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# The $(a, b)$ -monochromatic transversal game on clique-hypergraphs of powers of cycles

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## Abstract

We introduce the  $(a, b)$ -monochromatic transversal game where Alice and Bob alternately colours  $a$  and  $b$  vertices of a hypergraph, respectively. Alice, who colours the vertices with red, wins the game if she obtains a red transversal and Bob wins it if he does not let it happen. Both players are enabled to start the game and they play optimally. We analyze the game played on clique-hypergraphs of powers of cycles and we show many strategies that, depending on the choice of the parameters, allow a specific player to win the game.

*Keywords:* Combinatorial games, hypergraphs, powers of cycles, transversal.

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## 1. Introduction

The concept of *hypergraphs* has been extensively studied in many areas. It generalizes the definition of graphs allowing edges to have more than two incident vertices. Hypergraph theory is also applied as techniques that are used in modern mathematics and related research fields. For example, in

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<sup>☆</sup>Dedicated to healthcare professionals and essential workers amid COVID-19 pandemic.

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ergodic theory, the Furstenberg-Katznelson's theorem was proven using hypergraph methods as showed in [10]. Özturan [12] also used hypergraphs to model chemical reaction networks.

A *transversal* in a hypergraph is a set of vertices intersecting every hyperedge [3]. A *maximal clique* of a graph is a subset of vertices of this graph, which induces a complete graph, and it is not properly contained in any other clique. A *clique-hypergraph* of a graph  $G$  is a hypergraph with the same vertex set of  $G$  and whose hyperedges are the maximal cliques of  $G$ .

The study of clique-hypergraphs was firstly presented in 1991 by Duffus et al. [7], where the authors asked what is the smallest number of colours needed to colour the vertices of a clique-hypergraph such that no pair of adjacent vertices is monochromatic (*clique-chromatic number*). Later, Gravier et al. [9] showed that given a fixed graph  $F$ , there exists an integer  $f(F)$  such that the clique-hypergraph of any  $F$ -free graph can be  $f(F)$ -colored if and only if all components of  $F$  are paths. In 2013, the answer to Duffus' question in the case of clique-chromatic number of powers of cycles was presented by Campos et al. [6], where the authors showed that such number is equal to 2, except for odd cycles of size at least 5 where the answer is 3. After that, in 2014, Bacsó et al. [1] proved that this number is 3 for almost all perfect graphs.

Classic problems in graph theory, like coloring [11] or labeling [8], have been studied from the perspective of Combinatorial games. *Combinatorial games* [2] are alternating finite two-player games of pure strategy in which all the relevant information is public to both players, as well as no randomness is allowed. A combinatorial game using the concept of transversals in hypergraphs is the *transversal game*, presented by Bujtás et al. [4, 5]. In this game, two players, Edge-hitter and Staller, take turns choosing a vertex from  $\mathcal{H}$ . Each chosen vertex must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of selected vertices becomes a transversal in  $\mathcal{H}$ . The strategy of the Edge-hitter is to minimize the number of chosen vertices, while Staller wishes to maximize it. The authors defined the *game transversal number* of  $\mathcal{H}$  as the number of vertices chosen when Edge-hitter starts the transversal game and both players play optimally. Using this definition, they showed that the  $\frac{3}{4}$ -Game Total Conjecture is true over class of graphs with minimum degree at least 2, and compared this parameter with its transversal number.

In this work, we introduce a combinatorial game over hypergraphs: the  $(a, b)$ -*monochromatic transversal game*. In this game, Alice and Bob alternately colour  $a$  and  $b$  vertices of a hypergraph, respectively. Alice, who colours the vertices with red, wins the game if she obtains a red transversal and Bob wins it if he obtains a monochromatic blue hyperedge. Both players are enabled to start the game and they play optimally. We consider the game played on clique-hypergraphs of complete graphs, powers of cycles and paths. For each of these graphs, we show strategies and bounds to the parameters  $a$  and  $b$  so that one of the players wins the game.

