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# ON CLIQUE-INVERSE GRAPHS OF GRAPHS WITH BOUNDED CLIQUE NUMBER

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**ABSTRACT.** The *clique graph*  $K(G)$  of  $G$  is the intersection graph of the family of maximal cliques of  $G$ . For a family  $\mathcal{F}$  of graphs, the family of *clique-inverse graphs of  $\mathcal{F}$*  is defined as  $K^{-1}(\mathcal{F}) = \{H \mid K(H) \in \mathcal{F}\}$ . Let  $\mathcal{F}_p$  be the family of  $K_p$ -free graphs, that is, graphs with clique number at most  $p - 1$ , for an integer constant  $p \geq 2$ . Deciding whether a graph  $H$  is a clique-inverse graph of  $\mathcal{F}_p$  can be done in polynomial time; in addition, for  $p \in \{2, 3, 4\}$ ,  $K^{-1}(\mathcal{F}_p)$  can be characterized by a finite family of forbidden induced subgraphs. In [F. Protti and J. Szwarcfiter, Clique-inverse graphs of  $K_3$ -free and  $K_4$ -free graphs, J. Graph Theory 35 (2000) 257–272], the authors propose to extend such characterizations to higher values of  $p$ . A natural conjecture that then arises is: Is there a characterization of  $K^{-1}(\mathcal{F}_p)$  by means of a *finite* family of forbidden induced subgraphs, for any  $p \geq 2$ ? In this note we show that this conjecture is true.

**Keywords:** clique graph, clique-inverse graph.

## 1. INTRODUCTION

The *clique graph*  $K(G)$  of  $G$  is the intersection graph of the family of maximal cliques of  $G$ , i.e., vertices of  $K(G)$  correspond to maximal cliques of  $G$ , and an edge exists between two vertices in  $K(G)$  if and only if the corresponding maximal cliques in  $G$  intersect [5]. In the literature,  $K$  is often viewed as a unary *operator* that maps a graph  $G$  into its clique graph  $K(G)$  [11]. Clique graphs have been studied in several aspects, such as: structural characterizations [5, 16], complexity of algorithmic recognition [3], images of graph families under the clique operator [2, 6, 17], convergence/divergence of the clique operator [6, 7, 10], and theoretical aspects of clique-inverse graphs [9, 13, 14, 15], to name just a few. Several results on clique graphs can be found in the survey [18].

A graph  $G$  is a *clique-inverse graph* of a graph  $H$  if  $K(G) = H$ . Not every graph  $H$  admits a clique-inverse graph; this occurs precisely when  $H$  is not a clique graph. However, if  $H$  admits a clique-inverse graph  $G$  then  $H$  admits other clique-inverse graphs (for instance, any graph obtained from  $G$  by the addition of a simplicial vertex is also a clique-inverse graph of  $H$ ). Thus, the family  $K^{-1}(H) = \{G \mid K(G) = H\}$  (the *clique-inverse graphs of  $H$* ) either is empty or contains infinitely many graphs.

For a family  $\mathcal{F}$  of graphs, the family of *clique-inverse graphs* of  $\mathcal{F}$  is defined as  $K^{-1}(\mathcal{F}) = \{G \mid K(G) \in \mathcal{F}\}$ . For an integer  $p \geq 2$ , denote by  $\mathcal{F}_p$  the family of  $K_p$ -free graphs, that is, graphs with clique number at most  $p - 1$ . The problem of deciding whether a graph  $G$  is a clique-inverse graph of  $\mathcal{F}_p$ , when  $p$  is part of the input, is clearly in Co-NP, but it is still an open question to know whether it is Co-NP-complete. On the other hand, if  $p$  is a constant, the problem can be solved in polynomial time [14]. This can be easily seen by observing that if  $G \in K^{-1}(\mathcal{F}_p)$  then each vertex of  $G$  is in at most  $p - 1$  maximal cliques, i.e.,  $G$  contains at most  $(p - 1)n$  maximal cliques. Then, if  $p$  is a constant,  $K(G)$  can be determined in polynomial time by using any polynomial-delay algorithm for the generation of the maximal cliques of a graph, e.g. [12]. In addition, checking whether the clique number of  $K(G)$  is at most  $p - 1$  amounts to analyzing all the  $\binom{n'}{p}$  subsets of  $K(G)$  with  $p$  vertices, where  $n' = |V(K(G))|$ .

