2)| \\ & \leq \epsilon \int_0^t (|\nabla g|_{L^2}^2 + |\Delta f|_{L^2}^2) ds + C(\epsilon) \int_0^t (|\Delta u_1|_{L^2}^2 + |v_1|_{L^4}^4 + 1) (|g|_{L^2}^2 + |\nabla f|_{L^2}^2) ds. \end{aligned} \quad (6.18)$$

The remaining term can be estimated as follows:

$$\left| \iint_{[0,t] \times \mathbb{T}^2} (-\Delta f \cdot f) |\nabla u_1|^2 dx ds \right| \leq \epsilon \int |\Delta f|_{L^2}^2 ds + C(\epsilon) \int |\nabla u_1|_{L^4}^4 |f|_{L^\infty}^2 ds. \quad (6.19)$$

Concerning the stochastic integral, we have for any $m > 1$:

$$\mathbb{E} \left[\sup_{s \in [0,t]} |X(s)|^m \right] \leq C(m, |\nabla \psi|_{\mathcal{L}_2(L^2, L^2)}) \mathbb{E} \left[\int_0^t |\nabla f|_{L^2}^2 ds \right]^{m/2}. \quad (6.20)$$

Now, fix $k \geq 1$ and for simplicity in the notations let

$$(\hat{v}_\ell, \hat{u}_\ell) := (\hat{v}_{\ell,k}, \hat{u}_{\ell,k}).$$

Denote by

$$g_\ell := \hat{v} - \hat{v}_\ell, \quad \text{and} \quad f_\ell := \hat{u} - \hat{u}_\ell.$$

From (6.10), we infer that

$$(\hat{v}_\ell, \nabla \hat{u}_\ell) \rightarrow (\hat{v}, \nabla \hat{u}), \quad \mathbb{P} \otimes dt \text{ almost everywhere in } L^p, \quad p \in [1, \infty). \quad (6.21)$$

$$f_\ell \rightarrow 0, \quad \mathbb{P} \otimes dt \text{ almost everywhere in } L^\infty. \quad (6.22)$$

Next, upon choosing $\epsilon > 0$ sufficiently small, we can sum up the inequalities (6.14), (6.15), (6.18), (6.19) and (6.20) to obtain

$$\begin{aligned} & \mathbb{E} \left[\operatorname{ess\,sup}_{t \in [0, T]} (|g_\ell(t)|_{L^2}^2 + |\nabla f_\ell(t)|^2) + \int_0^T (|\nabla g_\ell|_{L^2}^2 + |\Delta f_\ell|_{L^2}^2) dt \right] \\ & \leq C \mathbb{E} \left[\int_0^T \left(|\nabla(\hat{v}_\ell, \hat{v})|_{L^2}^2 + |(\hat{v}_\ell, \hat{v})|_{L^4}^4 + |\Delta(\hat{u}_\ell, \hat{u})|_{L^2}^2 + |\nabla(\hat{u}_\ell, \hat{u})|_{L^4}^4 + 1 \right) \right. \\ & \quad \left. \times (|g_\ell|_{L^2}^2 + |\nabla f_\ell|_{L^2}^2 + |f_\ell|_{L^\infty}^2) dt \right]. \quad (6.23) \end{aligned}$$

We can now proceed to the proof of the main result of this paragraph.

Proof of Claim 6.2. We first need to show the following uniform integrability Property.

There is a full measure set $\Omega_1 \in \mathcal{A}$, such that for any $\omega \in \Omega_1$ the family $(t \mapsto |\nabla \hat{v}_\ell(\omega, t)|_{L^2}^2 + |\Delta \hat{u}_\ell(\omega, t)|_{L^2}^2)_{n \in \mathbb{N}}$ is uniformly integrable with respect to the Lebesgue measure dt . More precisely, for every $\omega \in \Omega_1$ and $\epsilon > 0$, there corresponds a $\delta(\epsilon, \omega) > 0$ such that:

$$\int_s^t (|\nabla \hat{v}_\ell(\omega, r)|_{L^2}^2 + |\Delta \hat{u}_\ell(\omega, r)|_{L^2}^2) dr \leq \epsilon, \quad (6.24)$$

for every $(s, t) \in [0, T]^2$ such that $0 \leq |t - s| \leq \delta(\epsilon, \omega)$.

Prior to show (6.24), note that ψ_{n_ℓ} is the covariance associated with \hat{W}_ℓ through (1.9). Thus

$$\operatorname{cov}(\hat{W}_\ell(1), \hat{W}(1)) = \psi_{n_\ell} \psi^*. \quad (6.25)$$

Next, for each $\ell \in \mathbb{N}$, and for every $(\omega, t) \in \Omega \times [0, T]$, by estimating as in (5.4), we see that there exists a constant $C(\rho, |\psi|_{\mathcal{L}_2(L^2, H^1)}) > 0$ such that \mathbb{P} -a.s., for any $0 \leq s \leq t \leq T$:

$$\begin{aligned} & \int_s^t |\nabla \hat{v}_\ell(r)|_{L^2}^2 + |\Delta \hat{u}_\ell(r)|_{L^2}^2 dr \\ & \leq C \sup_{\theta_1, \theta_2 \in [s, t]} |X_\ell(\theta_1) - X_\ell(\theta_2)| + C(t - s) \left(1 + \sup_{\theta \in [0, T]} |\nabla \hat{u}_\ell(\theta)|_{L^2}^2 \right), \quad (6.26) \end{aligned}$$

where we denote by

$$X_\ell(\theta) := \int_0^\theta \langle \nabla \hat{u}_\ell, \hat{u}_\ell \times d\nabla \hat{W}_\ell \rangle.$$

