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# ON TOTALLY PERIODIC $\omega$ -LIMIT SETS FOR MONOTONE MAPS ON REGULAR CURVES

AMIRA MCHAALIA

ABSTRACT. An  $\omega$ -limit set of a continuous self-mapping of a compact metric space  $X$  is said to be totally periodic if all of its points are periodic. In [2], Askri and Naghmouchi proved that if  $f$  is a homeomorphism on regular curve, then every totally periodic  $\omega$ -limit set is finite. In [1], Abdelli proved that for every monotone map on a local dendrite every totally periodic  $\omega$ -limit set is finite. In this paper we generalize these results to monotone maps on regular curves.

## 1. Introduction

The structure of  $\omega$ -limit sets plays an important role in studying dynamical system since it represents asymptotically the state of the orbit. In [7], it was introduced the notion of  $\omega$ -FTP property. An  $\omega$ -limit set is said to be totally periodic if it is composed of periodic points. A compact metric space  $X$  has the  $\omega$ -FTP property provided that for each self-continuous map  $f$  of  $X$ , every totally periodic  $\omega$ -limit set of  $f$  is finite. In [1], Abdelli proved that for a monotone map on local dendrite any totally periodic  $\omega$ -limit set is finite. However, it was shown in [7] that, there is a map on a dendrite having an infinite totally periodic  $\omega$ -limit sets. In [2], it was shown that any totally periodic  $\omega$ -limit set of a homeomorphism on a regular curve is finite. In this paper, we generalize the result to monotone maps on regular curves.

Our main result is the following theorem:

**Theorem 1.1.** *Let  $X$  be a regular curve and  $f : X \rightarrow X$  be a monotone map. Then any totally periodic  $\omega$ -limit set of  $f$  is finite.*

In other words, a monotone map on a regular curve has the  $\omega$ -FTP-property.

## 2. Preliminaries

Let  $X$  be a compact metric space with metric  $d$  and let  $f : X \rightarrow X$  be a continuous map. Let  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$  be the sets of integers, non-negative integers and positive integers, respectively. For  $n \in \mathbb{Z}_+$  denote by  $f^n$  the  $n$ -th iterate of  $f$ ; that is,  $f^0 = \text{identity}$  and  $f^n = f \circ f^{n-1}$  if  $n \in \mathbb{N}$ .

The point  $x \in X$  is called periodic of period  $n \in \mathbb{N}$  if  $f^n(x) = x$  and

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$f^i(x) \neq x$  for  $1 \leq i \leq n-1$ ; if  $n=1$ ,  $x$  is called a *fixed point* of  $f$  i.e.  $f(x) = x$ . The subset  $\text{Orb}_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$  is called the *orbit* of  $x$  (under  $f$ ). If  $x$  is periodic the orbit of  $x$  called periodic orbit. We denote respectively by  $P(f)$  and  $\text{Fix}(f)$  the set of periodic points and fixed point of  $f$  respectively.

A subset  $A \subset X$  is called *f-invariant* (resp. strongly *f-invariant*) if  $f(A) \subset A$  (resp  $f(A) = A$ ). The closure (respectively the boundary) of  $A$  is denoted by  $\overline{A}$  (respectively  $Bd(A)$ ). We denote by  $\text{int}(A)$  the interior of  $A$ . For  $A \subset B$ , we denote by  $\text{int}_B(A)$  the interior of  $A$  in  $B$ . The diameter of  $A$  is denoted by  $\text{diam}(A) = \sup_{x,y \in A} d(x,y)$ . We denote by  $B(x, \epsilon)$  the open ball centered at  $x$  of radius  $\epsilon$ .

We define the  $\omega$ -limit set of a point  $x \in X$  to be the set:

$$\begin{aligned} \omega_f(x) &= \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow +\infty, \lim_{i \rightarrow \infty} d(f^{n_i}(x), y) = 0\} \\ &= \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}. \end{aligned}$$

It is non-empty, closed and strongly invariant set (in [3], Chapter IV, Lemma 2).