The family  $K^{-1}(\mathcal{F}_p)$  can be characterized by a finite family of forbidden induced subgraphs for  $p \in \{2, 3, 4\}$ . Note that a graph in  $K^{-1}(\mathcal{F}_2)$  is a disjoint union of cliques, and thus  $G \in K^{-1}(\mathcal{F}_2)$  if and only if  $G$  contains no  $P_3$  as an induced subgraph. The cases  $p = 3$  and  $p = 4$  are described below.  $K_{1,3}$  is the graph with vertices  $a, b, c, d$  and edges  $ab, ac, ad$ . The *gem* is the graph with vertices  $a, b, c, d, e$  and edges  $ab, ac, ad, ae, bc, cd, de$ . The *4-wheel* is the graph with vertices  $a, b, c, d, e$  and edges  $ab, ac, ad, ae, bc, be, cd, de$ .

**Theorem 1.** [13] *A graph  $G$  is in  $K^{-1}(\mathcal{F}_3)$  if and only if  $G$  does not contain as an induced subgraph any of the following graphs:  $K_{1,3}$ , gem, 4-wheel.*

$K_{1,4}$  is the graph with vertices  $a, b, c, d, e$  and edges  $ab, ac, ad, ae$ . The *pyramid* is the graph with vertices  $a, b, c, d, e, f$  and edges  $ab, ac, bc, bd, be, ce, cf, de, ef$ . The *4-broom* is the graph with vertices  $a, b, c, d, e, f$  and edges  $ab, bc, bd, be, bf, cd, de, ef$ . The *5-wheel* is the graph with vertices  $a, b, c, d, e, f$  and edges  $ab, ac, ad, ae, af, bc, bf, cd, de, ef$ . The *5-fan* is the graph with vertices  $a, b, c, d, e, f$  and edges  $ab, ac, ad, ae, af, bc, cd, de, ef$ . The graph  $H_0^*$  is the graph obtained from the pyramid by replacing edge  $ac$  by edge  $ae$ . Finally, the graph  $Q_2$  is the graph with vertices  $a, b, c, d, e, f, g$  where  $a$  is a universal vertex and the remaining edges are  $bc, be, cd, ce, cf, df, dg, ef, fg$ .

**Theorem 2.** [13] *A graph  $G$  is in  $K^{-1}(\mathcal{F}_4)$  if and only if  $G$  does not contain as an induced subgraph any of the following graphs:  $K_{1,4}$ , pyramid, 4-broom, 4-wheel, 5-wheel, 5-fan,  $H_0^*$ ,  $Q_2$ .*

Let  $G \in K^{-1}(\mathcal{F}_p)$ , for  $p \geq 2$ , and let  $H$  be an induced subgraph of  $G$ . Clearly, every maximal clique of  $H$  is contained in some maximal clique of  $G$ . Suppose that there are  $p$  distinct, pairwise intersecting maximal cliques  $C_1, \dots, C_p$  in  $H$ , and let  $C'_i$  be a maximal clique of  $G$  such that  $C_i \subseteq C'_i$ ,  $1 \leq i \leq p$ . If  $C'_i = C'_j$  for distinct indices  $i$  and  $j$ , then  $C_i$  and  $C_j$  are completely adjacent, because  $H$  is an induced subgraph of  $G$ ; but then  $C_i$  and  $C_j$  are not maximal cliques in  $H$ . Thus,  $C'_1, \dots, C'_p$  are distinct and pairwise intersecting maximal cliques in  $G$ , i.e.,  $\omega(K(G)) \geq p$ . This is a contradiction. Therefore, no family of  $p$  distinct and pairwise intersecting maximal cliques can exist in  $H$ , and thus  $\omega(K(H)) \leq p - 1$ , that is,  $H \in K^{-1}(\mathcal{F}_p)$ . This shows that being a member of  $K^{-1}(\mathcal{F}_p)$  is an induced-hereditary property, and therefore (see [8])  $K^{-1}(\mathcal{F}_p)$  can be characterized by a

family of vertex-minimal graphs  $G$  such that  $\omega(K(G)) \geq p$ . Such vertex-minimal graphs are also called *forbidden induced subgraphs* or *minimal obstructions*.

