



HAL
open science

The p -completed cyclotomic trace in degree 2

Arthur-César Le Bras, Johannes Anschütz

► **To cite this version:**

Arthur-César Le Bras, Johannes Anschütz. The p -completed cyclotomic trace in degree 2. *Annals of K-Theory*, 2020. hal-03018441

HAL Id: hal-03018441

<https://hal.science/hal-03018441v1>

Submitted on 22 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE p -COMPLETED CYCLOTOMIC TRACE IN DEGREE 2

JOHANNES ANSCHÜTZ AND ARTHUR-CÉSAR LE BRAS

ABSTRACT. We prove that for a quasi-regular semiperfectoid $\mathbb{Z}_p^{\text{cycl}}$ -algebra R (in the sense of Bhatt-Morrow-Scholze), the cyclotomic trace map from the p -completed K -theory spectrum $K(R; \mathbb{Z}_p)$ of R to the topological cyclic homology $\text{TC}(R; \mathbb{Z}_p)$ of R identifies on π_2 with a q -deformation of the logarithm.

CONTENTS

1. Introduction	1
1.1. K -theory and topological cyclic homology	2
1.2. Main results	5
1.3. Plan of the paper	7
1.4. Acknowledgements	7
2. The p -completed Dennis trace in degree 2	7
3. Transversal prisms	14
4. The q -logarithm	18
5. Prismatic cohomology and topological cyclic homology	26
6. The p -completed cyclotomic trace in degree 2	27
References	34

1. INTRODUCTION

Fix a prime p . The aim of this paper is to concretely identify in degree 2, for a certain class of p -complete rings R , the p -completed cyclotomic trace

$$\text{ctr}: K(R; \mathbb{Z}_p) \rightarrow \text{TC}(R; \mathbb{Z}_p)$$

from the p -completed K -theory spectrum $K(R; \mathbb{Z}_p)$ of R to the topological cyclic homology $\text{TC}(R; \mathbb{Z}_p)$ of R . Our main result is that on π_2 the p -completed cyclotomic trace is given by a q -logarithm

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x-1)(x-q) \cdots (x-q^{n-1})}{[n]_q},$$

which is a q -deformation of the usual logarithm (where q is a parameter which will be defined later). Before stating a precise version of the theorem, let us try to put it in context and to explain what the involved objects are.

During this project, J.A. was partially supported by the ERC 742608, GeoLocLang.

1.1. K -theory and topological cyclic homology. We start with K -theory. For any commutative ring A , Quillen defined in [22] the algebraic K -theory space $K(A)$ of A as a generalization of the Grothendieck group $K_0(A)$ of vector bundles on the scheme $\mathrm{Spec}(A)$. The (connective) K -theory spectrum $K(A)$ of a ring A is obtained by group completing¹ the \mathbb{E}_∞ -monoid of vector bundles on $\mathrm{Spec}(A)$ whose addition is given by the direct sum. In other words, for the full K -theory one mimicks in a homotopy theoretic context the definition of $K_0(A)$ with the set of isomorphism classes of vector bundles replaced by the groupoid of vector bundles. Algebraic K -theory behaves like a cohomology theory but has the nice feature, compared to other cohomology theories, like étale cohomology, that it only depends on the category of vector bundles on the ring (rather than on the ring itself) and thus enjoys strong functoriality properties, which makes it a powerful invariant attached to A .

Unfortunately, the calculation of the homotopy groups

$$K_i(A) := \pi_i(K(A)), \quad i \geq 1,$$

is in general rather untractable. There is for example a natural embedding

$$A^\times \rightarrow \pi_1(K(A)),$$

which is an isomorphism if A is local, but the higher K -groups are much more mysterious. One essential difficulty comes from the fact that K -theory, although it is a Zariski (and even Nisnevich) sheaf of spaces (cf. [28]), does not satisfy étale descent. One could remedy this by étale sheafification, but one would lose the good properties of K -theory. This lead people to look for good *approximations* of K -theory, at least after profinite completion : by this, we mean invariants, still depending only on the category of vector bundles on the underlying ring, satisfying étale descent - and therefore, easier to compute - and close enough to (completed) K -theory, at least in some range.

The work of Thomason, [27], provides a good illustration of this principle. Thomason shows that the $K(1)$ -*localization* of K -theory, with respect to a prime ℓ invertible in A , satisfies étale descent² and coincides with ℓ -adically completed (for short: ℓ -adic) K -theory in high degrees under some extra assumptions, later removed by Rosenschon-Ostaver, [23], building upon the work of Voevodsky-Rost. When the prime p is not invertible in A , the situation is much more subtle. For instance, a theorem of Gabber [11] shows that ℓ -adic K -theory is insensitive to replacing A by A/I if (A, I) forms an henselian pair ; in particular, the computation of ℓ -adic K -theory of henselian rings (which form a basis of the Nisnevich topology) is reduced to the computation of the ℓ -adic K -theory of fields. This is not true anymore for p -adic K -theory. Nevertheless, the recent work of Clausen-Mathew-Morrow, [8], expresses this failure in terms of another non-commutative invariant attached to A , the *topological cyclic homology* of A , whose definition will be recalled below. Topological cyclic homology is related to K -theory via the *cyclotomic trace*, cf. [6, Section 10.3], [7, Section 5],

$$\mathrm{ctr}: K(A) \rightarrow \mathrm{TC}(A).$$

¹Cf. [19] for a discussion of homotopy-theoretic group completions and Quillen's $+$ -construction.

²In fact, it even coincides with ℓ -adic étale K -theory on connective covers.

Clausen, Mathew and Morrow prove, extending earlier work of Dundas, Goodwillie and McCarthy [9] in the nilpotent case³, that the cyclotomic trace induces, for any ideal $I \subseteq A$ such that the pair (A, I) is henselian, an isomorphism

$$K(A, I)/n \cong \mathrm{TC}(A, I)/n$$

from the relative K -theory

$$K(A, I)/n := \mathrm{fib}(K(A)/n \rightarrow K(A/I)/n)$$

to the relative topological cyclic homology

$$\mathrm{TC}(A, I)/n := \mathrm{fib}(\mathrm{TC}(A)/n \rightarrow \mathrm{TC}(A/I)/n),$$

for *any* integer n . This has the consequence that p -completed TC provides a good approximation of p -adic K -theory, at least for rings henselian along (p) : namely, it satisfies étale descent (because topological cyclic homology does) and coincides with p -adic K -theory in high degrees. Under additional hypotheses, one can even get better results: for instance, Clausen, Mathew and Morrow prove, among other things, that the cyclotomic trace induces an isomorphism

$$K(R; \mathbb{Z}_p) \cong \tau_{\geq 0} \mathrm{TC}(R; \mathbb{Z}_p)$$

for all rings R which are henselian along (p) and such that R/p is semiperfect (i.e., such that Frobenius is surjective), cf. [8, Corollary 6.9.].

Examples of such rings are the *quasi-regular semiperfectoid rings* of [4]. A ring R is called quasi-regular semiperfectoid, if R is p -complete with bounded p^∞ -torsion⁴, the p -completed cotangent complex $\widehat{L}_{R/\mathbb{Z}_p}$ has p -complete Tor-amplitude in $[-1, 0]$ and there exists a surjective morphism $R' \rightarrow R$ with R' (integral) perfectoid. This class of rings is interesting as for R quasi-regular semiperfectoid the topological cyclic homology $\pi_*(\mathrm{TC}(R; \mathbb{Z}_p))$ can be computed in more concrete terms.

Let us recall the description of topological cyclic homology $\pi_*(\mathrm{TC}(R; \mathbb{Z}_p))$ from [4], which builds heavily on the foundational work of Nikolaus and Scholze [20]. For this, we need to spell some definitions. From now on, all spectra will be assumed to be p -completed. One starts with the (p -completed) topological Hochschild homology spectrum $\mathrm{THH}(R; \mathbb{Z}_p)$ of R , which is equipped with a natural $\mathbb{T} = S^1$ -action and a \mathbb{T} -equivariant map, the cyclotomic Frobenius,

$$\varphi_{\mathrm{cycl}}: \mathrm{THH}(R; \mathbb{Z}_p) \rightarrow \mathrm{THH}(R; \mathbb{Z}_p)^{tC_p}$$

to the Tate fixed points of the cyclic group $C_p \subseteq \mathbb{T}$. Then one takes the homotopy fixed points, the *negative topological cyclic homology*,

$$\mathrm{TC}^-(R; \mathbb{Z}_p) := \mathrm{THH}(R; \mathbb{Z}_p)^{h\mathbb{T}}$$

and the Tate fixed points, the *periodic topological cyclic homology*,

$$\mathrm{TP}(R; \mathbb{Z}_p) := \mathrm{THH}(R; \mathbb{Z}_p)^{t\mathbb{T}}.$$

From the cyclotomic Frobenius on $\mathrm{THH}(R; \mathbb{Z}_p)$ one derives a map⁵

$$\varphi_{\mathrm{cycl}}^{h\mathbb{T}}: \mathrm{TC}^-(R; \mathbb{Z}_p) \rightarrow \mathrm{TP}(R; \mathbb{Z}_p).$$

³This is not a generalization though, since the result of Dundas-Goodwillie-McCarthy applies also to non-commutative rings and is not restricted to finite coefficients.

⁴This means that there exists $N \geq 0$ such that $R[p^\infty] = R[p^N]$. This technical condition is useful when dealing with derived completions.

⁵Here one needs [20, Lemma II.4.2.] which implies $\mathrm{TP}(R; \mathbb{Z}_p) \cong (\mathrm{THH}(R; \mathbb{Z}_p)^{tC_p})^{h\mathbb{T}}$.

Then the topological cyclic homology $\mathrm{TC}(R; \mathbb{Z}_p)$ of R is defined via the fiber sequence

$$\mathrm{TC}(R; \mathbb{Z}_p) \rightarrow \mathrm{TC}^-(R; \mathbb{Z}_p) \xrightarrow{\mathrm{can} - \varphi_{\mathrm{cycl}}^{h\mathbb{T}}} \mathrm{TP}(R; \mathbb{Z}_p),$$

where $\mathrm{can}: \mathrm{TC}^-(R; \mathbb{Z}_p) \rightarrow \mathrm{TP}(R; \mathbb{Z}_p)$ is the canonical map from homotopy to Tate fixed points. The ring

$$\widehat{\Delta}_R := \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p)) \cong \pi_0(\mathrm{TP}(R; \mathbb{Z}_p)).$$

is p -complete, p -torsion free⁶ and the cyclotomic Frobenius $\varphi_{\mathrm{cycl}}^{h\mathbb{T}}$ induces a Frobenius lift φ on $\widehat{\Delta}_R$ (cf. [5, Theorem 11.10]).

Remark 1.1. The prismatic perspective of [5] gives an alternative description of $\widehat{\Delta}_R$: it is the completion with respect to the Nygaard filtration of the (derived) prismatic cohomology Δ_R of R . In particular, using the theory of δ -rings, one can give, when R is a p -complete with bounded p^∞ -torsion quotient of a perfectoid ring by a regular sequence, a construction of $\widehat{\Delta}_R$ as the Nygaard completion of a concrete prismatic envelope (cf. [5, Proposition 3.12]).

The choice of a morphism $R' \rightarrow R$ with R' perfectoid yields a distinguished element $\tilde{\xi}$ (up to a unit) of the ring $\widehat{\Delta}_R$. Using $\tilde{\xi}$ one defines the Nygaard filtration

$$\mathcal{N}^{\geq i} \widehat{\Delta}_R := \varphi^{-1}((\tilde{\xi}^i))$$

on $\widehat{\Delta}_R$. The graded rings $\pi_*(\mathrm{TC}^-(R; \mathbb{Z}_p))$ and $\pi_*(\mathrm{TP}(R; \mathbb{Z}_p))$ are then concentrated in even degrees and

$$\begin{aligned} \pi_{2i}(\mathrm{TC}^-(R; \mathbb{Z}_p)) &\cong \mathcal{N}^{\geq i} \widehat{\Delta}_R \\ \pi_{2i}(\mathrm{TP}(R; \mathbb{Z}_p)) &\cong \widehat{\Delta}_R \end{aligned}$$

for $i \in \mathbb{Z}$ (cf. [5, Theorem 11.10]).⁷ Moreover, on π_{2i} the cyclotomic Frobenius

$$\varphi_{\mathrm{cycl}}^{h\mathbb{T}}: \pi_{2i}(\mathrm{TC}^-(R; \mathbb{Z}_p)) \rightarrow \pi_{2i}(\mathrm{TP}(R; \mathbb{Z}_p))$$

identifies with the divided Frobenius $\frac{\varphi}{\tilde{\xi}^i}$. Thus from the definition of $\mathrm{TC}(R; \mathbb{Z}_p)$ we obtain exact sequences

$$0 \rightarrow \pi_{2i}(\mathrm{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi = \tilde{\xi}^i} \rightarrow \mathcal{N}^{\geq i} \widehat{\Delta}_R \xrightarrow{1 - \frac{\varphi}{\tilde{\xi}^i}} \widehat{\Delta}_R \rightarrow \pi_{2i-1}(\mathrm{TC}(R; \mathbb{Z}_p)) \rightarrow 0.$$

As mentioned in Remark 1.1, the ring $\widehat{\Delta}_R$ tends to be computable. For example, if R is perfectoid, then $\widehat{\Delta}_R \cong A_{\mathrm{inf}}(R)$ is Fontaine's construction applied to R and if $pR = 0$, then $\widehat{\Delta}_R$ is the Nygaard completion of the universal PD-thickening $A_{\mathrm{crys}}(R)$ of R . Thus, for quasi-regular semiperfectoid rings the target of the cyclotomic trace is rather explicit.

⁶Indeed, any element killed by p is killed by φ , cf. the proof of [5, Lemma 2.28], and thus lies in all the steps of the Nygaard filtration.

