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Abstract

We introduce the (a, b)-monochromatic transversal game where Alice and Bob alternately colours a vertices in red and b vertices in blue of a hypergraph, respectively. Both players are enabled to start the game. Alice wins the game if she obtains a red transversal; otherwise, Bob wins if he obtains a monochromatic blue hyperedge. We analyze the game played on clique-hypergraphs of powers of cycles and we show many strategies that, depending on the choice of the parameters, allow a specific player to win the game.

1 Introduction

The concept of *hypergraphs* has been extensively studied in many areas. It generalizes the definition of graphs allowing edges to have more than two incident vertices. Hypergraph theory is also applied in modern mathematics and related research fields, such as Furstenberg-Katznelson's theorem in ergodic theory was proven using hypergraph modeling [10]. zturan [14] also used hypergraphs to model chemical reaction networks.

A *transversal* in a hypergraph is a set of vertices intersecting every hyperedge [3]. A *maximal clique* of a graph is a subset of its vertices that induces a complete graph, and it is not properly contained in any other clique. A *clique-hypergraph* of a graph G is a hypergraph with the same vertex set of G and whose hyperedges are the maximal cliques of G.

The study of clique-hypergraphs was firstly presented in 1991 by Duffus et al. [9], where the authors asked what is the smallest number of colours needed to colour the vertices of a clique-hypergraph such that no pair of adjacent vertices is monochromatic (*clique-chromatic number*). Later, in 2003, Gravier et al. [12] showed that, given a fixed graph F, there exists an integer $f(F)$ such that the clique-hypergraph of any F-free graph can be $f(F)$-colored if and only if all components of F are paths. After that, in 2004, Bacs et al. [2] proved that this number is 3 for almost all perfect graphs. In 2013, the answer to Duffus' question in the case of the clique-chromatic number of powers of cycles was presented by Campos et al. [7], where the authors showed that such number is equal to 2, except for odd cycles of size at least 5 where the answer is 3.

Classic problems in graph theory, like coloring [11], labeling [1] and domination [8], have been studied from the perspective of Combinatorial games. *Combinatorial games* [4] are alternating finite two-player games of pure strategy in which all the relevant information is public to both players, as well as no randomness or luck is allowed. A combinatorial game using the concept of tranversals in hypergraphs is the *transversal game*, presented by Bujts et al. [5, 6]. In this game, two players, Edge-hitter and Staller, take turns choosing a vertex from $\mathcal{H}$. Each chosen vertex must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of selected vertices becomes a transversal in $\mathcal{H}$. The strategy of Edge-hitter is to minimize the number of chosen vertices, while Staller wishes to maximize it. The authors defined the *game transversal number* of $\mathcal{H}$ as the number of vertices chosen when Edge-hitter starts the transversal game and both players play optimally. Using this definition, they showed that the $\frac{3}{4}$-Game Total Domination Conjecture is true over the graph classes with minimum degree at least 2, and compared this parameter with its transversal number.

In this work, we introduce a combinatorial game over hypergraphs: the (a, b)-*monochromatic transversal game*. In this game, Alice and Bob alternately colour a vertices in red and b vertices in blue of a hypergraph, respectively. Both players are enabled to start the game. Alice wins the game if she obtains a red transversal while Bob wins if he obtains a monochromatic blue hyperedge. We consider the game played on clique-hypergraphs of complete graphs, powers of cycles and paths. For each of these graphs, we show strategies and bounds to the parameters a and b so that one of the players wins the game.

We organize the paper as follows. First, in Section 2, we introduce the game and present standard definitions and notation. In Section 3, we start playing the game considering that Alice and Bob colour a single vertex at a time on complete graphs, powers of cycles and paths. In Section 4, we exhibit results when Alice and Bob are allowed to colour more vertices on powers of cycles, and show that Alice has advantage on the game played on the hypergraphs of these graphs when $a \geq b$ and $n > 3k$. In Section 5, we prove that for n sufficiently large Bob wins the game when $b > a$. In Section 6, we show how "small" n must be in order to guarantee Alice's victory. Finally, in Sections 7 and 8, we present our conclusions and acknowledgments.