From (6.26) and Proposition 4.2 we see that, in order to show the claimed uniform integrability, it is sufficient to show the existence of $\Omega_1 \in \mathcal{A}$ with full measure such that for each $\omega \in \Omega_1$:

$$X_\ell(\omega, \cdot) \text{ is uniformly equi-continuous.} \quad (6.27)$$

But from [10, Thm. 4.27 p. 103 & Thm. 3.15 p. 77] we see that, denoting by $X(\theta) := \int_0^\theta \langle \nabla \hat{u}, \hat{u} \times d\nabla \hat{W} \rangle$, we have by (6.25):

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_\ell(t) - X(t)| \right] &\leq C \mathbb{E} \left[\left(\int_0^T |\psi_{n_\ell}^* \operatorname{div}(\hat{u} \times \nabla(\hat{u}_\ell - \hat{u}))|_{L^2}^2 dr \right)^{1/2} \right. \\ &\left. + \left(\int_0^T |\psi_{n_\ell}^* \operatorname{div}((\hat{u}_\ell - \hat{u}) \times \nabla \hat{u})|_{L^2}^2 dr \right)^{1/2} + \left(\int_0^T |(\psi_{n_\ell}^* - \psi^*) \operatorname{div}(\hat{u} \times \nabla \hat{u})|_{L^2}^2 dr \right)^{1/2} \right]. \end{aligned}$$

By Hölder Inequality together with (6.10), we infer that the above right hand side converges to 0 as $n, p \rightarrow \infty$. This shows in particular the existence of $\tilde{\Omega} \in \mathcal{A}$ with full probability such that

$$\omega \in \tilde{\Omega} \quad \Rightarrow \quad X_\ell(\omega, \cdot) - X(\omega, \cdot) \rightarrow 0 \quad \text{strongly in } C(0, T; \mathbb{R}).$$

From the converse of Ascoli-Arzelà Theorem, we see that (6.27) holds if one lets $\Omega_1 := \tilde{\Omega}$. This proves uniform integrability and (6.24).

Now, making use of Proposition 4.1, we infer immediately that a similar uniform integrability as that of (6.24) holds for the family $t \mapsto |\hat{v}_\ell(t)|_{L^4}^4 + |\nabla \hat{u}_\ell(t)|_{L^4}^4$. Therefore, using Vitali's convergence theorem, we see thanks to (6.21), (6.22) and (6.24) that

$$\begin{aligned} \int_0^T (|\nabla(\hat{v}_\ell, \hat{v})|_{L^2}^2 + |(\hat{v}_\ell, \hat{v})|_{L^4}^4 + |\Delta(\hat{u}_\ell, \hat{u})|_{L^2}^2 + |\nabla(\hat{u}_\ell, \hat{u})|_{L^4}^4 + 1) \\ \times (|g_\ell|_{L^2}^2 + |\nabla f_\ell|_{L^2}^2 + |f_\ell|_{L^\infty}^2) dt \longrightarrow 0, \quad \mathbb{P}\text{-a.s.} \quad (6.28) \end{aligned}$$

By dominated convergence, we further see that (6.28) also holds in $L^1(\Omega)$.

Finally, we deduce (6.12) from the inequality (6.23). Claim 6.2 is now proved. \blacksquare

As a consequence of the above analysis, we observe the following.

Corollary 6.1. *The whole sequence $(u_n, v_n; \zeta_n)_{n \in \mathbb{N}}$ converges in probability to a local strong solution $(u, v; \zeta)$, in the space $\mathcal{V}^0 \times \mathcal{U}^1 \times [0, T]$. Namely, for any $\epsilon > 0$ it holds*

$$\mathbb{P} \left(\|v_{n,k} - v_k\|_{\mathcal{V}^0(0, \zeta_k)} + \|u_{n,k} - u_k\|_{\mathcal{U}^1(0, \zeta_k)} + |\zeta_{n,k} - \zeta_n| > \epsilon \right) \rightarrow_{n \rightarrow \infty} 0.$$

Proof. Step 1: convergence of the processes. Using the same arguments as above, it is immediately checked that given two sequences m_ℓ and n_ℓ , the sequence of random variable defined by

$$(X_\ell, Y_\ell) := ((\mathbf{v}_{n_\ell}, \mathbf{u}_{n_\ell}, \zeta_{n_\ell}), (\mathbf{v}_{m_\ell}, \mathbf{u}_{m_\ell}, \zeta_{m_\ell}))$$

contains a subsequence converging in distribution to a law ν supported in $E \times E$ where E denotes the Polish space $\prod_{k \in \mathbb{N}} \mathcal{V}_T^0 \times \prod_{k \in \mathbb{N}} \mathcal{U}_T^1 \times \prod_{k \in \mathbb{N}} [0, T]$.

