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► **To cite this version:**

Amira Mchaalia. ON TOTALLY PERIODIC  $\omega$ -LIMIT SETS FOR MONOTONE MAPS ON REGULAR CURVES. 2021. hal-03018900v2

**HAL Id: hal-03018900**

**<https://hal.science/hal-03018900v2>**

Preprint submitted on 10 Jun 2021

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# ON TOTALLY PERIODIC $\omega$ -LIMIT SETS FOR MONOTONE MAPS ON REGULAR CURVES

AMIRA MCHAALIA

ABSTRACT. An  $\omega$ -limit set of a continuous self-mapping of a compact metric space  $X$  is said to be totally periodic if all of its points are periodic. In [3] Askri and Naghmouchi proved that if  $f$  is one-to-one continuous self mapping of a regular curve, then every totally periodic  $\omega$ -limit set of  $f$  is finite. This also holds whenever  $f$  is a monotone map of a local dendrite by Abdelli in [1]. In this paper we generalize these results to monotone maps on regular curves. On the other hand, we give some remarks related to expansivity and totally periodic  $\omega$ -limit sets for any continuous map on compact metric space.

## 1. Introduction

The structure of  $\omega$ -limit sets plays an important role in studying dynamical system since it represents asymptotically the state of the orbit. A class of  $\omega$ -limit sets composed of periodic points called *totally periodic  $\omega$ -limit* is investigated. This notion was introduced by Marzougui and Naghmouchi in [11]. A compact metric space  $X$  is said to have the  $\omega$ -FTP property if for each self continuous map  $f$  of  $X$ , every totally periodic  $\omega$ -limit set of a point  $x \in X$  is finite. We mention that totally periodic  $\omega$ -limit set appeared in the definition of  $\omega$ -chaos, introduced by Li in [10]. It also appeared as  $\omega$ -limit set of “spiral points” as proved in [2, Proposition 3.2] by Artigue. In zero-dimension, it was shown in [11] that, a zero dimensional compact metric space has the  $\omega$ -FTP property. In one-dimension, it was shown in ([1]) that any local dendrite has the  $\omega$ -FTP property when we restricted to monotone maps; that is any totally periodic  $\omega$ -limit set of a monotone map on a local dendrite is finite. The later result does not hold when we remove the restriction of monotonicity; it was shown in [11] that, there is a continuous map on a dendrite having an infinite totally periodic  $\omega$ -limit set. In [3], it was shown that a regular curve has the  $\omega$ -FTP property when we restricted to one-to-one continuous maps. In this line, we generalize in the present paper the above results to monotone maps on regular curves.

Our main result is the following theorem.

**Theorem 1.1.** *Let  $X$  be a regular curve and  $f : X \rightarrow X$  be a monotone map. Then any totally periodic  $\omega$ -limit set of  $f$  is finite.*

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2000 *Mathematics Subject Classification.* 37B20, 37B45, 54H20.

*Key words and phrases.* Minimal set, regular curve,  $\omega$ -limit set, totally periodic, monotone map.

In other words, a regular curve has the  $\omega$ -FTP property when we restricted to monotone maps.

Notice that Theorem 1.1 cannot be extended to other classes of systems such as rational curves; for instance, it was constructed in [13, Theorem D] an example of a homeomorphism of a rational curve (in fact a dendroid) for which some points have infinite  $\omega$ -limit sets consisting only of fixed points.

On the other hand, further results are obtained with regard to expansivity, we prove that any totally periodic  $\omega$ -limit set for expansive continuous map on a compact metric space  $X$  is finite (see Proposition 5.2).

## 2. Preliminaries

Let  $X$  be a compact metric space with metric  $d$  and let  $f : X \rightarrow X$  be a continuous map. Let  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$  be the sets of integers, non-negative integers and positive integers, respectively. For  $n \in \mathbb{Z}_+$  denote by  $f^n$  the  $n$ -th iterate of  $f$ ; that is,  $f^0 = \text{id}_X$ : identity of  $X$  and  $f^n = f \circ f^{n-1}$  if  $n \in \mathbb{N}$ . A point  $x \in X$  is called periodic of period  $n \in \mathbb{N}$  if  $f^n(x) = x$  and  $f^i(x) \neq x$  for  $1 \leq i \leq n-1$ . If  $n = 1$ ,  $x$  is a *fixed point* of  $f$ . For any  $x \in X$ , the subset  $\text{Orb}_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$  is called the *orbit* of  $x$  (under  $f$ ). If  $x$  is a periodic point, the orbit of  $x$  is called periodic orbit. We denote by  $P(f)$  and  $\text{Fix}(f)$  the set of periodic points and fixed points of  $f$ , respectively. A subset  $A \subset X$  is called  *$f$ -invariant* if  $f(A) \subset A$ . It is called *strongly  $f$ -invariant* if  $f(A) = A$ . We define the  $\omega$ -limit set of a point  $x \in X$  to be the set

$$\begin{aligned} \omega_f(x) &= \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow +\infty, \lim_{i \rightarrow +\infty} d(f^{n_i}(x), y) = 0\} \\ &= \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}. \end{aligned}$$

It is non-empty, closed and strongly  $f$ -invariant set (cf. [4, Chapter IV, Lemma 2]).