In [13] the authors propose to extend the characterizations in Theorems 1 and 2 to higher values of  $p$ . A natural conjecture that then arises is: Is there a characterization of  $K^{-1}(\mathcal{F}_p)$  by means of a *finite* family of minimal obstructions, for any  $p \geq 2$ ? More formally, let  $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$  denote the family of minimal obstructions for a graph  $G$  to have  $\omega(K(G)) \leq p - 1$ .

**Conjecture 3.** For every  $p \geq 2$ ,  $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$  is finite.

In this note we show that the above conjecture is true by means of counting arguments on certain subsets of a maximum clique of a graph  $G \in \mathcal{F}orb(K^{-1}(\mathcal{F}_p))$ .

The remainder of this paper is organized as follows. In Section 2 we present the main result, in Section 3 an application of the main result to hypergraphs, and in Section 4 our conclusions.

## 2. THE MAIN RESULT

In this section, the term *clique* always means a maximal clique. We say that  $G$  is a *clique-critical graph* if  $K(G) \neq K(G - v)$  for all  $v \in V(G)$ . In what follows  $G$  is a clique-critical graph. Let  $\mathcal{C}(G)$  be the set of cliques of  $G$ . By [1], for every vertex  $v \in V(G)$ , there exist  $C$  and  $C'$  in  $\mathcal{C}(G)$  such that either  $\{v\} = C \setminus C'$  or  $\{v\} = C \cap C'$ .

Let  $C_1$  be a clique of  $G$  with at least 4 vertices. Given  $C_2, \dots, C_p$  cliques of  $G$  intersecting  $C_1$  (not necessarily all the cliques of  $G$  intersect  $C_1$ ), we define the following subsets of  $C_1$ :

$$\begin{aligned} I &= \{x \in C_1 : \exists i, j \in \{2, \dots, p\} \text{ s.t. } C_i \cap C_j = \{x\}\}; \\ D &= \{x \in C_1 \setminus I : \exists i, j \in \{2, \dots, p\} \text{ s.t. } C_i \setminus C_j = \{x\}\}; \\ I' &= \{x \in (C_1 \setminus I) \setminus D : \exists j \in \{2, \dots, p\} \text{ s.t. } C_1 \cap C_j = \{x\}\}; \\ D' &= \{x \in ((C_1 \setminus I) \setminus D) \setminus I' : \exists j \in \{2, \dots, p\} \text{ s.t. } C_1 \setminus C_j = \{x\}\}. \end{aligned}$$

**Lemma 4.** *Let  $G$  be a graph in  $\mathcal{F}orb(K^{-1}(\mathcal{F}_p))$  and let  $\mathcal{F} = \{C_1, C_2, \dots, C_p\}$  be a pairwise intersecting subfamily of  $\mathcal{C}(G)$ . Then  $C_1 = I \cup D \cup I' \cup D'$ .*

*Proof.* Suppose in order to obtain a contradiction that there exists  $x \in C_1 \setminus (I \cup D \cup I' \cup D')$ .

For every  $i \in \{1, 2, \dots, p\}$ , either  $C_i \setminus \{x\}$  is a clique of  $G - x$  or  $C_i \setminus \{x\}$  is contained in some other clique of  $G$ . In the former case, we let  $C'_i$  be  $C_i \setminus \{x\}$  (notice that in this case  $C'_i$  is a clique of  $G - x$  but it is not a clique of  $G$ ); and, in the latter, we let  $C'_i$  be the clique containing  $C_i \setminus \{x\}$  (in this case,  $C'_i$  is both a clique of  $G - x$  and a clique of  $G$ , but it does not belong to  $\{C_1, C_2, \dots, C_p\}$ ).