We organize the paper as follows. First, in Section 2, we introduce the game and present standard definitions and notation. In Section 3, we start playing the game where Alice and Bob colour a single vertex at a time. We illustrate this with complete graphs and powers of cycles, showing that Alice has advantage on this type of hypergraphs. We also show an example of the game played on paths. In Sections 4 and 5, we exhibit results for the case where Bob is allowed to colour more vertices than Alice, looking for cases where Alice's advantage disappears. Finally in Sections 6 and 7, we present our conclusions and acknowledgments.

## 2. Description of the game

A *hypergraph*  $\mathcal{H}$  is a pair  $(\mathcal{V}, \mathcal{E})$  where  $\mathcal{V}$  is a finite vertex set and  $\mathcal{E}$  is a family of non-empty subsets of  $V$  called *hyperedges*. The  $(a, b)$ -*monochromatic transversal game* is an avoider-enforcer game where two players, Alice and Bob, alternately colour the vertices of a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ . Alice colours  $a$  vertices in red at each turn and Bob colours  $b$  vertices in blue at each turn. Alice wins the game if she obtains a red *transversal*, that is, a subset of vertices in  $\mathcal{V}$  that has a nonempty intersection with every hyperedge of  $\mathcal{H}$ . Bob wins the game if he obtains a monochromatic blue hyperedge of  $\mathcal{E}$ . The game presents properties as the following:

**Remark 2.1.** *If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts the  $(a, b)$ -monochromatic transversal game on a given hypergraph, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts the game on that hypergraph.*

**Remark 2.2.** *If there exists a strategy that allows Alice (resp. Bob) to win the  $(a_0, b_0)$ -monochromatic transversal game played on a given hypergraph,*

independently of who starts it, then for any  $a > a_0$  (resp.  $b > b_0$ ) there exists a strategy that allows Alice (resp. Bob) to win the  $(a, b_0)$ -monochromatic transversal game (resp.  $(a_0, b)$ -game) on that hypergraph, independently of who starts it.

The *clique-hypergraph*  $\mathcal{H} = (V, \mathcal{E})$  of an undirected simple graph  $G = (V, E)$  is a pair  $\mathcal{H} = (V, \mathcal{E})$  such that  $V$  is the vertex set of  $G$  and the hyperedge set  $\mathcal{E}$  is the set of all maximal *cliques* in  $G$ , that is,  $\mathcal{E}$  is the set of all maximal subsets of  $V$  whose vertices induce a complete graph. In this work we consider clique-hypergraphs of powers of cycles. A  $k$ -th power of cycle of length  $n$ ,  $C_n^k$ , for  $k \geq 1$ , is a undirected simple graph on  $n$  vertices whose vertex set is  $V(C_n^k) = \{v_i; i \in \mathbb{Z}_n\}$  and whose edges  $\{v_i, v_j\}$ ,  $i, j \in \mathbb{Z}_n$  have the property that  $i = j \pm r \pmod{n}$  for some  $r \in \{1, 2, \dots, k\}$ . Observe that, with the previous definition,  $C_n^1$  is simply a cycle  $C_n$ . Furthermore,  $C_n^k$  (with  $k \geq \lfloor \frac{n}{2} \rfloor$ ) is isomorphic to the complete graph on  $n$  vertices  $K_n$  and for this reason we also consider the game played on clique hypergraphs of these graphs but we differ them according to this event.

According to [6], the maximal cliques of powers of cycles  $C_n^k$  can be classified into two types: an *external clique*, whose vertex set is composed by  $k+1$  vertices with consecutive indexes  $v_x, \dots, v_{x+k \pmod{n}}$  for some  $x \in \mathbb{Z}_n$ , and an *internal clique*, whose vertex set contains vertices with non-consecutive indexes.