**Definition 2.1** ([5], p. 131). *Let  $X, Y$  be two topological spaces. A map  $f : X \rightarrow Y$  is said to be monotone if for any connected subset  $C$  of  $Y$ ,  $f^{-1}(C)$  is connected.*

When  $f$  is closed and onto, Definition 2.1 is equivalent to that the preimage of any point by  $f$  is connected (cf. [5], p. 131). Notice that  $f^n$  is monotone for every  $n \in \mathbb{N}$  when  $f$  itself is monotone.

In the definitions below, we use the terminology from Nadler [8]:

- A *continuum* is a compact connected metric space.
- An *arc*  $I$  is any space homeomorphic to the compact interval  $[0, 1]$ . For an arc  $I = [a, b]$ , denote by  $[a, b) = I \setminus \{b\}$  and  $(a, b) = I \setminus \gamma(I) = I \setminus \{a, b\}$  where  $\gamma(I) = \{a, b\}$  the set of its endpoints.
- A space is called *degenerate* if it is a single point, otherwise it is *non-degenerate*.
- A *dendrite* is a locally connected continuum which contains no circle.
- A *local dendrite*  $X$  is a continuum such that every point of which has a dendrite neighborhood. A local dendrite is then a locally connected continuum containing only a finite number of circles ([5], Theorem 4, p. 303).
- A *regular curve* is a continuum every point of which  $x$  and each open neighborhood  $V$  of  $x$  in  $X$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $U \subset V$  and the boundary set  $Bd(U)$  of  $U$  is finite. Each regular curve is locally arcwise connected ( see [5] and [8]). Notice that every local dendrite is a regular curve (cf. [5], p. 303).
- A *hereditarily locally connected* is a continuum  $X$  such that every subcontinuum of  $X$  is locally connected.
- A *Peano continuum* is a locally connected continuum. Each regular curve is a Peano continuum.

Given a compact metric space  $X$  with metric  $d$ , we denote by  $2^X$  (resp.  $C(X)$ ) the set of all non-empty compact subsets (resp. continua) of  $X$ .

We endow  $2^X$  with the Hausdorff metric  $d_H$  defined as follows:

For any  $A, B \in 2^X$ ,  $d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A))$ , where

$$d(x, A) = \inf_{y \in A} d(x, y) \text{ for any } x \in X \text{ and } A \in 2^X.$$

It is known that  $(2^X, d_H)$  is a compact metric space (for more details, see [8]). For any  $B \in C(X)$ , we denote by

$$\text{mesh}(B) = \sup\{\text{diam}(C) : C \text{ is a connected component of } B\}.$$

### 3. Some results

**Lemma 3.1.** *Let  $X$  be a compact metric space and  $f$  be a continuous self-mapping of  $X$ . Then for all  $n \in \mathbb{N}$ ,  $P(f^n) = P(f)$ .*

**Lemma 3.2.** *Let  $X$  be a metric compact space and  $f$  a continuous self-mapping of  $X$ . Then for all  $x \in X$ ,  $\omega_f(f^i(x)) = \omega_f(x)$ , for all  $i \in \mathbb{N}$ .*

**Lemma 3.3.** ([3], p. 70) *Let  $X$  be a metric compact space and  $f$  a continuous self-mapping of  $X$ . Then for all  $N \in \mathbb{N}$  and  $x \in X$ ,  $\omega_f(x) = \bigcup_{i=0}^{N-1} \omega_{f^N(x)}(f^i(x))$ .*

**Lemma 3.4.** ([7], theorem 1.1) *Let  $X$  be a compact metric space,  $f$  be a continuous self-mapping of  $X$  and  $x \in X$ . If  $\omega_f(x)$  is totally periodic then  $\omega_f(x)$  has finitely many connected components that form a periodic cycle.*

**Proposition 3.5.** ([6], theorem 1.6) *Let  $(C_n)_n$  be a sequence of disjoint sub-continua of a regular curve  $X$ . Then  $\lim_{n \rightarrow +\infty} \text{diam}(C_n) = 0$ .*