⁷These identifications depend on the choice of a suitable generator $v \in \pi_{-2}(\mathrm{TC}^-(R; \mathbb{Z}_p))$. If R is an algebra over $\mathbb{Z}_p^{\mathrm{cycl}}$ we will clarify our choice in Section 6 carefully.

1.2. Main results. The results of [8] (together with those of [4]) therefore give a way of computing higher p -completed K -groups of quasi-regular semiperfectoid rings. But there is at least one degree (except 0) where one can be more explicit, without using the cyclotomic trace map: namely, after p -completion of $K(R)$ there is a canonical morphism

$$T_p(R^\times) \rightarrow \pi_2(K(R; \mathbb{Z}_p))$$

from the Tate module $T_p(R^\times)$ of the units of R , which is an isomorphism in many cases. The results explained in the previous paragraph show that the cyclotomic trace identifies $\pi_2(K(R; \mathbb{Z}_p))$ with

$$\pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}.$$

What does the composite map

$$T_p(R^\times) \rightarrow \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

look like? The main result of this paper, which we now state, provides a concrete description of it. Let R be a quasi-regular semiperfectoid ring which admits a compatible system of morphisms $\mathbb{Z}[\zeta_{p^n}] \rightarrow R$ for $n \geq 0$. These morphisms give rise to the elements

$$\varepsilon = (1, \zeta_p, \dots) \in R^b = \varprojlim_{x \mapsto x^p} R, \quad q := [\varepsilon]_\theta \in \widehat{\Delta}_R$$

and

$$\tilde{\xi} := \frac{q^p - 1}{q - 1}.$$

Here

$$[-]_\theta: R^b \rightarrow \Delta_R$$

is the Teichmüller lift coming from the surjection $\theta: \Delta_R \rightarrow R$ (see the proof of Lemma 2.4).

Theorem 1.2 (cf. Theorem 6.7). *The composition*⁸

$$T_p(R^\times) \rightarrow \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is given by the q -logarithm

$$x \mapsto \log_q([x]_\theta) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{([x]_\theta - 1)([x]_\theta - q) \cdots ([x]_\theta - q^{n-1})}{[n]_q}.$$

Here we embed

$$T_p(R^\times) \subseteq R^b, \quad (r_0 \in R^\times[p], r_1, \dots) \mapsto (1, r_0, r_1, \dots).$$

By

$$[n]_q := \frac{q^n - 1}{q - 1}$$

we denote the the q -analog of $n \in \mathbb{Z}$.

⁸Cf. Section 6 for a more precise description of the isomorphism $\pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$. We note that it depends on the choice of some compatible system $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive p^n -th roots of unity.

Remark 1.3. A similar result can be found in [12, Lemma 4.2.3.], but only before p -completion, on π_1 and in terms of TR_* , which is not enough to deduce Theorem 1.2 from their result.

As a consequence of [8] and Theorem 1.2, one gets the following result.

Corollary 1.4. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. The map*

$$\log_q([-]_\theta) : T_p(R^\times) \rightarrow \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection.

This corollary is used in [1], which studies a prismatic version of Dieudonné theory for p -divisible groups, and was our original motivation for proving Theorem 1.2.

Here is a short description of the proof of Theorem 1.2. By testing the universal case $R = \mathbb{Z}^{\mathrm{cycl}}\langle x^{1/p^\infty} \rangle / (x-1)$ one is reduced to the case where $(p, \tilde{\xi})$ form a regular sequence on $\widehat{\Delta}_R$, i.e., the prism $(\widehat{\Delta}_R, \tilde{\xi})$ is *transversal* (cf. Definition 3.2). In this situation, we prove that the reduction map

$$\widehat{\Delta}_R^{\varphi=\tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R$$

is *injective* (cf. Corollary 3.10). Thus it suffices to identify the composition

$$T_p(R^\times) \xrightarrow{\mathrm{ctr}} \widehat{\Delta}_R^{\varphi=\tilde{\xi}} \rightarrow \mathcal{N}^{\geq 1} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R.$$

Using the results [4] the quotient $\mathcal{N}^{\geq 1} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R$ identifies with the p -completed Hochschild homology $\pi_2(\mathrm{HH}(R; \mathbb{Z}_p))$ (cf. Section 5) and therefore the above composition identifies with the p -completed Dennis trace. A straightforward computation then identifies the p -completed Dennis trace (cf. Section 2), which allows us to conclude. We expect the results in Section 2 to be known, in some form, to the experts, but we did not find the results anywhere in the literature.

Let us end this introduction by a remark and a question. One could try to reverse the perspective from Corollary 1.4 and try to recover a (very) special case of the result of Clausen-Mathew-Morrow (cf. [8]) regarding the cyclotomic trace map using the concrete description furnished by Theorem 1.2. If R is of characteristic p , we have $q = 1$ and then the q -logarithm becomes the honest logarithm

$$\log([-]_\theta) : T_p(R^\times) \rightarrow A_{\mathrm{crys}}(R)^{\varphi=p}.$$

In [25], it is proven (using the exponential) that the map $\log([-])$ is an isomorphism, when R is the quotient of a perfect ring modulo a regular sequence. If R is the quotient of a perfectoid ring by a finite regular sequence and is p -torsion free, it is not difficult to deduce from Scholze-Weinstein's result that the map

$$\log_q([-]_\theta) : T_p(R^\times) \rightarrow \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection when p is odd. Is there a way to prove it directly in general, for any p and any quasi-regular semiperfectoid ring ?

1.3. Plan of the paper. In Section 2 we concretely identify the p -completed Dennis trace on the Tate module of units (cf. Proposition 2.5) in the form we need it. In Section 3 we prove the crucial injectivity statement, namely Corollary 3.10, for transversal prisms. In Section 4 we make sense of the q -logarithm. Finally, in Section 6 we prove our main result Theorem 1.2 and its consequence, Corollary 1.4.

1.4. Acknowledgements. The authors thank Peter Scholze for answering several questions and his suggestion to think about Lemma 3.9. Moreover, we thank Bhargav Bhatt, Akhil Mathew, Andreas Mihatsch, Emmanuele Dotto, Matthew Morrow for answers and interesting related discussions. Very special thanks go to Irakli Patchkoria, who helped the authors with all necessary topology and provided detailed comments on a first draft. The authors thank an anonymous referee for her/his excellent report, in particular for suggesting the shorter proof of Proposition 2.5, which we presented here.

The authors would like to thank the University of Bonn and the Institut de Mathématiques de Jussieu for their hospitality while this work was done.

2. THE p -COMPLETED DENNIS TRACE IN DEGREE 2

Fix some prime p and let $A = R/I$ be the quotient of a (p, I) -complete ring R . The aim of this section is to concretely describe in degree 2 the composition

$$T_p(A^\times) \rightarrow \pi_2(K(A; \mathbb{Z}_p)) \xrightarrow{\text{Dtr}} \pi_2(\text{HH}(A; \mathbb{Z}_p)) \rightarrow \pi_2(\text{HH}(A/R; \mathbb{Z}_p)).$$

Here

$$K(A; \mathbb{Z}_p)$$

denotes the p -completed (connective) K -theory spectrum of A ,

$$\text{HH}(A; \mathbb{Z}_p), \text{ resp. } \text{HH}(A/R; \mathbb{Z}_p)$$

the p -completed (derived) Hochschild homology of A as a \mathbb{Z} -algebra, resp. as an R -algebra, and Dtr is the Dennis trace map. Before stating precisely our result, let us start by some reminders on the objects and the maps involved in the previous composition.

Let us first recall the construction of the first map $T_p(A^\times) \rightarrow \pi_2(K(A; \mathbb{Z}_p))$. Let

$$\text{GL}(A) = \varinjlim_r \text{GL}_r(A)$$

be the infinite general linear group over A . There is a canonical inclusion

$$A^\times = \text{GL}_1(A) \rightarrow \text{GL}(A)$$

of groups which on classifying spaces induces a map

$$B(A^\times) \rightarrow B(\text{GL}(A)).$$

Composing with the morphism to Quillen's +-construction yields a canonical morphism

$$B(A^\times) \rightarrow \text{BGL}(A) \rightarrow K(A) := \text{BGL}(A)^+ \times K_0(A)$$

into the K -theory space $K(A)$ of A . After p -completion of spaces⁹ we obtain a canonical morphism

$$\iota: B(A^\times)_p^\wedge \rightarrow K(A; \mathbb{Z}_p) := K(A)_p^\wedge.$$

⁹We use space as a synonym for Kan complex.

We recall (cf. [18, Theorem 10.3.2]) that the space $B(A^\times)_p^\wedge$ has two non-trivial homotopy groups which are given by

$$\pi_1(B(A^\times)_p^\wedge) \cong H^0(R\varprojlim_n (A^\times \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n))$$

and

$$\pi_2(B(A^\times)_p^\wedge) \cong H^{-1}(R\varprojlim_n (A^\times \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n)) \cong T_p(A^\times).$$

In degree 2 we thus get a morphism

$$T_p(A^\times) = \pi_2(B(A^\times)_p^\wedge) \rightarrow \pi_2(K(A; \mathbb{Z}_p)),$$

which is the first constituent of the map

$$T_p(A^\times) \rightarrow \pi_2(K(A; \mathbb{Z}_p)) \xrightarrow{\text{Dtr}} \pi_2(\text{HH}(A; \mathbb{Z}_p)) \rightarrow \pi_2(\text{HH}(A/R; \mathbb{Z}_p))$$

we want to describe.

Now we turn to the construction of Hochschild homology and the Dennis trace

$$\text{Dtr}: K(A) \rightarrow \text{HH}(A).$$

Let R be a (commutative) ring and A a (commutative) R -algebra. Let

$$\mathbb{T} := S^1 \cong B\mathbb{Z}$$

be the circle group. Then the Hochschild homology spectrum

$$\text{HH}(A/R)$$

(simply denoted $\text{HH}(A)$ when $R = \mathbb{Z}$) is the initial \mathbb{T} -equivariant¹⁰ $E_\infty - R$ -algebra with a non-equivariant map $A \rightarrow \text{HH}(A/R)$ of $E_\infty - R$ -algebras, cf. [4, Remark 2.4]. For a comparison with classical definitions, we refer to [15].

The functor $A \mapsto \text{HH}(A/R)$ extends to all simplicial R -algebras and as such is left Kan extended (as it commutes with sifted colimits) from the category of finitely generated polynomial R -algebras. By left Kan extending the (decreasing) Postnikov filtration $\tau_{\geq \bullet} \text{HH}(A/R)$ on $\text{HH}(A/R)$ for A a finitely generated polynomial R -algebra one obtains the \mathbb{T} -equivariant HKR-filtration

$$\text{Fil}_{\text{HKR}}^n \text{HH}(A/R)$$

on $\text{HH}(A/R)$ for A a general R -algebra. The ∞ -category of \mathbb{T} -equivariant objects in the derived ∞ -category $\mathcal{D}(R)$ of R is equivalent to the ∞ -category of $R[\mathbb{T}]$ -modules, where

$$R[\mathbb{T}] = R \otimes \Sigma_+^\infty \mathbb{T}$$

is the group algebra of \mathbb{T} over R (cf. [15, page 5]). Let

$$\gamma \in H_1(\mathbb{T}, R) \cong \text{Hom}_{\mathcal{D}(R)}(R[1], R[\mathbb{T}])$$

be a generator¹¹. The multiplication by γ induces a differential

$$d: \text{HH}_i(A/R) \rightarrow \text{HH}_{i+1}(A/R)$$

which makes $\text{HH}_*(A/R)$ into a graded commutative dg-algebra over R all of whose elements of odd degree square to zero (cf. [16, Lemma 2.3]). By the universal

¹⁰For an ∞ -category \mathcal{C} the category of \mathbb{T} -equivariant objects of \mathcal{C} is by definition the ∞ -category of functors $B\mathbb{T} \rightarrow \mathcal{C}$.

¹¹We will mostly assume that γ is obtained by base change from some generator of $H_1(\mathbb{T}, \mathbb{Z})$.

property of the de Rham complex $\Omega_{A/R}^*$, the canonical morphism $A \rightarrow \mathrm{HH}_0(A/R)$ extends therefore to a morphism

$$\alpha_\gamma: \Omega_{A/R}^* \rightarrow \mathrm{HH}_*(A/R).$$

The Hochschild-Kostant-Rosenberg theorem affirms that α_γ is an isomorphism if $R \rightarrow A$ is smooth. By left Kan extension, one obtains for arbitrary $R \rightarrow A$ the natural description

$$\alpha_\gamma: \wedge^i L_{A/R}[i] \cong \mathrm{gr}_{\mathrm{HKR}}^i \mathrm{HH}(A/R)$$

of the graded pieces of the HKR-filtration via exterior powers of the cotangent complex of A over R (cf. [4, Section 2.2]).

In particular, we get after p -completion the following consequence in degree 2, which will be used to formulate our description of the Dennis trace below.

Lemma 2.1. *Let R be a ring and $I \subseteq R$ an ideal. Let $A = R/I$. Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. There is a natural isomorphism*

$$\alpha_\gamma: (I/I^2)_p^\wedge \cong \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)).$$

Here (and in the rest of the paper) we denote by M_p^\wedge the *derived* p -adic completion of an abelian group M , i.e.,

$$M_p^\wedge := H^0(R \varprojlim_n M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n).$$

Proof. The first assertion follows from the HKR-filtration on $\mathrm{HH}(A/R; \mathbb{Z}_p)$ described above and the fact there is a canonical isomorphism

$$(I/I^2)_p^\wedge \cong H^{-1}((L_{A/R})_p^\wedge)$$

which is implied by [26, Tag 08RA]. \square

The Dennis trace can be defined abstractly, cf. [6, Section 10.2], as the composition of the unique natural transformation

$$K \rightarrow \mathrm{THH}$$

of additive invariants of small stable ∞ -categories from K -theory to topological Hochschild homology, which induces the identity on

$$\mathbb{Z} \cong \pi_0(K(\mathbb{S})) \rightarrow \pi_0(\mathrm{THH}(\mathbb{S})) \cong \mathbb{Z},$$

and the natural transformation¹² $\mathrm{THH} \rightarrow \mathrm{HH}$.

