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ON TOTALLY PERIODIC ω -LIMIT SETS FOR MONOTONE MAPS ON REGULAR CURVES

AMIRA MCHAALIA

ABSTRACT. An ω -limit set of a continuous self-mapping of a compact metric space X is said to be totally periodic if all of its points are periodic. In [3] Askri and Naghmouchi proved that if f is one-to-one continuous self mapping of a regular curve, then every totally periodic ω -limit set of f is finite. This also holds whenever f is a monotone map of a local dendrite by Abdelli in [1]. In this paper we generalize these results to monotone maps on regular curves. On the other hand, we give some remarks related to expansivity and totally periodic ω -limit sets for any continuous map on compact metric space.

1. Introduction

The structure of ω -limit sets plays an important role in studying dynamical system since it represents asymptotically the state of the orbit. A class of ω -limit sets composed of periodic points called *totally periodic ω -limit* is investigated. This notion was introduced by Marzougui and Naghmouchi in [11]. A compact metric space X is said to have the ω -FTP property if for each self continuous map f of X , every totally periodic ω -limit set of a point $x \in X$ is finite. We mention that totally periodic ω -limit set appeared in the definition of ω -chaos, introduced by Li in [10]. It also appeared as ω -limit set of “spiral points” as proved in [2, Proposition 3.2] by Artigue. In zero-dimension, it was shown in [11] that, a zero dimensional compact metric space has the ω -FTP property. In one-dimension, it was shown in ([1]) that any local dendrite has the ω -FTP property when we restricted to monotone maps; that is any totally periodic ω -limit set of a monotone map on a local dendrite is finite. The later result does not hold when we remove the restriction of monotonicity; it was shown in [11] that, there is a continuous map on a dendrite having an infinite totally periodic ω -limit set. In [3], it was shown that a regular curve has the ω -FTP property when we restricted to one-to-one continuous maps. In this line, we generalize in the present paper the above results to monotone maps on regular curves.

Our main result is the following theorem.

Theorem 1.1. *Let X be a regular curve and $f : X \rightarrow X$ be a monotone map. Then any totally periodic ω -limit set of f is finite.*

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In other words, a regular curve has the ω -FTP property when we restricted to monotone maps.

Notice that Theorem 1.1 cannot be extended to other classes of systems such as rational curves; for instance, it was constructed in [13, Theorem D] an example of a homeomorphism of a rational curve (in fact a dendroid) for which some points have infinite ω -limit sets consisting only of fixed points.

On the other hand, further results are obtained with regard to expansivity, we prove that any totally periodic ω -limit set for expansive continuous map on a compact metric space X is finite (see Proposition 5.2).

2. Preliminaries

Let X be a compact metric space with metric d and let $f : X \rightarrow X$ be a continuous map. Let \mathbb{Z} , \mathbb{Z}_+ and \mathbb{N} be the sets of integers, non-negative integers and positive integers, respectively. For $n \in \mathbb{Z}_+$ denote by f^n the n -th iterate of f ; that is, $f^0 = \text{id}_X$: identity of X and $f^n = f \circ f^{n-1}$ if $n \in \mathbb{N}$. A point $x \in X$ is called periodic of period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. If $n = 1$, x is a *fixed point* of f . For any $x \in X$, the subset $\text{Orb}_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the *orbit* of x (under f). If x is a periodic point, the orbit of x is called periodic orbit. We denote by $P(f)$ and $\text{Fix}(f)$ the set of periodic points and fixed points of f , respectively. A subset $A \subset X$ is called *f -invariant* if $f(A) \subset A$. It is called *strongly f -invariant* if $f(A) = A$. We define the ω -limit set of a point $x \in X$ to be the set

$$\begin{aligned} \omega_f(x) &= \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow +\infty, \lim_{i \rightarrow +\infty} d(f^{n_i}(x), y) = 0\} \\ &= \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}. \end{aligned}$$

It is non-empty, closed and strongly f -invariant set (cf. [4, Chapter IV, Lemma 2]).

For any subset A of X , we denote by \overline{A} , $\text{Bd}(A)$ and $\text{int}(A)$ the closure, boundary and interior of A , respectively. The symbol $\text{card}(A)$ stands for the cardinality of A and $\text{diam}(A) = \sup_{x, y \in A} d(x, y)$ means the diameter of A .

We denote by $B(x, \varepsilon)$ the open ball centered at x of radius ε .