**Lemma 3.6.** ([5], theorem 4, p. 257) *Let  $X$  be a Peano continuum and  $\epsilon > 0$  be a given number. Then there exists  $\eta > 0$  such that for every pair of points  $(x, y)$  such that  $d(x, y) < \eta$ , there exists an arc  $I$  joining  $a$  and  $b$  with  $\text{diam}(I) < \epsilon$ .*

**Lemma 3.7.** ([4], Lemma 4.2) *If  $X$  is hereditarily locally connected, then for any sequence  $(A_n)_{n \geq 0}$  in  $C(X)$  that converges to  $A \in C(X)$  (with respect to Hausdorff metric), then we have  $\lim_{n \rightarrow +\infty} \text{mesh}(A_n \setminus A) = 0$ .*

**Lemma 3.8.** *Let  $X$  be a regular continuum. Then for each subcontinuum  $A$  of  $X$ , there is a decreasing sequence  $(U_n)_{n \in \mathbb{N}}$  of open connected neighborhoods of  $A$  with finite boundary such that  $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$ .*

*Proof.* It is clear that  $A = \bigcap_{n \in \mathbb{N}} B(A, \frac{1}{n})$ , where  $B(A, \frac{1}{n}) = \{x \in X, d(x, A) < \frac{1}{n}\}$ . Since  $A$  is compact and  $X$  is locally connected, for each  $n \in \mathbb{N}$ , there

exists an open connected neighborhood  $V_n$  of  $A$  in  $B(A, \frac{1}{n})$  with finite boundary such that  $A = \bigcap_{n \in \mathbb{N}} V_n$ . For each  $n \in \mathbb{N}$ , set  $W_n = \bigcap_{i=1}^n V_i$ . Then  $(W_n)_n$  is a decreasing sequence of open connected neighborhoods of  $A$  and  $A = \bigcap_{n \in \mathbb{N}} W_n$ . Moreover  $Bd(W_n)$  is finite since each  $Bd(V_i)$  is,  $1 \leq i \leq n$ . Now, let  $U_n$  be the connected component of  $W_n$  containing  $A$ . Then the  $(U_n)_n$  is a sequence of decreasing open connected neighborhoods of  $A$  with  $Bd(U_n)$  finite and we have  $A = \bigcap_{n \in \mathbb{N}} \overline{U_n}$ . Since  $X$  is locally connected,  $U_n$  is open and connected. Moreover the sequence of continua  $(\overline{U_n})_{n \in \mathbb{N}}$  converges to  $A$  in  $C(X)$  (with respect to Hausdorff metric). By lemma 3.7, we have that  $\lim_{n \rightarrow +\infty} \text{mesh}(\overline{U_n} \setminus A) = 0$  and therefore  $\lim_{n \rightarrow +\infty} \text{mesh}(U_n \setminus A) = 0$ .  $\square$

**Lemma 3.9.** *Let  $F$  be a non degenerate closed subset of  $X$ . Then  $\eta(F) := \inf_{a \in F} d_H(a, F) > 0$ .*

*Proof.* We have  $d_H(a, F) = \sup_{b \in F} d(a, b)$ . Assume that  $\eta(F) = 0$ ; then there exists a sequence  $(a_n)_n$  in  $F$  such that  $\lim_{n \rightarrow \infty} d(a_n, b) = 0$  for all  $b \in F$ . This implies that  $F$  is degenerate, a contradiction.  $\square$

From lemma 3.9, it follows that: for any  $a \in F$  there is  $c \in F$  such that  $d(a, c) \geq \eta(F)$ .

#### 4. Proof of theorem 1.2

Assume that  $\omega_f(x)$  is totally periodic, that is  $\omega_f(x) \subset P(f)$ . Suppose contrary to theorem 1.2 that  $\omega_f(x)$  is infinite.

**Lemma 4.1.** *Let  $x \in X$  such that  $\omega_f(x)$  is totally periodic. Then, there is  $N \in \mathbb{N}$  such that for all  $0 \leq i \leq N - 1$ :*

- (1)  $\text{int}_{\omega_{f^N}(f^i(x))} (\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset$ .
- (2)  $\omega_{f^N}(f^i(x))$  is connected.