2 Description of the game

In this work, the graphs considered are all undirected and simple. A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite vertex set, and $\mathcal{E}$ is a family of nonempty subsets of V called *hyperedges*. The (a, b)-*monochromatic transversal game* is an avoider-enforcer combinatorial game where two players, Alice and Bob, alternately colour the vertices of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. Both players are enabled to start the game and, in each turn, Alice colours $a \geq 1$ vertices in red and Bob colours $b \geq 1$ vertices in blue. Alice wins the game if she obtains a red transversal, that is, a subset of vertices in $\mathcal{V}$ that has a nonempty intersection with every hyperedge of $\mathcal{H}$. Bob wins the game if he prevents it by obtaining a monochromatic blue hyperedge of $\mathcal{E}$. The game presents properties as the following:

Remark 2.1. *If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts playing the* (a, b)*-monochromatic transversal game on a given hypergraph, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts playing the game on that hypergraph.*

Remark 2.2. *If there exists a strategy that allows Alice (resp. Bob) to win the (a_0, b_0)-monochromatic transversal game played on a given hypergraph, independently of who starts it, then for any $a > a_0$ (resp. $b > b_0$) there exists a strategy that allows Alice (resp. Bob) to win the (a, b_0)-monochromatic transversal game (resp. (a_0, b)-monochromatic transversal game) played on that hypergraph, independently of who starts it.*

The *clique-hypergraph* $\mathcal{H}(G) = (V, \mathcal{E})$ of a graph $G = (V, E)$ is a hypergraph such that V is the vertex set of G, and the hyperedge set $\mathcal{E}$ is the set of all *maximal cliques* in G, that is, $\mathcal{E}$ is the set of all maximal subsets of V whose vertices induce a complete graph. In this work, we consider clique-hypergraphs of powers of cycles.

A *k-th power of cycle of length n*, C_n^k, for $k \geq 1$, is a graph on n vertices whose vertex set is $V(C_n^k) = \{v_i : i \in \mathbb{Z}_n\}$, and whose edges $\{v_i, v_j\}$, $i, j \in \mathbb{Z}_n$ have the property that $i = j \pm r \pmod n$ for some $r \in \{1, 2, \ldots, k\}$. We take $(v_0, \ldots, v_{n-1})$ to be a cyclic order on the vertex set of C_n^k, and always perform arithmetic modulo n on vertex indexes. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of all the vertices adjacent to v. The *reach* of an edge $\{v_i, v_j\}$ is defined as $d_{ij} = \min\{(i - j) \bmod n, (j - i) \bmod n\}$. Observe that, with the previous definition, C_n^1 is a cycle C_n, and for $k \geq \lfloor \frac{n}{2} \rfloor$ is isomorphic to the complete graph.

A *k-th power of a path of order n*, P_n^k, for $k \geq 1$, is a graph on n vertices whose vertex set is $V(P_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$ and whose edges $\{v_i, v_j\} \in E(P_n^k)$ if and only if $|i - j| \leq k$. Note that, $P_n^1 \simeq P_n$ and $P_n^k \simeq K_n$ for $n \leq k + 1$. In a power of a path P_n^k, we take $(v_0, v_1, \ldots, v_{n-1})$ to be a *linear order* on the vertex set.

According to [7], the maximal cliques of powers of cycles C_n^k can be classified into two types: an *external clique*, whose vertex set is composed by $k+1$ vertices with consecutive indexes $v_x, \ldots, v_{x+k \pmod n}$ for some $x \in \mathbb{Z}_n$, and an *internal clique* that has non-consecutive vertex indexes. Figure 1 illustrates powers of cycles with and without internal maximal cliques.

We observe that if $b > k$, then the (a, b)-monochromatic transversal game played on clique-hypergraphs of powers of cycles becomes trivial because at his first turn Bob colours all the vertices of a maximal external clique Similarly, if $a \geq \lceil \frac{n}{k+1} \rceil$, and C_n^k has no internal maximal cliques, then Alice colours all the vertices of a transversal.

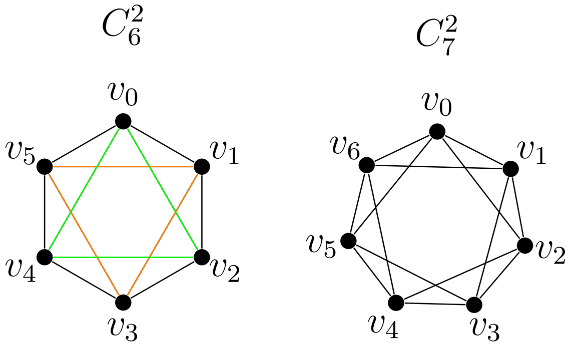

Figure 1: Graphs C_6^2, with internal maximal cliques in orange and green, and C_7^2, without internal maximal cliques.

Furthermore, observe that if Bob can not colour a clique of size k' while playing the (a, b)-monochromatic transversal game on a clique-hypergraph $\mathcal{H}(C_n^{k'})$, then he can not colour a clique of size k for $k > k'$. By this observation we obtain the following remark:

Remark 2.3. *Let C_n^k be a power of cycle with no internal maximal cliques and $2 \leq k' < k$. If there exists a strategy that allows Alice to win the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^{k'})$, then there exists a strategy that allows Alice to win the game played on the clique-hypergraph $\mathcal{H}(C_n^k)$.*

3 First plays

We begin this section analyzing the $(1, 1)$-monochromatic transversal game played on clique-hypergraphs of powers of cycles C_n^k. First, we show strategies that can be used when k has either its smallest or largest possible value, that is, C_n (when $k = 1$) or K_n (when $k \geq \lfloor \frac{n}{2} \rfloor$).

