HAL open science

On clique-inverse graphs of graphs with bounded clique number.

Liliana Alcón, Sylvain Gravier, Claudia Linhares Sales, Fábio Protti, Gabriela Ravenna

To cite this version:

Liliana Alcón, Sylvain Gravier, Claudia Linhares Sales, Fábio Protti, Gabriela Ravenna. On clique-inverse graphs of graphs with bounded clique number.. Journal of Graph Theory, 2020, 10.1002/jgt.22544 . hal-03015838

HAL Id: hal-03015838

https://hal.science/hal-03015838v1

Submitted on 20 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON CLIQUE-INVERSE GRAPHS OF GRAPHS WITH BOUNDED CLIQUE NUMBER

L. ALCÓN[1,2], S. GRAVIER[3], C. LINHARES SALES[4], F. PROTTI[5*], G. RAVENNA[1,2]

*Corresponding author
[1] Universidad Nacional de La Plata, Argentina.
[2] CONICET, Argentina.
[3] Université Joseph Fourier, Grenoble, France.
[4] Universidade Federal do Ceará, Brazil.
[5] Universidade Federal Fluminense, Niterói, Brazil.

E-mails: liliana@mate.unlp.edu.ar, Sylvain.Gravier@ujf-grenoble.fr, linhares@lia.ufc.br, fabio@ic.uff.br, gravenna@mate.unlp.edu.ar

ABSTRACT. The *clique graph* $K(G)$ of G is the intersection graph of the family of maximal cliques of G. For a family $\mathcal{F}$ of graphs, the family of *clique-inverse graphs of* $\mathcal{F}$ is defined as $K^{-1}(\mathcal{F}) = \{H \mid K(H) \in \mathcal{F}\}$. Let $\mathcal{F}_p$ be the family of K_p-free graphs, that is, graphs with clique number at most $p-1$, for an integer constant $p \geq 2$. Deciding whether a graph H is a clique-inverse graph of $\mathcal{F}_p$ can be done in polynomial time; in addition, for $p \in \{2,3,4\}$, $K^{-1}(\mathcal{F}_p)$ can be characterized by a finite family of forbidden induced subgraphs. In [F. Protti and J. Szwarcfiter, Clique-inverse graphs of K_3-free and K_4-free graphs, J. Graph Theory 35 (2000) 257–272], the authors propose to extend such characterizations to higher values of p. A natural conjecture that then arises is: Is there a characterization of $K^{-1}(\mathcal{F}_p)$ by means of a *finite* family of forbidden induced subgraphs, for any $p \geq 2$? In this note we show that this conjecture is true.

Keywords: clique graph, clique-inverse graph.

1. Introduction

The *clique graph* $K(G)$ of G is the intersection graph of the family of maximal cliques of G, i.e., vertices of $K(G)$ correspond to maximal cliques of G, and an edge exists between two vertices in $K(G)$ if and only if the corresponding maximal cliques in G intersect [5]. In the literature, K is often viewed as a unary *operator* that maps a graph G into its clique graph $K(G)$ [11]. Clique graphs have been studied in several aspects, such as: structural characterizations [5, 16], complexity of algorithmic recognition [3], images of graph families under the clique operator [2, 6, 17], convergence/divergence of the clique operator [6, 7, 10], and theoretical aspects of clique-inverse graphs [9, 13, 14, 15], to name just a few. Several results on clique graphs can be found in the survey [18].

A graph G is a *clique-inverse graph* of a graph H if $K(G) = H$. Not every graph H admits a clique-inverse graph; this occurs precisely when H is not a clique graph. However, if H admits a clique-inverse graph G then H admits other clique-inverse graphs (for instance, any graph obtained from G by the addition of a simplicial vertex is also a clique-inverse graph of H). Thus, the family $K^{-1}(H) = \{G \mid K(G) = H\}$ (the *clique-inverse graphs of* H) either is empty or contains infinitely many graphs.