*Proof.* (1) As  $\omega_f(x)$  is totally periodic, so  $\omega_f(x) = \bigcup_{n \in \mathbb{N}} \text{Fix}(f^n) \cap \omega_f(x)$ . By the Baire property, there is  $N \in \mathbb{N}$  such that  $V := \text{int}_{\omega_f(x)} (\text{Fix}(f^N) \cap \omega_f(x)) \neq \emptyset$ . As  $\omega_f(x) = \bigcup_{i=0}^{N-1} \omega_{f^N}(f^i(x))$  (lemma 3.3), so there exists  $0 \leq i \leq N - 1$  such that  $V \cap \omega_{f^N}(f^i(x)) \neq \emptyset$ . As  $V \cap \omega_{f^N}(f^i(x)) \subset \text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))$ , thus  $\text{int}_{\omega_{f^N}(f^i(x))} (\text{Fix}(f^N) \cap \omega_{f^N}(f^i(x))) \neq \emptyset$ .

(2) Set  $g = f^N$  and  $y = f^i(x)$ . From (1),  $\text{int}_{\omega_g(y)} (\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$ . We have  $\omega_g(y) \subset \omega_f(y) \subset \omega_f(x) \subset P(f)$  (lemma 3.2). As  $P(f) = P(g)$  (lemma 3.1), so  $\omega_g(y) \subset P(g)$ . Following lemma 3.4,  $\omega_g(y) = \bigcup_{i=0}^{m-1} C_i$  for

some  $m \in \mathbb{N}$  where the  $(C_i)_{0 \leq i \leq m-1}$  are the connected components of  $\omega_g(y)$  with  $g(C_i) = C_{i+1}$  for every  $i \in \{0, \dots, m-1\}$ . As  $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$ , there is  $a \in \omega_g(y) \cap \text{Fix}(g)$ . Therefore  $a \in C_i$  for every  $i \in \{0, \dots, m-1\}$ , which implies that  $\omega_g(y) = C_0$ . In result  $\omega_g(y)$  is connected.  $\square$

Following lemma 4.1, we have  $\text{int}_{\omega_g(y)}(\text{Fix}(g) \cap \omega_g(y)) \neq \emptyset$  and  $\omega_g(y)$  is connected. We can find an open connected set  $O \subset \text{Fix}(g) \cap \omega_g(y)$ . Moreover, since  $\omega_f(x)$  is infinite, so  $\omega_g(y)$  is also.

For the sequel, set  $S_a = \bigcup_{i \geq 0} g^{-i}(a)$ , for any  $a \in \overline{O}$ .

**Lemma 4.2.** *Let  $U$  be an open neighborhood of  $\omega_g(y)$  in  $X$  with finite boundary. Then there exists an open connected subset  $U_1$  such that  $\overline{U_1} \subset O$  and for all  $a \in \overline{U_1}$ ,  $S_a \subset U$ .*

*Proof.* First for any  $a \neq b \in \overline{O}$ , we have  $S_a \cap S_b = \emptyset$ ; indeed otherwise, there is  $c \in S_a \cap S_b$  such that  $g^j(c) = a$  and  $g^i(c) = b$  for some  $j \neq i \in \mathbb{N}$ , so  $a = g^{j-i}(g^i(c)) = g^{j-i}(b) = b$ , a contradiction. By monotonicity of  $g$ , the  $(S_a)_{a \in \overline{O}}$  are pairwise disjoint connected sets. Set  $\text{card}(\text{Bd}(U)) = k$ . So there is at most  $k$  point  $a_1, a_2, \dots, a_k \in \overline{O}$  such that  $S_{a_i} \not\subset U$  for every  $i \in \{1, \dots, k\}$ : Indeed, otherwise, there are at least  $(k+1)$  point  $a_1, a_2, \dots, a_{k+1} \in \overline{O}$  such that  $S_{a_i} \not\subset U$  for all  $1 \leq i \leq k+1$ . As  $a_i \in S_{a_i} \cap U$ , so  $S_{a_i}$  meets  $\text{Bd}(U)$  for more than  $k$  points, this contradicts  $\text{card}(\text{Bd}(U)) = k$ . As  $O \setminus \{a_1, a_2, \dots, a_k\}$  is open and non-empty, so we can find an open connected subset  $U_1$  of  $X$  such that  $\overline{U_1} \subset O \setminus \{a_1, a_2, \dots, a_k\}$ . This implies that for all  $a \in \overline{U_1}$ ,  $S_a \subset U$ . The proof is complete.  $\square$