We claim that if  $i \neq j$  then  $C'_i \neq C'_j$ . Indeed, assume they are equal and say  $C = C'_i = C'_j$ . If  $C$  is not a clique of  $G$  then  $C'_i = C_i \setminus \{x\}$  and  $C'_j = C_j \setminus \{x\}$ , hence  $C_i = C_j$ , a contradiction. If  $C$  is a clique of  $G$  then  $C_i \setminus \{x\} \subseteq C$  and  $C_j \setminus \{x\} \subseteq C$  which implies  $C_i \cup C_j$  is a clique of  $G$ , a contradiction.

It follows that  $C'_1, C'_2, \dots, C'_p$  are  $p$  cliques of  $G - x$ , thus, by hypothesis, they are not pairwise intersecting. Let  $C'_i$  and  $C'_j$  have empty intersection. Since  $C_i \cap C_j \neq \emptyset$ , we have that  $C_i \cap C_j = \{x\}$ , which contradicts the fact that  $x \notin I$ .  $\square$

**Lemma 5.** *In the conditions of Lemma 4,*

$$|V(C_1)| \leq \binom{p-1}{2} + 1.$$

*Proof.* By Lemma 4,  $C_1 = I \cup D \cup I' \cup D'$ .

For every vertex  $x \in I$  (resp.  $x \in D$ ) choose a pair of elements  $i, j \in \{2, \dots, p\}$  such that  $C_i \cap C_j = \{x\}$  ( $C_i \setminus C_j = \{x\}$  resp.) and let  $I_x = \{i, j\}$  ( $D_x = \{i, j\}$ , resp.).

For every vertex  $x \in I'$  (resp.  $x \in D'$ ) chose an element  $j \in \{2, \dots, p\}$  such that  $C_1 \cap C_j = \{x\}$  ( $C_1 \setminus C_j = \{x\}$  resp.) and let  $I'_x = \{j\}$  ( $D'_x = \{j\}$ , resp.).

Then the following statements easily hold.

- (1) If  $x$  and  $y$  belong to  $I$  then  $I_x \neq I_y$ .
- (2) If  $x$  and  $y$  belong to  $D$  then  $D_x \neq D_y$ . In fact, if  $D_x = D_y = \{i, j\}$  then  $\{x\} = C_i \setminus C_j$  and  $\{y\} = C_j \setminus C_i$ . Therefore  $C_i \setminus \{x\} \subseteq C_j$  and so  $y$  is adjacent to all the vertices of  $C_i \setminus \{x\}$ . Since, in addition,  $y$  is adjacent to  $x$  because both vertices belong to  $C_1$ , we have that  $C_i \cup \{y\}$  is a clique of  $G$ , contradicting the fact that  $C_i$  is a clique.
- (3) If  $x \in I$  and  $y \in D$  then  $I_x \neq D_y$ . In fact, if  $I_x = D_y = \{i, j\}$  then  $\{x\} = C_i \cap C_j$  and  $\{y\} = C_i \setminus C_j$ , and so  $C_i = \{x, y\}$ , which implies the contradiction  $C_i \subseteq C_1$ .
- (4) If  $x$  and  $y$  belong to  $I'$  then  $I'_x \neq I'_y$ . Let  $I'_x = \{i\}$  and  $I'_y = \{j\}$ . Then there is no vertex  $z \in I$  such that  $I_z = \{i, j\}$ , and there is no vertex  $w \in D$  such that  $D_w = \{i, j\}$ , because  $C_i \cap C_j \cap C_1 = \emptyset$ ,  $x \in C_i \setminus C_j$ , and  $y \in C_j \setminus C_i$ .
- (5) If  $x$  and  $y$  belong to  $D'$  then  $D'_x \neq D'_y$ . Let  $D'_x = \{i\}$  and  $D'_y = \{j\}$ . Then there is no vertex  $z \in I$  such that  $I_z = \{i, j\}$ , and there is no vertex  $w \in D$  such that  $D_w = \{i, j\}$ , because  $|C_i \cap C_j| > 1$  (otherwise,  $|C_1| = 3$ ),  $x \in C_j \setminus C_i$ , and  $y \in C_i \setminus C_j$ .
- (6) If  $x \in I'$  and  $y \in D'$  then  $I'_x \neq D'_y$ . In fact, if  $I'_x = D'_y = \{i\}$  then  $\{x\} = C_1 \cap C_i$  and  $\{y\} = C_1 \setminus C_i$ , and so  $C_1 = \{x, y\}$ , which implies the contradiction  $|C_1| = 2 < 4$ .