The only thing we will need to use as an input regarding the Dennis trace is the following explicit description in degree 1. Recall from above that if A is a ring, each choice of a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$ gives rise to an isomorphism

$$\alpha_\gamma: \pi_1(\mathrm{HH}(A/\mathbb{Z})) \cong \Omega_{A/\mathbb{Z}}^1$$

as $H^0(L_{A/\mathbb{Z}}) \cong \Omega_{A/\mathbb{Z}}^1$ for any A .

Lemma 2.2. *Let A be a commutative ring. There exists a unique bijection*

$$\delta_1: \{\text{generators of } H_1(\mathbb{T}, \mathbb{Z})\} \cong \{\pm 1\}$$

such that

$$A^\times \cong \pi_1(BA^\times) \xrightarrow{\mathrm{Dtr}} \pi_1(\mathrm{HH}(A)) \xrightarrow{\alpha_\gamma} \Omega_{A/\mathbb{Z}}^1, \quad a \mapsto \delta_1(\gamma) \mathrm{dlog}(a)$$

for any generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$.

¹²On rings.

Proof. Let A be any commutative ring. The Hochschild homology $\mathrm{HH}(A)$ can be calculated as the geometric realization

$$\mathrm{HH}(A) := \varinjlim_{\Delta^{\mathrm{op}}} A^{\otimes_{\mathbb{Z}}^{L_{n+1}}},$$

Note that this representation, which relies on the standard simplicial model of the circle $\Delta^1/\partial\Delta^1$, depends implicitly on the choice of a generator γ_0 of $H_1(\mathbb{T}, \mathbb{Z})$, cf. [15, Theorem 2.3]¹³. Replacing the derived tensor product by the non-derived one obtains the classical, non-derived Hochschild homology $\mathrm{HH}^{\mathrm{usual}}(A)$ of A . As

$$\pi_1(\mathrm{HH}(A)) \cong \pi_1(\mathrm{HH}^{\mathrm{usual}}(A))$$

we may argue using $\mathrm{HH}^{\mathrm{usual}}$ instead of HH .

Using the above description of the classical Hochschild homology, the Dennis trace can be described more concretely, cf. [7, Section 5], [17, Chapter 8.4]. It factors (on homotopy groups) through the integral group homology of $\mathrm{GL}(A)$, i.e., through $H_*(\mathrm{BGL}(A), \mathbb{Z})$, which is by definition (and the Dold-Kan correspondence) the homotopy of the space $\mathbb{Z}[\mathrm{BGL}(A)]$ obtained by taking the free simplicial abelian group on the simplicial $\mathrm{BGL}(A)$. As the $+$ -construction

$$\mathrm{BGL}(A) \rightarrow \mathrm{BGL}(A)^+$$

is an equivalence on integral homology (cf. [29, Chapter IV.Theorem 1.5]) the morphism

$$\mathbb{Z}[\mathrm{BGL}(A)] \simeq \mathbb{Z}[\mathrm{BGL}(A)^+]$$

is an equivalence of simplicial abelian groups and using the canonical inclusion

$$\mathrm{BGL}(A)^+ \rightarrow \mathbb{Z}[\mathrm{BGL}(A)^+]$$

we arrive at a canonical morphism

$$K(A) \rightarrow \mathrm{BGL}(A)^+ \rightarrow \mathbb{Z}[\mathrm{BGL}(A)^+] \simeq \mathbb{Z}[\mathrm{BGL}(A)].$$

We observe that for $r = 1$ the morphism $\mathrm{BGL}_1(A) \rightarrow \mathrm{BGL}_1(A)^+$ is an equivalence as $\mathrm{GL}_1(A) = A^\times$ is abelian. Thus there is a commutative diagram (up to homotopy)

$$\begin{array}{ccc} \mathrm{BGL}_1(A) & \longrightarrow & \mathbb{Z}[\mathrm{BGL}_1(A)] \\ \downarrow & & \downarrow \\ K(A) & \longrightarrow & \mathbb{Z}[\mathrm{BGL}(A)] \end{array}$$

with each morphism being the canonical one.

The Dennis trace factors as a composition

$$\mathrm{Dtr}: K(A) \rightarrow \mathbb{Z}[\mathrm{BGL}(A)] \xrightarrow{\mathrm{Dtr}'} \mathrm{HH}^{\mathrm{usual}}(A/\mathbb{Z}),$$

where by construction

$$\mathrm{Dtr}': \mathbb{Z}[\mathrm{BGL}(A)] \rightarrow \mathrm{HH}^{\mathrm{usual}}(A)$$

is given as the colimit of compatible maps¹⁴

$$\mathrm{Dtr}'_r: \mathbb{Z}[\mathrm{BGL}_r(A)] \rightarrow \mathrm{HH}^{\mathrm{usual}}(A).$$

¹³In this reference, γ_0 is called γ .

¹⁴Here compatible means up to some homotopy. To obtain strict compatibility one has to use the normalised Hochschild complex, cf. [17, Section 8.4.]

When $r = 1$, which is the only case relevant for us, the map Dtr'_1 is the linear extension of the map

$$BA^\times \rightarrow \text{HH}^{\text{usual}}(A)$$

which in simplicial degree n is given by

$$(a_1, \dots, a_n) \mapsto \frac{1}{a_1 \dots a_n} \otimes a_1 \otimes \dots \otimes a_n.$$

Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. The choice of γ gives the HKR-isomorphism

$$\alpha_\gamma : \pi_1(\text{HH}^{\text{usual}}(A)) \cong \pi_1(\text{HH}(A/\mathbb{Z})) \cong \Omega_{A/\mathbb{Z}}^1.$$

Using the above description of Hochschild homology as a geometric realization, the isomorphism α_γ is given by

$$\pi_1(\text{HH}^{\text{usual}}(A)) \cong \Omega_{A/\mathbb{Z}}^1, \quad a \otimes b \mapsto adb$$

with inverse $adb \mapsto a \otimes b$, if $\gamma = \gamma_0$, and by

$$\pi_1(\text{HH}^{\text{usual}}(A)) \cong \Omega_{A/\mathbb{Z}}^1, \quad a \otimes b \mapsto bda$$

with inverse $bda \mapsto a \otimes b$ if $\gamma = -\gamma_0$; this can be checked by analyzing compatibility with differentials and using [15, Theorem 2.3]. In the first case, we set $\delta_1(\gamma) = 1$; in the second case, we set $\delta_1(\gamma) = -1$. Then on homotopy groups the map Dtr_1 is given by

$$A^\times \cong \pi_1(BA^\times) \rightarrow \pi_1(\text{HH}(A)) \xrightarrow{\alpha_\gamma} \Omega_{A/\mathbb{Z}}^1, \quad a \mapsto \delta_1(\gamma) d\log(a) := \delta_1(\gamma) \frac{da}{a},$$

as claimed. \square

Remark 2.3. Let A be a flat \mathbb{Z} -algebra. The description of $\text{HH}(A) = \text{HH}^{\text{usual}}(A)$ as the geometric realization of the simplicial object

$$\text{HH}(A/\mathbb{Z}) := \lim_{\Delta^{\text{op}}} A^{\otimes_{\mathbb{Z}} n+1}$$

shows that $\text{HH}(A; \mathbb{Z}_p)$ is computed by the complex

$$\dots \rightarrow (A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A)_p^\wedge \rightarrow (A \otimes_{\mathbb{Z}} A)_p^\wedge \rightarrow A_p^\wedge.$$

One can then show that the p -completed Dennis trace $(BA^\times)_p^\wedge \rightarrow \text{HH}(A; \mathbb{Z}_p)$ sends an element

$$(a_1, a_2, \dots) \in T_p(A^\times) = \pi_2((BA^\times)_p^\wedge)$$

to the element represented, up to a sign, by the cycle

$$1 \otimes 1 \otimes 1 + \sum_{n=1}^{\infty} p^{n-1} \left(\frac{1}{a_n^2} \otimes a_n \otimes a_n + \frac{1}{a_n^3} \otimes a_n^2 \otimes a_n + \dots + \frac{1}{a_n^p} \otimes a_n^{p-1} \otimes a_n \right).$$

We omit the proof, since we will not use this result.

We can now state and prove the main result of this section. Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. We will describe the image of some element $T_p(A^\times)$ under the composition

$$T_p(A^\times) \xrightarrow{\text{Dtr}} \pi_2(\text{HH}(A; \mathbb{Z}_p)) \rightarrow \pi_2(\text{HH}(A/R; \mathbb{Z}_p)) \xrightarrow{\alpha_\gamma^{-1}} (I/I^2)_p^\wedge,$$

using the notation of Lemma 2.1. Recall first the following standard lemma.

Lemma 2.4. *Let R be a ring, $I \subseteq R$ an ideal and assume that R is (p, I) -adically complete. Then the canonical map*

$$R^{\flat} := \varprojlim_{x \mapsto x^p} R \rightarrow A^{\flat} := \varprojlim_{x \mapsto x^p} A$$

with $A = R/I$ is bijective.

Proof. It suffices to construct a well-defined, multiplicative map

$$[-]: A^{\flat} \rightarrow R$$

reducing to the first projection modulo I . Let

$$r := (r_0, r_1, \dots) \in A^{\flat}$$

be a p -power compatible system of elements in A with lifts $r'_i \in R$ of each r_i . Then the limit

$$\lim_{n \rightarrow \infty} (r'_n)^{p^n}$$

exists and is independent of the lift. Thus

$$[r] := \lim_{n \rightarrow \infty} (r'_n)^{p^n}$$

defines the desired map. \square

The morphism

$$[-]: A^{\flat} \rightarrow R$$

is the Teichmüller lift for the surjection $\pi: R \rightarrow R/I$. If we want to make its dependence of the surjection clear, we write $[-]_{\pi}$. Let

$$T_p A^{\times} = \varprojlim_{x \mapsto x^p} A^{\times}[p^n]$$

be the Tate module of A^{\times} . Then we embed $T_p A^{\times}$ into A^{\flat} as the sequences with first coordinate 1. For any $a \in A^{\flat}$ we define

$$[a] := r_0$$

where $r = (r_0, r_1, \dots) \in R^{\flat}$ is the unique element reducing to a . If $a = (1, a_1, a_2, \dots)$ lies in $T_p A^{\times}$, then $[a] \in 1 + I$.

Proposition 2.5. *Fix a generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$. Let R be a ring and $I \subseteq R$ an ideal such that R is (p, I) -adically complete. Let $A = R/I$. Then the composition*

$$T_p(A^{\times}) \cong \pi_2((BA^{\times})_p^{\wedge}) \xrightarrow{\text{Dtr}} \pi_2(\text{HH}(A/R; \mathbb{Z}_p)) \cong (I/I^2)_p^{\wedge}$$

is given by sending $a \in T_p(A^{\times})$ to

$$\delta_1(\gamma)([a] - 1),$$

with $\delta_1(\gamma) \in \{\pm 1\}$ is the sign from Lemma 2.2.

*Proof.*¹⁵ Fix $a \in T_p(A^{\times})$. Then there exists, by (p, I) -adic completeness of R , a unique morphism $\mathbb{Z}[1/p] \rightarrow R^{\times}$ of abelian groups such that

$$1/p^n \mapsto [a^{1/p^n}].$$

By naturality, it therefore suffices to check that for

$$R := \mathbb{Z}[t^{1/p^{\infty}}] \cong \mathbb{Z}[\mathbb{Z}[1/p]]$$

¹⁵The following argument is simpler than our original argument and was suggested by the referee. We thank her/him for allowing us to include it.

and

$$A := R/(t-1) \cong \mathbb{Z}[\mathbb{Q}_p/\mathbb{Z}_p],$$

then, under the morphism,

$$T_p A^\times \xrightarrow{\text{Dtr}} \text{HH}_2(A; \mathbb{Z}_p) \rightarrow \text{HH}_2(A/R; \mathbb{Z}_p) \cong L_{A/R}[-1] \cong (t-1)/(t-1)^2$$

the element $(1, t^{1/p}, t^{1/p^2}, \dots) \in A^\times$ is mapped to the class of $\delta_1(\gamma)(t-1)$. This is what we will do.

Observe first that the Hochschild homology

$$\text{HH}_2(A)$$

vanishes. Indeed, it is easy to see that $L_{A/\mathbb{Z}}$ is concentrated in degree 0. Moreover, $\Omega_{A/\mathbb{Z}}^1 \cong L_{A/\mathbb{Z}}$ is generated by one element. This implies that

$$\pi_0(\wedge^n L_{A/\mathbb{Z}}) = 0$$

for $n \geq 2$ (cf. the proof of [2, Corollary 3.13]). By the HKR-filtration, we get that $\text{HH}_2(A) = 0$. Passing to p -completions we can conclude that

$$\text{HH}_2(A; \mathbb{Z}_p) \cong T_p \text{HH}_1(A) \xrightarrow{\alpha_\gamma} T_p(\Omega_{A/\mathbb{Z}}^1),$$

where the last isomorphism is the HKR-isomorphism (for γ).