From the above lemmas,  $\omega_g(y)$  is a continuum (lemma 4.1). For  $F = \overline{O}$ , let  $\eta := \eta(F) > 0$  be given as in lemma 3.9. By lemma 3.8, there exists a sequence of open connected neighborhoods  $(U_n)_n$  of  $\omega_g(y)$  with finite boundary such that  $\text{mesh}(U_n \setminus \omega_g(y)) < \frac{\eta}{2}$ . We let  $U := U_N$  and set  $\text{card}(\text{Bd}(U)) = k$ . From lemma 4.2, there exists an open connected subset  $U_1 \subset O$  such that for all  $a \in \overline{U_1}$ ,  $S_a \subset U$ . Let  $a \in U_1$ . As  $\{a\} \not\subset \omega_g(y)$ , there is an open connected neighborhood  $W$  of  $a$  in  $X$  such that  $\overline{W} \cap \omega_g(y) \subset \overline{U_1}$  and  $\omega_g(y) \setminus \overline{W} \neq \emptyset$ . We can find two increasing sequences of integers  $(n_i)_i, (t_i)_i$  with the following properties:

- (1)  $g^{n_i}(y) \in W$  with  $\lim_{i \rightarrow +\infty} g^{n_i}(y) = a$ ,
- (2)  $g^{n_i+j}(y) \in W$  for each  $j = 0, \dots, t_i$ , with  $t_i \geq k$  and  $t_i \geq i$  for each  $i \in \mathbb{N}$ ,
- (3)  $\lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y) = b$  and  $g^{n_i+t_i+1}(y) \in X \setminus W$ .

Observe that  $a \in W \cap \omega_g(y)$  and  $b \in (\overline{W} \setminus W) \cap \omega_g(y)$ , so  $a \neq b$ .

We let  $\mathbb{N}_k = \bigcup_{i \in \mathbb{N}} \{n_i, \dots, n_i + t_i - k\}$ . As  $\lim_{i \rightarrow +\infty} t_i = +\infty$ , so  $\mathbb{N}_k$  is infinite. By definition, for each  $m \in \mathbb{N}_k$ ;  $g^m(y) \in W$ . As  $\lim_{m \rightarrow +\infty} d(g^m(y), \overline{W} \cap \omega_g(y)) = 0$ , so there is  $b_m \in \overline{W} \cap \omega_g(y)$  such that  $d(g^m(y), \overline{W} \cap \omega_g(y)) = d(g^m(y), b_m)$ . Since  $\overline{W}$  is a subcontinuum of  $X$ , so it is a Peano continuum. By lemma 3.6, one can find an arc  $T_m = [b_m, g^m(y)] \subset \overline{W}$  such that  $\lim_{\substack{m \rightarrow +\infty \\ m \in \mathbb{N}_k}} \text{diam}(T_m) = 0$ .

As  $g^m(y) \notin \omega_g(y)$  (because  $\omega_g(y) \subset P(g)$  is infinite), so let  $I_m$  be the connected component of  $T_m \setminus \omega_g(y)$  containing  $g^m(y)$ . Thus  $I_m = [g^m(y), a_m]$  with  $\bar{I}_m \cap \omega_g(y) = \{a_m\}$ .

**Claim 1.** Under the notations above, there exists  $N \in \mathbb{N}$  such that for all  $m \in \mathbb{N}_k$ ,  $m \geq N$  the sets  $(I_{m+j})_{0 \leq j \leq k}$  are not pairwise disjoint.