For any subset  $A$  of  $X$ , we denote by  $\overline{A}$ ,  $\text{Bd}(A)$  and  $\text{int}(A)$  the closure, boundary and interior of  $A$ , respectively. The symbol  $\text{card}(A)$  stands for the cardinality of  $A$  and  $\text{diam}(A) = \sup_{x, y \in A} d(x, y)$  means the diameter of  $A$ .

We denote by  $B(x, \varepsilon)$  the open ball centered at  $x$  of radius  $\varepsilon$ .

**Definition 2.1** ([8], p. 131). *Let  $X, Y$  be two topological spaces. A continuous map  $f : X \rightarrow Y$  is said to be monotone if for any connected subset  $C$  of  $Y$ ,  $f^{-1}(C)$  is connected.*

When  $f$  is closed and onto, Definition 2.1 is equivalent to that the preimage of any point by  $f$  is connected (cf. [8], p. 131). Notice that  $f^n$  is monotone for every  $n \in \mathbb{N}$  when  $f$  itself is monotone.

In the definitions below, we use the terminology from Nadler [12].

- A *continuum* is a compact connected metric space.
- An *arc*  $I$  (resp. a circle) is any space homeomorphic to the compact interval  $[0, 1]$  (resp. to the unit circle  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ ). For an arc

$I = [a, b]$ , denote by  $\gamma(I) = \{a, b\}$  the set of its endpoints,  $[a, b) = I \setminus \{b\}$  and  $(a, b) = I \setminus \gamma(I) = I \setminus \{a, b\}$ .

– A space is called *degenerate* if it is a single point, otherwise it is called *non-degenerate*.

– A *dendrite* is a locally connected continuum which contains no circle.

– A *local dendrite* is a continuum such that every point of which has a dendrite neighborhood. A local dendrite is then a locally connected continuum containing only a finite number of circles (cf. [8], Theorem 4, p. 303).

– A *regular curve* is a continuum every point of which  $x$  and each open neighborhood  $V$  of  $x$  in  $X$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $U \subset V$  and the boundary set  $\text{Bd}(U)$  of  $U$  is finite. Each regular curve is locally arcwise connected. Every sub-continuum of a regular curve is a regular curve (cf. [8] and [12]). Notice that every local dendrite is a regular curve (cf. [8], p. 303). There exists a regular curve which does not a local dendrites for example the Sierpiński triangle.

– A *hereditarily locally connected* is a continuum such that every sub-continuum of it is locally connected.

– A *Peano continuum* is a locally connected continuum. In particular, each regular curve is a Peano continuum.

– A continuum  $X$  is said to be a *dendroid* provided that  $X$  is an arcwise connected hereditarily unicoherent space; that is if  $X = A \cup B$  and  $A, B$  are subcontinua of  $X$ , then  $A \cap B$  is connected.

Given a compact metric space  $X$  with metric  $d$ , we denote by  $2^X$  (resp.  $C(X)$ ) the set of all non-empty compact subsets (resp. continua) of  $X$ .

We endow  $2^X$  with the Hausdorff metric  $d_H$  defined as follows:

For  $A, B \in 2^X$ ,  $d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A))$ , where

$d(x, M) = \inf_{y \in M} d(x, y)$  for any  $x \in X$  and  $M \in 2^X$ . For  $A = \{a\}$ , we write

simply,  $d_H(A, B) = d_H(a, B)$ . It is well known that  $(2^X, d_H)$  is a compact metric space (see [12], for more details). For any  $B \in C(X)$ , we denote by

$$\text{mesh}(B) = \sup\{\text{diam}(C) : C \text{ is a connected component of } B\}.$$

### 3. Some results

**Lemma 3.1.** *Let  $X$  be a compact metric space and  $f$  be a continuous self-mapping of  $X$ . Then for all  $n \in \mathbb{N}$ ,  $P(f^n) = P(f)$ .*

**Lemma 3.2.** *Let  $X$  be a metric compact space and  $f$  a continuous self mapping of  $X$ . Then for all  $x \in X$ ,  $\omega_f(f^i(x)) = \omega_f(x)$ , for all  $i \in \mathbb{N}$ .*

**Lemma 3.3.** ([4], p. 70) *Let  $X$  be a metric compact space and  $f$  a continuous self mapping of  $X$ . Then for all  $N \in \mathbb{N}$  and  $x \in X$ ,  $\omega_f(x) = \cup_{i=0}^{N-1} \omega_{f^N(x)}(f^i(x))$ .*

**Lemma 3.4.** ([11, Theorem 1.1]) *Let  $X$  be a compact metric space,  $f$  be a continuous self-mapping of  $X$  and  $x \in X$ . If  $\omega_f(x)$  is totally periodic, then  $\omega_f(x)$  has finitely many connected components that form a periodic cycle.*

Recall that a  $p$ -tuple  $(A_0, \dots, A_{p-1})$  of subsets of  $X$  is called a periodic cycle if  $f(A_0) = A_1, f(A_1) = A_2, \dots, f(A_{p-1}) = A_0$ .