Reasoning as in the above paragraph, we see that there exists a filtered probability space $\tilde{\mathfrak{P}} = (\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}}; \tilde{W})$ endowed with a Wiener process satisfying (1.9), and a sequence of random variables $(\tilde{X}_\ell, \tilde{Y}_\ell) \sim (X_\ell, Y_\ell)$, converging $\tilde{\mathbb{P}}$ -almost surely to an element

$$(\tilde{X}, \tilde{Y}) := \left((\hat{\mathbf{v}}, \hat{\mathbf{u}}, \hat{\zeta}), (\hat{\mathbf{v}}, \hat{\mathbf{u}}, \hat{\zeta}) \right) \in L^0(\tilde{\Omega}; E \times E)$$

Furthermore, for any $k \in \mathbb{N}$, the processes (\hat{v}_k, \hat{u}_k) and (\hat{v}_k, \hat{u}_k) are both martingale solutions defined on the same stochastic basis, up to the stopping time $\kappa_k := \hat{\zeta}_k \wedge \hat{\zeta}_k$. By Theorem 1.1, we infer that $(\hat{v}_k, \hat{u}_k)|_{[0, \kappa_k]} = (\hat{v}_k, \hat{u}_k)|_{[0, \kappa_k]}$.

Step 2: convergence of the stopping times. We now claim that

$$\kappa_k = \hat{\zeta}_k = \hat{\zeta}_k = \inf \left\{ t \in [0, T] : \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho_k)} |\nabla \hat{u}(t, y)|^2 dy \geq \varepsilon_1 \right\}. \quad (6.29)$$

This is indeed a consequence of Claim 6.2: for any $k \in \mathbb{N}$, since by (6.12) the convergence $\hat{u}_{\ell, k} \rightarrow_{\ell \rightarrow \infty} \hat{u}_k$ is strong in $C(H^1)$, we can write

$$\begin{aligned} & \sup_{t \in [0, \kappa_k]} \left| \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho_k)} |\nabla \hat{u}_{\ell, k}(t, y)|^2 dy - \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho_k)} |\nabla \hat{u}_k(t, y)|^2 dy \right| \\ & \leq \sup_{t \in [0, \kappa_k]} \sup_{x \in \mathbb{T}^2} \int_{B(x, \rho)} \left| |\nabla \hat{u}_{\ell, k}(t, y)|^2 - |\nabla \hat{u}_k(t, y)|^2 \right| dy \rightarrow 0, \end{aligned}$$

showing that

$$\hat{\zeta}_k \equiv \text{a.s.} - \lim_{\ell \rightarrow \infty} \hat{\zeta}_{\ell, k} = \zeta(\hat{v}_k, \hat{u}_k),$$

where the above r.h.s. denotes the stopping time defined from the solution (\hat{v}_k, \hat{u}_k) by (5.1). This shows the claimed property.

Conclusion. As a consequence, one sees that (X, Y) is supported on the diagonal of $E \times E$. By the well known Gyöny-Krylov Lemma [14], this implies that the original sequence $(\mathbf{v}_n, \mathbf{u}_n, \zeta_n)_{n \in \mathbb{N}}$ converges in probability to an element $(\mathbf{v}, \mathbf{u}, \zeta)$, in the space $\prod_{k \in \mathbb{N}} \mathcal{V}_T^{\delta-1} \times \prod_{k \in \mathbb{N}} \mathcal{U}_T^\delta \times \prod_{k \in \mathbb{N}} [0, T]$. \blacksquare

6.3. Controlling the bubbling time. We start with a lemma.

Lemma 6.1. *Consider a local strong solution (v, u) of (SEL) on $[0, \tau) \subset (0, T]$, where τ is a stopping time. For all $\rho > 0$, and $\delta \in [0, T]$, define the random variable*

$$\varepsilon(\delta, \rho) := \sup_{t \in [0, \delta \wedge \tau], x \in \mathbb{T}^2} \int_{B(x, \rho)} (|v(t, y)|^2 + |\nabla u(t, y)|^2) dy,$$

and assume that for some constant $\varepsilon_1 > 0$, it holds

$$\mathbb{P}(\varepsilon(0, 3\rho) < \varepsilon_1) = 1. \quad (6.30)$$

Furthermore, denote by $\tilde{\pi}$ the zero-mean pressure term obtained from equation (4.11).

Then, for all $\delta \in [0, T]$ and every $\lambda > 0$, we have the estimate

$$\begin{aligned} & \mathbb{P}(\{\varepsilon(\delta, \rho) \geq \varepsilon_1 + \lambda\} \cap \{\delta < \tau\}) \\ & \leq \frac{C(\rho, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)})}{\lambda} \mathbb{E} \left[\delta + \int_0^\delta (|v|_{L^4}^4 + |\nabla u|_{L^4}^4 + |\tilde{\pi}|_{L^{4/3}}^{4/3}) ds \right] \quad (6.31) \end{aligned}$$

where the above constant depends on the indicated quantities, but not on the individual element (v, u) such that $\mathbb{E}[\sup_{t \in [0, T]} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2)] \leq E_0$.