*Proof.* Suppose the contrary, so there is an increasing sequence of positive integers  $(m_i)_i$  such that the family  $(I_{m_i+j})_{0 \leq j \leq k}$  are pairwise disjoint. By passing to a subsequence of  $(m_i)_i$ , one can assume that  $(\bar{I}_{m_i})_i$  converges to  $I$  in  $C(X)$  (in Hausdorff metric). As  $\lim_{i \rightarrow +\infty} \text{diam}(T_{m_i}) = 0$ , then  $\lim_{i \rightarrow +\infty} \text{diam}(\bar{I}_{m_i}) = 0$  and so  $I$  is degenerate i.e.  $I = \{c\}$ . As  $g^{m_i}(y) \in I_{m_i} \subset W$  and  $\lim_{i \rightarrow +\infty} g^{m_i}(y) = c \in \omega_g(y)$ , so  $c \in \bar{W} \cap \omega_g(y)$ . As  $\bar{W} \cap \omega_g(y) \subset \bar{U}_1 \subset \bar{O} \subset \text{Fix}(g)$ , then for all  $0 \leq j \leq k$ ,  $\lim_{i \rightarrow +\infty} g^{m_i+j}(y) = g^j(c) = c$ . In result  $\lim_{i \rightarrow +\infty} \bar{I}_{m_i+j} = \{c\}$  and therefore  $\lim_{i \rightarrow +\infty} \bigcup_{0 \leq j \leq k} \bar{I}_{m_i+j} = \{c\}$ . Apply lemma 3.9 for  $F = \bar{O}$ . So there is  $c' \in \bar{O}$  such that  $d(c, c') \geq \eta$ . Then we can find two open neighborhoods  $O_c$  and  $O_{c'}$  of  $c$  and  $c'$  respectively in  $U$  such that  $d(O_c, O_{c'}) > \frac{\eta}{2}$ . As  $c' \in \bar{O} \subset \omega_g(y) \cap \text{Fix}(g)$ , so there is  $m > 0$  such that  $\{g^m(y), g^{m+1}(y), \dots, g^{m+k}(y)\} \subset O_{c'}$ . Let  $m_p > m$  such that  $\bigcup_{0 \leq j \leq k} \bar{I}_{m_p+j} \subset O_c$ . Set  $C_j = g^{-(m_p-m)}(\bar{I}_{m_p+j})$ ;  $0 \leq j \leq k$ . By monotonicity of  $g$ ,  $C_j$  is a sub-continuum of  $X$ . Then there is an arc  $J_j = [g^{m+j}(y), a_{m_p+j}] \subset C_j$ . Each  $C_j$  meets  $O_{c'}$  in at least a point  $g^{m+j}(y)$  and meets  $O_c$  in at least  $a_{m_p+j}$  (since  $a_{m_p+j} \in \text{Fix}(g)$ ). Moreover,  $J_j \setminus \omega_g(y) = J_j \setminus \{a_{m_p+j}\}$ : Indeed, it clear that  $J_j \setminus \omega_g(y) \subset J_j \setminus \{a_{m_p+j}\}$ . Conversely, if  $z \in J_j \cap \omega_g(y)$ , then  $g^{m_p-m}(z) \in \bar{I}_{m_p+j} \cap \omega_g(y) = \{a_{m_p+j}\}$ . As  $a_{m_p+j} \in \text{Fix}(g)$  then  $g^{m_p-m}(a_{m_p+j}) = g^{m_p-m}(z)$ . As  $\omega_g(y) \subset P(g)$ , then the restriction  $g|_{\omega_g(y)}^{m_p-m}$  is a homeomorphism and therefore  $z = a$ . In result,  $J_j \setminus \{a_{m_p+j}\} \subset J_j \setminus \omega_g(y)$  and hence the equality holds. As a consequence  $J_j \setminus \{a_{m_p+j}\}$  is connected. In addition,  $\text{diam}(J_j) \geq \frac{\eta}{2}$ . We have that  $C_j \not\subset U$ ; otherwise,  $J_j \setminus \{a_{m_p+j}\} \subset U \setminus \omega_g(y)$  and so  $\text{diam}(J_j \setminus \{a_{m_p+j}\}) \leq \text{mesh}(U \setminus \omega_g(y)) < \frac{\eta}{2}$ , absurd. As  $a_{m_p+j} \in C_j \cap \omega_g(y)$ , we conclude that  $C_j \cap \text{Bd}(U) \neq \emptyset$ , for each  $0 \leq j \leq k$ .