There is a commutative diagram

$$\begin{array}{ccc} \text{HH}_2(A; \mathbb{Z}_p) & \xrightarrow{\cong} & \text{HH}_2(A/R; \mathbb{Z}_p) \\ \downarrow \cong & & \downarrow \cong \\ T_p \Omega_{A/\mathbb{Z}}^1 \cong \pi_1((L_{A/\mathbb{Z}})_p^\wedge) & \xrightarrow{\cong} & \pi_1((L_{A/R})_p^\wedge) \cong ((t-1)/(t-1)^2)_p^\wedge. \end{array}$$

Using Lemma 2.2 the element

$$(1, t^{1/p}, t^{1/p^2}, \dots) \in T_p A^\times$$

is mapped to the element

$$\delta_1(\gamma)(0, d \log(t^{1/p}), d \log(t^{1/p^2}), \dots) \in T_p(\Omega_{A/\mathbb{Z}}^1).$$

The effect of the bottom row can be calculated using the exact triangle

$$L_{R/\mathbb{Z}} \otimes_R^{\mathbb{L}} A \rightarrow L_{A/\mathbb{Z}} \xrightarrow{\beta} L_{A/R}$$

and applying p -completions. More precisely, rotating plus the isomorphisms

$$L_{R/\mathbb{Z}} \cong \Omega_{R/\mathbb{Z}}^1, \quad L_{A/R} \cong (t-1)/(t-1)^2[1]$$

yield the exact triangle

$$(t-1)/(t-1)^2 \xrightarrow{d} \Omega_{R/\mathbb{Z}}^1 \otimes_R A \rightarrow \Omega_{A/R}^1 \rightarrow (t-1)/(t-1)^2[1]$$

where the first morphism is the differential. Now apply (derived) p -completion to this exact triangle, the resulting connecting morphism

$$T_p(\Omega_{A/\mathbb{Z}}^1) \rightarrow (t-1)/(t-1)^2$$

sends to $(0, d \log(t^{1/p}), d \log(t^{1/p^2}), \dots)$ to $t-1$ as $t-1 \equiv \frac{t-1}{t} \pmod{(t-1)^2}$ and

$$\frac{d(t-1)}{t} = d \log(t) = p^n d \log(t^{1/p^n})$$

for all $n \geq 0$ ¹⁶. Thus,

$$\beta((0, d \log(t^{1/p}), d \log(t^{1/p^2}), \dots)) = t - 1$$

as claimed. \square

We recall the following lemma. For a perfect ring S we denote its ring of Witt vectors by $W(S)$.

Lemma 2.6. *Let S be a perfect ring and let A be an $W(S)$ -algebra. Then the canonical morphism*

$$\mathrm{HH}(A; \mathbb{Z}_p) \rightarrow \mathrm{HH}(A/W(S); \mathbb{Z}_p)$$

is an equivalence.

Proof. By the HKR-filtration, it suffices to see that the canonical morphism

$$L_{A/\mathbb{Z}} \rightarrow L_{A/W(S)}$$

of cotangent complexes is a p -adic equivalence, i.e., an equivalence after $-\otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p$. Computing the right hand side by polynomial algebras over $W(S)$ we see that it suffices to consider the case that A is p -torsion free. Then by base change

$$L_{A/\mathbb{Z}} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p \cong L_{(A/p)/\mathbb{F}_p}$$

resp.

$$L_{A/W(S)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p \cong L_{(A/p)/S}$$

and the claim follows from the transitivity triangle

$$A/p \otimes_S^{\mathbb{L}} L_{S/\mathbb{F}_p} \rightarrow L_{(A/p)/\mathbb{F}_p} \rightarrow L_{(A/p)/S}$$

using that S is perfect which implies that the cotangent complex L_{S/\mathbb{F}_p} of S over \mathbb{F}_p vanishes. \square

3. TRANSVERSAL PRISMS

In this section we want to prove the crucial injectivity statement (Corollary 3.10) mentioned in the introduction. Let us recall the following definition from [5].

Definition 3.1. A δ -ring is a pair (A, δ) , where A is a commutative ring, $\delta: A \rightarrow A$ a map of sets, with $\delta(0) = 0$, $\delta(1) = 0$, and

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}; \quad \delta(xy) = x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y),$$

for all $x, y \in A$.

A *prism* (A, I) is a δ -ring A with an ideal I defining a Cartier divisor on $\mathrm{Spec}(A)$, such that A is derived (p, I) -adically complete and $p \in (I, \varphi(I))$.

Here, the map

$$\varphi: A \rightarrow A, \quad x \mapsto \varphi(x) := x^p + p \delta(x)$$

denotes the lift of Frobenius induced from δ -structure on A . We will make the (usually harmless) assumption that $I = (\tilde{\xi})$ is generated by some distinguished element $\tilde{\xi} \in A$, i.e., $\tilde{\xi}$ is a non-zero divisor and $\delta(\tilde{\xi})$ is a unit.

Definition 3.2. We call a prism *transversal* if $(p, \tilde{\xi})$ is a regular sequence on A .

¹⁶If $0 \rightarrow M \rightarrow N \rightarrow Q \rightarrow 0$ is a short exact sequence of abelian groups, then the boundary map $T_p Q \rightarrow M_p^\wedge$ has the following description: take $x := (q_i)_{i \geq 0} \in T_p Q$ and lift each q_i to some $n_i \in N$. Then $p^i n_i \in M$ and the limit $\varprojlim p^i n_i \in M_p^\wedge$ exists and is the image of x .

Let us fix a transversal prism (A, I) . In particular, A is p -torsion free. Moreover, A is classically (p, I) -adically complete. Indeed, $(p, \tilde{\xi})$ being a regular sequence implies that

$$A \otimes_{\mathbb{Z}[x, y]}^{\mathbb{L}} \mathbb{Z}[x, y]/(x^n, y^n) \cong A/(p^n, \tilde{\xi}^n)$$

and therefore

$$A \cong R \varprojlim_n (A \otimes_{\mathbb{Z}[x, y]}^{\mathbb{L}} \mathbb{Z}[x, y]/(x^n, y^n) \cong R \varprojlim_n (A/(p^n, \tilde{\xi}^n)) \cong \varprojlim_n A/(p^n, \tilde{\xi}^n)$$

using Mittag-Leffler for the last isomorphism.

We set

$$I_r := I\varphi(I) \dots \varphi^{r-1}(I)$$

for $r \geq 1$ (where $\varphi^0(I) := I$). Then $I_r = (\tilde{\xi}_r)$ with

$$\tilde{\xi}_r := \tilde{\xi}\varphi(\tilde{\xi}) \dots \varphi^{r-1}(\tilde{\xi}).$$

Lemma 3.3. *For all $r \geq 1$ the element*

$$\varphi^r(\tilde{\xi})$$

is a non-zero divisor and $(\varphi^r(\tilde{\xi}), p)$ is again a regular sequence. In particular, the elements $\tilde{\xi}_r$, $r \geq 1$, are non-zero divisors.

Proof. The regularity of the sequence $(p, \varphi^r(\tilde{\xi}))$, or equivalently of $(p, \tilde{\xi}^{p^r})$, follows from the one of $(p, \tilde{\xi})$. The regularity of $(\varphi(\tilde{\xi}^{p^r}), p)$ follows from this and the fact that in any ring R with a regular sequence (r, s) such that R is r -adically complete the sequence (s, r) is again regular¹⁷. \square

Lemma 3.4. *The ring A is complete for the topology induced by the ideals I_r , i.e.,*

$$A \cong \varprojlim_r A/I_r.$$

Proof. Each A/I_r is p -torsion free by Lemma 3.3. Therefore both sides are p -complete and p -torsion free. Hence, it suffices to check the statement modulo p (note that by p -torsion free of each A/I_r modding out p commutes with the inverse limit). But modulo p the topology defined by the ideals I_r is just the $\tilde{\xi}$ -adic topology and A/p is $\tilde{\xi}$ -adically complete. \square

Lemma 3.5. *For $r \geq 1$ there is a congruence*

$$\varphi^r(\tilde{\xi}) \equiv pu \text{ modulo } (\tilde{\xi})$$

with $u \in A^\times$ some unit.

Proof. For $r = 1$ this follows from

$$\varphi(\tilde{\xi}) = \tilde{\xi}^p + p\delta(\xi)$$

because by definition of distinguishedness the element $\delta(\xi) \in A^\times$ is a unit. For $r \geq 2$ we compute

$$\varphi^r(\tilde{\xi}) = \varphi^{r-1}(\tilde{\xi}^p + p\delta(\tilde{\xi})) = \varphi^{r-1}(\tilde{\xi})^p + p\varphi^{r-1}(\delta(\tilde{\xi})).$$

¹⁷Passing to the inverse limit of the injections $R/r^n \xrightarrow{s} R/r^n$ implies that $s \in R$ is a non-zero divisor. Thus, (r, s) is regular and s is regular, which implies that (s, r) is regular.

By induction we may write $\varphi^{r-1}(\tilde{\xi}) = pu + a\tilde{\xi}$ with $u \in A^\times$ some unit and thus modulo $\tilde{\xi}$ we calculate

$$\varphi^r(\tilde{\xi}) \equiv (pu)^p + p\varphi(\delta(\tilde{\xi})) = p(\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p)$$

with $\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p \in A^\times$ some unit. \square

Lemma 3.6. *For all $r \geq 1$ the sequences $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ and $(\tilde{\xi}, \varphi^r(\tilde{\xi}))$ are again regular.*

Moreover, $I_r = \bigcap_{i=0}^{r-1} \varphi^i(I)$ for all $r \geq 1$.

Proof. We can write $\varphi(\tilde{\xi}) = p\delta(\tilde{\xi}) + \tilde{\xi}^p$, where $\delta(\tilde{\xi}) \in A^\times$ is a unit. By Lemma 3.5 we get $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $(\tilde{\xi})$ with $u \in A^\times$ a unit. As $(\tilde{\xi}, p)$ is a regular sequence we conclude (using [26, Tag 07DW] and Lemma 3.3) that $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ is a regular sequence. To prove the last statement we proceed by induction on r . First note the following general observation: If R is some ring and (f, g) a regular sequence in R , then $(f) \cap (g) = (fg)$. In fact, if $r = sg \in (f) \cap (g)$, then $sg \equiv 0$ modulo f , hence $s \equiv 0$ modulo f as desired. Thus, it suffices to prove that $(\tilde{\xi}_r, \varphi^r(\tilde{\xi}))$ is a regular sequence for $r \geq 1$ (recall that $\tilde{\xi}_r = \tilde{\xi}\varphi(\tilde{\xi}) \cdots \varphi^{r-1}(\tilde{\xi})$). By induction the morphism

$$A/(\tilde{\xi}_r) \rightarrow \prod_{i=0}^{r-1} A/(\varphi^i(\tilde{\xi}))$$

is injective. Hence, it suffices to show that for each $i = 0, \dots, r-1$ the element $\varphi^r(\tilde{\xi})$ maps to a non-zero divisor in $A/(\varphi^i(\tilde{\xi}))$. But this follows from Lemma 3.5 which implies $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $\varphi^i(\tilde{\xi})$ for some unit $u \in A^\times$. \square

We can draw the following corollary.

Lemma 3.7. *Define $\rho: A \rightarrow \prod_{r \geq 0} A/\varphi^r(I)$, $x \mapsto (x \bmod \varphi^r(I))$. Then ρ is injective.*

Proof. This follows from Lemma 3.4 and Lemma 3.6 as the kernel of ρ is given by $\bigcap_{r=1}^{\infty} \varphi^r(I) = \bigcap_{r=1}^{\infty} I_r = 0$. \square

We now define the Nygaard filtration of the prism (A, I) (cf. [5, Definition 11.1]).

Definition 3.8. Define

$$\mathcal{N}^{\geq n} A := \{x \in A \mid \varphi(x) \in I^n A\},$$

the n -th filtration step of the Nygaard filtration.

By definition the Frobenius on A induces a morphism

$$\varphi: \mathcal{N}^{\geq 1} A \rightarrow I.$$

Note that we do not divide the Frobenius by $\tilde{\xi}$. Moreover, we define

$$\sigma: \prod_{i \geq 0} A/\varphi^i(I) \rightarrow \prod_{i \geq 0} A/\varphi^i(I), (x_0, x_1, \dots) \mapsto (0, \varphi(x_0), \varphi(x_1), \dots).$$

Here we use that if $a \equiv b \pmod{\varphi^i(I)}$, then $\varphi(a) \equiv \varphi(b) \pmod{\varphi^{i+1}(I)}$ to get that σ is well-defined. Then the diagram

$$(1) \quad \begin{array}{ccc} \mathcal{N}^{\geq 1}A & \xrightarrow{\rho} & \prod_{i \geq 0} A/\varphi^i(I) \\ \downarrow \varphi & & \downarrow \sigma \\ I & \xrightarrow{\rho} & \prod_{i \geq 0} A/\varphi^i(I) \end{array}$$

commutes where ρ is the homomorphism from Lemma 3.7.

Lemma 3.9. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \rightarrow A/I, \quad x \mapsto x \pmod{(\tilde{\xi})}$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap I$. We want to prove that $x = 0$. Clearly, $x \in \mathcal{N}^{\geq 1}A$. By Lemma 3.7 it suffices to prove that

$$x \equiv 0 \pmod{\varphi^i(I)}$$

for all $i \geq 0$. Write

$$\rho(x) = (x_0, x_1, \dots)$$

By the commutativity of the square (Equation (1)) we get

$$\rho(\varphi(x)) = \sigma(\rho(x)) = (0, \varphi(x_0), \varphi(x_1), \dots).$$

As $\varphi(x) = \tilde{\xi}x$ and therefore $\rho(\varphi(x)) = \tilde{\xi}\rho(x)$ we thus get

$$(\tilde{\xi}x_0, \tilde{\xi}x_1, \dots) = (0, \varphi(x_0), \varphi(x_1), \dots).$$

We assumed that $x \in I$, thus $x_0 = 0 \in A/I$. Now we use that $\tilde{\xi}$ is a non-zero divisor modulo $\varphi^i(I)$ (cf. Lemma 3.6) for $i > 0$. Hence, if $x_i = 0$, then

$$0 = \varphi(x_i) = \tilde{\xi}x_{i+1} \in A/\varphi^{i+1}(I)$$

implies $x_{i+1} = 0$. Beginning with $x_0 = 0$ this shows that $x_i = 0$ for all $i \geq 0$, which implies our claim. \square

The same proof shows that also the reduction map

$$A^{\varphi=\tilde{\xi}^n} \rightarrow A/I$$

is injective for $n \geq 1$.