We observe that if  $b > k$  or  $a \geq \lceil \frac{n}{k+1} \rceil$  then the  $(a, b)$ -monochromatic transversal game on clique-hypergraphs of powers of cycles becomes trivial. For that purpose, along this work we always assume that  $b \leq k$  and  $a \leq \lceil \frac{n}{k+1} \rceil - 1$ . Furthermore, observe that if Bob can not colour a clique of size  $k'$  while playing the  $(a, b)$ -monochromatic transversal game on a clique-hypergraph  $\mathcal{H}(G)$ , then he can not colour a clique of size  $k$  for  $k > k'$ . By this observation we obtain the following remark:

**Remark 2.3.** *Let  $C_n^k$  be with no internal maximal cliques and  $2 \leq k' < k$ . If there exists a strategy that allows Alice to win the  $(a, b)$ -monochromatic transversal game in  $\mathcal{H}(C_n^{k'})$ , then there exists a strategy that allows Alice to win the game in  $\mathcal{H}(C_n^k)$ .*

### 3. The (1, 1)-monochromatic transversal game

In this section we obtain a complete analysis of the (1, 1)-monochromatic transversal game played on clique-hypergraphs of powers of cycles, and before we do so, we show simple strategies that can be used when  $k$  has either its smallest or largest possible value, that is,  $C_n$  (when  $k = 1$ ) or  $K_n$  ( $k \geq \lfloor \frac{n}{2} \rfloor$ ).

#### 3.1. Playing on clique-hypergraphs of complete graphs, cycles and paths

In order to make the reader familiar with the game, we start playing it on clique-hypergraphs of complete graphs  $K_n$  and on clique-hypergraphs of cycles  $C_n$ , on  $n$  vertices.

**Proposition 3.1.** *If  $K_n$  is a complete graph with  $n \geq 2$ , then there exists a strategy that allows Alice to win the (1, 1)-monochromatic transversal game played on the clique-hypergraph  $\mathcal{H}(K_n)$ , independently of who starts playing the game.*

*Proof.* Firstly, by Remark 2.1, we may assume that Bob starts playing the game. Next, observe that the clique-hypergraph  $\mathcal{H}(K_n)$  of the complete graph on  $n$  vertices contains a unique hyperedge with  $n$  vertices. Therefore, Alice obtains a red transversal in her first turn.  $\square$

Now we analyze the game played on the clique-hypergraph of a cycle. Since  $C_3$  is isomorphic to  $K_3$  (whose result is contained in Proposition 3.1), we consider the case  $n \geq 4$ .

**Proposition 3.2.** *Let  $C_n$  be a cycle of length  $n \geq 4$ . There exists a strategy that allows Bob to win the (1, 1)-monochromatic transversal game on the clique-hypergraph  $\mathcal{H}(C_n)$ , independently of who starts playing the game.*

*Proof.* Firstly, observe that the hyperedges of  $\mathcal{H}(C_n)$  are the edges of  $C_n$ . By Remark 2.1 we may assume that Alice starts the game. Let  $v_j$  denote the vertex that is coloured red in Alice's first turn. If Bob colours blue a vertex  $v_k$  that is not adjacent to  $v_j$  then, independently of which vertex Alice colours red in her next move, in Bob's next turn there exists an uncoloured vertex  $v_l$ ,  $l \in \{k - 1 \pmod{n}, k + 1 \pmod{n}\}$ , that is adjacent to  $v_k$ . Colouring  $v_l$  with blue, Bob obtains a monochromatic blue hyperedge  $\{v_k, v_l\}$  and wins the game.  $\square$

