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NOTES ON THE HISTORICAL BIBLIOGRAPHY OF THE GAMMA FUNCTION

RICARDO PÉREZ-MARCO

Abstract. Telegraphic notes on the historical bibliography of the Gamma function and Eulerian integrals. We correct some classical references. Some topics of the interest of the author. We provide some extensive (but not exhaustive) bibliography. Feedback is welcome, notes will be updated and some references need completion.

1. Correction of references.

1.1. **Integral formula for Euler-Mascheroni constant.** Whittaker-Whatson ([71] 12.3 Example 2, p. 248) attributes the formula to Dirichlet (presumabibly to [17], 1836), but this formula is already in Euler (1770) [22] section 25. Moreover, he also gives the formula in [24] (1785) and devotes a full article [26] (1789) to it.

1.2. **Frullani integral.** According to Binet [11] the integral is used before by Euler.

1.3. **Hadamard-Weierstrass product formula for the Gamma function.** According to Nörlund [55] and to Remmert p.41 [60],the product formula is due to Schlömilch [62] p.171. According to Aycock [6] p.6, the product formula was essentially found by Euler in Chapter 16 and 17 of the second part of [21] and in [25]. Ph.J. Davis [16] attributes the product formula to F.W. Newman (1848) (no reference provided, but it is given in [33] and in [60] as [49] p.57).

Weierstrass theory of factorization is dated of 1876 [68].

1.4. **Malmsten formula.** Usually attributed to Malmsten (1849) [45], it is already in Binet (1839) [11], and Nörlund [55] attributes the formula to Plana.

1.5. **Gauss multiplication formula.** Apparently it is also already in Euler [23] according to A. Aycock (see [5] for the details). Hermite knows that is due to Euler (see his cours [35], Leçon 15, p.145). Artin attributes it to Gauss [7] p.24.

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1.6. **Laplace integral formula.** Pribitkin article starts with the historically assertion that "In 1812 Laplace establishes..." citing [41] p.134 when indeed the formula was established 30 years earlier in 1782 in [40].

1.7. **Kummer trigonometric expansion.** Attributed to Kummer (1847) but apparently found first by Malmstén (1846) (published in 1849 [44], but the submission date is 1st May 1846), according to [4].

2. Different definitions of the Gamma function.

2.1. **Bohr-Mollerup.** Bohr and Mollerup [13] gave the characterization of the Gamma function as exercice in his calculus book. Emil Artin made this the starting point of the definition of the Gamma function in his monograph [7]. This definition became popular. Bourbaki, [15] Chapter VII, takes also this approach to define the Gamma function. This definition is used and generalized by Vigneras [66] to define higher Barnes Gamma functions.

2.2. **Wielandt's criterion.** Based on the functional equation, plus estimate on vertical strip. See Wielandt [70] and Remmert [59] (see Aycock [6] p.12, see also Birkhoff [12]).

2.3. **Weierstrass product.** Attributed to Weierstrass in Whittaker-Whatson [71].

2.4. **Functional equation plus asymptotic estimates.** See Prym [58].

Weierstrass [67] (see Aycock [6] where he remarks that Weierstrass added the condition $\Gamma(1) = 1$ which is not necessary in view of the estimate) added the condition:

$$\lim_{n\to+\infty} \frac{\Gamma(x+n)}{(n-1)!n^x}$$

Usually the functional-difference equation is solved explicitly (see [6] p.19, see also the Norlund approach). See also [Eu613] for the same ideas. Norlund fundamental solution for the difference equation [51] and [53] provides also a unique characterization of the Gamma function.

2.5. **Functional equation plus more symmetries.** Davis [16] p.867 writes, without providing a reference,

" By the middle of the 19th century it was recognized that Euler's gamma function was the only continuous function which satisfied simultaneously the recurrence relationship, the reflection formula and the multiplication formula."

2.6. **Gauss product formula.** Product formula known to Euler. Gauss [30], Liouville [43].