For a family $\mathcal{F}$ of graphs, the family of *clique-inverse graphs of* $\mathcal{F}$ is defined as $K^{-1}(\mathcal{F}) = \{G \mid K(G) \in \mathcal{F}\}$. For an integer $p \geq 2$, denote by $\mathcal{F}_p$ the family of K_p-free graphs, that is, graphs with clique number at most $p-1$. The problem of deciding whether a graph G is a clique-inverse graph of $\mathcal{F}_p$, when p is part of the input, is clearly in Co-NP, but it is still an open question to know whether it is Co-NP-complete. On the other hand, if p is a constant, the problem can be solved in polynomial time [14]. This can be easily seen by observing that if $G \in K^{-1}(\mathcal{F}_p)$ then each vertex of G is in at most $p-1$ maximal cliques, i.e., G contains at most $(p-1)n$ maximal cliques. Then, if p is a constant, $K(G)$ can be determined in polynomial time by using any polynomial-delay algorithm for the generation of the maximal cliques of a graph, e.g. [12]. In addition, checking whether the clique number of $K(G)$ is at most $p-1$ amounts to analyzing all the $\binom{n'}{p}$ subsets of $K(G)$ with p vertices, where $n' = \mid V(K(G)) \mid$.

The family $K^{-1}(\mathcal{F}_p)$ can be characterized by a finite family of forbidden induced subgraphs for $p \in \{2, 3, 4\}$. Note that a graph in $K^{-1}(\mathcal{F}_2)$ is a disjoint union of cliques, and thus $G \in K^{-1}(\mathcal{F}_2)$ if and only if G contains no P_3 as an induced subgraph. The cases $p = 3$ and $p = 4$ are described below. $K_{1,3}$ is the graph with vertices a, b, c, d and edges ab, ac, ad. The *gem* is the graph with vertices a, b, c, d, e and edges ab, ac, ad, ae, bc, cd, de. The 4-*wheel* is the graph with vertices a, b, c, d, e and edges ab, ac, ad, ae, bc, be, cd, de.

Theorem 1. [13] *A graph G is in $K^{-1}(\mathcal{F}_3)$ if and only if G does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, gem, 4-wheel.*

$K_{1,4}$ is the graph with vertices a, b, c, d, e and edges ab, ac, ad, ae. The *pyramid* is the graph with vertices a, b, c, d, e, f and edges ab, ac, bc, bd, be, ce, cf, de, ef. The 4-*broom* is the graph with vertices a, b, c, d, e, f and edges ab, bc, bd, be, bf, cd, de, ef. The 5-*wheel* is the graph with vertices a, b, c, d, e, f and edges ab, ac, ad, ae, af, bc, bf, cd, de, ef. The 5-*fan* is the graph with vertices a, b, c, d, e, f and edges ab, ac, ad, ae, af, bc, cd, de, ef. The graph H_0^* is the graph obtained from the pyramid by replacing edge ac by edge ae. Finally, the graph Q_2 is the graph with vertices a, b, c, d, e, f, g where a is a universal vertex and the remaining edges are bc, be, cd, ce, cf, df, dg, ef, fg.

Theorem 2. [13] *A graph G is in $K^{-1}(\mathcal{F}_4)$ if and only if G does not contain as an induced subgraph any of the following graphs: $K_{1,4}$, pyramid, 4-broom, 4-wheel, 5-wheel, 5-fan, H_0^*, Q_2.*

Let $G \in K^{-1}(\mathcal{F}_p)$, for $p \geq 2$, and let H be an induced subgraph of G. Clearly, every maximal clique of H is contained in some maximal clique of G. Suppose that there are p distinct, pairwise intersecting maximal cliques $C_1, \ldots, C_p$ in H, and let C'_i be a maximal clique of G such that $C_i \subseteq C'_i$, $1 \leq i \leq p$. If $C'_i = C'_j$ for distinct indices i and j, then C_i and C_j are completely adjacent, because H is an induced subgraph of G; but then C_i and C_j are not maximal cliques in H. Thus, $C'_1, \ldots, C'_p$ are distinct and pairwise intersecting maximal cliques in G, i.e., $\omega(K(G)) \geq p$. This is a contradiction. Therefore, no family of p distinct and pairwise intersecting maximal cliques can exist in H, and thus $\omega(K(H)) \leq p-1$, that is, $H \in K^{-1}(\mathcal{F}_p)$. This shows that being a member of $K^{-1}(\mathcal{F}_p)$ is an induced-hereditary property, and therefore (see [8]) $K^{-1}(\mathcal{F}_p)$ can be characterized by a

family of vertex-minimal graphs G such that $\omega(K(G)) \geq p$. Such vertex-minimal graphs are also called *forbidden induced subgraphs* or *minimal obstructions*.