Proposition 3.1. *If K_n is a complete graph with $n \geq 2$, then there exists a strategy that allows Alice to win the $(1, 1)$-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(K_n)$, independently of who starts playing the game.*

Proof. Suppose that Bob starts playing the game. We observe that the clique-hypergraph $\mathcal{H}(K_n)$ of the complete graph on n vertices, K_n, contains a unique hyperedge with n vertices. Therefore, Alice obtains a red transversal in her first turn. The result follows by Remark 2.1. □

Now we analyze the game played on the clique-hypergraph of a cycle. Since C_3 is isomorphic to K_3 (whose result is contained in Proposition 3.1), we consider the case $n \geq 4$.

Proposition 3.2. *If C_n is a cycle of length $n \geq 4$, then exists a strategy that allows Bob to win the $(1,1)$-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n)$, independently of who starts playing the game.*

Proof. First, we observe that the hyperedges of $\mathcal{H}(C_n)$ are the edges of C_n. Suppose that Alice starts the game and colours red the vertex v_i. Bob wins the game colouring blue a vertex v_j that is not adjacent to v_i. Indeed, independently of which vertex Alice colours red in her next move, Bob obtains a monochromatic blue hyperedge coloring vertex v_l adjacent to v_j, with $l \in \{j-1 \, (\text{mod } n), j+1 \, (\text{mod } n)\}$. The result follows by Remark 2.1. □

The game played on clique-hypergraphs of paths, $\mathcal{H}(P_n)$, $n \geq 3$, is similar to the game over $\mathcal{H}(C_n)$, $n \geq 4$. If $n = 2$, since P_2 is isomorphic to K_2, then the result follows by Proposition 3.1. If $3 \leq n \leq 5$, then there exists a strategy that allows the player who had the first turn to win the game. Indeed, if Bob starts playing, he colours blue vertex $v_{\lfloor \frac{n}{2} \rfloor}$. Now, Alice must colour $v_{\lfloor \frac{n}{2} \rfloor - 1}$ or $v_{\lfloor \frac{n}{2} \rfloor + 1}$. On his next turn, Bob obtains a blue hyperedge colouring $v_{\lfloor \frac{n}{2} \rfloor + 1}$ or $v_{\lfloor \frac{n}{2} \rfloor - 1}$. If Alice starts playing, she colours red vertex $v_{\lfloor \frac{n}{2} \rfloor}$. On the next turns, independently of which vertex Bob colours blue, Alice wins the game colouring red the vertex adjacent to Bob's last coloured vertex, and prevent him to make any monochromatic blue P_2. If $n \geq 6$, an argument analogous to the proof of Proposition 3.2 shows that there exists a strategy that allows Bob to win the game, independently of who starts playing the game.

4 Alice's dream

Now, we consider the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$. We recall that the maximal cliques of powers of cycles can be classified into external or internal cliques. The next result presents a lower bound to the size of n that guarantees the non-existence of internal maximal cliques.

Lemma 4.1. *If $n > 3k$ and $k \geq 2$, then C_n^k has no internal maximal clique.*

Proof. Let K be a maximal clique of C_n^k. Without loss of generality we may assume that $v_0 \in K$. Let $i, j \in \mathbb{N}^*$, $i \neq j$, where: (1) $i \leq k$ and $j \geq k$ are largest as possible and such that (2) v_{n-j}, $v_i \in K$. Since $n > 3k$, and by (2), the reach between the vertices v_{n-j} and v_i is $d_{n-j,i} = i + j \leq k$. By (1), $v_s \notin K$ for each $s \in]i, k+1[$ and $v_{n-s} \notin K$ for each $s \in]k+1, j[$. Moreover, for each $s \in [k+1, n-(k+1)]$, since $v_0 \in K$ and $d_{0,s} > k$, we have that $v_s \notin K$. Now, since the subgraph H induced by $\{v_{n-j}, \ldots, v_0, \ldots, v_i\}$ is a clique, it follows that $K = H$ and $i + j = k$. □

Now, the following theorem shows that the condition $a \geq b$ ensures that Alice wins the game when C_n^k has no internal cliques.