The proof of the above Lemma relies partially on the following argument, whose proof can be found in [15]:

For every $\rho > 0$, there exist a positive integer N_ρ , and points $\{x_\rho^1, x_\rho^2, \dots, x_\rho^{N_\rho}\} \subset \mathbb{T}^2$ fulfilling the property:

“for all $x \in \mathbb{T}^2$, there exists $i \in \{1, \dots, N_\rho\}$ with $B(x, \rho) \subset B(x_\rho^i, 2\rho)$.” (6.32)

Proof of Lemma 6.1. Let x_ρ^i , $1 \leq i \leq N_\rho$ be as in (6.32), and consider test functions $\varphi = \varphi_{\rho,i} \in C_0^\infty(\mathbb{T}^2)$ with $\text{Supp } \varphi \subset B(x_\rho^i, 3\rho)$ such that

$$\mathbb{1}_{B(x_\rho^i, 2\rho)} \leq \varphi \leq \mathbb{1}_{B(x_\rho^i, 3\rho)}, \quad \sup_{x \in B(x_\rho^i, 3\rho)} |\nabla \varphi(x)| \leq \frac{C}{\rho}, \quad (6.33)$$

for some $C > 0$ independent of i, ρ .

By (6.30) together with (6.33), we have for every $i \in \{1, \dots, N_\rho\}$:

$$E^{i,\rho}(0) := \mathbb{E}[|\varphi_{\rho,i} v(0)|_{L^2}^2 + |\varphi_{\rho,i} \nabla u(0)|_{L^2}^2] < \varepsilon_1$$

By (6.32), we see that for $\delta \in [0, \tau)$, it holds

$$\begin{aligned} \{\varepsilon(\delta, \rho) \geq \varepsilon_1 + \lambda\} &\subset \{\exists i, 1 \leq i \leq N_\rho, \sup_{t \in [0, \delta]} E^{i,\rho}(t) \geq \varepsilon_1 + \lambda\} \\ &\subset \cup_{i=1}^{N_\rho} \left\{ \sup_{t \in [0, \delta]} E^{i,\rho}(t) - E^{i,\rho}(0) \geq \lambda \right\}. \end{aligned} \quad (6.34)$$

But from Proposition 4.3, we have:

$$\begin{aligned} &\frac{1}{2}(E^{i,\rho}(t) - E^{i,\rho}(0)) \\ &\leq C\left(1 + \frac{1}{\rho^2}\right) \iint_{[0,t] \times \mathbb{T}^2} (|v|^3 + (|v| + 1)|\nabla u|^2 + |v||\pi - \bar{\pi}|) \, ds \\ &\quad + X^{i,\rho}(t) + t|\varphi \nabla \psi|_{\mathcal{L}_2(L^2, L^2)}^2 \\ &=: V^{i,\rho}(t), \end{aligned} \quad (6.35)$$

for $t \in [0, \tau]$, a.s., where we let $X^{i,\rho}(t) := \int_0^t \langle \varphi \nabla u, \varphi u \times \nabla dW \rangle$.

Next, from (6.34), we have the estimate

$$\begin{aligned} \mathbb{P}(\varepsilon(\delta, \rho) \geq \varepsilon_1 + \lambda) &\leq \mathbb{P}\left(\exists i \in \{1, \dots, N_\rho\}, \sup_{t \in [0, \delta]} E^{i,\rho}(t) - E^{i,\rho}(0) \geq \lambda\right) \\ &\leq \sum_{i=1}^{N_\rho} \mathbb{P}\left(\sup_{t \in [0, \delta]} V^{i,\rho}(t) \geq \frac{\lambda}{2}\right) \\ &\leq \sum_{i=1}^{N_\rho} \frac{2}{\lambda} \mathbb{E}\left[\sup_{t \in [0, \delta]} V^{i,\rho}(t)\right]. \end{aligned} \quad (6.36)$$

Using Hölder Inequality, and then Burkholder-Davis-Gundy Inequality to estimate the martingale term, we end up with the bound

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq \delta} V^{i,\rho}(t) \right] \\
& \leq C(1 + \frac{1}{\rho^2}) \mathbb{E} \left[\delta \sup_{t \in [0, \inf(\delta, \tau)]} (|v(t)|_{L^2}^2 + |\nabla u(t)|_{L^2}^2) + \int_0^\delta (|v|_{L^3}^3 + |v|_{L^2}^2) ds \right. \\
& \quad \left. + \int_0^\delta |\nabla u|_{L^4}^4 ds + \left(\int_0^\delta |v|_{L^4}^4 ds \right)^{1/4} \left(\int_0^\delta |\tilde{\pi}|_{L^{4/3}}^{4/3} ds \right)^{3/4} \right] \\
& \quad + C(|\psi|_{\mathcal{L}_2(L^2, H^1)}) \mathbb{E} \left[\iint_{[0, \delta] \times \mathbb{T}^2} (\varphi^{i,\rho})^4 |\nabla u|^2 ds \right]^{1/2} + \delta |\varphi^{i,\rho} \nabla \psi|_{\mathcal{L}_2(L^2, L^2)}^2 \\
& \leq C(\rho, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}) \mathbb{E} \left[\delta + \int_0^\delta (|v|_{L^4}^4 + |\nabla u|_{L^4}^4 + |\tilde{\pi}|_{L^{4/3}}^{4/3}) ds \right] \quad (6.37)
\end{aligned}$$