Definition 2.1 ([8], p. 131). *Let X, Y be two topological spaces. A continuous map $f : X \rightarrow Y$ is said to be monotone if for any connected subset C of Y , $f^{-1}(C)$ is connected.*

When f is closed and onto, Definition 2.1 is equivalent to that the preimage of any point by f is connected (cf. [8], p. 131). Notice that f^n is monotone for every $n \in \mathbb{N}$ when f itself is monotone.

In the definitions below, we use the terminology from Nadler [12].

- A *continuum* is a compact connected metric space.
- An *arc* I (resp. a circle) is any space homeomorphic to the compact interval $[0, 1]$ (resp. to the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$). For an arc

$I = [a, b]$, denote by $\gamma(I) = \{a, b\}$ the set of its endpoints, $[a, b) = I \setminus \{b\}$ and $(a, b) = I \setminus \gamma(I) = I \setminus \{a, b\}$.

– A space is called *degenerate* if it is a single point, otherwise it is called *non-degenerate*.

– A *dendrite* is a locally connected continuum which contains no circle.

– A *local dendrite* is a continuum such that every point of which has a dendrite neighborhood. A local dendrite is then a locally connected continuum containing only a finite number of circles (cf. [8], Theorem 4, p. 303).

– A *regular curve* is a continuum every point of which x and each open neighborhood V of x in X , there exists an open neighborhood U of x in X such that $U \subset V$ and the boundary set $\text{Bd}(U)$ of U is finite. Each regular curve is locally arcwise connected. Every sub-continuum of a regular curve is a regular curve (cf. [8] and [12]). Notice that every local dendrite is a regular curve (cf. [8], p. 303). There exists a regular curve which does not a local dendrites for example the Sierpiński triangle.

– A *hereditarily locally connected* is a continuum such that every sub-continuum of it is locally connected.

– A *Peano continuum* is a locally connected continuum. In particular, each regular curve is a Peano continuum.

– A continuum X is said to be a *dendroid* provided that X is an arcwise connected hereditarily unicoherent space; that is if $X = A \cup B$ and A, B are subcontinua of X , then $A \cap B$ is connected.

Given a compact metric space X with metric d , we denote by 2^X (resp. $C(X)$) the set of all non-empty compact subsets (resp. continua) of X .

We endow 2^X with the Hausdorff metric d_H defined as follows:

For $A, B \in 2^X$, $d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A))$, where

$d(x, M) = \inf_{y \in M} d(x, y)$ for any $x \in X$ and $M \in 2^X$. For $A = \{a\}$, we write

simply, $d_H(A, B) = d_H(a, B)$. It is well known that $(2^X, d_H)$ is a compact metric space (see [12], for more details). For any $B \in C(X)$, we denote by

$$\text{mesh}(B) = \sup\{\text{diam}(C) : C \text{ is a connected component of } B\}.$$

3. Some results

Lemma 3.1. *Let X be a compact metric space and f be a continuous self-mapping of X . Then for all $n \in \mathbb{N}$, $P(f^n) = P(f)$.*

Lemma 3.2. *Let X be a metric compact space and f a continuous self mapping of X . Then for all $x \in X$, $\omega_f(f^i(x)) = \omega_f(x)$, for all $i \in \mathbb{N}$.*

Lemma 3.3. ([4], p. 70) *Let X be a metric compact space and f a continuous self mapping of X . Then for all $N \in \mathbb{N}$ and $x \in X$, $\omega_f(x) = \cup_{i=0}^{N-1} \omega_{f^N(x)}(f^i(x))$.*

Lemma 3.4. ([11, Theorem 1.1]) *Let X be a compact metric space, f be a continuous self-mapping of X and $x \in X$. If $\omega_f(x)$ is totally periodic, then $\omega_f(x)$ has finitely many connected components that form a periodic cycle.*

Recall that a p -tuple (A_0, \dots, A_{p-1}) of subsets of X is called a periodic cycle if $f(A_0) = A_1, f(A_1) = A_2, \dots, f(A_{p-1}) = A_0$.