The following corollary is crucially used in Theorem 6.7.

Corollary 3.10. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \rightarrow \mathcal{N}^{\geq 1}A/\mathcal{N}^{\geq 2}A$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap \mathcal{N}^{\geq 2}A$. Then

$$\tilde{\xi}x = \varphi(x) = \tilde{\xi}^2y$$

for some $y \in A$. As $\tilde{\xi}$ is a non-zero divisor in A we get $x \in I = (\tilde{\xi})$. But then $x = 0$ by Lemma 3.9. \square

Similarly, for each $n \geq 0$ the morphism

$$(2) \quad A^{\varphi=\hat{\xi}^n} \rightarrow \mathcal{N}^{\geq i} A / \mathcal{N}^{\geq i+1} A$$

is injective. Let R be a quasi-regular semiperfectoid ring (cf. [4, Definition 4.19]) which is p -torsion free. In this case,

$$A := \widehat{\Delta}_R$$

is transversal and (Equation (2)) implies that for $i \geq 0$

$$\pi_{2i}(\mathrm{TC}(R)) \rightarrow \pi_{2i}(\mathrm{THH}(R))$$

is injective (cf. [4, Theorem 1.12]). We ignore if there exists a direct topological proof. Note that the p -torsion freeness is necessary. Indeed, by [4, Remark 7.20] $\pi_{2i}(\mathrm{TC}(R))$ is always p -torsion free.

4. THE q -LOGARITHM

In this section we recall the definition of the q -logarithm and prove some properties of it. Several statements in q -mathematics that we use are probably standard; cf. e.g. [24] for more on q -mathematics. Recall that the q -analog of the integer $n \in \mathbb{Z}$ is defined to be

$$[n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q^{\pm 1}].$$

If $n \geq 1$, then we can rewrite

$$[n]_q = 1 + q + \dots + q^{n-1}$$

and then the q -number actually lies in $\mathbb{Z}[q]$. For $n \geq 0$ we moreover get the relation

$$(3) \quad [-n]_q = \frac{q^{-n} - 1}{q - 1} = q^{-n} \frac{1 - q^n}{q - 1} = -q^{-n} [n]_q.$$

The q -numbers satisfy some basic relations, for example

$$(4) \quad [n + k]_q = q^k [n]_q + [k]_q$$

for $n, k \in \mathbb{Z}$, or

$$[m]_q = \frac{(q^n)^k - 1}{q^n - 1} \frac{q^n - 1}{q - 1} = \frac{(q^n)^k - 1}{q^n - 1} [n]_q,$$

if $n|m$. As further examples of q -analogs let us define the q -factorial for $n \geq 1$ as

$$[n]_q! := [1]_q \cdot [2]_q \cdot \dots \cdot [n]_q \in \mathbb{Z}[q]$$

(with the convention that $[0]_q! := 1$) and, for $0 \leq k \leq n$, the q -binomial coefficient as

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Lemma 4.1. 1) For $0 \leq k \leq n$ the q -binomial $\binom{n}{k}_q \in \mathbb{Z}[q]$.

2) For $1 \leq k \leq n$ the analog

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

of Pascal's identity holds.

Proof. 1) follows from 2) using induction and the easy case $\binom{n}{0}_q = 1$. Then 2) can be proved as follows: Let $1 \leq k \leq n$, then

$$\begin{aligned} q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q &= \frac{[n-1]_q!}{[k-1]_q! [n-1-k]_q!} \left(\frac{q^k}{[k]_q} + \frac{1}{[n-k]_q} \right) \\ &= \frac{[n-1]_q!}{[k-1]_q! [n-1-k]_q!} \left(\frac{q^k [n-k]_q + [k]_q}{[k]_q [n-k]_q} \right) \\ &= \frac{[n-1]_q!}{[k-1]_q! [n-1-k]_q!} \frac{[n]_q}{[k]_q [n-k]_q} \\ &= \binom{n}{k}_q \end{aligned}$$

using the addition rule (Equation (4)). \square

Let us define a generalized q -Pochhammer symbol by

$$(x, y; q)_n := (x+y)(x+yq) \dots (x+yq^{n-1}) \in \mathbb{Z}[q^{\pm 1}, x, y]$$

for $n \geq 1$ (setting $x = 1$ and $y := -a$ recovers the known q -Pochhammer symbol

$$(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1}) = (1, -a; q)_n.$$

Moreover we make the convention

$$(x, y; q)_0 := 1.$$

In the q -world the generalized q -Pochhammer symbol replaces the polynomial

$$(x+y)^n.$$

For example one can show (using Lemma 4.1) the following q -binomial formula

$$(5) \quad (x, y; q)_n = \sum_{k=0}^n q^{k(k-1)/2} \binom{n}{k}_q x^{n-k} y^k.$$

Let us now come to q -derivations. We recall that the q -derivative $\nabla_q f$ of some polynomial $f \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}]$ is defined by

$$\nabla_q f(x) := \frac{f(qx) - f(x)}{qx - x} \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}].$$

Thus for example, if $f(x) = x^n$, $n \in \mathbb{Z}$, then we can calculate

$$\nabla_q(x^n) = \frac{q^n x^n - qx}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q x^{n-1}.$$

The q -derivative satisfies an analog of the Leibniz rule, namely

$$\nabla_q(f(x)g(x)) = \nabla_q(f(x))g(qx) + f(x)\nabla_q(g(x)).$$

Similarly to the classical rule

$$\nabla_x((x+y)^n) = n\nabla_x((x+y)^{n-1})$$

we obtain the following relation for the generalized q -Pochhammer symbol.

Lemma 4.2. *Let $\nabla_q := \nabla_{q,x}$ denote the q -derivative with respect to x . Then the formula*

$$\nabla_q((x, y; q)_n) = [n]_q (x, y; q)_{n-1}$$

holds in $\mathbb{Z}[q^{\pm 1}][x^{\pm 1}, y^{\pm 1}]$.

Proof. We proceed by induction on n . Let $n = 1$. Then $(x, y; q)_n = x + y$ and

$$\nabla_q((x + y)) = 1.$$

Now let $n \geq 2$. We calculate using induction

$$\begin{aligned} \nabla_q((x, y; q)_n) &= \nabla_q((x, y; q)_{n-1}(x + yq^{n-1})) \\ &= (x, y, q)_{n-1} \nabla_q(x + yq^{n-1}) + (qx + q^{n-1}y) \nabla_q((x, y; q)_{n-1}) \\ &= (x, y; q)_{n-1} \cdot 1 + q(x + q^{n-2}y)[n-1]_q (x, y; q)_{n-2} \\ &= (1 + q[n-1]_q)(x, y; q)_{n-1} \\ &= [n]_q (x, y; q)_{n-1} \end{aligned}$$

where we used the q -Leibniz rule and (Equation (4)). \square

Similarly as the polynomials

$$1, x - 1, \frac{(x-1)^2}{2!}, \dots, \frac{(x-1)^n}{n!}, \dots$$

are useful for developing some function into a Taylor series around $x = 1$ (because the derivative of one polynomial is the previous one) the q -polynomials

$$1, (x, -1; q)_1, \frac{(x, -1; q)_2}{[2]_q!}, \dots, \frac{(x, -1; q)_n}{[n]_q!}, \dots$$

are useful for developing a q -polynomial into some “ q -Taylor series” around $x = 1$. However, for this to make sense we have to pass to suitable completions and localize at $\{[n]_q\}_{n \geq 1}$. Let us be more precise about this. The $(q-1, x-1)$ -completion $\mathbb{Z}[[q-1, x-1]]$ of $\mathbb{Z}[q, x]$ contains expressions of the form

$$\sum_{n=0}^{\infty} a_n (x, -1; q)_n$$

with $a_n \in \mathbb{Z}[[q-1]]$ because

$$(x, -1; q)_n = (x-1)(x-1+1-q) \dots (x-1+(1-q)\frac{1-q^{n-1}}{1-q}) \in (q-1, x-1)^n.$$

Finally, the next calculations will take place in the ring¹⁸

$$\mathbb{Q}[[q-1, x-1]] \cong \mathbb{Z}[[q-1, x-1]][1/[n]_q | n \geq 1]_{(q-1, x-1)}^{\wedge}$$

because

$$\frac{(x, -1; q)_n}{[n]_q!} \in (q-1, x-1)_{\mathbb{Q}[[q-1, x-1]]}.$$

The ring $\mathbb{Q}[[q-1, x-1]]$ admits a surjection to

$$\mathbb{Q}[[q-1, x-1]] \rightarrow \mathbb{Q}[[x-1]]$$

with kernel generated by $q-1$. Similarly, there is a morphism

$$\text{ev}_1: \mathbb{Q}[[q-1, x-1]] \rightarrow \mathbb{Q}[[q-1]]$$

with kernel generated by $x-1$. Finally, the q -derivative ∇_q extends to a q -derivation on $\mathbb{Q}[[q-1, x-1]]$ and it induces the usual derivative after modding out $q-1$. We denote by ∇_q^n the n -fold decomposition of ∇_q and by

$$f(x)|_{x=1} := \text{ev}_1(f(x))$$

¹⁸Note that inverting $[n]_q$ for $n \geq 0$ and then $q-1$ -adically completing is the same as inverting n for $n \geq 0$ and then $q-1$ -adically completing.

the evaluation at $x = 1$ of an element $f \in \mathbb{Q}[[q - 1, x - 1]]$.

Lemma 4.3. *Let $f(x) \in \mathbb{Q}[[q - 1, x - 1]]$. If $\nabla_q^n(f(x))|_{x=1} = 0$ for all $n \geq 0$, then $f(x) = 0$.*

Proof. As ∇_q reduces to the usual derivative modulo $q - 1$, we see that f must be divisible by $q - 1$, i.e., we can write $f(x) = (q - 1)g(x)$ with $g(x) \in \mathbb{Q}[[q - 1, x - 1]]$. But then $\nabla_q^n(g(x))|_{x=1} = 0$ for all $n \geq 0$ and we can conclude as before that $q - 1 | g(x)$ which in the end implies

$$f(x) \in \bigcap_{k=1}^{\infty} (q - 1)^k = \{0\}$$

because $\mathbb{Q}[[q - 1, x - 1]]$ is $(q - 1)$ -adically separated. \square

Now we can state the q -Taylor expansion around $x = 1$ for elements in $\mathbb{Q}[[q - 1, x - 1]]$.

Proposition 4.4. *For any $f(x) \in \mathbb{Q}[[q - 1, x - 1]]$ there is the Taylor expansion*

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))|_{x=1} \frac{(x, -1; q)_n}{[n]_q!}.$$

Proof. Because

$$\nabla_q \left(\frac{(x, -1; q)_n}{[n]_q!} \right) = \frac{(x, -1; q)_{n-1}}{[n-1]_q!}$$

we can directly calculate that both sides have equal higher derivatives at $x = 1$. Thus they agree by Lemma 4.3. \square

Using this we can in Lemma 4.6 motivate the following formula for the q -logarithm.

Definition 4.5. We define the q -logarithm as

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x, -1; q)_n}{[n]_q} \in \mathbb{Q}[[q - 1, x - 1]].$$

Note that the element $\log_q(x)$ is contained in a much smaller subring of $\mathbb{Q}[[q - 1, x - 1]]$, it suffices to adjoin the elements $\frac{(x, -1; q)_n}{[n]_q}$ for $n \geq 0$ to $\mathbb{Z}[q^{\pm 1}, x^{\pm 1}]$ and $(x - 1)$ -adically complete.

In the ring $\mathbb{Q}[[q - 1, x - 1]]$ the element x is invertible, as

$$\frac{1}{x} = \frac{1}{1 - (1 - x)} = 1 + (1 - x) + (1 - x)^2 + \dots$$

The q -derivative of the q -logarithm is $1/x$, similarly to the usual logarithm.

Lemma 4.6. *The q -logarithm $\log_q(x)$ is the unique $f(x) \in \mathbb{Q}[[q - 1, x - 1]]$ satisfying $f(1) = 0$ and $\nabla_q(f(x)) = \frac{1}{x}$. Moreover,*

$$\log_q(x) = \frac{q - 1}{\log(q)} \log(x)$$

as elements in $\mathbb{Q}[[q - 1, x - 1]]$.