Inserting this estimate into (6.36), we obtain (6.31). \blacksquare

6.4. End of the proof of Theorem 1.2.

Step 1. We first prove the following assertion:

there exists an integer k_0 such that for each $k \geq k_0$, the limit point ζ_k of the sequence $\{\zeta_{\ell, k}, \ell \in \mathbb{N}\}$ defined in (6.5) verifies

$$\mathbb{P}(\zeta_k > 0) = 1. \quad (6.38)$$

To prove (6.38), observe that since the sequence ρ_k converges to 0 and since the sequence $(v_{n,k}(0), u_{n,k}(0))$ converges to $(v_k, u_k)(0)$ strongly in $\mathbb{L}^2 \times \mathbf{H}^1$, there is an integer k_0 such that for each $k \geq k_0$:

$$\sup_{x \in \mathbb{T}^2} \int_{B(x, 3\rho_k)} (|v_{n,k}(0)|^2 + |\nabla u_{n,k}(0)|^2) dy \leq \frac{\mu_1^{-1}}{2}, \quad \text{uniformly in } n \in \mathbb{N}. \quad (6.39)$$

For such $k \geq k_0$ we have in particular

$$\mathbb{P}(\zeta_k > 0) = 1 - \mathbb{P}(\zeta_k = 0) = 1 - \lim_{\delta \rightarrow 0} \mathbb{P}(\zeta_k \leq \delta) \quad (6.40)$$

and, according to the definition of $\zeta_{n,k}$ in (6.5), and τ_n in (6.4),

$$\{\zeta_{n,k} \leq \delta\} \subset \{\zeta_{n,k} < \tau_n \text{ and } \zeta_{n,k} \leq \delta\} \cup \{\zeta_{n,k} = \tau_n \text{ and } \tau_n \leq \delta\} =: \Omega_1 \cup \Omega_2. \quad (6.41)$$

However, by Theorem 5.1, we have $\mathbb{P}(\Omega_2) = 0$ and therefore, Lemma 6.1 with $\varepsilon_1 = \frac{1}{2}\mu_1^{-1}$ yields:

$$\begin{aligned}
\mathbb{P}(\{\zeta_{n,k} \leq \delta\}) &= \mathbb{P}(\{\varepsilon(\delta, \rho_k) \geq \mu_1^{-1}\} \cap \{\zeta_{n,k} < \tau_n\}) \\
&\leq 2\mu_1 C(k, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}) \mathbb{E} \left[\delta + \int_0^\delta (|v_{n,k}|_{L^4}^4 + |\nabla u_{n,k}|_{L^4}^4 + |\tilde{\pi}_{n,k}|_{L^{4/3}}^{4/3}) ds \right].
\end{aligned}$$

Estimating the $L_t^{4/3}(L_x^{4/3})$ -norm of $\tilde{\pi}$ by its $L_t^2(L_x^2)$ -norm, and making use of the estimate (4.12), we further obtain

$$\mathbb{P}(\{\zeta_{n,k} \leq \delta\}) \leq C(\mu_1, k, T, E_0, |\psi|_{\mathcal{L}_2(L^2, H^1)}) \mathbb{E} \left[\delta + \int_0^\delta (|v_{n,k}|_{L^4}^4 + |\nabla u_{n,k}|_{L^4}^4) ds \right],$$

which by (4.5), (5.3) and (2.1) goes to 0 as $\delta \rightarrow 0$, uniformly in $n \in \mathbb{N}$.

Now, writing that for each $n \in \mathbb{N}$:

$$\{\zeta_k \leq \delta\} \subset \{|\zeta_{n,k} - |\zeta_{n,k} - \zeta_k|| \leq \delta\},$$

it is seen that

$$\mathbb{P}(\zeta_k \leq \delta) \leq \mathbb{P}(\zeta_{n,k} \leq 2\delta) + \mathbb{P}(|\zeta_{n,k} - \zeta_k| \geq \delta).$$

The conclusion follows by the fact that $|\zeta_{n,k} - \zeta_k| \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$, and the uniform convergence of $\mathbb{P}(\zeta_{n,k} \leq \delta)$ as $\delta \rightarrow 0$. This shows (6.38).

Step 2. Definition of the solution. We now define

$$(v_\star, u_\star)(v_0, u_0; t) := \begin{cases} (v_k(t), u_k(t)) & \text{if } t \in [0, \zeta_k) \text{ for some } k \in \mathbb{N}. \\ 0 & \text{otherwise.} \end{cases} \quad (6.42)$$

Taking the limit as $n \rightarrow \infty$ in the identity $\zeta_{n,k} \leq \zeta_{n,k+1}$ (which holds because the sequence ρ_k is non-increasing, see (6.5)), one obtains that for each $k \in \mathbb{N}$:

$$\zeta_k \leq \zeta_{k+1},$$

and hence by Theorem 1.1, one sees that the definition (6.42) is not ambiguous. Hence, we have defined a local strong solution $(v_\star(v_0, u_0), u_\star(v_0, u_0); \zeta_\star(v_0, u_0))$, where

$$\zeta_\star(v_0, u_0) := \lim_{k \rightarrow \infty} \zeta_k = \sup_{k \in \mathbb{N}} \zeta_k.$$

The proof of the property (P3), as well as the fact that the random variable J is finite a.s., follows exactly the same steps as that of [15], and therefore we leave it to the reader.