Proposition 3.5. ([9, Theorem 1.6]) *Let $(C_n)_{n \in \mathbb{N}}$ be a sequence of disjoint sub-continua of a regular curve X . Then $\lim_{n \rightarrow +\infty} \text{diam}(C_n) = 0$.*

Lemma 3.6. ([8, Theorem 4, p. 257]) *Let X be a Peano continuum and $\varepsilon > 0$ be a given number. Then there exists $\eta > 0$ such that for every pair of points (x, y) such that $d(x, y) < \eta$, there exists an arc I joining x and y with $\text{diam}(I) < \varepsilon$.*

Lemma 3.7. ([5, Lemma 4.2]) *If X is hereditarily locally connected, then for any sequence $(A_n)_{n \geq 0}$ in $C(X)$ that converges to $A \in C(X)$ (with respect to Hausdorff metric), we have $\lim_{n \rightarrow +\infty} \text{mesh}(A_n \setminus A) = 0$.*

Lemma 3.8. *Let X be a regular continuum. Then for each sub-continuum A of X , there is a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of open connected neighborhoods of A with finite boundary such that $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$.*

Proof. It is clear that

$$A = \bigcap_{n \in \mathbb{N}} B(A, \frac{1}{n}), \quad \text{where } B(A, \frac{1}{n}) = \{x \in X : d(x, A) < \frac{1}{n}\}.$$

Since A is compact and X is locally connected, there exists, for each $n \in \mathbb{N}$, an open connected neighborhood V_n of A in $B(A, \frac{1}{n})$ with finite

boundary such that $A = \bigcap_{n \in \mathbb{N}} V_n$. For each $n \in \mathbb{N}$, set $W_n = \bigcap_{i=1}^n V_i$. Then

$(W_n)_{n \in \mathbb{N}}$ is a decreasing sequence of open connected neighborhoods of A and $A = \bigcap_{n \in \mathbb{N}} W_n$. Moreover $\text{Bd}(W_n)$ is finite since each $\text{Bd}(V_i)$ is finite,

$1 \leq i \leq n$. Now, let $U_{n \in \mathbb{N}}$ be the connected component of W_n containing A . Then the $(U_n)_{n \in \mathbb{N}}$ is a sequence of decreasing open connected neighborhoods of A with $\text{Bd}(U_n)$ finite and we have $A = \bigcap_{n \in \mathbb{N}} \overline{U}_n$. Since X

is locally connected, U_n is open and connected. Moreover the sequence of continua $(\overline{U}_n)_{n \in \mathbb{N}}$ converges to A in $C(X)$ (with respect to Hausdorff metric). By Lemma 3.7, we have that $\lim_{n \rightarrow +\infty} \text{mesh}(\overline{U}_n \setminus A) = 0$ and

therefore $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$. □

Lemma 3.9. *Let F be a non degenerate closed subset of X . Then*

$$(1) \quad \eta(F) := \inf_{a \in F} d_H(a, F) > 0.$$

(2) *For any $a \in F$, there is $c \in F$ such that $d(a, c) \geq \eta(F)$.*

Proof. (1) Since $d(a, F) = 0$, for $a \in F$, so $d_H(a, F) = \sup_{b \in F} d(a, b)$. There exists $a_0 \in F$ and $b_0 \in F$ such that $\inf_{a \in F} d_H(a, F) = d_H(a_0, F) = d(a_0, b_0)$. If $a_0 = b_0$, so $d(a_0, b) = 0, \forall b \in F$. This implies that $F = \{a_0\}$, a

contradiction since F is non degenerate. We conclude that

$$\eta(F) = \inf_{a \in F} d_H(a, F) > 0.$$

(2) For any $a \in F$, $d_H(a, F) \geq \eta(F)$ and so there is $c \in F$ such that $d(a, c) = d_H(a, F) \geq \eta(F)$. \square

4. Proof of Theorem 1.1

Suppose the theorem does not hold. Then we could find $x \in X$ such that $\omega_f(x)$ totally periodic (that is $\omega_f(x) \subset P(f)$) and infinite.

Lemma 4.1. *Let $x \in X$ such that $\omega_f(x)$ is totally periodic. Then, there are $N \in \mathbb{N}$ and $0 \leq i \leq N - 1$ such that:*

- (1) $\text{int}_{\omega_{f^N}(f^i(x))}(\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset$.
- (2) $\omega_{f^N}(f^i(x))$ is connected.

Proof. (1) As $\omega_f(x)$ is totally periodic, so $\omega_f(x) = \bigcup_{n \in \mathbb{N}} \text{Fix}(f^n) \cap \omega_f(x)$. By the Baire property, there is $N \in \mathbb{N}$ such that

$$V := \text{int}_{\omega_f(x)}(\text{Fix}(f^N) \cap \omega_f(x)) \neq \emptyset.$$

As

$$\omega_f(x) = \bigcup_{i=0}^{N-1} \omega_{f^N}(f^i(x)) \quad (\text{Lemma 3.3}),$$

so there exists $0 \leq i \leq N - 1$ such that $V \cap \omega_{f^N}(f^i(x)) \neq \emptyset$.