Let us show that  $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$  for all  $0 \leq j \neq j' \leq k$ . We have  $\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'} \subset \{a_{m_p+j}\}$  for  $j \neq j'$ . Then  $C_j \cap C_{j'} = g^{-(m_p-m)}(\bar{I}_{m_p+j} \cap \bar{I}_{m_p+j'}) \subset g^{-(m_p-m)}(a_{m_p+j}) \subset S_{a_{m_p+j}}$ . By lemma 4.2,  $S_{a_{m_p+j}} \subset U$ , hence  $C_j \cap C_{j'} \subset U$  and therefore  $C_j \cap C_{j'} \cap \text{Bd}(U) = \emptyset$ . We conclude that for each  $0 \leq j \leq k$ ,  $C_j$  meets  $\text{Bd}(U)$  in at least  $k+1$  points, which contradicts  $\text{Bd}(U) = k$ . This completes the proof of Claim 2.  $\square$

From Claim 1, there are some  $m \in \mathbb{N}_k$  with  $0 \leq j_1(m) < j_2(m) \leq k$ , such that  $I_{m+j_1(m)} \cap I_{m+j_2(m)} \neq \emptyset$ . As  $(I_{m+j_1(m)} \cup I_{m+j_2(m)})$  is arcwise connected,

so we can find an arc  $J_m = [g^{m+j_1(m)}(y), g^{m+j_2(m)}(y)] \subset I_{m+j_1(m)} \cup I_{m+j_2(m)}$ . We have that  $J_m \cap \omega_g(y) = \emptyset$  and  $\lim_{\substack{m \rightarrow +\infty \\ m \in \mathbb{N}_k}} \text{diam}(J_m) = 0$ .

**Claim 2.** Let  $B_m = g^{-j_1(m)}(J_m)$ ,  $m \in \mathbb{N}_k$ . We have

$$\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(B_m) = 0.$$

*Proof.* Suppose that  $\lim_{m \rightarrow +\infty} \text{diam}(B_m) > 0$ , then there is  $\delta > 0$  and a sequence of positive integers  $(m_l)_l$  such that  $\text{diam}(B_{m_l}) > \delta$ , for all  $l \in \mathbb{N}$ . Recall  $\lim_{m \rightarrow +\infty, m \in \mathbb{N}_k} \text{diam}(I_m) = 0$ , where  $I_m = [g^m(y), a_m]$ . Therefore  $\lim_{l \rightarrow +\infty} \text{diam}(J_{m_l}) = 0$ . By passing to a subsequence if necessary, one can assume that  $(J_{m_l})_l$  converges necessarily to a degenerate set, say  $\{c\}$ . Hence,  $\lim_{l \rightarrow +\infty} f^{m_l+j_1(m_l)}(y) = c$ , so  $c \in \omega_g(y)$ . As  $J_{m_l} \cap \omega_g(y) = \emptyset$  for all  $l \in \mathbb{N}$ , so  $c \notin J_{m_l}$ . Then we can construct a subsequence  $(J_{m_{l_q}})_{q \geq 0}$  which are pairwise disjoint. Since  $0 \leq j_1(m_{l_q}) \leq k$  we can assume that  $j_1(m_{l_q}) = j_0$ , for all  $q \in \mathbb{N}$ . Then the  $B_{m_{l_q}} = g^{-j_0}(J_{m_{l_q}})$ ,  $q \in \mathbb{N}$ , are pairwise disjoint connected sets and hence, by proposition 3.5,  $\lim_{q \rightarrow +\infty} \text{diam}(B_{m_{l_q}}) = 0$ . This contradicts the fact that  $\text{diam}(B_{m_l}) > \delta$ . This completes the proof.  $\square$