It is sufficient to show (P2) for $j = 0$, hence we let for $k \in \mathbb{N}$:

$$V_k := v_k(\sigma_k) \quad \text{and} \quad U_k := u_k(\sigma_k),$$

where $\sigma_k \nearrow \tau^1 := \zeta_\star(v_0, u_0)$, \mathbb{P} -a.s. Notice by Proposition 4.2 that for any $m \geq 1$, the sequence $(V_k)_{k \in \mathbb{N}}$ is bounded in $L^m(\Omega; \mathbb{L}^2)$, while $(U_k)_{k \in \mathbb{N}}$ is bounded in $L^m(\Omega; \mathbf{H}^1)$. On the other hand using the equation and the fact that ζ_k converges a.s. to ζ_\star , it is easily seen that (V_k, U_k) is a Cauchy sequence in $L^2(\Omega; \mathbb{H}^{-1}) \times L^2(\Omega; \mathbf{L}^2)$. In particular, the sequence (V_k, U_k) can have only one limit point in $L^m(\Omega; \mathbb{L}^2) \times L^m(\Omega; \mathbf{H}^1)$, hence showing that the whole sequence converges towards a uniquely determined couple (U, V) . This shows (P2).

In order to prove (P4), by Theorem 1.1 and the above step, it is sufficient to show that for each $j \in \{0, \dots, J-1\}$,

$$(v, u) \text{ belongs to } L^4(\Omega; L^4(\tau_j, \tau'_{j+1}; \mathbb{L}^4 \times \mathbf{W}^{1,4})), \quad (6.43)$$

where τ'_j is an arbitrary stopping time such that $\tau_j \leq \tau'_{j+1} < \tau_{j+1}$. But (6.43) holds thanks to (2.1). This finishes the proof of Theorem 1.2. \blacksquare

APPENDIX A. APPENDIX: COMPUTATION OF SOME TRACE TERMS

In this appendix, we denote by (v, u) a strong solution of (SEL), and we furthermore assume that it is supported in $\mathcal{V}_T^2 \times \mathcal{U}_T^3$. By the expression ‘‘trace term’’ we refer to Itô-Stratonovitch corrections of the form (1.10).

Recalling the notation (1.8), it is easy to see that for any vector $\zeta \in \mathbb{R}^3$:

$$\sum_{i=1}^3 (\zeta \times \mathbf{e}_i) \times \mathbf{e}_i = -2\zeta. \quad (\text{A.1})$$

Similarly, the norm constraint yields that for any index $\alpha \in \{1, 2\}$:

$$\partial_\alpha \frac{|u|^2}{2} = \partial_\alpha u \cdot u \equiv 0. \quad (\text{A.2})$$

We can now proceed to the computations.

Proof of (4.8). Using (A.1), and recalling (1.6), (1.7), we have for the first term

$$\begin{aligned} A^1(t) &= \frac{1}{2} \sum_{l,j,\alpha} \langle \partial_\alpha u, (u \times \psi_l^j) \times \partial_\alpha \psi_l^j \rangle \\ &= - \sum_{l,j,\alpha} \langle \partial_\alpha u, u(\psi_l \partial_\alpha \psi_l) \rangle \\ &= 0, \end{aligned} \quad (\text{A.3})$$

where we have also used (A.2).

Now, from the skew-symmetry of the vector product and properties (A.1) and (A.2), we have for the second term:

$$\begin{aligned} A^2(t) &\equiv \frac{1}{2} \sum_{l,j,\alpha} [\langle u \times \partial_\alpha \psi_l^j, \partial_\alpha u \times \psi_l^j \rangle + \langle u \times \partial_\alpha \psi_l^j, u \times \partial_\alpha \psi_l^j \rangle] \\ &= -\frac{1}{2} \sum_{l,j,\alpha} [\langle (u \times \partial_\alpha \psi_l^j) \times \psi_l^j, \partial_\alpha u \rangle + 2\langle (u \times \partial_\alpha \psi_l^j) \times \partial_\alpha \psi_l^j, u \rangle] \\ &= \sum_{l,\alpha} [\langle u(\partial_\alpha \psi_l) \psi_l, \partial_\alpha u \rangle + 2|\partial_\alpha \psi_l|_{L^2}^2] \\ &= 2 \sum_l |\nabla \psi_l|_{L^2}^2, \end{aligned} \quad (\text{A.4})$$

which is the claim. ■

We now aim to compute the Itô-Stratonovitch correction for the semimartingale Y defined in (5.15). We have, using coordinates:

$$\begin{aligned} Y(t) &= 2 \int_0^t \langle \hat{\mathcal{F}}, \partial_\beta u \times \text{od} \partial_\beta W \rangle + \int_0^t \langle \hat{\mathcal{F}}, u \times \text{od} \Delta W \rangle - \int_0^t \langle \hat{\mathcal{F}}, v^i (u \times \text{od} \partial_i W) \rangle \\ &\quad + 2 \int_0^t \langle \Delta u, \partial_\beta u \times \text{od} \partial_\beta W \rangle + \int_0^t \langle \Delta u, u \times \text{od} \Delta W \rangle =: \sum_{\gamma=1}^5 Y^\gamma(t), \end{aligned} \quad (\text{A.5})$$

where we recall that $\hat{\mathcal{F}} = \mathcal{F}_u - v \cdot \nabla u = \Delta u + u|\nabla u|^2 - v \cdot \nabla u$.