In [13] the authors propose to extend the characterizations in Theorems 1 and 2 to higher values of p. A natural conjecture that then arises is: Is there a characterization of $K^{-1}(\mathcal{F}_p)$ by means of a *finite* family of minimal obstructions, for any $p \geq 2$? More formally, let $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ denote the family of minimal obstructions for a graph G to have $\omega(K(G)) \leq p-1$.

Conjecture 3. For every $p \geq 2$, $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ is finite.

In this note we show that the above conjecture is true by means of counting arguments on certain subsets of a maximum clique of a graph $G \in \mathcal{F}orb(K^{-1}(\mathcal{F}_p))$.

The remainder of this paper is organized as follows. In Section 2 we present the main result, in Section 3 an application of the main result to hypergraphs, and in Section 4 our conclusions.

2. The main result

In this section, the term *clique* always means a maximal clique. We say that G is a *clique-critical graph* if $K(G) \neq K(G-v)$ for all $v \in V(G)$. In what follows G is a clique-critical graph. Let $\mathcal{C}(G)$ be the set of cliques of G. By [1], for every vertex $v \in V(G)$, there exist C and C' in $\mathcal{C}(G)$ such that either $\{v\} = C \setminus C'$ or $\{v\} = C \cap C'$.

Let C_1 be a clique of G with at least 4 vertices. Given C_2, ..., C_p cliques of G intersecting C_1 (not necessarily all the cliques of G intersect C_1), we define the following subsets of C_1:

$$\begin{aligned}
I &= \{x \in C_1 : \exists\, i,j \in \{2,\ldots,p\} \text{ s.t. } C_i \cap C_j = \{x\}\};\\
D &= \{x \in C_1 \setminus I : \exists\, i,j \in \{2,\ldots,p\} \text{ s.t. } C_i \setminus C_j = \{x\}\};\\
I' &= \{x \in (C_1 \setminus I) \setminus D : \exists\, j \in \{2,\ldots,p\} \text{ s.t. } C_1 \cap C_j = \{x\}\};\\
D' &= \{x \in ((C_1 \setminus I) \setminus D) \setminus I' : \exists\, j \in \{2,\ldots,p\} \text{ s.t. } C_1 \setminus C_j = \{x\}\}.
\end{aligned}$$

Lemma 4. *Let G be a graph in $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ and let $\mathcal{F} = \{C_1, C_2, ..., C_p\}$ be a pairwise intersecting subfamily of $\mathcal{C}(G)$. Then $C_1 = I \cup D \cup I' \cup D'$.*

Proof. Suppose in order to obtain a contradiction that there exists $x \in C_1 \setminus (I \cup D \cup I' \cup D')$.

For every $i \in \{1, 2, ..., p\}$, either $C_i \setminus \{x\}$ is a clique of $G - x$ or $C_i \setminus \{x\}$ is contained in some other clique of G. In the former case, we let C'_i be $C_i \setminus \{x\}$ (notice that in this case C'_i is a clique of $G - x$ but it is not a clique of G); and, in the latter, we let C'_i be the clique containing $C_i \setminus \{x\}$ (in this case, C'_i is both a clique of $G - x$ and a clique of G, but it does not belong to $\{C_1, C_2, ..., C_p\}$).

We claim that if $i \neq j$ then $C'_i \neq C'_j$. Indeed, assume they are equal and say $C = C'_i = C'_j$. If C is not a clique of G then $C'_i = C_i \setminus \{x\}$ and $C'_j = C_j \setminus \{x\}$, hence $C_i = C_j$, a contradiction. If C is a clique of G then $C_i \setminus \{x\} \subseteq C$ and $C_j \setminus \{x\} \subseteq C$ which implies $C_i \cup C_j$ is a clique of G, a contradiction.