Theorem 4.2. *Let $n > 3k$ and $k \geq 2$. If $a \geq b$ and $b < k$, then there exists a strategy that allows Alice to win the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.*

Before proving this result, we present a new way of seeing the evolution of the game throughout the turns. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and set $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{E}_0 = \mathcal{E}$. For each $t \geq 1$, $\mathcal{B}_t$ (resp. $\mathcal{V} \setminus \mathcal{V}_t$) is the set of blue (resp. red) vertices after t turns. We assume that $\mathcal{B}_0 = \emptyset$. Let $S_t \subseteq (\mathcal{V}_{t-1} \setminus \mathcal{B}_{t-1})$ be the set of vertices coloured in turn t.

Definition 4.3. *A hypergraph $\mathcal{H}_t = (\mathcal{V}_t, \mathcal{E}_t)$ is the hypergraph at turn t, where $\mathcal{V}_t$ and $\mathcal{E}_t$ are obtained according following rules:*

1. *If turn t is played by Alice then $\mathcal{V}_t = \mathcal{V}_{t-1} \setminus S_t$, $\mathcal{E}_t$ is the subset of hyperedges e in $\mathcal{E}_{t-1}$ such that $e \cap S_t = \emptyset$, and $\mathcal{B}_t = \mathcal{B}_{t-1}$.*

2. *If turn t is played by Bob then $\mathcal{V}_t = \mathcal{V}_{t-1}$, $\mathcal{E}_t = \mathcal{E}_{t-1}$, and $\mathcal{B}_t = \mathcal{B}_{t-1} \cup S_t$.*

The game ends at some t^* such that $\mathcal{V}_{t^*} = \mathcal{B}_{t^*}$. Finally, Alice wins if and only if $\mathcal{E}_{t^*} = \emptyset$.

For example, let $t = 0$ and consider the clique-hypergraph $\mathcal{H}_0 = \mathcal{H}(C_{10}^2) = (\mathcal{V}_0, \mathcal{E}_0)$, where $\mathcal{V}_0 = \{v_0, v_1, \ldots, v_9\}$, and $\mathcal{E}_0 = \{e_1, e_2, \ldots, e_{10}\}$ such that $e_1 = \{v_0, v_1, v_2\}$, $e_2 = \{v_1, v_2, v_3\}$, $e_3 = \{v_2, v_3, v_4\}$, $e_4 = \{v_3, v_4, v_5\}$, $e_5 = \{v_4, v_5, v_6\}$, $e_6 = \{v_5, v_6, v_7\}$, $e_7 = \{v_6, v_7, v_8\}$, $e_8 = \{v_7, v_8, v_9\}$, $e_9 = \{v_8, v_9, v_0\}$, and $e_{10} = \{v_9, v_0, v_1\}$. Note that, $\mathcal{B}_0 = \emptyset$. We refer to Figure 2.

Suppose that, in $t = 1$, Bob colours blue vertices v_3 and v_4, thus $\mathcal{V}_1 = \mathcal{V}_0$, $\mathcal{E}_1 = \mathcal{E}_0$, and $\mathcal{B}_1 = \mathcal{B}_0 \cup S_1 = \emptyset \cup \{v_3, v_4\} = \{v_3, v_4\}$. Therefore, $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{B}_1 = \{v_3, v_4\}$.

Now, in $t = 2$, if Alice colours red vertices v_2 and v_5 then $S_2 = \{v_2, v_5\}$, $\mathcal{V}_2 = \mathcal{V}_1 \setminus \{v_2, v_5\}$, and $\mathcal{E}_2 = \{e_7, e_8, e_9, e_{10}\}$, since $e_i \cap S_2 = \emptyset$ for $i \in \{7, 8, 9, 10\}$. Therefore, $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ and $\mathcal{B}_2 = \mathcal{B}_1$ (see Figure 2).

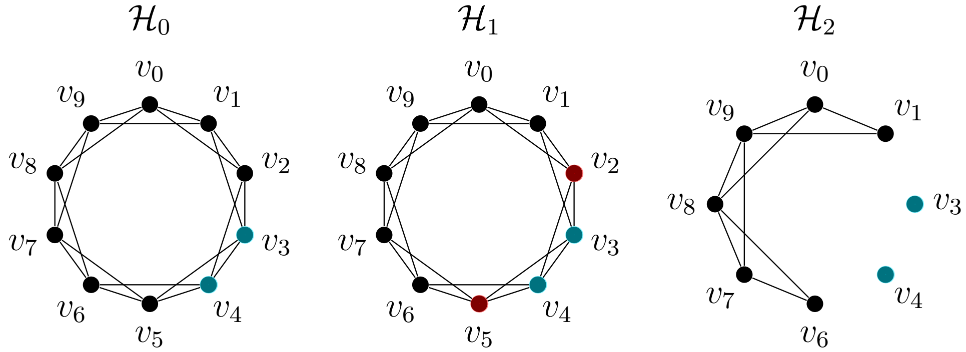

Figure 2: Hypergraphs $\mathcal{H}_0 = C_{10}^2$, $\mathcal{H}_1$ after Bob colours blue v_3 and v_4 in H_0, and $\mathcal{H}_2$ after Alice colours red v_2 and v_5 in H_1.