**Proposition 3.5.** ([9, Theorem 1.6]) *Let  $(C_n)_{n \in \mathbb{N}}$  be a sequence of disjoint sub-continua of a regular curve  $X$ . Then  $\lim_{n \rightarrow +\infty} \text{diam}(C_n) = 0$ .*

**Lemma 3.6.** ([8, Theorem 4, p. 257]) *Let  $X$  be a Peano continuum and  $\varepsilon > 0$  be a given number. Then there exists  $\eta > 0$  such that for every pair of points  $(x, y)$  such that  $d(x, y) < \eta$ , there exists an arc  $I$  joining  $x$  and  $y$  with  $\text{diam}(I) < \varepsilon$ .*

**Lemma 3.7.** ([5, Lemma 4.2]) *If  $X$  is hereditarily locally connected, then for any sequence  $(A_n)_{n \geq 0}$  in  $C(X)$  that converges to  $A \in C(X)$  (with respect to Hausdorff metric), we have  $\lim_{n \rightarrow +\infty} \text{mesh}(A_n \setminus A) = 0$ .*

**Lemma 3.8.** *Let  $X$  be a regular continuum. Then for each sub-continuum  $A$  of  $X$ , there is a decreasing sequence  $(U_n)_{n \in \mathbb{N}}$  of open connected neighborhoods of  $A$  with finite boundary such that  $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$ .*

*Proof.* It is clear that

$$A = \bigcap_{n \in \mathbb{N}} B(A, \frac{1}{n}), \quad \text{where } B(A, \frac{1}{n}) = \{x \in X : d(x, A) < \frac{1}{n}\}.$$

Since  $A$  is compact and  $X$  is locally connected, there exists, for each  $n \in \mathbb{N}$ , an open connected neighborhood  $V_n$  of  $A$  in  $B(A, \frac{1}{n})$  with finite

boundary such that  $A = \bigcap_{n \in \mathbb{N}} V_n$ . For each  $n \in \mathbb{N}$ , set  $W_n = \bigcap_{i=1}^n V_i$ . Then

$(W_n)_{n \in \mathbb{N}}$  is a decreasing sequence of open connected neighborhoods of  $A$  and  $A = \bigcap_{n \in \mathbb{N}} W_n$ . Moreover  $\text{Bd}(W_n)$  is finite since each  $\text{Bd}(V_i)$  is finite,

$1 \leq i \leq n$ . Now, let  $U_n \in \mathbb{N}$  be the connected component of  $W_n$  containing  $A$ . Then the  $(U_n)_{n \in \mathbb{N}}$  is a sequence of decreasing open connected neighborhoods of  $A$  with  $\text{Bd}(U_n)$  finite and we have  $A = \bigcap_{n \in \mathbb{N}} \overline{U}_n$ . Since  $X$

is locally connected,  $U_n$  is open and connected. Moreover the sequence of continua  $(\overline{U}_n)_{n \in \mathbb{N}}$  converges to  $A$  in  $C(X)$  (with respect to Hausdorff metric). By Lemma 3.7, we have that  $\lim_{n \rightarrow +\infty} \text{mesh}(\overline{U}_n \setminus A) = 0$  and

therefore  $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$ . □

**Lemma 3.9.** *Let  $F$  be a non degenerate closed subset of  $X$ . Then*

$$(1) \quad \eta(F) := \inf_{a \in F} d_H(a, F) > 0.$$

(2) *For any  $a \in F$ , there is  $c \in F$  such that  $d(a, c) \geq \eta(F)$ .*

*Proof.* (1) Since  $d(a, F) = 0$ , for  $a \in F$ , so  $d_H(a, F) = \sup_{b \in F} d(a, b)$ . There exists  $a_0 \in F$  and  $b_0 \in F$  such that  $\inf_{a \in F} d_H(a, F) = d_H(a_0, F) = d(a_0, b_0)$ . If  $a_0 = b_0$ , so  $d(a_0, b) = 0, \forall b \in F$ . This implies that  $F = \{a_0\}$ , a

contradiction since  $F$  is non degenerate. We conclude that

$$\eta(F) = \inf_{a \in F} d_H(a, F) > 0.$$

(2) For any  $a \in F$ ,  $d_H(a, F) \geq \eta(F)$  and so there is  $c \in F$  such that  $d(a, c) = d_H(a, F) \geq \eta(F)$ .  $\square$

#### 4. Proof of Theorem 1.1

Suppose the theorem does not hold. Then we could find  $x \in X$  such that  $\omega_f(x)$  totally periodic (that is  $\omega_f(x) \subset P(f)$ ) and infinite.

**Lemma 4.1.** *Let  $x \in X$  such that  $\omega_f(x)$  is totally periodic. Then, there are  $N \in \mathbb{N}$  and  $0 \leq i \leq N - 1$  such that:*

- (1)  $\text{int}_{\omega_{f^N}(f^i(x))}(\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset$ .
- (2)  $\omega_{f^N}(f^i(x))$  is connected.