It follows that C'_1, C'_2, ..., C'_p are p cliques of $G - x$, thus, by hypothesis, they are not pairwise intersecting. Let C'_i and C'_j have empty intersection. Since $C_i \cap C_j \neq \emptyset$, we have that $C_i \cap C_j = \{x\}$, which contradicts the fact that $x \notin I$. $\square$

Lemma 5. *In the conditions of Lemma 4,*

$$| V(C_1) | \leq \binom{p-1}{2} + 1.$$

Proof. By Lemma 4, $C_1 = I \cup D \cup I' \cup D'$.

For every vertex $x \in I$ (resp. $x \in D$) choose a pair of elements $i, j \in \{2, \ldots, p\}$ such that $C_i \cap C_j = \{x\}$ ($C_i \setminus C_j = \{x\}$ resp.) and let $I_x = \{i, j\}$ ($D_x = \{i, j\}, resp.$).

For every vertex $x \in I'$ (resp. $x \in D'$) chose an element $j \in \{2, \ldots, p\}$ such that $C_1 \cap C_j = \{x\}$ ($C_1 \setminus C_j = \{x\}$ resp.) and let $I'_x = \{j\}$ ($D'_x = \{j\}, resp.$).

Then the following statements easily hold.

(1) If x and y belong to I then $I_x \neq I_y$.

(2) If x and y belong to D then $D_x \neq D_y$. In fact, if $D_x = D_y = \{i, j\}$ then $\{x\} = C_i \setminus C_j$ and $\{y\} = C_j \setminus C_i$. Therefore $C_i \setminus \{x\} \subseteq C_j$ and so y is adjacent to all the vertices of $C_i \setminus \{x\}$. Since, in addition, y is adjacent to x because both vertices belong to C_1, we have that $C_i \cup \{y\}$ is a clique of G, contradicting the fact that C_i is a clique.

(3) If $x \in I$ and $y \in D$ then $I_x \neq D_y$. In fact, if $I_x = D_y = \{i, j\}$ then $\{x\} = C_i \cap C_j$ and $\{y\} = C_i \setminus C_j$, and so $C_i = \{x, y\}$, which implies the contradiction $C_i \subseteq C_1$.

(4) If x and y belong to I' then $I'_x \neq I'_y$. Let $I'_x = \{i\}$ and $I'_y = \{j\}$. Then there is no vertex $z \in I$ such that $I_z = \{i, j\}$, and there is no vertex $w \in D$ such that $D_w = \{i, j\}$, because $C_i \cap C_j \cap C_1 = \emptyset$, $x \in C_i \setminus C_j$, and $y \in C_j \setminus C_i$.

(5) If x and y belong to D' then $D'_x \neq D'_y$. Let $D'_x = \{i\}$ and $D'_y = \{j\}$. Then there is no vertex $z \in I$ such that $I_z = \{i, j\}$, and there is no vertex $w \in D$ such that $D_w = \{i, j\}$, because $|C_i \cap C_j| > 1$ (otherwise, $|C_1| = 3$), $x \in C_j \setminus C_i$, and $y \in C_i \setminus C_j$.

(6) If $x \in I'$ and $y \in D'$ then $I'_x \neq D'_y$. In fact, if $I'_x = D'_y = \{i\}$ then $\{x\} = C_1 \cap C_i$ and $\{y\} = C_1 \setminus C_i$, and so $C_1 = \{x, y\}$, which implies the contradiction $| C_1 | = 2 < 4$.

Let $I'_x = \{i\}$ and $D'_y = \{j\}$. Then there is no vertex $z \in I$ such that $I_z = \{i, j\}$, and there is no vertex $w \in D$ such that $D_w = \{i, j\}$, because $x \in C_i \cap C_j$; $C_1 \cap (C_i \setminus C_j) = \emptyset$ and $C_j \setminus C_i \geq 2$.

Therefore, if the cardinality of the sets I, D, I' and D' are denoted by n_I, n_D, $n_{I'}$ and $n_{D'}$, respectively, we have the following.

By (1), (2), and (3),

$$n_I + n_D \leq \binom{p-1}{2}. \tag{2.1}$$

And adding (4), (5), and (6):

$$n_I + n_D \leq \binom{p-1}{2} - \binom{n_{D'}}{2} - \binom{n_{I'}}{2} - n_{I'} n_{D'}. \tag{2.2}$$

Let $\binom{a}{b} = 0$ whenever $a < b$.