Now, we are ready to prove our result.

Proof of Theorem 4.2. Suppose that Bob starts playing. Bob's turns consists in colouring blue ℓ disjoint paths P_j ($j = 1, ..., \ell$; $\ell \leq a$) in the cycle C_n. By definition, we have that $\sum_{j=1}^{\ell} |P_j| \leq b < k$. Alice's strategy consists in applying the following rules:

1. If P_j has one vertex, say v_i, then Alice colours red v_{i+1} if is not coloured yet; otherwise she colours red v_{i-1}.
2. If P_j has two distinct extremities v_i and v_s with $i < s$, then Alice colours vertices v_{i-1} and v_{s+1}, if they are not coloured yet.

First, observe that Alice colours at most $a \geq b$ vertices by turn. Second, to check that these rules ensure that Alice wins the game, it is enough to verify that, after each Alice's turn $2t \geq 2$, the hypergraph $\mathcal{H}_{2t}$ is a disjoint union of k-power of paths having the following property (P): each connected component of $\mathcal{H}_{2t}$ contains at most one blue vertex. Moreover, if there exists one such blue vertex then it is the right extremity (clockwise) of this component.

It is easy to see that $\mathcal{H}_{2t}$, $2t > 2$, satisfies (P). Suppose that $\mathcal{H}_{2t-2}$ satisfies (P), we prove that $\mathcal{H}_{2t}$ satisfies (P) whenever Alice apply the rules above.

The set $\mathcal{B}_{2t-1} \setminus \mathcal{B}_{2t-3}$ corresponds to the vertices coloured by Bob at turn $2t-1$ which induces a disjoint union of paths P_j ($j = 1, ..., \ell$) in the cycle C_n. Remark that each P_j must belong to one connected component O_j of $\mathcal{H}_{2t-2}$. We analyse two cases:

1. P_j has one vertex, say v_i: if Alice colours red v_{i+1} then $O_j \setminus \{v_{i+1}\}$ induces two disjoint paths satisfying (P); else, Alice colours red v_{i-1} that is either it is an extremity of O_j or the neighbor of the blue extremity of O_j. In both cases, $O_j \setminus \{v_{i-1}\}$ induces the disjoint union of a path of length $|O_j| - c$, with $c = 1$ or 2, satisfying (P) plus c isolated blue vertices.

2. P_j has two distinct extremities v_i and v_s with $i < s$: if Alice colours red vertices v_{i-1} and v_{s+1}, then $O_j \setminus \{v_{i-1}, v_{s+1}\}$ induces $s - i + 1 \leq b$ isolated blue vertices and at most two disjoint paths satisfying (P). In the other cases, it means that P_j is closed to the extremities of O_j. The worst case occur when $v_{s+1} \in \mathcal{B}_{2t-2}$, thus colouring v_{i-1} (if it exists in $\mathcal{V}_{2t-2}$), Alice splits O_j into at most one connected component satisfying (P) (the one containing vertices of O_j with index smaller than $i-2$) plus $s - i + 1 + 1 \leq b + 1$ vertices which are isolated since $b < k$.

The result follows by Remark 2.1. □

5 Bob is the winner

In the previous sections, we have shown that Alice has advantage with respect to Bob when they play the (a, b)-monochromatic transversal game with $a \geq b$ on clique-hypergraphs of powers of cycles $\mathcal{H}(C_n^k)$. In this section, we consider $1 \leq a < b$ and define conditions so that Bob wins the (a, b)-monochromatic transversal game played on these hypergraphs.

Theorem 5.1. *Let $1 \leq a < b$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$. If $n \geq (a+1)^\alpha (k+1)$, then Bob wins the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, when he starts playing.*

Proof. Let $a, b, n, k \in \mathbb{N}^*$ with $a < b$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$.

We describe Bob's strategy on a well chosen subset of hyperedges of $\mathcal{H}(C_n^k)$. Let $\mathcal{H}_0 = \mathcal{H}(C_n^k)$. Let $\mathcal{W}_0$ be a set of $(a+1)^\alpha$ disjoint hyperedges of $\mathcal{H}(C_n^k)$. Since the size of the hyperedge in $\mathcal{H}(C_n^k)$ is $k+1$ and $n \geq (a+1)^\alpha.(k+1)$, such a set $\mathcal{W}_0$ exists. For each hyperedge $e \in \mathcal{W}_0$, we associate a weight $w(e)$. Before Bob's first turn, we set $w(e) = 0$ for all $e \in \mathcal{W}_0$. We say that Bob *marks* a hyperedge e if he adds 1 to $w(e)$.