As $V \cap \omega_{f^N}(f^i(x)) \subset \text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))$, thus

$$\text{int}_{\omega_{f^N}(f^i(x))}(\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset.$$

(2) Set $g = f^N$ and $y = f^i(x)$. From (1), $\text{int}_{\omega_g(y)}(\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$. We have $\omega_g(y) \subset \omega_f(y) = \omega_f(x)$ (Lemma 3.2). As $P(f) = P(g)$ (Lemma 3.1),

so $\omega_g(y) \subset P(g)$. Following Lemma 3.4, $\omega_g(y) = \bigcup_{i=0}^{m-1} C_i$, for some $m \in \mathbb{N}$,

where the $(C_i)_{0 \leq i \leq m-1}$ are the connected components of $\omega_g(y)$ with $g(C_i) = C_{i+1}$ for every $i \in \{0, \dots, m-1\}$. As $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$, so there is $a \in \omega_g(y) \cap \text{Fix}(g)$. Therefore $a \in C_i$ for every $i \in \{0, \dots, m-1\}$, which implies that $\omega_g(y) = C_0$. In result $\omega_g(y)$ is connected. \square

Set $g = f^N$ and $y = f^i(x)$. From Lemma 4.1, we have the following properties.

- $\omega_g(y) \subset P(g)$.
- $\text{int}_{\omega_g(y)}(\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$.
- $\omega_g(y)$ is a regular curve (since $\omega_g(y)$ is a sub-continuum of X).

It follows that we can find an open connected set $O \subset \text{Fix}(g) \cap \omega_g(y)$. For the sequel, we let

$$S_a = \bigcup_{i \geq 0} g^{-i}(a), \text{ for any } a \in \overline{O}.$$

Lemma 4.2. *Let U be an open neighborhood of $\omega_g(y)$ in X with finite boundary. Then there exists an open connected subset V of $\omega_g(y)$ such that $\overline{V} \subset O$ and $S_a \subset U$, for all $a \in \overline{V}$.*

Proof. First, the sets $(S_a)_{a \in \overline{O}}$ are pairwise disjoint connected sets; for $a \neq b \in \overline{O}$, we have $S_a \cap S_b = \emptyset$; indeed otherwise, there is $c \in S_a \cap S_b$, so $g^j(c) = a$ and $g^i(c) = b$ for some $j > i \in \mathbb{N}$. As $a, b \in \text{Fix}(g)$, $a = g^{j-i}(g^i(c)) = g^{j-i}(b) = b$, a contradiction. By monotonicity of g , S_a is connected. Set $\text{card}(\text{Bd}(U)) = k$. So there are at most k points $a_1, a_2, \dots, a_k \in \overline{O}$ such that $S_{a_i} \not\subset U$ for every $i \in \{1, \dots, k\}$: Otherwise, there are at least $k+1$ points $a_1, a_2, \dots, a_{k+1} \in \overline{O}$ such that $S_{a_i} \not\subset U$ for all $1 \leq i \leq k+1$. As $a_i \in S_{a_i} \cap U$ and S_{a_i} is connected, so S_{a_i} meets $\text{Bd}(U)$ for every $1 \leq i \leq k+1$. Hence $\text{card}(\text{Bd}(U)) \geq k+1$, a contradiction. As $O \setminus \{a_1, a_2, \dots, a_k\}$ is open in $\omega_g(y)$ and non-empty (since $\omega_g(y)$ is infinite), so we can find a non-empty open connected subset V of $\omega_g(y)$ such that $\overline{V} \subset O \setminus \{a_1, a_2, \dots, a_k\}$. This implies that for all $a \in \overline{V}$, $S_a \subset U$. The proof is complete. \square

Step 1. For $F = \overline{O}$, we denote by $\eta := \eta(F) > 0$ be given as in Lemma 3.9. By Lemma 3.8, there exists a sequence of open connected neighborhoods $(U_n)_{n \in \mathbb{N}}$ of $\omega_g(y)$ in X with finite boundary such that $\text{mesh}(U_n \setminus \omega_g(y)) < \frac{\eta}{2}$. We let $U := U_N$ and set $\text{card}(\text{Bd}(U)) = k$. From Lemma 4.2, there exists an open connected subset $V \subset O$ of $\omega_g(y)$ such that for all $a \in \overline{V}$, $S_a \subset U$. Let $a \in V$. As $\{a\} \subsetneq \omega_g(y)$, there is an open connected neighborhood W of a in X such that $\overline{W} \cap \omega_g(y) \subset \overline{V}$ and $\omega_g(y) \setminus \overline{W} \neq \emptyset$. We can find two increasing sequences of integers $(n_i)_{i \in \mathbb{N}}, (t_i)_{i \in \mathbb{N}}$ with the following properties:

- (1) $g^{n_i}(y) \in W$ with $\lim_{i \rightarrow +\infty} g^{n_i}(y) = a$,
- (2) $g^{n_i+j}(y) \in W$ for each $j = 0, \dots, t_i$, with $t_i \geq k$ and $t_i \geq i, \forall i \in \mathbb{N}$,
- (3) $g^{n_i+t_i+1}(y) \in X \setminus W$ and since b is a fixed point of g ,

$$\lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y) = b = \lim_{i \rightarrow +\infty} g^{n_i+t_i+1}(y) \in \overline{W} \setminus W.$$

Observe that $a \in W \cap \omega_g(y)$ and $b \in (\overline{W} \setminus W) \cap \omega_g(y)$. So $a \neq b$.

We let $\mathbb{N}_k = \bigcup_{i \in \mathbb{N}} \{n_i, \dots, n_i + t_i - k\}$. Since $\lim_{i \rightarrow +\infty} t_i = +\infty$, \mathbb{N}_k is infinite.

By definition, $g^m(y) \in W$, for each $m \in \mathbb{N}_k$ and

$\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} d(g^m(y), \overline{W} \cap \omega_g(y)) = 0$. For each $m \in \mathbb{N}_k$, there is

$b_m \in \overline{W} \cap \omega_g(y)$ such that $d(g^m(y), \overline{W} \cap \omega_g(y)) = d(g^m(y), b_m)$. Since \overline{W} is a sub-continuum of X , so it is a Peano continuum. By Lemma 3.6, one can find an arc $T_m = [b_m, g^m(y)] \subset \overline{W}$ such that $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(T_m) = 0$.

Since $\omega_g(y)$ is infinite and $\omega_g(y) \subset P(g)$, so $g^m(y) \notin \omega_g(y)$, for every $m \in \mathbb{N}_k$.

Let $I_m, m \in \mathbb{N}_k$, be the connected component of $T_m \setminus \omega_g(y)$ containing $g^m(y)$. Then one can write $I_m = [g^m(y), a_m)$ with $\bar{I}_m \cap \omega_g(y) = \{a_m\}$. We see that $a_m \in \bar{W} \cap \omega_g(y) \subset \bar{V} \subset \bar{O} \subset \text{Fix}(g) \cap \omega_g(y)$, for any $m \in \mathbb{N}_k$.

Claim 1. Under the notations above, there exists $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}_k, m \geq N$, the family $F_m = \{I_{m+j} : 0 \leq j \leq k, m+j \in \mathbb{N}_k\}$ is not pairwise disjoint.

Proof. Suppose that, there exists an increasing sequence of positive integers $(m_i)_{i \geq 1}$ such that the family F_{m_i} is pairwise disjoint. By passing to a subsequence of $(m_i)_{i \geq 1}$, one can assume that $(\bar{I}_{m_i})_{i \geq 1}$ converges to $I \in C(X)$ (in Hausdorff metric). As $\lim_{i \rightarrow +\infty} \text{diam}(T_{m_i}) = 0$, then

$\lim_{i \rightarrow +\infty} \text{diam}(\bar{I}_{m_i}) = 0 = \text{diam}(I)$, so I is degenerate i.e. $I = \{c\}$. As

$g^{m_i}(y) \in I_{m_i} \subset W$ and $\lim_{i \rightarrow +\infty} d_H(c, \bar{I}_{m_i}) = 0$, so

$\lim_{i \rightarrow +\infty} g^{m_i}(y) = c \in \bar{W} \cap \omega_g(y)$. Since $\bar{W} \cap \omega_g(y) \subset \bar{V} \subset \bar{O} \subset \text{Fix}(g)$, so for all $0 \leq j \leq k$, $\lim_{i \rightarrow +\infty} g^{m_i+j}(y) = g^j(c) = c$. In result, we have that

$\lim_{i \rightarrow +\infty, m_i+j \in \mathbb{N}_k} \bar{I}_{m_i+j} = \{c\}$ (in Hausdorff metric). Now apply Lemma 3.9 for