By Lemma 4 and inequality 2.2, we have:

$$| V(C_1) | = n_I + n_D + n_{I'} + n_{D'} \leq$$

$$\binom{p-1}{2} - \binom{n_{D'}}{2} - \binom{n_{I'}}{2} - n_{I'}n_{D'} + n_{I'} + n_{D'} =$$
$$\binom{p-1}{2} + \frac{1}{2}(3(n_{I'} + n_{D'}) - (n_{I'} + n_{D'})^2).$$

Since $n_{I'} + n_{D'}$ is a nonnegative integer, it is easy to check that $3(n_{I'} + n_{D'}) - (n_{I'} + n_{D'})^2 \leq 2$. Thus the proof is complete. □

Lemma 6. *Let G be a graph in $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$. Then a maximum clique of G contains at most $\binom{p-1}{2} + 1$ vertices, and every stable set of G contains at most p vertices.*

Proof. By Lemma 5, any clique of G has size at most $\binom{p-1}{2} + 1$, then a maximum clique satisfies the same bound.

Now suppose that there is a stable set S in G with $p + 1$ vertices. Since G is minimal, every vertex of S must belong to one of the cliques $C_1, C_2, ..., C_p$. Thus, two vertices of S belong to the same clique. This is a contradiction. □

Now, consider the Ramsey number $r(\binom{p-1}{2}+2, p+1) = k$. This means that every graph with at least k vertices has a clique of size $\binom{p-1}{2} + 2$ or a stable set of size $p + 1$. In other words, every graph with no such a clique or stable set must have at most $k - 1$ vertices. Therefore, by applying the previous lemmas, we conclude that every graph $G \in \mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ has at most $k - 1$ vertices. Therefore:

Theorem 7. *For every $p \geq 2$, $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ is finite.*

3. On intersecting families of hypergraphs

Let $\mathcal{H} = (V, \mathcal{C})$ be a hypergraph whose set of vertices is V and whose set of hyperedges (or simply, *edges*) is $\mathcal{C} = \{C_1, C_2, ..., C_m\}$. A *Sperner* or *simple* hypergraph is a hypergraph such that $C_i \subseteq C_j$ implies $i = j$.

Following the terminology used in [4], we define a 2-section of $\mathcal{H}$, denoted by $[\mathcal{H}]_2$, as the graph G where $V(G) = V(\mathcal{H})$ and

$$E(G) = \{(x, y) \mid x \neq y \text{ and } \{x, y\} \subseteq C_i, \text{ for some } 1 \leq i \leq m\}.$$

A hypergraph $\mathcal{H}$ is a *conformal* hypergraph when every maximal clique of $G = [\mathcal{H}_2]$ is an edge of $\mathcal{H}$. Thus, a hypergraph $\mathcal{H}$ is Sperner and conformal if and only if its edges correspond exactly to the family of maximal cliques of the graph $G = [\mathcal{H}]_2$.

The *rank* of a hypergraph $\mathcal{H}$, denoted by $r(\mathcal{H})$, is the maximum cardinality of an edge of $\mathcal{H}$. We define an intersecting family to be a subfamily of edges of a hypergraph $\mathcal{H}$ having non-empty pairwise intersection.

We denote by $\triangle_0(\mathcal{H})$ the maximum cardinality of an intersecting family of $\mathcal{H}$. We say that $\mathcal{H}$ is $\triangle_0$-minimal when $\triangle_0(\mathcal{H} - x) < \triangle_0(\mathcal{H})$, for all $x \in \mathcal{H}$.

Finally, observe that given a vertex-minimal graph G containing p pairwise intersecting maximal cliques $C_1, ..., C_p$, the hypergraph $\mathcal{H} = (C_1, C_2, ..., C_p)$ on $V(G)$ is simple, conformal and p-minimal ($\triangle_0(\mathcal{H}) = p$).

Thus, another version of Lemma 6 is:

Theorem 8. *Every simple, conformal, and $\triangle_0$-minimal hypergraph $\mathcal{H}$ satisfies*

$$r(\mathcal{H}) \leq \binom{\triangle_0(\mathcal{H}) - 1}{2} + 1.$$

4. Concluding remarks

We remark that the Ramsey numbers provide loose upper bounds for the number of vertices of a graph in $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$. For instance, if $p = 4$ then a graph $G \in \mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ must have at least four pairwise intersecting maximal cliques and its number of vertices is bounded according to the inequalities $|V(G)| \leq r(\binom{4-1}{2} + 2, 4 + 1) - 1 = r(5, 5) - 1 \leq 48$. However, by Theorem 2, each graph in $\mathcal{F}orb(K^{-1}(\mathcal{F}_4))$ has at most 7 vertices. Hence, an interesting question is to obtain better upper bounds for the number of vertices of a graph in $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$.