We prove that Bob wins the game in at most $\frac{(a+1)^\alpha - 1}{a}$ turns. First, we introduce additional definitions. Let $T_i = [i^-, i^+]$, $1 \leq i \leq \alpha$, be the interval of indices that represents the partition of the turns played by Bob where

$$i^- = 1 + \textstyle\sum_{j=\alpha-(i-1)}^{\alpha-1} (a+1)^j \text{ and } i^+ = \textstyle\sum_{j=\alpha-i}^{\alpha-1} (a+1)^j.$$

We define $1^- = 1$. Remark that $|T_i| = i^+ - i^- + 1 = (a+1)^{\alpha - i}$. Also, note that $\alpha^- = \alpha^+ = \frac{(a+1)^\alpha - 1}{a}$. Now, let $\mathcal{W}_i \subseteq \mathcal{W}_0 \cap \mathcal{E}_{2.i^+}$, for all $i \in [1, .., \alpha]$, be the set of hyperedge in $\mathcal{W}_0$ which has $p.i^+$ blue vertices and no red vertex after $2.i^+$ turns, i.e., the set of the hyperedges coloured by Bob in a turn in T_{2i+1}.

Let $t \in \mathbb{N}^*$ such that $\lceil \frac{t}{2} \rceil \leq \frac{(a+1)^\alpha - 1}{a}$. Let $\alpha_t \in \mathbb{N}^*$ such that $\lceil \frac{t}{2} \rceil \in T_{\alpha_t}$, i.e., the index of the set T_i for which Bob is playing. Since Bob starts playing the game, in order to describe his strategy we focus on odd turns:

(1) If $\alpha_t < \alpha$, then Bob selects $a + 1$ hyperedges e in $\mathcal{W}_{\alpha_t - 1}$ with weight $w(e) = \alpha_t - 1$. Bob marks all these hyperedges and for each one that does not have a red vertex, Bob colours blue p vertices in e.

(2) Else, Bob colours all the noncoloured vertices of a hyperedge in $\mathcal{W}_\alpha$.

Observe that by the definition of $\mathcal{W}_{\alpha_t}$, any hyperedge e in this set with no red vertex has exactly $p \cdot \alpha_t$ blue vertices. Now, since $b+(\alpha-1)p < k+1$, then, when $\alpha_t < \alpha$, the hyperedge e has more than b noncoloured vertices. Since $1 \leq a < b$ follows that $b > p$. So, Bob colours p noncoloured vertices of e. Moreover, by rule (1), Bob selects $a+1$ hyperedges of $\mathcal{W}_{\alpha_t}$, thus he colours at most $(a+1)p$ vertices, which is less than b. So, Bob can apply rule (1).

Second, we claim that Bob's strategy provides the following property:

Claim 5.2. *For all $i \in [0, ..., \alpha]$, we have that $|\mathcal{W}_i| \geq (a+1)^{\alpha-i}$.*

Proof. Our proof works by induction on i. By definition, we have that $|\mathcal{W}_0| \geq (a+1)^{\alpha}$. Now suppose, for some $j > 0$, that $|\mathcal{W}_i| \geq (a+1)^{\alpha-i}$ holds for all $i < j$. Observe that $\mathcal{W}_j \subset \mathcal{W}_{j-1}$. During the turns between $(j-1)^-$ and $(j-1)^+$, i.e. the turns in T_{j-1}, Bob applies rule (1) which guarantees that a marked hyperedge with no red vertex has exactly pj blue vertices. Moreover, he marked $|T_{j-1}|(a+1) = (a+1)^{\alpha-j+1}$ hyperedges while Alice could not colour red more than $|T_{j-1}|a = a(a+1)^{\alpha-j}$ hyperedges. Therefore, $|\mathcal{W}_j| \geq (a+1)^{\alpha-j+1} - a(a+1)^{\alpha-j} = (a+1)^{\alpha-j}$. □

The previous claim shows that $|\mathcal{W}_\alpha| \geq 1$. Moreover, any hyperedge in $\mathcal{W}_\alpha$ has αp blue vertices and contains no red one. So, when Bob applies rule (2) on such hyperedge, he colours exactly $k+1-\alpha p$ vertices which is less than b by the hypotheses. □

Corollary 5.3. *Let $1 \leq a < b$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b + (\alpha-1)p < k+1 \leq b + \alpha p$. If $n \geq a(a+1)^\alpha(k+1)$, then there exists a strategy that allows Bob to win the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.*

Proof. By Theorem 5.1, it is enough to consider the case when Alice starts. Since $n \geq a(a+1)^\alpha(k+1)$, after Alice's turn, there exists a connected component of $\mathcal{H}_1$ having at least $(a+1)^\alpha(k+1)$ vertices. Applying the same strategy described in the proof of Theorem 5.1, in this component, ensures Bob's victory. The result follows by Remark 2.1. □

For fixed α, the bound for n given in Corollary 5.3 may be improved. For instance, a slight modification of the proof (not given here) allows to show that, for $\alpha = 2$, the bound $n \geq 2(a+1)(k+1)$ is enough to guarantee Bob's victory whenever he starts.