$c \in \bar{O}$: there is $c' \in \bar{O}$ such that $d(c, c') \geq \eta$. Then we can find two open neighborhoods O_c and $O_{c'}$ of c and c' in U , respectively such that $d(O_c, O_{c'}) > \frac{\eta}{2}$. As $c' \in \bar{O} \subset \omega_g(y) \cap \text{Fix}(g)$, so there is $m \geq 1$ such that $\{g^m(y), g^{m+1}(y), \dots, g^{m+k}(y)\} \subset O_{c'}$. Let $m_p > m$ such that

$\bigcup_{m_p+j \in \mathbb{N}_k} \bar{I}_{m_p+j} \subset O_c$ and set

$$C_j = g^{-(m_p-m)}(\bar{I}_{m_p+j}), \quad m_p + j \in \mathbb{N}_k.$$

By monotonicity of g , C_j is a sub-continuum of X . Then there is an arc $J_j = [g^{m_p+j}(y), a_{m_p+j}] \subset C_j$. Each J_j meets $O_{c'}$ in at least $g^{m_p+j}(y)$ and meets O_c in at least $a_{m_p+j} \in \text{Fix}(g) \cap \omega_g(y)$. Let us show that

$J_j \setminus \omega_g(y) = J_j \setminus \{a_{m_p+j}\}$:

It is clear that $J_j \setminus \omega_g(y) \subset J_j \setminus \{a_{m_p+j}\}$. Conversely, if $z \in J_j \cap \omega_g(y)$, then $g^{m_p-m}(z) \in \bar{I}_{m_p+j} \cap \omega_g(y) = \{a_{m_p+j}\}$. As $a_{m_p+j} \in \text{Fix}(g)$, then

$g^{m_p-m}(a_{m_p+j}) = g^{m_p-m}(z)$. As $\omega_g(y) \subset P(g)$, then the restriction $g|_{\omega_g(y)}^{m_p-m}$ is a homeomorphism and therefore $z = a_{m_p+j}$. In result,

$J_j \setminus \{a_{m_p+j}\} \subset J_j \setminus \omega_g(y)$ and hence the equality holds.

We conclude that $J_j \setminus \{a_{m_p+j}\}$ is connected and hence $\text{diam}(J_j) > \frac{\eta}{2}$.

Moreover we have that $J_j \not\subset U$ (since otherwise, $J_j \setminus \{a_{m_p+j}\} \subset U \setminus \omega_g(y)$ and so $\text{diam}(J_j \setminus \{a_{m_p+j}\}) \leq \text{mesh}(U \setminus \omega_g(y)) < \frac{\eta}{2}$. A contradiction).

Therefore $C_j \cap \text{Bd}(U) \neq \emptyset$, for each $m_p + j \in \mathbb{N}_k$.

Let us show that $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$ for all $j \neq j'$. We have

$\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'} \subset \{a_{m_p+j}\}$ for $j \neq j'$ (since $a_{m_p+j} = a_{m_p+j'}$). Then

$C_j \cap C_{j'} = g^{-(m_p-m)}(\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'}) \subset g^{-(m_p-m)}(a_{m_p+j}) \subset S_{a_{m_p+j}}$. As

$a_{m_p+j} \in \bar{V}$, so by Lemma 4.2, $S_{a_{m_p+j}} \subset U$. Hence $C_j \cap C_{j'} \subset U$ and so $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$. We conclude that $\text{Bd}(U)$ has at least $k+1$ points, which contradicts $\text{Bd}(U) = k$. This completes the proof of Claim 1. \square

Step 2. From Claim 1, there is some $m \in \mathbb{N}_k$ with $m+j_1(m), m+j_2(m) \in \mathbb{N}_k$ such that $I_{m+j_1(m)} \cap I_{m+j_2(m)} \neq \emptyset$. Hence $I_{m+j_1(m)} \cup I_{m+j_2(m)}$ is arcwise connected and so we can find an arc

$$P_m = [g^{m+j_1(m)}(y), g^{m+j_2(m)}(y)] \subset I_{m+j_1(m)} \cup I_{m+j_2(m)}.$$

We have that $P_m \cap \omega_g(y) = \emptyset$. Since $P_m \subset I_{m+j_1(m)} \cup I_{m+j_2(m)}$, we have $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(P_m) = 0$.