*Proof.* (1) As  $\omega_f(x)$  is totally periodic, so  $\omega_f(x) = \bigcup_{n \in \mathbb{N}} \text{Fix}(f^n) \cap \omega_f(x)$ . By the Baire property, there is  $N \in \mathbb{N}$  such that

$$V := \text{int}_{\omega_f(x)}(\text{Fix}(f^N) \cap \omega_f(x)) \neq \emptyset.$$

As

$$\omega_f(x) = \bigcup_{i=0}^{N-1} \omega_{f^N}(f^i(x)) \quad (\text{Lemma 3.3}),$$

so there exists  $0 \leq i \leq N - 1$  such that  $V \cap \omega_{f^N}(f^i(x)) \neq \emptyset$ .

As  $V \cap \omega_{f^N}(f^i(x)) \subset \text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))$ , thus

$$\text{int}_{\omega_{f^N}(f^i(x))}(\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset.$$

(2) Set  $g = f^N$  and  $y = f^i(x)$ . From (1),  $\text{int}_{\omega_g(y)}(\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$ . We have  $\omega_g(y) \subset \omega_f(y) = \omega_f(x)$  (Lemma 3.2). As  $P(f) = P(g)$  (Lemma 3.1),

so  $\omega_g(y) \subset P(g)$ . Following Lemma 3.4,  $\omega_g(y) = \bigcup_{i=0}^{m-1} C_i$ , for some  $m \in \mathbb{N}$ ,

where the  $(C_i)_{0 \leq i \leq m-1}$  are the connected components of  $\omega_g(y)$  with  $g(C_i) = C_{i+1}$  for every  $i \in \{0, \dots, m-1\}$ . As  $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$ , so there is  $a \in \omega_g(y) \cap \text{Fix}(g)$ . Therefore  $a \in C_i$  for every  $i \in \{0, \dots, m-1\}$ , which implies that  $\omega_g(y) = C_0$ . In result  $\omega_g(y)$  is connected.  $\square$

Set  $g = f^N$  and  $y = f^i(x)$ . From Lemma 4.1, we have the following properties.

- $\omega_g(y) \subset P(g)$ .
- $\text{int}_{\omega_g(y)}(\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$ .
- $\omega_g(y)$  is a regular curve (since  $\omega_g(y)$  is a sub-continuum of  $X$ ).

It follows that we can find an open connected set  $O \subset \text{Fix}(g) \cap \omega_g(y)$ . For the sequel, we let

$$S_a = \bigcup_{i \geq 0} g^{-i}(a), \text{ for any } a \in \overline{O}.$$

**Lemma 4.2.** *Let  $U$  be an open neighborhood of  $\omega_g(y)$  in  $X$  with finite boundary. Then there exists an open connected subset  $V$  of  $\omega_g(y)$  such that  $\overline{V} \subset O$  and  $S_a \subset U$ , for all  $a \in \overline{V}$ .*

*Proof.* First, the sets  $(S_a)_{a \in \overline{O}}$  are pairwise disjoint connected sets; for  $a \neq b \in \overline{O}$ , we have  $S_a \cap S_b = \emptyset$ ; indeed otherwise, there is  $c \in S_a \cap S_b$ , so  $g^j(c) = a$  and  $g^i(c) = b$  for some  $j > i \in \mathbb{N}$ . As  $a, b \in \text{Fix}(g)$ ,  $a = g^{j-i}(g^i(c)) = g^{j-i}(b) = b$ , a contradiction. By monotonicity of  $g$ ,  $S_a$  is connected. Set  $\text{card}(\text{Bd}(U)) = k$ . So there are at most  $k$  points  $a_1, a_2, \dots, a_k \in \overline{O}$  such that  $S_{a_i} \not\subset U$  for every  $i \in \{1, \dots, k\}$ : Otherwise, there are at least  $k+1$  points  $a_1, a_2, \dots, a_{k+1} \in \overline{O}$  such that  $S_{a_i} \not\subset U$  for all  $1 \leq i \leq k+1$ . As  $a_i \in S_{a_i} \cap U$  and  $S_{a_i}$  is connected, so  $S_{a_i}$  meets  $\text{Bd}(U)$  for every  $1 \leq i \leq k+1$ . Hence  $\text{card}(\text{Bd}(U)) \geq k+1$ , a contradiction. As  $O \setminus \{a_1, a_2, \dots, a_k\}$  is open in  $\omega_g(y)$  and non-empty (since  $\omega_g(y)$  is infinite), so we can find a non-empty open connected subset  $V$  of  $\omega_g(y)$  such that  $\overline{V} \subset O \setminus \{a_1, a_2, \dots, a_k\}$ . This implies that for all  $a \in \overline{V}$ ,  $S_a \subset U$ . The proof is complete.  $\square$