Claim A.1. Under the condition $\psi \in \mathcal{L}_2(L^2, H^4)$, the process Y is a well-defined semi-martingale, whose decomposition writes as

$$\begin{aligned} Y(t) = \sum_{l \in \mathbb{N}} \iint_{[0,t] \times \mathbb{T}^2} & \left[2(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u)(u \cdot \partial_\beta u \times \partial_\beta \psi_l^j) \right. \\ & + 4(\partial_\alpha \psi_l \partial_\beta \psi_l) \partial_\alpha u \cdot \partial_\beta u + |\Delta \psi_l|^2 - (v \cdot \nabla \psi_l)^2 \\ & \left. + |\nabla u|^2 |\nabla \psi_l|^2 + 2 \sum_{\alpha, \beta} (\partial_\alpha \psi_l \partial_\beta \psi_l) \partial_\alpha u \cdot \partial_\beta u + |\Delta \psi_l|^2 \right] dx ds + \hat{Y}(t) \quad (\text{A.6}) \end{aligned}$$

where $\hat{Y}(t)$ is a martingale.

Moreover, its quadratic variation is estimated above as

$$\begin{aligned} \langle Y \rangle(t) \leq C \int_0^t & \left\{ |\psi^* \operatorname{div}(u \times \nabla \hat{\mathcal{T}})|_{L^2}^2 + |\psi^* \Delta(u \times \hat{\mathcal{T}})|_{L^2}^2 + |\psi^* \partial_i(v^i u \times \hat{\mathcal{T}})|_{L^2}^2 \right. \\ & \left. + |\psi^* \Delta(-u \times \Delta u)|_{L^2}^2 + |\psi^* \operatorname{div}(u \times \nabla \Delta u)|_{L^2}^2 \right\} ds \quad (\text{A.7}) \end{aligned}$$

Proof. Computation of Y^1 . Similarly as above, we have, recalling (1.10),

$$\begin{aligned} Y^1(t) & - 2 \int_0^t \langle \hat{\mathcal{T}}, \partial_\beta u \times d\partial_\beta W \rangle \\ & = \sum_{l,j} \int_0^t \left[\langle \hat{\mathcal{T}}, (\partial_\beta u \times \psi_l^j) \times \partial_\beta \psi_l^j \rangle + \langle \hat{\mathcal{T}}, (u \times \partial_\beta \psi_l^j) \times \partial_\beta \psi_l^j \rangle \right. \\ & \quad + \langle \Delta u \times \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle + 2 \langle \partial_\alpha u \times \partial_\alpha \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle \\ & \quad + \langle u \times \Delta \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle + \langle u \times \psi_l^j |\nabla u|^2, \partial_\beta u \times \partial_\beta \psi_l^j \rangle \\ & \quad + 2 \langle u(\partial_\alpha u \times \psi_l^j \cdot \partial_\alpha u), \partial_\beta u \times \partial_\beta \psi_l^j \rangle + 2 \langle u(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u), \partial_\beta u \times \partial_\beta \psi_l^j \rangle \\ & \quad \left. - \langle v^i \partial_i u \times \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle - \langle v^i u \times \partial_i \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle \right] ds \\ & = \sum_l \int_0^t \left[\sum_j \left\langle \hat{\mathcal{T}}, (\partial_\beta u \times \psi_l^j) \times \partial_\beta \psi_l^j - (\partial_\beta u \times \partial_\beta \psi_l^j) \times \psi_l^j \right\rangle \right. \\ & \quad \left. + 4 \langle \partial_\alpha u \partial_\alpha \psi_l, \partial_\beta u \partial_\beta \psi_l \rangle + 2 \langle u(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u), \partial_\beta u \times \partial_\beta \psi_l^j \rangle \right] ds \\ & = \sum_l \int_0^t \left[4 \langle \partial_\alpha u \partial_\alpha \psi_l, \partial_\beta u \partial_\beta \psi_l \rangle + 2 \langle u(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u), \partial_\beta u \times \partial_\beta \psi_l^j \rangle \right] ds, \end{aligned}$$

thanks to (A.1).

Computation of Y^2 . We have

$$\begin{aligned}
Y^2(t) &= \int_0^t \langle \hat{\mathcal{F}}, u \times d\Delta W \rangle \\
&= \frac{1}{2} \sum_{l,j} \int_0^t \left[\langle \hat{\mathcal{F}}, (u \times \psi_l^j) \times \Delta \psi_l^j \rangle + \langle \Delta u \times \psi_l^j, u \times \Delta \psi_l^j \rangle \right. \\
&\quad + 2\langle \partial_\alpha u \times \partial_\alpha \psi_l^j, u \times \Delta \psi_l^j \rangle + \langle u \times \Delta \psi_l^j, u \times \Delta \psi_l^j \rangle \\
&\quad + \langle u \times \psi_l^j |\nabla u|^2, u \times \Delta \psi_l^j \rangle + 2\langle u(\partial_\alpha u \times \psi_l^j \cdot \partial_\alpha u), u \times \Delta \psi_l^j \rangle \\
&\quad + \langle u(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u), u \times \Delta \psi_l^j \rangle - \langle v^i \partial_i u \times \psi_l^j, u \times \Delta \psi_l^j \rangle \\
&\quad \left. - \langle v^i u \times \partial_i \psi_l^j, u \times \Delta \psi_l^j \rangle \right] ds \\
&= \sum_l \int_0^t \left[\int_{\mathbb{T}^2} |\Delta \psi_l|^2 dx - \langle v^i \partial_i \psi_l, \Delta \psi_l \rangle \right] ds.
\end{aligned}$$