**Remark 3.3.** *The game played on clique-hypergraphs of paths is not much different. Indeed,  $P_2$  is isomorphic to  $K_2$  (whose result is contained in Proposition 3.1). If  $3 \leq n \leq 5$  then there exists a strategy that allows the player who had the first turn to win the game. Colouring vertex  $v_{\lfloor \frac{n}{2} \rfloor}$  at their first turn, at the second turn of that player they must colour  $v_{\lfloor \frac{n}{2} \rfloor - 1}$  or  $v_{\lfloor \frac{n}{2} \rfloor + 1}$  if it is Bob, or colour a vertex that is adjacent to Bob's last coloured vertex if it is Alice. In the case of  $\mathcal{H}(P_n)$ ,  $n \geq 6$ , an argument analogous to the proof of Proposition 3.2 shows that there exists a strategy that allows Bob to win the game, independently of who starts playing the game.*

### 3.2. Playing on clique-hypergraphs of powers of cycles

Since graphs  $C_n^2$  are isomorphic to the complete graphs  $K_n$  on  $n$  vertices whenever  $n \leq 5$ , and these graphs have been analyzed in Proposition 3.1, we consider the second powers of cycles  $C_n^2$  with  $n \geq 6$ . We recall that the maximal cliques in clique-hypergraphs powers of cycles can be classified in external or internal cliques. It can be quickly verified that the unique hypergraph  $\mathcal{H}(C_n^2)$  which contains these internal maximal cliques is the  $\mathcal{H}(C_6^2)$  and therefore we start with the analysis of the game played on this hypergraph.

**Proposition 3.4.** *There exists a strategy that allows Bob to win the  $(1, 1)$ -monochromatic transversal game played in the clique-hypergraph  $\mathcal{H}(C_6^2)$ , independently of who starts playing the game.*

*Proof.* By Remark 2.1, we may assume that Alice starts playing the game. Before we exhibit Bob's strategy, let us label the hyperedges of  $\mathcal{H}(C_6^2)$  as follows:  $e_1 = \{v_0, v_2, v_4\}$ ,  $e_2 = \{v_1, v_3, v_5\}$ ,  $e_3 = \{v_5, v_0, v_1\}$ ,  $e_4 = \{v_2, v_3, v_4\}$ ,  $e_5 = \{v_4, v_5, v_0\}$ ,  $e_6 = \{v_1, v_2, v_3\}$ ,  $e_7 = \{v_0, v_1, v_2\}$  e  $e_8 = \{v_3, v_4, v_5\}$ . Observe that  $e_1$  and  $e_2$  are the unique internal maximal cliques.

Without loss of generality, we may assume that Alice colours the vertex  $v_0$  with the colour red at her first turn. Since  $v_0$  is adjacent to every vertex in  $V(C_n^2)$  except  $v_3$ , it makes  $v_3$  be the vertex whose adjacent hyperedges have no red vertex. Therefore, Bob's best option is to colour  $v_3$  blue. Thus, by the symmetry of  $C_n^2$ , Alice has two options for her second turn: she can either colour  $v_1$  (or  $v_5$ ) or  $v_2$  (or  $v_4$ ).

If she colours  $v_1$  (resp.  $v_5$ ) red then Bob colours vertex  $v_4$  (resp.  $v_2$ ) blue, making her choose one of the vertices  $v_2$  and  $v_5$  (resp.  $v_1$  and  $v_4$ ). Hence, in her next turn, she intersects only one of the hyperedges  $e_4$  and  $e_8$

(resp.  $e_4$  and  $e_6$ ) and so, Bob wins the game obtaining a monochromatic blue hyperedge. Similarly, if Alice colours  $v_2$  (resp.  $v_4$ ) red at her second turn, Bob will colour vertex  $v_5$  (resp.  $v_1$ ) making her colour one of the vertices  $v_1$  and  $v_4$  (resp.  $v_2$  and  $v_5$ ), which forbids her to intersect both hyperedges  $e_2$  and  $e_8$  (resp.  $e_2$  and  $e_6$ ) and so, Bob wins the game again.  $\square$

Now we analyze the game played on the hypergraphs  $\mathcal{H}(C_n^2)$  for  $n > 6$ :