6 Alice's revenge

From previous results, the only way for Alice to win is to consider a "small" n. But how "small" n can be? The following results give an element of the answer.

Theorem 6.1. *Let $1 \leq a \leq b < k+1$ and $\beta = \lceil \frac{k+1}{b} \rceil \geq 2$. If $n \in \mathbb{N}^*$ such that $3k < n \leq a(\beta-1)(k+1)$, then there exists a strategy that allows Alice to win the (a,b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.*

Proof. Suppose that Bob starts playing and assume that $n \leq a(\beta-1)(k+1)$. Let $\mathcal{H}_t$ be the hypergraph as in Definition 4.3. We recall that S_{2t} is the set of vertices coloured red by Alice in the turn $2t$; and $\mathcal{B}_{2t-1}$ is the set of vertices coloured blue by Bob after $2t-1$ turns, for $t \geq 1$ (Bob plays in the odd turns). Without loss of generality, one may assume that $v_0 \notin \mathcal{B}_1$.

At turn $2t$ and $1 \leq t < \beta$, Alice's strategy consists in colouring red a vertices obtaining a partition the hypergraph $\mathcal{H}_{2t-1}$ into $ta-1$ connected components of length at most k, and one connected component with the remaining vertices R_t of length at most $n - (ta(k+1) - tb)$.

Therefore, at turn $2t$ with $1 \leq t < \beta$, Alice colours red the vertices $v_i \in S_{2t}$ with $i \in \{i_{(t-1)a}, \dots, i_{ta-1}\}$ defined by :

(i) $i_0 = 0$ and $i_{(t-1)a}$ is the smallest index $i > i_{(t-1)a-1}$ such that $v_i \notin \mathcal{B}_{2t-1}$.

(ii) For all $1 \leq j < a$, the integer $i_{(t-1)a+j}$ is the largest index $i \leq i_{(t-1)a} + j(k+1)$ such that $v_i \notin \mathcal{B}_{2t-1}$.

Remark that, since $(\beta-1)b < k+1$, then for all $1 \leq j < a$, we have that $i_{(t-1)a} + (j-1)(k+1) \leq i_{(t-1)a+j} \leq i_{(t-1)a} + j(k+1)$. Hence, $\mathcal{H}_{2t-1}$ can be partitioned into $ta-1$ sets of size $i_\ell - i_{\ell-1} \leq k+1$, for $\ell = 1, ..., ta-1$, plus a set of vertices R_t having at most $n - (ta(k+1) - tb)$. Hence $|R_{\beta-1}| \leq (\beta-1)b < k+1$ since $n \leq a(\beta-1)(k+1)$. Therefore, Alice wins the game since the hyperedge set $\mathcal{E}_{2(\beta-1)} = \emptyset$. The result follows by Remark 2.1.

□

We note that in Theorem 6.1 the bound on n is linear in a despite of the one of Theorem 5.1 which is in a^α. Next result shows that the two bounds are close whenever $\alpha = \beta - 1 = 1$.

Corollary 6.2. *Let a, b and k be integers such that $b \geq p.a$ and $k+1 = b+p$. There exists a strategy that allows Alice to win the (a,b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$ when Bob starts playing if and only if $n < a(k+1)+1$.*

Proof. First assume that $n \geq a(k+1)+1$. Bob's strategy is a slight improvement of the one given in Theorem 5.1. For $i = 0, \ldots, a$, let e_i be the hyperedge of $\mathcal{H}(C_n^k)$ induced by vertices of indexes $\{i(k+1), \ldots, i(k+1)+k\}$. Bob's first turn consists to colour blue the b vertices of indices $\{i(k+1), \ldots, i(k+1)+p\}$ for all $i = 0, \ldots, a$. Now, for $i = 0, \ldots, a$, Alice has to colour red at least one vertex on each e_i, otherwise, Bob colours blue all the $k+1-p$ remaining vertices in the third turn. Since there exists a such hyperedges then Alice colours exactly one vertex on each e_i. Let $v_{i(k+1)+r_i}$ be the red vertex in e_i.