Proof. That $\log_q(x)$ has q -derivative $1/x$ can be checked using Proposition 4.4 after writing $1/x$ in its q -Taylor expansion. Moreover, $\log_q(1) = 0$. For the converse pick f as in the statement. By Proposition 4.4 we can write

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))|_{x=1} \frac{(x, -1; q)_n}{[n]_q!}.$$

and thus we have to determine

$$a_n := \nabla_q^n(f(x))|_{x=1}$$

for $n \geq 0$. By assumption we must have $a_0 = f(1) = 0$. Moreover, for $n \geq 1$

$$a_n = \nabla_q^n(f(x))|_{x=1} = \nabla_q^{n-1}(x^{-1})|_{x=1} = [-n+1]_q \dots [-1]_q.$$

Using $[-k]_q = -q^{-k}[k]_q$ for $k \in \mathbb{Z}$ the last expression simplifies to

$$[-n+1]_q \dots [-1]_q = (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q!.$$

Thus we can conclude

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x, -1; q)_n}{[n]_q} = \log_q(x).$$

For the last statement note that

$$f(x) := \frac{q-1}{\log(q)} \log(x)$$

exists in $\mathbb{Q}[[q-1, x-1]]$ (because $n \in R_q^\times$ for all $n \geq 1$) and satisfies $f(1) = 0$. Moreover,

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x} = \frac{q-1}{\log(q)} \frac{\log(q) + \log(x) - \log(x)}{(q-1)x} = \frac{1}{x}$$

which implies $f(x) = \log_q(x)$ by the proven uniqueness of the q -logarithm. \square

We now turn to prisms again. Define

$$\tilde{\xi} := [p]_q = 1 + q + \dots + q^{p-1}$$

and

$$\tilde{\xi}_r = \tilde{\xi} \varphi(\tilde{\xi}) \dots \varphi^{r-1}(\tilde{\xi})$$

for $r \geq 1$. Here, φ is the Frobenius lift on $\mathbb{Z}[q^{\pm 1}]$ satisfying $\varphi(q) = q^p$. Then $\tilde{\xi}$ is a distinguished element in the prism $\mathbb{Z}_p[[q-1]]$. The $\tilde{\xi}_r$ are again q -numbers, namely

$$\tilde{\xi}_r = [p^r]_q.$$

Let us recall the following situation from crystalline cohomology. Assume that A is a p -complete ring with an ideal $J \subseteq A$ equipped with divided powers

$$\gamma_n: J \rightarrow J, \quad n \geq 1.$$

In this situation the logarithm

$$\log(x) := \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \gamma_n(x-1)$$

converges in A for every element $x \in 1 + J$. We now want to prove an analogous statement for the q -logarithm. Recall that for a prism (A, I) we defined the Nygaard filtration

$$\mathcal{N}^{\geq n} A := \{x \in A \mid \varphi(x) \in I^n\}, \quad n \geq 0$$

in Definition 3.8. From now on, we assume that the prism (A, I) lives over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$. The expression

$$\gamma_{n,q}(x-y) := \frac{(x-y)(x-xy)\cdots(x-q^{n-1}y)}{[n]_q!} \in \mathbb{Z}_p[[q-1]][x,y][1/[m]_q \mid m \geq 0]$$

is called the n -th q -divided power of $x-y$ (cf. [21, Rem. 1.4])¹⁹. We will study the divisibility of

$$(x-y)(x-xy)\cdots(x-q^{n-1}y)$$

by

$$\tilde{\xi}, \varphi(\tilde{\xi}), \dots$$

The following statement is clear.

Lemma 4.7. *For $r \geq 1$ the polynomial (in q)*

$$\varphi^{r-1}(\tilde{\xi}) = \frac{q^{p^r} - 1}{q^{p^{r-1}} - 1}$$

is the minimal polynomial of a p^r -th root of unity ζ_{p^r} , i.e., the morphism

$$\mathbb{Z}[q]/(\varphi^{r-1}(\tilde{\xi})) \rightarrow \mathbb{Z}[\zeta_{p^r}], \quad q \mapsto \zeta_{p^r}$$

is injective.

Thus reducing modulo $\varphi^{r-1}(\tilde{\xi})$ is the same as setting $q = \zeta_{p^r}$. Moreover, in $\mathbb{Z}[\zeta_{p^r}]$ there is the equality

$$z^{p^r} - 1 = \prod_{i=0}^{p^r-1} (z - \zeta_{p^r}^i).$$

Setting $z = \frac{x}{y}$ one thus arrives at the congruence

$$(6) \quad x^{p^r} - y^{p^r} \equiv (x-y)(x-xy)\cdots(x-q^{p^r-1}y) \pmod{\varphi^{r-1}(\tilde{\xi})},$$

which will be useful.

Lemma 4.8. *Let $n \geq 1$ and for $r \geq 1$ write $n = a_r p^r + b_r$ with $a_r, b_r \geq 0$ and $b_r < p^r$. Then in $\mathbb{Z}_p[[q-1]]$*

$$[n]_q! = u \prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in \mathbb{Z}_p[[q-1]]^\times$.

Proof. We may prove the statement by induction on n . Thus let us assume that it is true for $m = n-1$ and for $r \geq 1$ write $m = c_r p^r + d_r$ with $c_r, d_r \geq 0$ and $d_r < p^r$. If n is prime to p , then $[n]_q$ is a unit in $\mathbb{Z}_p[[q-1]]$ and it suffices to see that the righthand side is equal (up to some unit in $\mathbb{Z}_p[[q-1]]$) to

$$\prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r}.$$

But n prime to p implies that $b_r > 0$ for all $r \geq 1$. Thus $c_r = a_r$ and $d_r = b_r - 1$, which implies that both products are equal. Now assume that p divides n and write

¹⁹This terminology is, however, quite bad. The q -divided power depends on the pair (x, y) and not simply their difference $x-y$.

$n = p^s n'$ with n' prime to p . Moreover, write $m = n - 1 = c_r p^r + d_r$ as above. Then we can conclude $a_r = c_r$ for $r > s$ while $c_r = a_r - 1$ for $1 \leq r \leq s$ (as $d_r = p^r - 1$ for such r). Altogether we therefore arrive at

$$\begin{aligned} [n]_q! &= [n]_q [n-1]_q! \\ &= u' [n]_q \prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r}, \\ &= u' v \prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r} \end{aligned}$$

$u' \in \mathbb{Z}_p[[q-1]]^\times$, where we used that

$$[n]_q = v [p^s]_q = v \varphi^{s-1}(\tilde{\xi}) \dots \tilde{\xi}$$

for some unit $v \in \mathbb{Z}_p[[q-1]]$. \square

Proposition 4.9. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$ and let $x, y \in A$ be elements of rank 1 such that $\varphi(x-y) = x^p - y^p \in \tilde{\xi}A$. Then for all $n \geq 1$ the ring A contains a q -divided power*

$$\gamma_{n,q}(x-y) = \frac{(x-y)(x-xy) \cdots (x-q^{n-1}y)}{[n]_q!}$$

of $x-y$ ²⁰. Moreover, $\gamma_{n,q}$ lies in fact in the n -th step $\mathcal{N}^{\geq n}A$ of the Nygaard filtration of A .

Proof. Replacing A, x, y by the universal case we may assume that A is flat over $\mathbb{Z}_p[[q-1]]$. In particular, this implies that $\tilde{\xi}, \varphi(\tilde{\xi}), \dots$ are pairwise regular sequences (cf. Lemma 3.6). Fix $n \geq 1$. For $r \geq 1$ we write n as

$$n = a_r p^r + b^r$$

with $a_r, b_r \geq 0$ and $0 \leq b^r < p^r$. We claim that for each $r \geq 0$

$$\varphi^{r-1}(\tilde{\xi})^{a_r}$$

divides

$$(x-y)(x-xy) \cdots (x-q^{n-1}y).$$

This implies the proposition, namely by Lemma 4.8 we have

$$[n]_q! = u \prod_{r \geq 1} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in A^\times$ while furthermore the morphism

$$A/([n]_q!) \rightarrow \prod_{r \geq 1} A/(\varphi^{r-1}(\tilde{\xi})^{a_r})$$

is injective by the proof of Lemma 3.6. Thus fix $r \geq 1$. To prove our claim we may replace n by $n - b_r$ as

$$(x-y)(x-xy) \cdots (x-q^{n-b_r-1}y)$$

divides

$$(x-y)(x-xy) \cdots (x-q^{n-1}y).$$

²⁰By this we mean that there exists an element, called $\gamma_{n,q}(x-y)$, such that $[n]_q! \gamma_{n,q}(x-y) = (x-y)(x-xy) \cdots (x-q^{n-1}y)$. The element $\gamma_{n,q}(x-y)$ need not be unique, but it is if A is $[n]_q$ -torsion free for any $n \geq 0$. Note that even in this torsion free case $\gamma_{n,q}(x-y)$ depends on the pair (x, y) and not merely on the difference $x-y$.

Thus let us assume that $n = a_r p^r$. We claim that each of the following a_r many elements (note that their product is $(x - y) \cdots (x - q^{n-1}y)$)

$$\begin{aligned} & (x - y)(x - qy) \cdots (x - q^{p^r-1}y), \\ & (x - q^{p^r}y)(x - q^{p^r+1}y) \cdots (x - q^{2p^r-1}y), \\ & \vdots \\ & (x - q^{(a_r-1)p^r}y)(x - q^{(a_r-1)p^r+1}y) \cdots (x - q^{a_r p^r-1}y), \end{aligned}$$

is divisible by $\varphi^{r-1}(\tilde{\xi})$. For this recall the congruence (Equation (6))

$$x^{p^r} - y^{p^r} \equiv (x - y)(x - qy) \cdots (x - q^{p^r-1}y) \pmod{\varphi^{r-1}(\tilde{\xi})}.$$

Replacing in this congruence y by $q^{p^r}y, \dots, q^{(a_r-1)p^r}y$ shows that each of the above a_r elements is congruent modulo $\varphi^{r-1}(\tilde{\xi})$ to an element of the form

$$x^{p^r} - q^k y^{p^r}$$

with $k \geq 0$ divisible by p^r . But we have

$$x^{p^r} - q^k y^{p^r} = (x^{p^r} - y^{p^r}) + y^{p^r}(1 - q^k)$$

and we claim that under our assumptions both summands are divisible by $\varphi^{r-1}(\tilde{\xi})$. For the first summand we use that x, y are of rank 1 to write

$$x^{p^r} - y^{p^r} = \varphi^{r-1}(x^p - y^p) = \varphi^{r-1}(\tilde{\xi})\varphi^{r-1}\left(\frac{x^p - y^p}{\tilde{\xi}}\right)$$

which makes sense as we assumed that

$$x^p - y^p \in \tilde{\xi}A.$$

For the second summand we note that

$$1 - q^k = \frac{1 - q^k}{1 - q^{p^r}} \varphi^{r-1}(\tilde{\xi})(1 - q^{p^r-1})$$

with all factors in $\mathbb{Z}_p[[q-1]]$ as p^r divides k . It remains to prove that

$$\gamma_{n,q}(x - y) = \frac{(x - y)(x - qy) \cdots (x - q^{n-1}y)}{[n]_q!}$$

lies in $\mathcal{N}^{\geq n}A$. But

$$\varphi(\gamma_{n,q}) = \frac{(x^p - y^p)(x^p - q^p y^p) \cdots (x^p - q^{p(n-1)} y^p)}{\varphi([n]_q!)}$$

and as we saw above $\tilde{\xi}$ divides each of the n factors

$$(x^p - y^p), (x^p - q^p y^p), \dots, (x^p - q^{p(n-1)} y^p).$$

But $\tilde{\xi}$ and $\varphi([n]_q!)$ form a regular sequence by Lemma 3.6 which implies that

$$(x^p - y^p)(x^p - q^p y^p) \cdots (x^p - q^{p(n-1)} y^p)$$

is divisible by $\tilde{\xi}^n \varphi([n]_q)$ as was to be proven. This finishes the proof of the proposition. \square

As the proof shows there exists unique choice of a q -divided power

$$\gamma_{n,q}(x-y)$$

which is functorial in the triple (A, x, y) (with $x, y \in A$ satisfying the assumptions in Proposition 4.9). From now on we will always assume that these q -divided powers are chosen.

Moreover, we get the following lemma concerning the convergence of the q -logarithm.

Lemma 4.10. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$. Then for every element $x \in 1 + \mathcal{N}^{\geq 1}A$ of rank 1 the series*

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q! \gamma_{n,q}(x-1)$$

is well-defined and converges in A . Moreover, $\log_q(x) \in \mathcal{N}^{\geq 1}A$ and

$$\log_q(x) \equiv x-1 \pmod{\mathcal{N}^{\geq 2}A}$$

and

$$\log_q(xy) = \log_q(x) + \log_q(y)$$

for any $x, y \in 1 + \mathcal{N}^{\geq 1}A$ of rank 1.

Proof. By our assumption on x we get $\varphi(x-1) \in \tilde{\xi}A$ and thus we may apply Proposition 4.9 to $x = x$ and $y = 1$. Thus the (canonical choice of) q -divided powers

$$\gamma_{n,q}(x-1) = \frac{(x-1)(x-q) \cdots (x-q^{n-1})}{[n]_q!}$$

in A are well-defined. Moreover, as

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q! \gamma_{n,q}(x-1)$$

and the elements $[n-1]_q!$ tend to zero in A for the (p, I) -adic topology we can conclude that the series $\log_q(x)$ converges because A is $\tilde{\xi}$ -adically complete. The claim concerning the Nygaard filtrations follows directly from $\gamma_{n,q}(x-1) \in \mathcal{N}^{\geq n}A$, which was proven in Proposition 4.9. That \log_q is a homomorphism can be seen in the universal case in which A is flat over $\mathbb{Z}_p[[q-1]]$ (by [5, Proposition 3.13]). Then the formula $\log_q(xy) = \log_q(x) + \log_q(y)$ can be checked after base change to $\mathbb{Q}_p[[q-1]]$ where it follows from Lemma 4.6 as the usual logarithm is a homomorphism. \square

5. PRISMATIC COHOMOLOGY AND TOPOLOGICAL CYCLIC HOMOLOGY

This section is devoted to the relation of the prismatic cohomology developed by Bhatt and Scholze [5] with topological cyclic homology (as described by Bhatt, Morrow and Scholze [4]) following [5, Section 11.5].

Let R be a quasi-regular semiperfectoid ring (cf. [4, Definition 4.19.]), and let S be any perfectoid ring with a map $S \rightarrow R$.

Proposition 5.1. *The category of prisms (A, I) with a map $R \rightarrow A/I$ admits an initial object $(\Delta_R^{\text{init}}, I)$, which is a bounded prism. Moreover, Δ_R^{init} identifies with the derived prismatic cohomology $\Delta_{R/A_{\text{inf}}(S)}$, for any choice of S as before.*

Proof. See [5, Proposition 7.2, Proposition 7.10] or [1, Proposition 3.4.2]. \square

In the following, we simply write $\Delta_R = \Delta_R^{\text{init}} = \Delta_{R/A_{\text{inf}}(S)}$.