**Theorem 3.5.** *Let  $n > 6$ . There exists a strategy that allows Alice to win the  $(1, 1)$ -monochromatic transversal game played on the clique-hypergraph  $\mathcal{H}(C_n^2)$ , independently of who starts playing the game.*

*Proof.* Recall that all hyperedges of  $\mathcal{H}(C_n^2)$ ,  $n > 6$ , are external. In order to obtain his monochromatic hyperedge, Bob must colour blue 3 consecutive vertices. Therefore, Alice wins the game playing according to the following rules which are stated in decreasing importance (she only follows rule  $(j)$  if its not possible to follow any rule  $(i)$  for  $i < j$  with  $i, j \in \{1, 2, 3, 4\}$ ): (1) she colours vertex  $v_{k-1}$  whenever there are two blue vertices  $v_{k \pmod n}$  and  $v_{k+1 \pmod n}$ ; (2) she colours vertex  $v_k$  whenever there are two blue vertices  $v_{k-1 \pmod n}$  and  $v_{k+1 \pmod n}$ ; (3) if Bob coloured vertex  $v_{k \pmod n}$  then Alice colours vertex  $v_{k+1 \pmod n}$ ; (4) if Bob coloured vertex  $v_{k \pmod n}$  then Alice colours vertex  $v_{k-1 \pmod n}$ .  $\square$

Now we use Remark 2.3 to extend the validity of Theorem 3.5 for the case when the game is played on  $\mathcal{H}(C_n^k)$ , with  $k \geq 3$ . In order to do so, we need to identify which are the powers of cycles that contains internal maximal cliques. The next lemma tells us when  $\mathcal{H}(C_n^k)$  has internal maximal cliques.

**Lemma 3.6.** *Let  $K = \{v_{a_1}, v_{a_2}, \dots, v_{a_{k+1}}\}$  be a clique of  $C_n^k$ .  $K$  is an internal maximal clique of  $C_n^k$ , if and only if,  $k$  is even,  $n = 2(k+1)$  and  $|a_{j+1} - a_j| = 2$  for every  $j \in \{1, \dots, k\}$ .*

*Proof.* Assume that there exists  $i, j \in \{1, \dots, k\}$  such that  $i \neq j$  and  $|a_{i+1} - a_i| \neq |a_{j+1} - a_j|$ . Thus, since  $v_\ell$  and  $v_r$  are adjacent for every  $\ell < r$  with  $r - \ell \leq k$ ,  $K$  is contained in an external clique which is not maximal. Therefore, if  $K$  is an internal maximal clique of  $C_n^k$ , then  $k + 1$  divides  $n$ , that is, there exists  $q \in \mathbb{N}$  such that  $n = q(k + 1)$ . Now we will prove that  $q = 2$ . Indeed, assume that  $q > 2$  and without loss of generality, let us also assume that



$v_0 \in K$ . Since we are not dealing with a complete graph, it is not possible to have  $v_{n/2} \in C_n^k$ . Therefore,  $k$  must be even and the vertices of  $K$  which maximizes their distances with respect to  $v_0$  are  $v_{-q(k/2) \pmod n}$  and  $v_{q(k/2)}$ . Since both  $v_0$  and  $v_{q(k/2)}$  belongs to  $K$ , they share a common edge and therefore  $k \geq q(k/2)$ , which contradicts the hypothesis  $q > 2$ . The converse is immediate.  $\square$

**Theorem 3.7.** *Let  $n > 6$ . Let  $k \geq 2$ . If  $C_n^k$  does not contain internal maximal cliques, i.e.,  $n > 2(k+1)$ , then there exists a strategy that allows Alice to win the (1,1)-monochromatic transversal game played in  $\mathcal{H}(C_n^k)$ .*