Let  $I'_x = \{i\}$  and  $D'_y = \{j\}$ . Then there is no vertex  $z \in I$  such that  $I_z = \{i, j\}$ , and there is no vertex  $w \in D$  such that  $D_w = \{i, j\}$ , because  $x \in C_i \cap C_j$ ;  $C_1 \cap (C_i \setminus C_j) = \emptyset$  and  $C_j \setminus C_i \geq 2$ .

Therefore, if the cardinality of the sets  $I$ ,  $D$ ,  $I'$  and  $D'$  are denoted by  $n_I$ ,  $n_D$ ,  $n_{I'}$  and  $n_{D'}$ , respectively, we have the following.

By (1), (2), and (3),

$$n_I + n_D \leq \binom{p-1}{2}. \quad (2.1)$$

And adding (4), (5), and (6):

$$n_I + n_D \leq \binom{p-1}{2} - \binom{n_{D'}}{2} - \binom{n_{I'}}{2} - n_{I'}n_{D'}. \quad (2.2)$$

Let  $\binom{a}{b} = 0$  whenever  $a < b$ .

By Lemma 4 and inequality 2.2, we have:

$$|V(C_1)| = n_I + n_D + n_{I'} + n_{D'} \leq$$

$$\binom{p-1}{2} - \binom{n_{D'}}{2} - \binom{n_{I'}}{2} - n_{I'}n_{D'} + n_{I'} + n_{D'} = \binom{p-1}{2} + \frac{1}{2}(3(n_{I'} + n_{D'}) - (n_{I'} + n_{D'})^2).$$

Since  $n_{I'} + n_{D'}$  is a nonnegative integer, it is easy to check that  $3(n_{I'} + n_{D'}) - (n_{I'} + n_{D'})^2 \leq 2$ . Thus the proof is complete.  $\square$

**Lemma 6.** *Let  $G$  be a graph in  $\text{Forb}(K^{-1}(\mathcal{F}_p))$ . Then a maximum clique of  $G$  contains at most  $\binom{p-1}{2} + 1$  vertices, and every stable set of  $G$  contains at most  $p$  vertices.*

**Proof.** By Lemma 5, any clique of  $G$  has size at most  $\binom{p-1}{2} + 1$ , then a maximum clique satisfies the same bound.

Now suppose that there is a stable set  $S$  in  $G$  with  $p + 1$  vertices. Since  $G$  is minimal, every vertex of  $S$  must belong to one of the cliques  $C_1, C_2, \dots, C_p$ . Thus, two vertices of  $S$  belong to the same clique. This is a contradiction.  $\square$

Now, consider the Ramsey number  $r(\binom{p-1}{2} + 2, p + 1) = k$ . This means that every graph with at least  $k$  vertices has a clique of size  $\binom{p-1}{2} + 2$  or a stable set of size  $p + 1$ . In other words, every graph with no such a clique or stable set must have at most  $k - 1$  vertices. Therefore, by applying the previous lemmas, we conclude that every graph  $G \in \text{Forb}(K^{-1}(\mathcal{F}_p))$  has at most  $k - 1$  vertices. Therefore:

**Theorem 7.** *For every  $p \geq 2$ ,  $\text{Forb}(K^{-1}(\mathcal{F}_p))$  is finite.*

### 3. ON INTERSECTING FAMILIES OF HYPERGRAPHS

Let  $\mathcal{H} = (V, \mathcal{C})$  be a hypergraph whose set of vertices is  $V$  and whose set of hyperedges (or simply, *edges*) is  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ . A *Sperner* or *simple* hypergraph is a hypergraph such that  $C_i \subseteq C_j$  implies  $i = j$ .