Claim 2. Let $B_m = g^{-j_1(m)}(P_m), m \in \mathbb{N}_k$. Then we have

$$\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) = 0.$$

Proof. Suppose that $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) > 0$. Then there is $\delta > 0$ and a sequence of positive integers $(m_l)_l \subset \mathbb{N}_k$ such that $\text{diam}(B_{m_l}) > \delta$, for all $l \in \mathbb{N}$. Recall $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(I_m) = 0$, where $I_m = [g^m(y), a_m)$ and $\lim_{l \rightarrow +\infty, m_l \in \mathbb{N}_k} \text{diam}(P_{m_l}) = 0$. By passing to a subsequence if necessary, one can assume that $(P_{m_l})_{m_l \in \mathbb{N}_k}$ converges necessarily to a degenerate set, say $\{c\}$. Hence, $\lim_{l \rightarrow +\infty, m_l \in \mathbb{N}_k} g^{m_l+j_1(m_l)}(y) = c \in \omega_g(y)$. As $P_{m_l} \cap \omega_g(y) = \emptyset$ for all $l \in \mathbb{N}$, so $c \notin P_{m_l}$. Then we can construct a subsequence $(P_{m_{l_q}})_{q \geq 0, m_{l_q} \in \mathbb{N}_k}$ which is pairwise disjoint: indeed, as $(P_{m_l})_{m_l \in \mathbb{N}_k}$ converges to $\{c\}$, then for each neighborhood U_0 of c , there is $n_0 \in \mathbb{N}$ such that for all $l \geq n_0$, $P_{m_l} \subset U_0$. Since $c \notin P_{m_l}$, for each $m_l \in \mathbb{N}_k$, so we can find an open neighborhood $U_1 \subset U_0$ of c such that $P_{m_{n_0}} \not\subset U_1$. Moreover, since $(P_{m_l})_{m_l \in \mathbb{N}_k}$ converges to $\{c\}$, so there is $n_1 \in \mathbb{N}$ such that for all $l \geq n_1$, $P_{m_l} \in U_1$, we obtain $P_{m_{n_0}} \cap P_{m_{n_1}} = \emptyset$. We continue in this fashion obtaining a subsequence $(P_{m_{l_q}})_{q \geq 0, m_{l_q} \in \mathbb{N}_k}$ which is pairwise disjoint. Since $0 \leq j_1(m_{l_q}) \leq k$, we can assume that $j_1(m_{l_q}) = j_0$, for all $q \in \mathbb{N}$. Then the $B_{m_{l_q}} = g^{-j_0}(P_{m_{l_q}}), q \in \mathbb{N}$, are connected and pairwise disjoint sets. Hence, by Proposition 3.5, $\lim_{q \rightarrow +\infty, m_{l_q} \in \mathbb{N}_k} \text{diam}(B_{m_{l_q}}) = 0$. This contradicts the fact that $\text{diam}(B_{m_l}) > \delta$, for all $l \in \mathbb{N}, m_l \in \mathbb{N}_k$. This completes the proof. \square

End of the proof of Theorem 1.1. Recall that $\omega_g(y) \subset P(g)$ and $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$ is infinite. From the notation given in Step 1, we have that $\lim_{i \rightarrow +\infty} g^{n_i}(y) = a$ and $\lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y) = b$ with $a \neq b \in \omega_g(y) \cap \text{Fix}(g)$.

From Claim 2, $B_m = g^{-j_1(m)}(P_m)$ is connected for each $m \in \mathbb{N}_k$ and contains the points $g^m(y)$ and $g^{m+j_2(m)-j_1(m)}(y)$. Moreover, for each $m \in \mathbb{N}_k$, we have $B_m \cap \omega_g(y) = \emptyset$; otherwise, there is $m \in \mathbb{N}_k$ such that $z \in B_m \cap \omega_g(y)$ and so $g^{j_1(m)}(z) \in P_m \cap \omega_g(y)$, this leads to a contradiction

with $P_m \cap \omega_g(y) = \emptyset$. By Claim 2, $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) = 0$. So let W_a, W_b be two open neighborhoods of a and b , respectively such that $d(W_a, W_b) > \frac{d(a,b)}{2}$. By Lemma 3.8, there exists a connected neighborhood $U' := U_{N'}$ of $\omega_g(y)$ such that $\text{mesh}(U' \setminus \omega_g(y)) < \frac{d(a,b)}{2}$. Since $\lim_{m \rightarrow +\infty} d(g^m(y), \omega_g(y)) = 0$, so there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $m \in \mathbb{N}_k$, $B_m \subset U'$. Thus there exists $q \in \mathbb{N}$ such that for $n_q \geq m_0$, $g^{n_q}(y) \in W_a$ and $\{g^{n_q+t_q-k}(y), \dots, g^{n_q+t_q}(y)\} \subset W_b$.