*Proof.* It follows from Remark 2.3, Lemma 3.6 and Theorem 3.5.  $\square$

#### 4. The $(a, a)$ -monochromatic transversal game on $\mathcal{H}(C_n^k)$

In this section we consider the  $(a, a)$ -monochromatic game played on  $\mathcal{H}(C_n^k)$  for any  $a \geq 2$ . The following result generalizes Theorem 3.7 for these new choices on the parameters, showing the advantage of Alice playing on this hypergraphs:

**Theorem 4.1.** *Let  $n > 6$ . Let  $k \geq 2$ . Assume that  $C_n^k$  does not contain internal maximal cliques, i.e.,  $n > 2(k+1)$ . Let  $a < \min \{ \lfloor k/2 \rfloor + 1, \lceil \frac{n}{k+1} \rceil \}$ . Thus, there exists a strategy that allows Alice to win the  $(a, a)$ -monochromatic transversal game played in  $\mathcal{H}(C_n^k)$ .*

In order to prove the above theorem, we introduce a new proof technique for the game. More precisely, we see the evolution of the game along the turns through a different perspective when it is played on a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ . Indeed, let  $\mathcal{V}_0 = \mathcal{V}$  and let  $\mathcal{E}_0 = \mathcal{E}$ . For each  $t \geq 1$  let  $S_t \subset \mathcal{V}$  denote the set of vertices that were coloured at turn  $t$ . For each  $t \geq 1$  let  $E_t$  denote the set of hyperedges in  $\mathcal{E}_{t-1}$  that are intercepted by vertices in  $S_t$ , i.e., for any  $e \in E_t$  there exists a vertex  $v \in S_t$  with  $v \in e$ . For each hyperedge  $e \in E_t$  let  $\tilde{e}$  be the set  $e \setminus \{e \cap S_t\}$ . Define  $\tilde{E}_t := \{\tilde{e}; e \in E_t\}$ . Now, for each  $t \geq 1$  let us define  $\mathcal{V}_t$  and  $\mathcal{E}_t$  according to the following rules:

- if turn  $t$  is played by Alice then  $\mathcal{V}_t = \mathcal{V}_{t-1} \setminus S_t$  and  $\mathcal{E}_t = \mathcal{E}_{t-1} \setminus E_t$ ;
- if turn  $t$  is played by Bob then  $\mathcal{V}_t = \mathcal{V}_{t-1}$  and  $\mathcal{E}_t = \{\mathcal{E}_{t-1} \setminus E_t\} \cup \tilde{E}_t$ .

Above, we have defined a process  $\{(\mathcal{V}_t, \mathcal{E}_t); t \geq 0\}$  which, at each turn, changes the structure of the hypergraph. Precisely, after Alice's turns, we remove all the vertices that she coloured and all hyperedges incident to them, while after Bob's turns, the vertex set remains the same but we remove the vertices that he coloured from the hyperedges of the previous hypergraph.

Observe that the process above defined allows the existence of empty hyperedges, that is,  $\lim_{t \rightarrow |\mathcal{V}|} \mathbb{P}(\emptyset \in \mathcal{E}_t) > 0$ . Furthermore, through this perspective, Alice wins the game if there exists  $t \in \mathbb{N}$  such that  $\mathcal{E}_t = \emptyset$ , which is equivalent to obtain a monochromatic transversal on  $\mathcal{H}$ ; and Bob does it whether there exists  $t \in \mathbb{N}$  such that  $\emptyset \in \mathcal{E}_t$ , which is equivalent to obtain a monochromatic hyperedge on  $\mathcal{H}$ . Now we are finally ready to prove Theorem 4.1:

*Proof of Theorem 4.1.* By Remark 2.1 we assume that Bob has the first turn. Thus, Alice can only play at even turns. Recall that Bob's strategy is to obtain a monochromatic induced path on  $C_n$  of length  $k + 1$  and that for each  $t \geq 0$ ,  $S_{2t+1}$  denotes the set of vertices that Bob coloured at turn  $2t + 1$ . Since Alice colours  $|S_{2t+1}|$  vertices per turn (with a possible exception if it is the last turn), for each  $v \in S_{2t+1}$  she can colour at least one vertex  $w = w(v)$  such that  $v$  and  $w$  are adjacent, at turn  $S_{2t+2}$ . Moreover, if  $S_{2t+1}$  is a path or a finite number of paths, she tries to colour the endpoints of these paths. At turn  $2t + 3$ , Bob must try to either decrease the number of vertices that belong to the edges in  $\tilde{E}_{2t+1} \cap \mathcal{E}_{2t+2}$ , otherwise he restarts his initial strategy. However, since  $a \leq \lfloor k/2 \rfloor$ , Bob will not be able to make  $\tilde{E}_{2t+1} \cap \mathcal{E}_{2t+2} = \emptyset$  because he obtains a monochromatic path on the induced  $C_n$  of length at most  $2a$ . At Alice's next turn, she colours the other endpoint of that path and makes him start his strategy again.  $\square$

## 5. Helping Bob to win the game on $\mathcal{H}(C_n^k)$

In the previous section we have shown that Alice has a strong advantage with respect to Bob when they play the game on clique-hypergraphs of powers of cycles  $\mathcal{H}(C_n^k)$ . In this section, we show for which how large  $b$  must be with respect to  $a$  so that Bob wins the  $(a, b)$ -monochromatic transversal game on these hypergraphs. We prove the following:

**Theorem 5.1.** *Let  $n > 6$ . Let  $k \geq 2$ . Assume that  $C_n^k$  does not contain internal maximal cliques, i.e.,  $n > 2(k + 1)$ . Let  $a < b$  be two positive*

integers. Define the function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by

$$f(a, b) := \left\lfloor \frac{b}{\left\lceil \frac{a}{2} \right\rceil + 1} \right\rfloor + b.$$

If  $f(a, b) > k$  then there exists a strategy that allows Bob to win the  $(a, b)$ -monochromatic transversal game on  $\mathcal{H}(C_n^k)$ , independently of who has the first turn.

*Proof.* Let us assume that Alice starts the game and  $f(a, b) > k$ . We show that there exists a strategy that allows Bob to win the game in this regime, and by Remark 2.1, the result is the same if he starts playing. Let  $r = \lceil \frac{n}{a} \rceil$ . Since Alice wants to maximize the number of hyperedges of  $\mathcal{H}(C_n^k)$  that she obtains per turn, she starts colouring vertices  $v_0, v_r, \dots, v_{(a-2)r}$  and an  $a$ -th vertex between  $v_{(a-2)r}$  and  $v_0$ . In this way, she spreads her colouring uniformly along the hypergraph. Observe that when Bob colours a vertex, or a path on the cycle, Alice immediately colours the vertices which are adjacent to the endpoints of that path. Thus, for each path coloured by Bob along the cycle, Alice must spend 2 of her vertex choices to prevent that Bob increases the lengths of those paths. For that purpose, take  $\ell = \lceil \frac{a}{2} \rceil + 1$  and  $L = \lfloor \frac{b}{\ell} \rfloor$ . Notice that  $b > \ell$  and therefore  $L \geq 1$ . Now, for each  $j \in \{0, 1, \dots, \ell - 1\}$  let  $P_j$  be the path on the cycle which is between vertices  $v_{jr}$  and  $v_{(j+1)r}$ . In his turn, for each  $j \in \{0, 1, \dots, \ell - 1\}$  Bob colours the  $L$  vertices of  $P_j$  which are furthest from  $v_{jr}$  and  $v_{(j+1)r}$ , and the remaining vertices elsewhere. In her second turn, Alice tries to preclude Bob to increase his  $\ell$  paths of length  $L$ . However, since  $2\ell > a$ , she is not able to do so. Thus, in his second turn, Bob may colour  $b$  adjacent vertices next to a survivor path, making a path of length  $f(a, b)$  on the cycle. Notice that this is possible due to the fact that  $f(a, b) < r$ . Hence, since  $f(a, b) > k$ , Bob got his monochromatic hyperedge.  $\square$

Now, one might immediately ask themselves whether Bob loses if  $f(a, b) \leq k$ . Trying to give an answer for this question, we prove the following result, which asserts that it is true if  $a \geq 2$ .