Following the terminology used in [4], we define a 2-section of  $\mathcal{H}$ , denoted by  $[\mathcal{H}]_2$ , as the graph  $G$  where  $V(G) = V(\mathcal{H})$  and

$$E(G) = \{(x, y) \mid x \neq y \text{ and } \{x, y\} \subseteq C_i, \text{ for some } 1 \leq i \leq m\}.$$

A hypergraph  $\mathcal{H}$  is a *conformal* hypergraph when every maximal clique of  $G = [\mathcal{H}]_2$  is an edge of  $\mathcal{H}$ . Thus, a hypergraph  $\mathcal{H}$  is Sperner and conformal if and only if its edges correspond exactly to the family of maximal cliques of the graph  $G = [\mathcal{H}]_2$ .

The *rank* of a hypergraph  $\mathcal{H}$ , denoted by  $r(\mathcal{H})$ , is the maximum cardinality of an edge of  $\mathcal{H}$ . We define an intersecting family to be a subfamily of edges of a hypergraph  $\mathcal{H}$  having non-empty pairwise intersection.

We denote by  $\Delta_0(\mathcal{H})$  the maximum cardinality of an intersecting family of  $\mathcal{H}$ . We say that  $\mathcal{H}$  is  $\Delta_0$ -minimal when  $\Delta_0(\mathcal{H} - x) < \Delta_0(\mathcal{H})$ , for all  $x \in \mathcal{H}$ .

Finally, observe that given a vertex-minimal graph  $G$  containing  $p$  pairwise intersecting maximal cliques  $C_1, \dots, C_p$ , the hypergraph  $\mathcal{H} = (C_1, C_2, \dots, C_p)$  on  $V(G)$  is simple, conformal and  $p$ -minimal ( $\Delta_0(\mathcal{H}) = p$ ).

Thus, another version of Lemma 6 is:

**Theorem 8.** *Every simple, conformal, and  $\Delta_0$ -minimal hypergraph  $\mathcal{H}$  satisfies*

$$r(\mathcal{H}) \leq \binom{\Delta_0(\mathcal{H}) - 1}{2} + 1.$$

#### 4. CONCLUDING REMARKS

We remark that the Ramsey numbers provide loose upper bounds for the number of vertices of a graph in  $\mathcal{Forb}(K^{-1}(\mathcal{F}_p))$ . For instance, if  $p = 4$  then a graph  $G \in \mathcal{Forb}(K^{-1}(\mathcal{F}_p))$  must have at least four pairwise intersecting maximal cliques and its number of vertices is bounded according to the inequalities  $|V(G)| \leq r(\binom{4-1}{2} + 2, 4 + 1) - 1 = r(5, 5) - 1 \leq 48$ . However, by Theorem 2, each graph in  $\mathcal{Forb}(K^{-1}(\mathcal{F}_4))$  has at most 7 vertices. Hence, an interesting question is to obtain better upper bounds for the number of vertices of a graph in  $\mathcal{Forb}(K^{-1}(\mathcal{F}_p))$ .

Although  $|\mathcal{Forb}(K^{-1}(\mathcal{F}_p))|$  seems to be exponential in  $p$ , another interesting question is to know whether it is possible to devise a systematic method for constructing  $\mathcal{Forb}(K^{-1}(\mathcal{F}_{p+1}))$  from  $\mathcal{Forb}(K^{-1}(\mathcal{F}_p))$  by the addition of new structures to each graph  $G$  in  $\mathcal{Forb}(K^{-1}(\mathcal{F}_p))$  in all possible ways, in order to obtain vertex-minimal graphs  $G'$  such that  $\omega(K(G')) \geq p + 1$ .

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