We claim that if $0 \leq i < j \leq a$ then $r_i \geq r_j$. Indeed, it is enough to observe that if there is some $i < a$ with $r_i < r_{i+1}$, then the subset of vertices with indexes in $\{i(k+1)+r_i+1, \ldots, (i+1)(k+1)+r_i\}$ is an hyperedge e of $\mathcal{H}(C_n^k)$ with no red vertex and containing the p blue vertices of e_{i+1}, so in the third turn, Bob wins colouring blue the b remaining vertices of e.

Now, since $r_0 \geq r_a$ and $n \geq a(k+1)+1$, then $n - r_0 > r_a$. Hence, the set of vertices with indexes in $\{n-r_0, \ldots, 0, \ldots, r_0\}$ is an hyperedge e of $\mathcal{H}(C_n^k)$ with no red vertex and containing the p blue vertices of e_0. So, again, in the third turn, Bob wins colouring blue the b remaining vertices of e.

Finally, Theorem 6.1 shows that Alice wins whenever $n \leq a(k+1)$. □

7 Conclusion

Combining Remark 2.2 with Proposition 3.1, Proposition 3.2 and by Theorem 4.2, it is possible to complete the following table of results:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(K_n)$	$1 \le a \le n-1$	$b < n$	Alice
$\mathcal{H}(C_n)$, $n \ge 4$	1	$b < n$	Bob
$\mathcal{H}(C_n^k)$, $n > 3k$, $k \ge 2$	$a \ge b$	$b < k$	Alice

Furthermore, by Corollary 5.3 taking $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b + (\alpha - 1)p < k + 1 \le b + \alpha p$, we have:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(C_n^k)$, $n \ge a(a+1)^\alpha(k+1)$	$1 \le a < b$	$b \le k$	Bob

Also, by Theorem 6.1 taking $\beta = \lceil \frac{k+1}{b} \rceil \ge 2$ for $n \in \mathbb{N}^*$, we have:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(C_n^k)$, $3k < n \le a(\beta-1)(k+1)$	$1 \le a \le b$	$b \le k$	Alice

We observe that Theorem 6.1 gives an improvement of Theorem 5.1 when $\alpha = 1$. We conjecture that when b is closed to a and far from k, Alice can win even for "large" n.

The case when C_n^k has internal maximal cliques seems challenging. Next result illustrates a way to study this case. It consists to give an upper bound on a transversal. Therefore, if a is larger than this bound, this will ensure Alice's winning whenever she starts.

Theorem 7.1. *Let $n = 3k - \varepsilon$ with $k \ge 3$ and $0 \le \varepsilon < k-1$. The hypergraph $\mathcal{H}(C_n^k)$ admits a transversal of size $k - \varepsilon$.*

Proof. Set $T = \mathcal{K} \cup \{v_{2k-1-\varepsilon}\}$ with $\mathcal{K} = \{v_0, \dots, v_{k-2-\varepsilon}\}$. We prove that T is a transversal of $\mathcal{H}(C_n^k)$. Let K be a maximal clique such that $K \cap \mathcal{K} = \emptyset$. Since $n = 3k - \varepsilon$, we have that $N(v_{2k-1-\varepsilon}) \supseteq V(C_n^k) \setminus \mathcal{K}$. Thus, K must contain the red vertex $v_{2k-1-\varepsilon}$. Therefore, any hyperedge of $\mathcal{H}(C_n^k)$ has at least one vertex belonging to T, i.e., T is a transversal of $\mathcal{H}(C_n^k)$ of size

$$\underbrace{((k-2+\varepsilon) - (0) + 1)}_{\text{vertices in } \kappa} + \underbrace{(1)}_{v_{2k-1-\varepsilon}} = k - \varepsilon.$$

□

For the case $n = 3k$ and $k \geq 2$, the gap ε is vanished, i.e., $\varepsilon = 0$, which proves that this result is optimal.

First, we evoke a key lemma of Meidanis [13]:

Lemma 7.2. *[13] Let E^k be the set of edges of C_n^k with reach k. If $n = 3k$, $k \geq 2$, then the subgraph induced by E^k has k connected components, each of which a cycle of length* 3.

Theorem 7.3. *Let $n = 3k$ with $k \geq 2$, and let $b \geq 3$. There exists a strategy that allows Alice to win the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$ if and only if Alice starts playing and $a \geq k$.*

Proof. By Lemma 7.2, C_{3k}^k admits k disjoint triangles. It is easy to prove that these triangles are internal maximal cliques. Therefore, when $b \geq 3$, Alice loses if $a < k$ or Bob starts. Indeed, wherever Alice plays, she misses at least one of these triangles, and in both cases, in Bob's turn, he colours blue 3 vertices of one of the k triangles, which is a maximal clique, that ensures his victory. The converse follows from applying Theorem 7.1 with $\varepsilon = 0$. □

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