*Step 1.* For  $F = \overline{O}$ , we denote by  $\eta := \eta(F) > 0$  be given as in Lemma 3.9. By Lemma 3.8, there exists a sequence of open connected neighborhoods  $(U_n)_{n \in \mathbb{N}}$  of  $\omega_g(y)$  in  $X$  with finite boundary such that  $\text{mesh}(U_n \setminus \omega_g(y)) < \frac{\eta}{2}$ . We let  $U := U_N$  and set  $\text{card}(\text{Bd}(U)) = k$ . From Lemma 4.2, there exists an open connected subset  $V \subset O$  of  $\omega_g(y)$  such that for all  $a \in \overline{V}$ ,  $S_a \subset U$ . Let  $a \in V$ . As  $\{a\} \subsetneq \omega_g(y)$ , there is an open connected neighborhood  $W$  of  $a$  in  $X$  such that  $\overline{W} \cap \omega_g(y) \subset \overline{V}$  and  $\omega_g(y) \setminus \overline{W} \neq \emptyset$ . We can find two increasing sequences of integers  $(n_i)_{i \in \mathbb{N}}, (t_i)_{i \in \mathbb{N}}$  with the following properties:

- (1)  $g^{n_i}(y) \in W$  with  $\lim_{i \rightarrow +\infty} g^{n_i}(y) = a$ ,
- (2)  $g^{n_i+j}(y) \in W$  for each  $j = 0, \dots, t_i$ , with  $t_i \geq k$  and  $t_i \geq i, \forall i \in \mathbb{N}$ ,
- (3)  $g^{n_i+t_i+1}(y) \in X \setminus W$  and since  $b$  is a fixed point of  $g$ ,

$$\lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y) = b = \lim_{i \rightarrow +\infty} g^{n_i+t_i+1}(y) \in \overline{W} \setminus W.$$

Observe that  $a \in W \cap \omega_g(y)$  and  $b \in (\overline{W} \setminus W) \cap \omega_g(y)$ . So  $a \neq b$ .

We let  $\mathbb{N}_k = \bigcup_{i \in \mathbb{N}} \{n_i, \dots, n_i + t_i - k\}$ . Since  $\lim_{i \rightarrow +\infty} t_i = +\infty$ ,  $\mathbb{N}_k$  is infinite.

By definition,  $g^m(y) \in W$ , for each  $m \in \mathbb{N}_k$  and

$\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} d(g^m(y), \overline{W} \cap \omega_g(y)) = 0$ . For each  $m \in \mathbb{N}_k$ , there is

$b_m \in \overline{W} \cap \omega_g(y)$  such that  $d(g^m(y), \overline{W} \cap \omega_g(y)) = d(g^m(y), b_m)$ . Since  $\overline{W}$  is a sub-continuum of  $X$ , so it is a Peano continuum. By Lemma 3.6, one can find an arc  $T_m = [b_m, g^m(y)] \subset \overline{W}$  such that  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(T_m) = 0$ .

Since  $\omega_g(y)$  is infinite and  $\omega_g(y) \subset P(g)$ , so  $g^m(y) \notin \omega_g(y)$ , for every  $m \in \mathbb{N}_k$ .

Let  $I_m$ ,  $m \in \mathbb{N}_k$ , be the connected component of  $T_m \setminus \omega_g(y)$  containing  $g^m(y)$ . Then one can write  $I_m = [g^m(y), a_m)$  with  $\bar{I}_m \cap \omega_g(y) = \{a_m\}$ . We see that  $a_m \in \bar{W} \cap \omega_g(y) \subset \bar{V} \subset \bar{O} \subset \text{Fix}(g) \cap \omega_g(y)$ , for any  $m \in \mathbb{N}_k$ .

*Claim 1.* Under the notations above, there exists  $N \in \mathbb{N}$  such that for all  $m \in \mathbb{N}_k$ ,  $m \geq N$ , the family  $F_m = \{I_{m+j} : 0 \leq j \leq k, m+j \in \mathbb{N}_k\}$  is not pairwise disjoint.

*Proof.* Suppose that, there exists an increasing sequence of positive integers  $(m_i)_{i \geq 1}$  such that the family  $F_{m_i}$  is pairwise disjoint. By passing to a subsequence of  $(m_i)_{i \geq 1}$ , one can assume that  $(\bar{I}_{m_i})_{i \geq 1}$  converges to  $I \in C(X)$  (in Hausdorff metric). As  $\lim_{i \rightarrow +\infty} \text{diam}(T_{m_i}) = 0$ , then

$\lim_{i \rightarrow +\infty} \text{diam}(\bar{I}_{m_i}) = 0 = \text{diam}(I)$ , so  $I$  is degenerate i.e.  $I = \{c\}$ . As

$g^{m_i}(y) \in I_{m_i} \subset W$  and  $\lim_{i \rightarrow +\infty} d_H(c, \bar{I}_{m_i}) = 0$ , so

$\lim_{i \rightarrow +\infty} g^{m_i}(y) = c \in \bar{W} \cap \omega_g(y)$ . Since  $\bar{W} \cap \omega_g(y) \subset \bar{V} \subset \bar{O} \subset \text{Fix}(g)$ , so for all  $0 \leq j \leq k$ ,  $\lim_{i \rightarrow +\infty} g^{m_i+j}(y) = g^j(c) = c$ . In result, we have that