**End of the proof of theorem 1.1.** From hypothesis,  $\omega_g(y) \subset P(g)$  and  $\omega_g(y) \cap \text{Fix}(g) \neq \emptyset$ . Let recall the construction of the points  $a, b$  given in the notation of page 5: We have that  $a = \lim_{i \rightarrow +\infty} g^{n_i}(y)$ ,  $b = \lim_{i \rightarrow +\infty} g^{n_i+t_i-k}(y)$ ,  $a \neq b \in \omega_g(y) \cap \text{Fix}(g)$ . So let  $W_a, W_b$  be two open neighborhoods of  $a$  and  $b$  respectively such that  $d(W_a, W_b) > \frac{d(a,b)}{2}$ . Choose a connected neighborhood  $U$  of  $\omega_g(y)$  such that  $\text{mesh}(U \setminus \omega_g(y)) < \frac{d(a,b)}{2}$ . From Claim 2, let  $B_m = g^{-j_1(m)}(J_m)$ ,  $m \in \mathbb{N}_k$  where  $J_m = [g^{m+j_1(m)}(y), g^{m+j_2(m)}(y)]$ . Then  $B_m$  is connected and contains the points  $g^m(y)$  and  $g^{m+j_2(m)-j_1(m)}(y)$ . We denote by  $s(m) = m + j_2(m) - j_1(m)$ ,  $0 \leq j_1(m) < j_2(m) \leq k$ . We have  $B_m \cap \omega_g(y) = \emptyset$  for all  $m \in \mathbb{N}_k$ ; otherwise there is  $m \in \mathbb{N}_k$  such that  $B_m \cap \omega_g(y) \neq \emptyset$ , so for  $z \in B_m \cap \omega_g(y)$ , we have  $g^{j_1(m)}(z) \in J_m \cap \omega_g(y)$ , a contradiction with  $J_m \cap \omega_g(y) = \emptyset$ . By Claim 2,  $\lim_{\substack{m \rightarrow +\infty \\ m \in \mathbb{N}_k}} \text{diam}(B_m) = 0$ .

Since  $\lim_{m \rightarrow +\infty} d(g^m(y), \omega_g(y)) = 0$ , there is  $m_0$  such that for all  $m \geq m_0$ ,  $m \in \mathbb{N}_k$ ,  $B_m \subset U$ . Therefore, let  $q \in \mathbb{N}$  such that for  $n_q \geq m_0$ ,  $g^{n_q}(y) \in W_a$  and  $\{g^{n_q+t_q-k}(y), \dots, g^{n_q+t_q}(y)\} \subset W_b$ . Set  $K_q = B_{n_q} \cup B_{s(n_q)} \cup \dots \cup B_{s^{p_{n_q}-1}(n_q)}$  where  $p_{n_q} = \inf \{n \in \mathbb{N}; s^n(n_q) \in \{n_q + t_q - k + 1, \dots, n_q + t_q\}\}$ . It is clear that  $K_q$  is connected. Furthermore,  $K_q \cap \omega_g(y) = \emptyset$  since  $B_{s^i(n_q)} \cap \omega_g(y) = \emptyset$ , for all  $0 \leq i \leq p_{n_q}$ ,  $s^i(n_q) \in \mathbb{N}_k$ . Moreover  $K_q$  meets  $W_a$  in at least in a point  $g^{n_q}(y)$  and meets  $W_b$  in at least in point  $g^{s^{p_{n_q}}(n_q)}$ . Then  $\text{diam}(K_q) > \frac{d(a,b)}{2}$ . As  $K_q \subset U \setminus \omega_g(y)$  is connected, then  $\text{diam}(K_q) \leq \text{mesh}(U \setminus \omega_g(y)) < \frac{d(a,b)}{2}$ . This contradicts the fact that  $\text{diam}(K_q) > \frac{d(a,b)}{2}$ .

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