2.7. **Integral representation.** This is a very common approach in all modern analysis books and the original definition by Euler (1729-1730) who uses the integral:

$$\Gamma(s) = \int_0^1 (\log(1/t))^{s-1}\, dt$$

The most popular version nowadays of integral form is:

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t}\, dt$$

Legendre, Liouville, Pringsheim [58].

3. Various notes.

3.1. **Weierstrass "factorielle" function.** Apparently ([16] p.862), Weierstrass preferred to work with the function $Fc(u) = 1/\Gamma(1+u)$ that he called "factorielle" function (probably in [67]).

3.2. **Bourget function.** See Godefroy [33] p.24. Bourget fonction $T(s)$ satisfies

$$T(s+1) - T(s) = -\mathrm{Fc}(s+1)$$

where $\mathrm{Fc}(s)$ is Weierstrass factorielle function. Hence, $e^T(s)$ is $\Gamma_1^f(s)$ for $f(s) = e^{-\mathrm{Fc}(s+1)}$.

We have

$$T(s) = \frac{eP(s)}{\Gamma(s)}$$

where $P(s)$ is the Prym function

$$eP(s) = \frac{1}{s} + \frac{1}{s(s+1)} + \frac{1}{s(s+1)(s+2)} + \ldots =$$

also

$$P(s) = \frac{1}{s} - \frac{1}{1}\frac{1}{s+1} + \frac{1}{1.2}\frac{1}{s+2} + \ldots + (-1)^n \frac{1}{n!}\frac{1}{s+n} + \ldots S$$

The Prym function is the polar (Mittag-Leffler) part of the Γ function:

$$Q(s) = \Gamma(s) - P(s)$$

is an entire function.

Problem mentioned in [33] p.23: Prove that P has no more than 4 complex zeros. Not known if Q has zeros.

3.3. **Stirling series.** According to [16] p.862 (no reference provided), C. Hermite (1900) wrote down the Stirling series for $\log\Gamma(1+s)$, convergent for $\Re s > 0$,

$$\log\Gamma(1+s) = \binom{s}{2}\log 2 + \binom{s}{3}(\log 3 - 2\log 2) + \dots$$

In the same place he also attributes to M.A. Stern (1847) (no reference provided) the Stirling series for the ψ function, convergent for $\Re s > 0$,

$$\psi(s) = \frac{d}{ds}\log\Gamma(s) = -\gamma + \binom{s}{1} - \frac{1}{2}\binom{s}{2} + \frac{1}{3}\binom{s}{3} + \dots$$

3.4. **Hadamard interpollation of the factorial.** According to [16] p.865, J. Hadamard (1894, no reference provided) gave an entire function interpollation of the factorial:

$$H(s) = \frac{1}{\Gamma(1-s)}\frac{d}{ds}\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\right)$$

It satisfies the functional equation

$$H(s+1) = sH(s) + \frac{1}{\Gamma(1-s)}$$

3.5. **Mellin Higher Gamma functions.** According to Godefroy p.81, Mellin [47] defined general Gamma functions satisfying

$$F(s+1) = R(s)F(s)$$

where R is a rational function.

3.6. **Davis pseudo-Gamma function.** Davis [16] p.867 gives the following pseudo-Gamma function Γ_S,

$$\begin{aligned}
&\Gamma_S(s) = 1/s \quad \text{for } 0 < s < 1\\
&\Gamma_S(s) = 1 \quad \text{for } 1 < s < 2\\
&\Gamma_S(s) = s-1 \quad \text{for } 2 < s < 3\\
&\Gamma_S(s) = (s-1)(s-2) \quad \text{for } 3 < s < 4\\
&\quad \dots\dots\\
&\Gamma_S(s) = (s-1)^{\underline{k-1}} \quad \text{for } k < s < k+1
\end{aligned}$$

It is a convex function in $\mathbb{R}_+^*$, and satisfies the functional equation

$$\Gamma_S(s+1) = s\,\Gamma_S(s)$$

3.7. **Bendersky Gamma function.** Bendersky (1933, [10]) studies another hierarchy of Gamma functions different from Barnes'. Bendersky's Gamma functions have been rediscovered by Milnor 50 years later (1983, [48]) arising as higher partial derivatives of Hurwitz zeta function à la Lerch.

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