$\lim_{i \rightarrow +\infty, m_i+j \in \mathbb{N}_k} \bar{I}_{m_i+j} = \{c\}$  (in Hausdorff metric). Now apply Lemma 3.9 for

$c \in \bar{O}$ : there is  $c' \in \bar{O}$  such that  $d(c, c') \geq \eta$ . Then we can find two open neighborhoods  $O_c$  and  $O_{c'}$  of  $c$  and  $c'$  in  $U$ , respectively such that  $d(O_c, O_{c'}) > \frac{\eta}{2}$ . As  $c' \in \bar{O} \subset \omega_g(y) \cap \text{Fix}(g)$ , so there is  $m \geq 1$  such that  $\{g^m(y), g^{m+1}(y), \dots, g^{m+k}(y)\} \subset O_{c'}$ . Let  $m_p > m$  such that

$\bigcup_{m_p+j \in \mathbb{N}_k} \bar{I}_{m_p+j} \subset O_c$  and set

$$C_j = g^{-(m_p-m)}(\bar{I}_{m_p+j}), \quad m_p + j \in \mathbb{N}_k.$$

By monotonicity of  $g$ ,  $C_j$  is a sub-continuum of  $X$ . Then there is an arc  $J_j = [g^{m_p+j}(y), a_{m_p+j}] \subset C_j$ . Each  $J_j$  meets  $O_{c'}$  in at least  $g^{m_p+j}(y)$  and meets  $O_c$  in at least  $a_{m_p+j} \in \text{Fix}(g) \cap \omega_g(y)$ . Let us show that

$J_j \setminus \omega_g(y) = J_j \setminus \{a_{m_p+j}\}$ :

It is clear that  $J_j \setminus \omega_g(y) \subset J_j \setminus \{a_{m_p+j}\}$ . Conversely, if  $z \in J_j \cap \omega_g(y)$ , then  $g^{m_p-m}(z) \in \bar{I}_{m_p+j} \cap \omega_g(y) = \{a_{m_p+j}\}$ . As  $a_{m_p+j} \in \text{Fix}(g)$ , then

$g^{m_p-m}(a_{m_p+j}) = g^{m_p-m}(z)$ . As  $\omega_g(y) \subset P(g)$ , then the restriction  $g|_{\omega_g(y)}^{m_p-m}$  is a homeomorphism and therefore  $z = a_{m_p+j}$ . In result,

$J_j \setminus \{a_{m_p+j}\} \subset J_j \setminus \omega_g(y)$  and hence the equality holds.

We conclude that  $J_j \setminus \{a_{m_p+j}\}$  is connected and hence  $\text{diam}(J_j) > \frac{\eta}{2}$ .

Moreover we have that  $J_j \not\subset U$  (since otherwise,  $J_j \setminus \{a_{m_p+j}\} \subset U \setminus \omega_g(y)$  and so  $\text{diam}(J_j \setminus \{a_{m_p+j}\}) \leq \text{mesh}(U \setminus \omega_g(y)) < \frac{\eta}{2}$ . A contradiction).

Therefore  $C_j \cap \text{Bd}(U) \neq \emptyset$ , for each  $m_p + j \in \mathbb{N}_k$ .

Let us show that  $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$  for all  $j \neq j'$ . We have

$\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'} \subset \{a_{m_p+j}\}$  for  $j \neq j'$  (since  $a_{m_p+j} = a_{m_p+j'}$ ). Then

$C_j \cap C_{j'} = g^{-(m_p-m)}(\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'}) \subset g^{-(m_p-m)}(a_{m_p+j}) \subset S_{a_{m_p+j}}$ . As



$a_{m_p+j} \in \overline{V}$ , so by Lemma 4.2,  $S_{a_{m_p+j}} \subset U$ . Hence  $C_j \cap C_{j'} \subset U$  and so  $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$ . We conclude that  $\text{Bd}(U)$  has at least  $k+1$  points, which contradicts  $\text{Bd}(U) = k$ . This completes the proof of Claim 1.  $\square$

*Step 2.* From Claim 1, there is some  $m \in \mathbb{N}_k$  with  $m+j_1(m), m+j_2(m) \in \mathbb{N}_k$  such that  $I_{m+j_1(m)} \cap I_{m+j_2(m)} \neq \emptyset$ . Hence  $I_{m+j_1(m)} \cup I_{m+j_2(m)}$  is arcwise connected and so we can find an arc

$$P_m = [g^{m+j_1(m)}(y), g^{m+j_2(m)}(y)] \subset I_{m+j_1(m)} \cup I_{m+j_2(m)}.$$

We have that  $P_m \cap \omega_g(y) = \emptyset$ . Since  $P_m \subset I_{m+j_1(m)} \cup I_{m+j_2(m)}$ , we have  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(P_m) = 0$ .