For each $m \in \mathbb{N}_k$, denote by

- $s(m) = m + j_2(m) - j_1(m)$, where $m + j_1(m), m + j_2(m) \in \mathbb{N}_k$,
- $p_{n_q} = \inf\{i \in \mathbb{N} : s^i(n_q) \in \{n_q + t_q - k + 1, \dots, n_q + t_q\}\}$,
- $K_q = B_{n_q} \cup B_{s(n_q)} \cup \dots \cup B_{s^{p_{n_q}-1}(n_q)}$.

It is easy to see that K_q is connected. Since for all $0 \leq i < p_{n_q}$, $s^i(n_q) \in \mathbb{N}_k$, so $B_{s^i(n_q)} \cap \omega_g(y) = \emptyset$ and then $K_q \cap \omega_g(y) = \emptyset$. Moreover as K_q meets W_a in at least $g^{n_q}(y)$ and meets W_b in at least $g^{s^{p_{n_q}}(y)}$, thus $\text{diam}(K_q) > \frac{d(a,b)}{2}$. Since for all $n_q \geq m_0$, $B_{n_q} \subset U'$, $K_q \subset U'$ and $K_q \cap \omega_g(y) = \emptyset$, thus $K_q \subset U' \setminus \omega_g(y)$. Therefore $\text{diam}(K_q) \leq \text{mesh}(U' \setminus \omega_g(y)) < \frac{d(a,b)}{2}$. A contradiction.

5. Expansivity and totally periodic ω -limit sets

In this section, we consider the notion of expansivity of continuous maps on compact metric space.

Definition 5.1. [14] *A continuous map $f : X \rightarrow X$ on a compact metric space X is said to be expansive if there is real constant $c > 0$ such that for any $x, y \in X$ and $x \neq y$, there is an integer $n \in \mathbb{Z}_+$ such that $d(f^n(x), f^n(y)) > c$.*

It is well known that the Cantor set and the torus \mathbb{T}^2 admit expansive homeomorphisms [15]. Kawamura showed [7] that a Peano continuum which contains a free arc admits no expansive homeomorphisms. In particular, there is no expansive homeomorphism on an arc and a circle. In [6], Kato showed that no dendroid admits an expansive homeomorphism. With respect to expansivity, we show the following.

Proposition 5.2. *If $f : X \rightarrow X$ is an expansive continuous map on a compact metric space X , then any totally periodic ω -limit set is finite.*

Lemma 5.3. ([4, Lemma 4]) *Let (X, f) be a dynamical system and let $x \in X$ such that $\omega_f(x)$ is infinite. If z is an isolated point in $\omega_f(x)$, then it is not periodic.*

Proof of Proposition 5.2. Let $x \in X$ such that $\omega_f(x) \subset P(f)$ and assume that $\omega_f(x)$ is infinite. We have that $\omega_f(x) = \bigcup_{n \in \mathbb{N}} \text{Fix}(f^n) \cap \omega_f(x)$. Then we claim that there exists $N \in \mathbb{N}$ such that $\text{Fix}(f^N) \cap \omega_f(x)$ is infinite.

Otherwise, $\omega_f(x)$ would be compact countable, so it contains an isolated point a and therefore by Lemma 5.3, $a \notin P(f)$. A contradiction. Now let $(x_n)_{n \in \mathbb{N}} \subset \text{Fix}(f^N) \cap \omega_f(x)$ be an infinite sequence which converges to some point y . From expansivity of f , there is a constant $c > 0$ such that for any $n \in \mathbb{N}$, there is an integer k_n such that $d(f^{k_n}(x_n), f^{k_n}(x_{n+1})) > c$. One can write $k_n = s_n N + r_n$ with $0 \leq r_n < N$. We may assume, by passing to a subsequence if necessary, that $r_n = r$, for every n . It follows that $d(f^r(x_n), f^r(x_{n+1})) > c$, for every n and so letting $n \rightarrow +\infty$, we obtain that $d(f^r(y), f^r(y)) = 0 \geq c$. A contradiction. \square

In [2], Artigue introduced the notion of turning point for a continuous map $f : X \rightarrow X$ on a metric space (X, d) as follows: we say that $x \in X$ is a spiral point of f if there is $m \in \mathbb{N}$ such that $\lim_{n \rightarrow +\infty} d(f^n(x), f^{n+m}(x)) = 0$.

From [2, Proposition 3.2] and Proposition 5.2, we obtain:

Corollary 5.4. *If $f : X \rightarrow X$ is an expansive continuous map on a metric space (X, d) and $x \in X$ is a spiral point of f , then $\omega_f(x)$ is finite.*

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