Computation of Y^3 . Proceeding similarly, we have

$$\begin{aligned}
Y^3(t) &= \int_0^t \langle \hat{\mathcal{F}}, v^i u \times d\partial_i W \rangle \\
&= \frac{1}{2} \sum_{l,j} \int_0^t \left[\langle \hat{\mathcal{F}}, v^i (u \times \psi_l^j) \times \partial_i \psi_l^j \rangle + \langle \Delta u \times \psi_l^j, v^i u \times \partial_i \psi_l^j \rangle \right. \\
&\quad + 2\langle \partial_\alpha u \times \partial_\alpha \psi_l^j, v^i u \times \partial_i \psi_l^j \rangle + \langle u \times \Delta \psi_l^j, v^i u \times \partial_i \psi_l^j \rangle \\
&\quad + \langle u \times \psi_l^j |\nabla u|^2, v^i u \times \partial_i \psi_l^j \rangle + 2\langle u(\partial_\alpha u \times \psi_l^j \cdot \partial_\alpha u), v^i u \times \partial_i \psi_l^j \rangle \\
&\quad + \langle u(u \times \partial_\alpha \psi_l^j \cdot \partial_\alpha u), v^i u \times \partial_i \psi_l^j \rangle - \langle v^k \partial_k u \times \psi_l^j, v^i u \times \partial_i \psi_l^j \rangle \\
&\quad \left. - \langle v^k u \times \partial_k \psi_l^j, v^i u \times \partial_i \psi_l^j \rangle \right] ds
\end{aligned}$$

Simplifications lead finally to

$$\begin{aligned}
Y^3(t) &= \int_0^t \langle \hat{\mathcal{F}}, v^i u \times d\partial_i W \rangle \\
&= \frac{1}{2} \sum_l \int_0^t \left[\langle \Delta u \cdot u, \psi_l v^i \partial_i \psi_l \rangle + 2\langle \Delta \psi_l, v^i \partial_i \psi_l \rangle + 2\langle |\nabla u|^2, v^i (\partial_i \psi_l) \psi_l \rangle \right. \\
&\quad \left. - 2\langle v^k \partial_k \psi_l, v^i \partial_i \psi_l \rangle \right] ds \\
&= \sum_l \int_0^t \left[\langle \Delta \psi_l, v^i \partial_i \psi_l \rangle - \langle v^k \partial_k \psi_l, v^i \partial_i \psi_l \rangle \right] ds.
\end{aligned}$$

Computation of Y^4 .. We have

$$\begin{aligned}
Y^4(t) &= \int_0^t \langle \Delta u, \partial_\beta u \times d\partial_\beta W \rangle \\
&= \frac{1}{2} \sum_{l,j} \int_0^t \left[\langle \Delta u, (\partial_\beta u \times \psi_l^j) \times \partial_\beta \psi_l^j \rangle + \langle \Delta u, (u \times \partial_\beta \psi_l^j) \times \partial_\beta \psi_l^j \rangle \right. \\
&\quad \left. + \langle \Delta u \times \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle + 2 \langle \partial_\alpha u \times \partial_\alpha \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle \right. \\
&\quad \left. + \langle u \times \Delta \psi_l^j, \partial_\beta u \times \partial_\beta \psi_l^j \rangle \right] ds, \\
&= \frac{1}{2} \sum_l \int_0^t \left[2 \langle |\nabla u|^2, (\partial_\beta \psi_l)^2 \rangle + 4 \langle \partial_\alpha u \partial_\alpha \psi_l, \partial_\beta u \partial_\beta \psi_l \rangle \right] ds.
\end{aligned}$$

Computation of Y^5 . Similarly:

$$\begin{aligned}
Y^5(t) &= \int_0^t \langle \Delta u, u \times d\Delta W \rangle \\
&= \frac{1}{2} \sum_{l,j} \int_0^t \left[\langle \Delta u, (u \times \psi_l^j) \times \Delta \psi_l^j \rangle \right. \\
&\quad \left. + \langle \Delta u \times \psi_l^j, u \times \Delta \psi_l^j \rangle + 2 \langle \partial_\alpha u \times \partial_\alpha \psi_l^j, u \times \Delta \psi_l^j \rangle \right. \\
&\quad \left. + |u \times \Delta \psi_l^j|_{L^2}^2 \right] ds \\
&= \frac{1}{2} \sum_l \int_0^t 2 |\Delta \psi_l|_{L^2}^2 ds.
\end{aligned}$$

Summing the above contributions, we end up with (A.6).

Quadratic variation of Y . The estimate (A.7) is a simple consequence of [10, Theorem 4.27] applied to each $Y^\gamma, \gamma = 1, \dots, 5$, together with the fact that $\langle Y \rangle(t) \leq C \sum_{\gamma=1}^5 \langle Y^\gamma \rangle(t)$. This finishes the proof. \blacksquare

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