Theorem 5.2. *Let R be a quasi-regular semiperfectoid ring. There is a functorial (in R) δ -ring structure on $\widehat{\Delta}_R^{\text{top}} := \pi_0(\text{TC}^-(R; \mathbb{Z}_p))$ refining the cyclotomic Frobenius. The induced map $\Delta_R = \Delta_R^{\text{init}} \rightarrow \widehat{\Delta}_R^{\text{top}}$ identifies $\widehat{\Delta}_R^{\text{top}}$ with the completion with respect to the Nygaard filtration (Definition 3.8) of Δ_R , and is compatible with the Nygaard filtration on both sides.*

Proof. See [5, Theorem 11.10]. \square

The Nygaard filtration on $\widehat{\Delta}_R^{\text{top}}$ is defined as the double-speed abutment filtration for the (degenerating) homotopy fixed point spectral sequence

$$E_2^{ij} := H^i(\mathbb{T}, \pi_{-j}(\text{THH}(R; \mathbb{Z}_p))) \Rightarrow \pi_{-i-j}(\text{TC}^-(R; \mathbb{Z}_p))$$

for the $\mathbb{T} = S^1$ -action on $\text{THH}(R; \mathbb{Z}_p)$. If $\eta \in H^2(\mathbb{T}, \mathbb{Z})$ is a generator, then multiplication by any lift $v \in \pi_{-2}(\text{TC}^-(R; \mathbb{Z}_p))$ of the image of η in $H^2(\mathbb{T}, \pi_0(\text{THH}(R; \mathbb{Z}_p)))$ induces isomorphisms

$$\pi_{2i}(\text{TC}^-(R; \mathbb{Z}_p)) \cong \mathcal{N}^{\geq i} \widehat{\Delta}_R^{\text{top}}$$

for $i \in \mathbb{Z}$.

Remark 5.3. We will only use the fact that $\widehat{\Delta}_R$ is a prism in this paper (as we will apply the results of Section 3 to $\pi_0(\text{TC}^-(R; \mathbb{Z}_p))$) and that the topological Nygaard filtration, defined via the homotopy fixed point spectral sequence, agrees with the Nygaard filtration from Definition 3.8, but the way one proves this is by showing the stronger statement that $\widehat{\Delta}_R^{\text{top}}$ is the Nygaard completion of Δ_R . We ignore if there is a more direct way to produce the δ -structure on $\widehat{\Delta}_R$ (cf. [5, Remark 1.14.]).

6. THE p -COMPLETED CYCLOTOMIC TRACE IN DEGREE 2

Now we are settled to prove our main theorem on the identification of the p -completed cyclotomic trace. Recall that for any ring A the cyclotomic trace

$$\text{ctr}: K(A) \rightarrow \text{TC}(A)$$

from the algebraic K -theory of A to its topological cyclic homology is a natural morphism²¹ refining the Dennis trace $\text{Dtr}: K(A) \rightarrow \text{HH}(A)$ introduced in Section 2, cf. [6, Section 10.3], [7, Section 5]. Let us carefully fix some notation. For the whole section we fix a generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$, but note that the formulas in Theorem 6.7 will be independent of this choice. Set $\mathbb{Z}_p^{\text{cycl}}$ as the p -completion of $\mathbb{Z}_p[\mu_{p^\infty}]$ and choose some p -power compatible system of p -power roots of unity

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in (\mathbb{Z}_p^{\text{cycl}})^{\flat}$$

with $\zeta_p \neq 1$. This choice determines several elements as we will now discuss. Set

$$q := [\varepsilon]_{\theta} \in A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}}) := W((\mathbb{Z}_p^{\text{cycl}})^{\flat}) \cong \pi_0(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)),$$

$$\mu := q - 1,$$

$$\tilde{\xi} := [p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \dots + q^{p-1}.$$

²¹When upgraded to a natural transformation of functors on small stable ∞ -categories the cyclotomic trace is uniquely determined by these properties, cf. [6, Section 10.3].

and

$$\xi = \varphi^{-1}(\tilde{\xi}).$$

Note that the ring $A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}})$ is the $(p, q-1)$ -adic completion of $\mathbb{Z}_p[q^{1/p^\infty}]$. We now construct elements

$$\begin{aligned} u &\in \pi_2(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)), \\ v &\in \pi_{-2}(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \end{aligned}$$

such that $uv = \xi \in \pi_0(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$ ²². The elements u, v will be uniquely determined by ε . Let

$$\text{ctr}: T_p(\mathbb{Z}_p^{\text{cycl}})^\times \rightarrow \pi_2(\text{TC}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

be the cyclotomic trace in degree 2. We denote by the same symbol the composition

$$\text{ctr}: T_p(\mathbb{Z}_p^{\text{cycl}})^\times \rightarrow \pi_2(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

with the canonical morphism $\text{TC}(-; \mathbb{Z}_p) \rightarrow \text{TC}^(-; \mathbb{Z}_p)$. Let

$$\text{can}: \text{TC}^(-; \mathbb{Z}_p) \rightarrow \text{TP}(-; \mathbb{Z}_p)$$

be the canonical morphism (from homotopy to Tate fixed points).

Lemma 6.1. *The element*

$$\text{can}(\text{ctr}(\varepsilon)) \in \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

is divisible by μ .

A similar statement (in terms of TF) is proven in [13, Proposition 2.4.2] (cf. [14, Definition 4.1]) using the explicit description of the cyclotomic trace in degree 1 via TR from [12, Lemma 4.2.3.].

Proof. Fix a generator

$$\sigma' \in \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)).$$

It suffices to show that $\text{can} \circ \text{ctr}(\varepsilon)$ maps to 0 under the composition

$$\pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \xrightarrow{\sigma'^{-1}} \pi_0(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \cong A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}}) \rightarrow W(\mathbb{Z}_p^{\text{cycl}})$$

because the kernel of $A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}}) \rightarrow W(\mathbb{Z}_p^{\text{cycl}})$ is generated by μ (cf. [3, Lemma 3.23]). It therefore suffices to prove the statement for \mathcal{O}_C for $C/\mathbb{Q}_p^{\text{cycl}}$ an algebraically closed, complete non-archimedean extension. Over \mathcal{O}_C we can (after changing σ') find

$$\begin{aligned} u' &\in \pi_2(\text{TC}^-(\mathcal{O}_C; \mathbb{Z}_p)), \\ v' &\in \pi_{-2}(\text{TC}^-(\mathcal{O}_C; \mathbb{Z}_p)) \end{aligned}$$

such that

$$\begin{aligned} u'v' &= \xi = \frac{\mu}{\varphi^{-1}(\mu)}, \\ \text{can}(v') &= \sigma'^{-1} \end{aligned}$$

and the cyclotomic Frobenius maps u' to σ' , cf. [4, Proposition 6.2., Proposition 6.3.]. Then multiplication by v' induces an isomorphism

$$\pi_2(\text{TC}(\mathcal{O}_C; \mathbb{Z}_p)) \cong A_{\text{inf}}(\mathcal{O}_C)^{\varphi=\xi}.$$

²²We need a finer statement than [4, Proposition 6.2 and Proposition 6.3] which asserts the existence of some u, v as above with $uv = a\xi$ for some unspecified unit $a \in A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}})^\times$.

By [10, Proposition 6.2.10.]

$$(A_{\text{inf}}(\mathcal{O}_C)[1/p])^{\varphi=\tilde{\xi}}$$

is 1-dimensional over \mathbb{Q}_p and thus generated by μ (as $\mu \neq 0$ and $\varphi(\mu) = \tilde{\xi}\mu$). But μ is not divisible by p in $A_{\text{inf}}(\mathcal{O}_C)$ as it maps to a unit in $W(C)$. This proves that $A_{\text{inf}}(\mathcal{O}_C)^{\varphi=\tilde{\xi}} = \mathbb{Z}_p\mu$, which implies the claim. \square

Let us define

$$\sigma := \frac{\text{ctr}(\varepsilon)}{\mu} \in \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

and

$$u := \xi\sigma \in \pi_2(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}})).$$

More precisely, the element u is defined via $\text{can}(u) = \xi\sigma$ (note that $\xi\sigma$ lies indeed in the image of

$$\text{can}: \pi_2(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \rightarrow \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

as the abutment filtration for the Tate fixed point spectral sequence on $\pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$ is the ξ -adic filtration).

Lemma 6.2. *The element u defined above lifts the class of*

$$\delta_1(\gamma)\xi \in \pi_2(\text{THH}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \cong \pi_2(\text{HH}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \xrightarrow{\alpha_\gamma} (\xi)/(\xi^2).$$

Proof. By definition

$$\text{can}(u) = \frac{\xi}{\mu} \text{ctr}(\varepsilon) \in \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)).$$

Now

$$\frac{\xi}{\mu} = \frac{1}{\varphi^{-1}(\mu)}$$

and $(\xi)/(\xi^2)$ is $\varphi^{-1}(\mu)$ -torsion free as a module over $A_{\text{inf}}(\mathbb{Z}_p^{\text{cycl}})$ (because

$$\theta(\varphi^{-1}(\mu)) = \zeta_p - 1 \neq 0 \in \mathbb{Z}_p^{\text{cycl}}).$$

Moreover, the cyclotomic trace lifts the Dennis trace in Hochschild homology. Thus by Proposition 2.5,

$$\alpha_\gamma(\text{Dtr}(\varepsilon)) \equiv \delta_1(\gamma)([\varepsilon] - 1) \in (\xi)/(\xi^2)$$

and therefore

$$u \equiv \delta_1(\gamma) \frac{[\varepsilon] - 1}{\varphi^{-1}(\mu)} = \delta_1(\gamma)\xi \in (\xi)/(\xi^2)$$

as desired. \square

In particular, we see that the element

$$\sigma \in \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$$

is a generator. Set

$$v := \sigma^{-1} \in \pi_{-2}(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)) \xrightarrow{\text{can}} \pi_2(\text{TP}(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p)).$$

Then

$$uv = \xi.$$

Recall that for any morphism of rings $R \rightarrow A$ the negative cyclic homology is defined to be

$$\mathrm{HC}^-(A/R) := \mathrm{HH}(A/R)^{h\mathbb{T}}$$

where $(-)^{h\mathbb{T}} := \varprojlim_{B\mathbb{T}}(-)$, cf. [15] for a comparison with the classical definition in [17,

Definition 5.1.3]. The homotopy fixed point spectral sequence

$$H^i(B\mathbb{T}, \pi_{-j}(\mathrm{HH}(A/R))) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(A/R))$$

endows $\pi_*(\mathrm{HC}^-(A/R))$ with a (multiplicative) decreasing filtration which we denote by

$$\mathcal{N}^{\geq \bullet} \mathrm{HC}^-(A/R).$$

We note that each generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ defines canonically a generator $\eta_\gamma \in H^2(B\mathbb{T}, \mathbb{Z})$. We will do the abuse of notation and denote by $\gamma \in H_1(\mathbb{T}, R)$ the image of $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$; similarly, for η_γ .

Proposition 6.3. *Let $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ be a generator and assume $A = R/(f)$ for some non-zero divisor $f \in R$. Then*

- (1) $\mathrm{HH}_*(A/R)$ is concentrated in even degrees and the homotopy fixed point spectral sequence

$$H^i(B\mathbb{T}, \pi_{-j}(\mathrm{HH}(A/R))) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(A/R))$$

degenerates.

- (2) There exists a unique element $\delta_2 \in \{\pm 1\}$, independent of the choice of γ , such that the morphism

$$(f)/(f)^2 \xrightarrow{\alpha_\gamma} \pi_2(\mathrm{HH}(A/R)) \xrightarrow{\tilde{\eta}_\gamma} \pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2} \mathrm{HC}^-(A/R)$$

sends the class of f to $\delta_2 f \cdot 1_{\pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2} \mathrm{HC}^-(A/R)}$. Here the first isomorphism is the one of Lemma 2.1. The second morphism is the multiplication by some lift $\tilde{\eta}_\gamma \in \pi_{-2}(\mathrm{HC}^-(A/R))$ of $\eta_\gamma \in H^2(B\mathbb{T}, \pi_0(\mathrm{HH}(A/R)))^{23}$.

Proof. The first claim follows from the HKR-filtration as the exterior powers

$$\wedge^i L_{A/R}[i]$$

are concentrated in even degrees for all $i \geq 0$. For the second claim we can reduce by naturality to the universal case $R = \mathbb{Z}[x]$, $f = x$ in which case it is well-known that the elements

$$\tilde{\eta}_\gamma(\alpha_\gamma(f)), f \cdot 1_{\pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2} \mathrm{HC}^-(A/R)}$$

are generators of the free A -module $\mathcal{N}^{\geq 1} \pi_0(\mathrm{HC}^-(A/R)/\mathcal{N}^{\geq 2} \pi_0(\mathrm{HC}^-(A/R)))$ of rank 1. This implies the existence of δ_2 as $A \cong \mathbb{Z}$. As the composition $\tilde{\eta}_\gamma \circ \alpha_\gamma$ is independent of the choice of $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ (because both α_γ and $\tilde{\eta}_\gamma$ will be changed by a sign), the proof is finished. \square

Remark 6.4. We expect that $\delta_2 = 1$, but did not make the explicit computation, since we will not need it.

We need the following relation of v to η_γ .

²³As we mod out by $\mathcal{N}^{\geq 2}$ and the spectral sequence degenerates, the second morphism does not depend on the choice of a lift.