*Claim 2.* Let  $B_m = g^{-j_1(m)}(P_m), m \in \mathbb{N}_k$ . Then we have

$$\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) = 0.$$

*Proof.* Suppose that  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) > 0$ . Then there is  $\delta > 0$  and a sequence of positive integers  $(m_l)_l \subset \mathbb{N}_k$  such that  $\text{diam}(B_{m_l}) > \delta$ , for all  $l \in \mathbb{N}$ . Recall  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(I_m) = 0$ , where  $I_m = [g^m(y), a_m)$  and  $\lim_{l \rightarrow +\infty, m_l \in \mathbb{N}_k} \text{diam}(P_{m_l}) = 0$ . By passing to a subsequence if necessary, one can assume that  $(P_{m_l})_{m_l \in \mathbb{N}_k}$  converges necessarily to a degenerate set, say  $\{c\}$ . Hence,  $\lim_{l \rightarrow +\infty, m_l \in \mathbb{N}_k} g^{m_l+j_1(m_l)}(y) = c \in \omega_g(y)$ . As  $P_{m_l} \cap \omega_g(y) = \emptyset$  for all  $l \in \mathbb{N}$ , so  $c \notin P_{m_l}$ . Then we can construct a subsequence  $(P_{m_{l_q}})_{q \geq 0, m_{l_q} \in \mathbb{N}_k}$  which is pairwise disjoint: indeed, as  $(P_{m_l})_{m_l \in \mathbb{N}_k}$  converges to  $\{c\}$ , then for each neighborhood  $U_0$  of  $c$ , there is  $n_0 \in \mathbb{N}$  such that for all  $l \geq n_0$ ,  $P_{m_l} \subset U_0$ . Since  $c \notin P_{m_l}$ , for each  $m_l \in \mathbb{N}_k$ , so we can find an open neighborhood  $U_1 \subset U_0$  of  $c$  such that  $P_{m_{n_0}} \not\subset U_1$ . Moreover, since  $(P_{m_l})_{m_l \in \mathbb{N}_k}$  converges to  $\{c\}$ , so there is  $n_1 \in \mathbb{N}$  such that for all  $l \geq n_1$ ,  $P_{m_l} \in U_1$ , we obtain  $P_{m_{n_0}} \cap P_{m_{n_1}} = \emptyset$ . We continue in this fashion obtaining a subsequence  $(P_{m_{l_q}})_{q \geq 0, m_{l_q} \in \mathbb{N}_k}$  which is pairwise disjoint. Since  $0 \leq j_1(m_{l_q}) \leq k$ , we can assume that  $j_1(m_{l_q}) = j_0$ , for all  $q \in \mathbb{N}$ . Then the  $B_{m_{l_q}} = g^{-j_0}(P_{m_{l_q}}), q \in \mathbb{N}$ , are connected and pairwise disjoint sets. Hence, by Proposition 3.5,  $\lim_{q \rightarrow +\infty, m_{l_q} \in \mathbb{N}_k} \text{diam}(B_{m_{l_q}}) = 0$ . This contradicts the fact that  $\text{diam}(B_{m_l}) > \delta$ , for all  $l \in \mathbb{N}, m_l \in \mathbb{N}_k$ . This completes the proof.  $\square$

*End of the proof of Theorem 1.1.* Recall that  $\omega_g(y) \subset P(g)$  and  $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$  is infinite. From the notation given in Step 1, we have that  $\lim_{i \rightarrow +\infty} g^{n_i}(y) = a$  and  $\lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y) = b$  with  $a \neq b \in \omega_g(y) \cap \text{Fix}(g)$ .

From Claim 2,  $B_m = g^{-j_1(m)}(P_m)$  is connected for each  $m \in \mathbb{N}_k$  and contains the points  $g^m(y)$  and  $g^{m+j_2(m)-j_1(m)}(y)$ . Moreover, for each  $m \in \mathbb{N}_k$ , we have  $B_m \cap \omega_g(y) = \emptyset$ ; otherwise, there is  $m \in \mathbb{N}_k$  such that  $z \in B_m \cap \omega_g(y)$  and so  $g^{j_1(m)}(z) \in P_m \cap \omega_g(y)$ , this leads to a contradiction

with  $P_m \cap \omega_g(y) = \emptyset$ . By Claim 2,  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) = 0$ . So let  $W_a, W_b$  be two open neighborhoods of  $a$  and  $b$ , respectively such that  $d(W_a, W_b) > \frac{d(a,b)}{2}$ . By Lemma 3.8, there exists a connected neighborhood  $U' := U_{N'}$  of  $\omega_g(y)$  such that  $\text{mesh}(U' \setminus \omega_g(y)) < \frac{d(a,b)}{2}$ . Since  $\lim_{m \rightarrow +\infty} d(g^m(y), \omega_g(y)) = 0$ , so there is  $m_0 \in \mathbb{N}$  such that for all  $m \geq m_0$ ,  $m \in \mathbb{N}_k$ ,  $B_m \subset U'$ . Thus there exists  $q \in \mathbb{N}$  such that for  $n_q \geq m_0$ ,  $g^{n_q}(y) \in W_a$  and  $\{g^{n_q+t_q-k}(y), \dots, g^{n_q+t_q}(y)\} \subset W_b$ .