**Theorem 5.2.** *Let  $n > 6$ . Let  $k \geq 2$ . Assume that  $C_n^k$  does not contain internal maximal cliques, i.e.,  $n > 2(k + 1)$ . Let  $2 \leq a < b$  be two positive*

integers. Define the function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by

$$f(a, b) := \left\lfloor \frac{b}{\left\lceil \frac{a}{2} \right\rceil + 1} \right\rfloor + b.$$

If  $f(a, b) \leq k$  then there exists a strategy that allows Alice to win the  $(a, b)$ -monochromatic transversal game on  $\mathcal{H}(C_n^k)$ , independently of who has the first turn.

*Proof.* The proof is very similar to the one of Theorem 5.1. Indeed, by Remark 2.1 we assume that Bob starts the game. If he colours  $b$  consecutive vertices  $v_0, \dots, v_{b-1}$  then, since  $a \geq 2$  and  $b \leq k$ , Alice colours the vertices  $v_{n-1}$  and  $v_b$ , and he has to restart his strategy with an inferior number of possibilities. Therefore, in his first turn, Bob must colour disjoint paths on the cycle such that Alice cannot colour the endpoints of these paths. Moreover, as shown in the proof of last theorem, he must colour  $\ell = \left\lceil \frac{a}{2} \right\rceil + 1$  paths of length  $L = \lfloor \frac{b}{\ell} \rfloor$ . At her first turn, Alice colours the endpoints of these paths with exception of one among them. Let  $P$  be such path. Thus, at his second turn, even if Bob increases the length of this path with other  $b$  vertices, the fact that  $f(a, b) \leq k$  does not allow him to obtain a monochromatic hyperedge. Therefore, he must keep creating this paths. Meanwhile, Alice continues preventing their growth, prioritizing the largest ones. The prove follows from the fact that, at each  $t$  turns,  $t \leq 3$ , all the paths which were coloured by Bob in the two previous turns have already been forbidden to grow by Alice's colouring.  $\square$

## 6. Conclusion

Combining Remark 2.2 with Proposition 3.1, Proposition 3.2 and Theorem 4.1, it is possible to complete the following table of results:

Hypergraph	Value of $a$	Value of $b$	Who wins (indep. of who started)
$\mathcal{H}(K_n)$	$a \in \mathbb{N}$	$b < n$	Alice
$\mathcal{H}(C_n)$ $n \geq 4$	1	$b \leq n$	Bob
$\mathcal{H}(C_n^k)$ $k \geq 2$ without internal hyperedges	$a \geq b$	$b \leq \lfloor k/2 \rfloor$	Alice

Furthermore, the combination of Remark 2.2 with Theorem 5.1 and Theorem 5.2 completes the second table:

Hypergraph	Who wins if $b > a$ and $\lfloor \frac{b}{\lfloor \frac{a}{2} \rfloor + 1} \rfloor + b > k$	Who wins if $b > a \geq 2$ and $\lfloor \frac{b}{\lfloor \frac{a}{2} \rfloor + 1} \rfloor + b \leq k$
$\mathcal{H}(C_n^k)$ $k \geq 2$ without internal hyperedges	Bob	Alice

From the above tables, one can conclude that it is done a thorough analysis about the  $(a, b)$ -monochromatic transversal game played on clique-hypergraphs of powers of cycles without internal maximal cliques,  $\mathcal{H}(C_n^k)$ , for every  $k \geq 2$ .

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