Although $|\mathcal{F}orb(K^{-1}(\mathcal{F}_p))|$ seems to be exponential in p, another interesting question is to know whether it is possible to devise a systematic method for constructing $\mathcal{F}orb(K^{-1}(\mathcal{F}_{p+1}))$ from $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ by the addition of new structures to each graph G in $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ in all possible ways, in order to obtain vertex-minimal graphs G' such that $\omega(K(G')) \geq p + 1$.

Acknowledgements

C. Linhares-Sales is partially supported by CNPq, Brazil. F. Protti is partially supported by CNPq and FAPERJ, Brazil.

References

[1] Alcón, L. Clique-critical graphs: Maximum size and recognition. Discrete Applied Mathematics 154 (2006) 1799–1802.
[2] Alcón, L. and Gutierrez, M. Clique graphs of planar graphs. Ars Combinatoria 71 (2004) 257–265.
[3] Alcón, L., Faria, L., de Figueiredo, C. M. H., Gutierrez, M. The complexity of clique graph recognition. Theoretical Computer Science 410 (2009) 2072–2083.
[4] Berge, C. Hypergraphs. North-Holland, 1989.
[5] Hamelink, R. C. A partial characterization of clique graphs. Journal of Combinatorial Theory 5 (1968) 192–197.
[6] Hedetniemi S. T. and Slater P. J. Line graphs of triangleless graphs and iterated clique graphs. Graph Theory and Applications, Springer (1972), pp. 139–147.
[7] Hedge, S. M. and Suresh Dara. On clique convergence of graphs. AKCE International Journal of Graphs and Combinatorics 13:3 (2016) 261–266.
[8] Lewis, J. M. and Yannakakis, M. The node-deletion problem for hereditary properties is NP-Complete. Journal of Computer and System Sciences 20 (1980) 219–230.
[9] Lucchesi, C. L., Mello, C. P., and Szwarcfiter, J. L. On clique-complete graphs. Discrete Mathematics 183 (1998) 247–254.
[10] Neumann-Lara, V. On clique-divergent graphs. Problems Combinatoires et Thorie des Graphes, Colloques internationaux du CNRS, Paris, 260 (1978), pp. 313–315.
[11] Prisner, E. Graph Dynamics. Pitman Research Notes in Mathematics Series 338, Longman, 1995.

[12] Protti, F., França, F. M. G., Szwarcfiter, J. L. On computing all maximal cliques distributedly. In: Bilardi G., Ferreira A., Lling R., Rolim J. (eds) Solving Irregularly Structured Problems in Parallel. IRREGULAR 1997. Lecture Notes in Computer Science 1253 (1997) 37–48, Springer, Berlin, Heidelberg.

[13] Protti, F. and Szwarcfiter, J. L. Clique-Inverse Graphs of K_3-free and K_4-free Graphs. Journal of Graph Theory 35 (2000) 257–272.

[14] Protti, F. and Szwarcfiter, J. L. On clique graphs with linear size. Congressus Numerantium 143 (2000) 207–219.

[15] Protti, F. and Szwarcfiter, J. L. Clique-inverse graphs of bipartite graphs. Journal of Combinatorial Mathematics and Combinatorial Computing 40 (2002) 193–203.

[16] Roberts, F. S ans Spencer, J. H. A characterization of clique graphs. Journal of Combinatorial Theory Series B 10 (1971) 102–108.

[17] Szwarcfiter, J. L. and Bornstein, C. F. Clique graphs of chordal and path graphs. SIAM Journal on Discrete Mathematics. 7 (1994) 331–336.

[18] Szwarcfiter, J. A Survey on Clique Graphs. In B. Reed and C. Linhares Sales, Eds., "Recent advances in Algorithms and Combinatorics", Springer, pp. 109–136, 2003.