Lemma 6.5. *Let $\eta_\gamma, \underline{v} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)))$ be the images of $\eta_\gamma \in H^2(\mathbb{T}, \mathbb{Z})$ resp. $v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$ under the canonical morphisms*

$$H^2(\mathbb{T}, \mathbb{Z}) \rightarrow H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)))$$

resp.

$$\pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \rightarrow H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))).$$

Then

$$\underline{v} = \delta_2 \delta_1(\gamma) \eta_\gamma.$$

Proof. By Lemma 6.2 we know that the image of u in

$$\pi_2(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \cong (\xi)/(\xi^2)$$

is

$$\alpha_\gamma(\delta_1(\gamma)\xi).$$

As $\eta_\gamma, \underline{v} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)))$ there exists some unit $r \in \mathbb{Z}_p^{\mathrm{cycl}}$ such that $r\eta_\gamma = \underline{v}$. We can calculate in $\pi_0(\mathrm{HC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$

$$\begin{aligned} \xi &= uv = v\alpha_\gamma(\delta_1(\gamma)\xi) \\ &= r\eta_\gamma\alpha_\gamma(\delta_1(\gamma)\xi) \\ &= r\delta_2\delta_1(\gamma)\xi \end{aligned}$$

using Proposition 6.3. Thus, $r = \delta_2\delta_1(\gamma)$. \square

One has the following (important) additional property (which, up to changing ξ by some unit, is implied by the conjunction of [4, Proposition 6.2., Proposition 6.3.]).

Lemma 6.6. *The cyclotomic Frobenius*

$$\varphi^{h\mathbb{T}}: \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \rightarrow \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

sends u to σ .

Proof. The cyclotomic Frobenius $\varphi^{h\mathbb{T}}$ is linear over the Frobenius on A_{inf} . Thus we can calculate (note $\frac{\xi}{\mu} = \frac{1}{\varphi^{-1}(\mu)}$)

$$\varphi^{h\mathbb{T}}(u) = \varphi\left(\frac{\xi}{\mu}\right)\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)) = \frac{1}{\mu}\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)).$$

But

$$\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)) = \mathrm{can}(\mathrm{ctr}(\varepsilon))$$

as the cyclotomic trace has image in $\pi_2(\mathrm{TC}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$. This implies that

$$\varphi^{h\mathbb{T}}(u) = \frac{\mathrm{ctr}(\varepsilon)}{\mu} = \sigma$$

as desired. \square

By Lemma 6.6 one can conclude that there is a commutative diagram, whose vertical arrows are isomorphisms,

$$\begin{array}{ccccc} \pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) & \longrightarrow & \pi_2(\mathrm{TC}^-(R; \mathbb{Z}_p)) & \xrightarrow{\varphi^{h\mathbb{T}}-\mathrm{can}} & \pi_2(\mathrm{TP}(R; \mathbb{Z}_p)) \\ \downarrow \beta_\varepsilon & & \downarrow v & & \downarrow \sigma^{-1} \\ \widehat{\Delta}_R^{\varphi=\tilde{\xi}} & \longrightarrow & \mathcal{N}^{\geq 1}\widehat{\Delta}_R & \xrightarrow{\frac{\xi}{\xi}-1} & \widehat{\Delta}_R \end{array}$$

for any quasi-regular semiperfectoid $\mathbb{Z}_p^{\text{cycl}}$ -algebra R . We remind the reader that the induced isomorphism

$$\beta_\varepsilon : \pi_2(\text{TC}(R; \mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

depends only on ε , not on γ .

For a quasi-regular semiperfectoid ring R we denote by

$$[-]_{\tilde{\theta}} : R^{\flat} = \varprojlim_{x \rightarrow x^p} R \rightarrow \Delta_R$$

the Teichmüller lift. More precisely, the canonical morphism $R \rightarrow \overline{\Delta}_R$ induces a morphism $\iota : R^{\flat} \rightarrow \overline{\Delta}_R^{\flat}$ and $[-]_{\tilde{\theta}}$ is the composition of ι with the Teichmüller lift for the surjection

$$\Delta_R \rightarrow \overline{\Delta}_R.$$

We set²⁴

$$[-]_{\theta} := [(-)^{1/p}]_{\tilde{\theta}}.$$

We will consider the p -adic Tate module

$$T_p R^{\times} = \varprojlim_{n \geq 0} R^{\times}[p^n]$$

of R^{\times} as being embedded into R^{\flat} as the elements with first coordinate equal to 1.

We are ready to state and prove our main theorem.

Theorem 6.7. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\text{cycl}}$ -algebra. Then the composition*

$$T_p R^{\times} \rightarrow \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\text{ctr}} \pi_2(\text{TC}(R; \mathbb{Z}_p)) \xrightarrow{\beta_\varepsilon} \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is given by sending $x \in T_p(R^{\times})$ to

$$\log_q([x]_{\theta}) = \frac{\sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} ([x]_{\theta} - 1)([x]_{\theta} - q) \cdots ([x]_{\theta} - q^{n-1})}{[n]_q}.$$

Proof. Replacing R by the universal case $\mathbb{Z}_p^{\text{cycl}}\langle x^{1/p^\infty} \rangle / (x-1)$ we may assume that R is p -torsion free and (thus) that $(\widehat{\Delta}_R, (\tilde{\xi}))$ is transversal (by Lemma 3.3 it suffices to see that (p, ξ) is a regular sequence which follows as $\widehat{\Delta}_R / \xi \cong L\widehat{\Omega}_{R/\mathbb{Z}_p^{\text{cycl}}}$, by [4, Theorem 7.2.(5)], is p -torsion free).

Let us define

$$\text{ctr}_2 : T_p R^{\times} \rightarrow \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\text{ctr}} \pi_2(\text{TC}(R; \mathbb{Z}_p)).$$

By Theorem 5.2 the canonical morphism

$$\iota : \Delta_R \rightarrow \pi_0(\text{TC}^-(R; \mathbb{Z}_p))$$

is compatible with the Nygaard filtrations and identifies $\pi_0(\text{TC}^-(R; \mathbb{Z}_p))$ with the Nygaard completion $\widehat{\Delta}_R$ of Δ_R . By Corollary 3.10 the morphism

$$\Delta_R^{\varphi=\tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1} \Delta_R / \mathcal{N}^{\geq 2} \Delta_R \cong \mathcal{N}^{\geq 1} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R$$

is injective. Hence it suffices to show that the two morphisms $\log_q([-]_{\theta})$ and $\beta_\varepsilon \circ \text{ctr}$ agree modulo $\mathcal{N}^{\geq 2} \widehat{\Delta}_R$. Multiplication by the element $v \in \pi_{-2}(\text{TC}^-(\mathbb{Z}_p^{\text{cycl}}; \mathbb{Z}_p))$

²⁴This agrees with the definition of $[-]_{\theta}$ made in the introduction.

constructed after Lemma 6.2 and the HKR-isomorphism (which depends on γ) induce an isomorphism

$$J/J^2 \stackrel{\alpha_\gamma}{\cong} \pi_2(\mathrm{THH}(R; \mathbb{Z}_p)) \stackrel{v}{\cong} \mathcal{N}^{\geq 1} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R$$

where J is the kernel of the surjection

$$\theta: W(R^\flat) \rightarrow R.$$

By Proposition 6.3 and Lemma 6.5 this isomorphism sends the class of $j \in J$ to

$$\delta_2^2 \delta_1(\gamma) \cdot j \cdot 1_{\widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R} = \delta_1(\gamma) \cdot j \cdot 1_{\widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R}.$$

for the canonical $W(R^\flat)$ -algebra structure on

$$\begin{aligned} \widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R &\cong \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p)) / \mathcal{N}^{\geq 2} \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p)) \\ &\cong \pi_0(\mathrm{HC}^-(R/W(R^\flat))) / \mathcal{N}^{\geq 2} \pi_0(\mathrm{HC}^-(R/W(R^\flat))) \end{aligned}$$

(which lifts the morphism θ). Let $x \in T_p R^\times$. By Lemma 4.10

$$\log_q([x]_\theta) \equiv [x]_\theta - 1 \pmod{\mathcal{N}^{\geq 2} \widehat{\Delta}_R}.$$

On the other hand, as the cyclotomic trace reduces to the Dennis trace Dtr , we can calculate using Proposition 2.5 and Lemma 6.5

$$\begin{aligned} \beta_\varepsilon(\mathrm{ctr}(x)) &\equiv v \mathrm{Dtr}(x) \\ &= v \delta_1(\gamma)([x]_\theta - 1) = \delta_1(\gamma)^2([x]_\theta - 1) \cdot 1_{\widehat{\Delta}_R / \mathcal{N}^{\geq 2} \widehat{\Delta}_R} \pmod{\mathcal{N}^{\geq 2} \widehat{\Delta}_R} \\ &= ([x]_\theta - 1) \pmod{\mathcal{N}^{\geq 2} \widehat{\Delta}_R} \end{aligned}$$

Thus we can conclude

$$\log_q([x]_\theta) = \beta_\varepsilon \circ \mathrm{ctr}(x)$$

as desired. \square

Corollary 6.8. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. The map*

$$\log_q([-]_\theta): T_p(R^\times) \rightarrow \widehat{\Delta}_R^{\varphi=\xi}$$

is a bijection.

Proof. Since both sides satisfy quasi-syntomic descent²⁵, one can assume, as in [4, Proposition 7.17], that R is w -local and such that R^\times is divisible. In this case, the map

$$T_p(R^\times) \rightarrow \pi_2(K(R; \mathbb{Z}_p))$$

is a bijection. Moreover, [8, Corollary 6.9] shows that

$$\mathrm{ctr}: \pi_2(K(R; \mathbb{Z}_p)) \rightarrow \pi_2(\mathrm{TC}(R; \mathbb{Z}_p))$$

is also bijective. As by Theorem 6.7, the composite of these two maps is the map $\log_q([-]_\theta)$, this proves the corollary. \square

²⁵For $T_p(-)^\times$ this follows from p -completely faithfully flat descent on p -complete rings with bounded p^∞ -torsion, cf. [1, Appendix], for $\widehat{\Delta}_R^{\varphi=\xi}$ this is proven in [4].

Remark 6.9. As explained at the end of the introduction, one can give a direct and more elementary proof of Corollary 6.8 when R is the quotient of a perfect ring by a finite regular sequence ([25]) or when R is a p -torsion free quotient of a perfectoid ring by a finite regular sequence and p is odd. But we do not know how to prove it directly in general.

REFERENCES

- [1] Johannes Anschütz and Arthur-César Le Bras. Prismatic Dieudonné theory, Preprint 2019.
- [2] Bhargav Bhatt. p -adic derived de Rham cohomology. *arXiv preprint arXiv:1204.6560*, 2012.
- [3] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p -adic Hodge theory. *Publications mathématiques de l’IHÉS*, 128(1):219–397, 2018.
- [4] Bhargav. Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p -adic Hodge theory. *ArXiv e-prints*, February 2018.
- [5] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. available at <http://www.math.uni-bonn.de/people/scholze/Publikationen.html>.
- [6] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K -theory. *Geometry & Topology*, 17(2):733–838, 2013.
- [7] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. The cyclotomic trace and algebraic K -theory of spaces. *Inventiones mathematicae*, 111(1):465–539, 1993.
- [8] Dustin Clausen, Akhil Mathew, and Matthew Morrow. K -theory and topological cyclic homology of henselian pairs. *arXiv e-prints*, page arXiv:1803.10897, March 2018.
- [9] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18. Springer Science & Business Media, 2012.
- [10] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p -adique. preprint on webpage at <https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html>.
- [11] Ofer Gabber. K -theory of henselian local rings and henselian pairs. *Contemp. Math.*, 126:59–70, 1992.
- [12] Thomas Geisser and Lars Hesselholt. Topological cyclic homology of schemes. In *Proceedings of Symposia in Pure Mathematics*, volume 67, pages 41–88. Providence, RI; American Mathematical Society; 1998, 1999.
- [13] Lars Hesselholt. On the topological cyclic homology of the algebraic closure of a local field. In *An alpine anthology of homotopy theory*, volume 399 of *Contemp. Math.*, pages 133–162. Amer. Math. Soc., Providence, RI, 2006.
- [14] Lars Hesselholt. Topological Hochschild homology and the Hasse-Weil zeta function. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 157–180. Amer. Math. Soc., Providence, RI, 2018.
- [15] Marc Hoyois. The homotopy fixed points of the circle action on Hochschild homology. *arXiv e-prints*, page arXiv:1506.07123, Jun 2015.
- [16] Achim Krause and Thomas Nikolaus. Lectures on topological Hochschild homology and cyclotomic spectra. available at <https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html>.
- [17] Jean-Louis Loday. *Cyclic homology*, volume 301. Springer Science & Business Media, 2013.
- [18] J Peter May and Kate Ponto. *More concise algebraic topology: localization, completion, and model categories*. University of Chicago Press, 2011.
- [19] Thomas Nikolaus. The group completion theorem via localizations of ring spectra. Available at <https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html>.
- [20] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *arXiv e-prints*, page arXiv:1707.01799, Jul 2017.
- [21] Jon P. Pridham. On q -de Rham cohomology via λ -rings. *Mathematische Annalen*, pages 1–28, 2016.
- [22] Daniel Quillen. Higher algebraic K -theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.
- [23] Andreas Rosenschon and Paul Arne Østvær. Descent for k -theories. *Journal of Pure and Applied Algebra*, 206(1-2):141–152, 2006.

- [24] Peter Scholze. Canonical q -deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(5):1163–1192, 2017.
- [25] Peter Scholze and Jared Weinstein. Moduli of p -divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.
- [26] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2017.
- [27] Robert W Thomason. Algebraic K -theory and étale cohomology. In *Annales scientifiques de l'École Normale Supérieure*, volume 18, pages 437–552, 1985.
- [28] Robert W Thomason and Thomas Trobaugh. Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift*, pages 247–435. Springer, 1990.
- [29] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, 4, PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
E-mail address: `ja@math.uni-bonn.de`

UNIVERSITÉ SORBONNE PARIS NORD, LAGA, C.N.R.S., UMR 7539, 93430 VILLETANEUSE, FRANCE
E-mail address: `lebras@math.univ-paris13.fr`