For each  $m \in \mathbb{N}_k$ , denote by

- $s(m) = m + j_2(m) - j_1(m)$ , where  $m + j_1(m), m + j_2(m) \in \mathbb{N}_k$ ,
- $p_{n_q} = \inf\{i \in \mathbb{N} : s^i(n_q) \in \{n_q + t_q - k + 1, \dots, n_q + t_q\}\}$ ,
- $K_q = B_{n_q} \cup B_{s(n_q)} \cup \dots \cup B_{s^{p_{n_q}-1}(n_q)}$ .

It is easy to see that  $K_q$  is connected. Since for all  $0 \leq i < p_{n_q}$ ,  $s^i(n_q) \in \mathbb{N}_k$ , so  $B_{s^i(n_q)} \cap \omega_g(y) = \emptyset$  and then  $K_q \cap \omega_g(y) = \emptyset$ . Moreover as  $K_q$  meets  $W_a$  in at least  $g^{n_q}(y)$  and meets  $W_b$  in at least  $g^{s^{p_{n_q}}(y)}$ , thus  $\text{diam}(K_q) > \frac{d(a,b)}{2}$ . Since for all  $n_q \geq m_0$ ,  $B_{n_q} \subset U'$ ,  $K_q \subset U'$  and  $K_q \cap \omega_g(y) = \emptyset$ , thus  $K_q \subset U' \setminus \omega_g(y)$ . Therefore  $\text{diam}(K_q) \leq \text{mesh}(U' \setminus \omega_g(y)) < \frac{d(a,b)}{2}$ . A contradiction.

## 5. Expansivity and totally periodic $\omega$ -limit sets

In this section, we consider the notion of expansivity of continuous maps on compact metric space.

**Definition 5.1.** [14] *A continuous map  $f : X \rightarrow X$  on a compact metric space  $X$  is said to be expansive if there is real constant  $c > 0$  such that for any  $x, y \in X$  and  $x \neq y$ , there is an integer  $n \in \mathbb{Z}_+$  such that  $d(f^n(x), f^n(y)) > c$ .*

It is well known that the Cantor set and the torus  $\mathbb{T}^2$  admit expansive homeomorphisms [15]. Kawamura showed [7] that a Peano continuum which contains a free arc admits no expansive homeomorphisms. In particular, there is no expansive homeomorphism on an arc and a circle. In [6], Kato showed that no dendroid admits an expansive homeomorphism. With respect to expansivity, we show the following.

**Proposition 5.2.** *If  $f : X \rightarrow X$  is an expansive continuous map on a compact metric space  $X$ , then any totally periodic  $\omega$ -limit set is finite.*

**Lemma 5.3.** ([4, Lemma 4]) *Let  $(X, f)$  be a dynamical system and let  $x \in X$  such that  $\omega_f(x)$  is infinite. If  $z$  is an isolated point in  $\omega_f(x)$ , then it is not periodic.*

*Proof of Proposition 5.2.* Let  $x \in X$  such that  $\omega_f(x) \subset P(f)$  and assume that  $\omega_f(x)$  is infinite. We have that  $\omega_f(x) = \bigcup_{n \in \mathbb{N}} \text{Fix}(f^n) \cap \omega_f(x)$ . Then we claim that there exists  $N \in \mathbb{N}$  such that  $\text{Fix}(f^N) \cap \omega_f(x)$  is infinite.

Otherwise,  $\omega_f(x)$  would be compact countable, so it contains an isolated point  $a$  and therefore by Lemma 5.3,  $a \notin P(f)$ . A contradiction. Now let  $(x_n)_{n \in \mathbb{N}} \subset \text{Fix}(f^N) \cap \omega_f(x)$  be an infinite sequence which converges to some point  $y$ . From expansivity of  $f$ , there is a constant  $c > 0$  such that for any  $n \in \mathbb{N}$ , there is an integer  $k_n$  such that  $d(f^{k_n}(x_n), f^{k_n}(x_{n+1})) > c$ . One can write  $k_n = s_n N + r_n$  with  $0 \leq r_n < N$ . We may assume, by passing to a subsequence if necessary, that  $r_n = r$ , for every  $n$ . It follows that  $d(f^r(x_n), f^r(x_{n+1})) > c$ , for every  $n$  and so letting  $n \rightarrow +\infty$ , we obtain that  $d(f^r(y), f^r(y)) = 0 \geq c$ . A contradiction.  $\square$

In [2], Artigue introduced the notion of turning point for a continuous map  $f : X \rightarrow X$  on a metric space  $(X, d)$  as follows: we say that  $x \in X$  is a spiral point of  $f$  if there is  $m \in \mathbb{N}$  such that  $\lim_{n \rightarrow +\infty} d(f^n(x), f^{n+m}(x)) = 0$ .

From [2, Proposition 3.2] and Proposition 5.2, we obtain:

**Corollary 5.4.** *If  $f : X \rightarrow X$  is an expansive continuous map on a metric space  $(X, d)$  and  $x \in X$  is a spiral point of  $f$ , then  $\omega_f(x)$  is finite.*

**Acknowledgements.** This work was supported by the research unit: “Dynamical systems and their applications”, (UR17ES21), Ministry of Higher Education and Scientific Research, Faculty of Science of Bizerte, Bizerte, Tunisia.

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