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Pierre Renaud

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CLIFFORD ALGEBRAS

LECTURE NOTES ON APPLICATIONS IN PHYSICS

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PIERRE RENAUD

*University of Canterbury  
Christchurch, New Zealand*

## Preface

This book ('lecture notes' seems a more appropriate name) has just one aim; that is to show *by examples* how some of the most basic and important ideas in modern physics, can be more deeply revealed by the geometry of Clifford algebras.

So it is not a detailed treatment of the theory of Clifford algebras, followed by diverse applications. There are already many such books, the author's favourite is still Lounesto's classic text - Clifford Algebras and Spinors. It would be a brave author who would attempt a better book along those lines.

Rather, this book is written with physicists firmly in mind. These gentle souls only rarely feel the need to plough through a mathematician's text, in all its depth and abstractness. This is not to decry these works - they are very often of great depth and beauty. But they are not what physicists usually need. A physicist, when confronted with a new field to study, or perhaps just an idea which seems critically important, will look for a mathematical theory which underpins this area. He or she will usually be satisfied if the mathematics can be used as a tool to explain the physics and to make new predictions.

Many mathematicians will be disappointed with this work. The theory of Clifford algebras is only very sketchily drawn here and many important results are completely omitted. Even the very definition of a Clifford algebra is not one which would satisfy a pure mathematician. Instead of starting with tensor products and quotient subspaces, Clifford algebras are defined in terms of bases. This is not at all the done thing. On the other hand, it is usually precisely what physicists need. Look at the way physicists use tensors. What is their definition of a tensor? Answer: A tensor is an object which transforms according to ... Such a definition is nonsense to a mathematician but is perfectly acceptable to a physicist *because it works*.

So, physicists use mathematics as a tool and also as a language. It is used to describe ideas. It follows that whenever possible, these ideas should be given a geometric interpretation since geometry is the very heart of description. This is a personal viewpoint - many mathematicians and physi-

cists will be content with an algebraic understanding of ideas and results. But in any case, a geometric viewpoint will always be of some use, even to algebraists.

This book is subtitled ‘lecture notes’ and this is for two reasons. The first is that the choice of topics can be more idiosyncratic and will reflect the author’s preferences and prejudices. This is certainly brought out here - you only have to look at the way some topics are handled and even why they are included at all. The topics chosen are just those the author found interesting and important. Readers would no doubt have chosen their own different ones.

The second advantage is that a careful trawling through the literature is not expected in a set of lecture notes. This is of real benefit to those of us who are somewhat careless in hunting down references and to give others the due recognition which they deserve. So here, the author must point out that the very large majority of the ideas and results given here are not new and in any case, no claim is made for originality.

The author has been particularly fortunate in working with a number of experts in this field and in discussing many of these ideas with them. In particular he is grateful to Eric Lehmann at the University of Caen, for his extensive proof reading, discussions and polite listening to the author’s more eccentric ideas. Many thanks are also owed to Guy Laville (Caen), Phil Butler (Canterbury, New Zealand) and Niels Gresnigt (Xi’an Jiaotong-Liverpool University, China) for their contributions to the author’s understanding of this beautiful area of research.

Readers will no doubt find any number of errors, missing references which should really have been given (even in lecture notes), topics where the explanation could be clearer and topics entirely left out which should have been there. The author would be grateful for any (positive or negative!) feedback and such should be sent to the email address

[peter.renaud@canterbury.ac.nz](mailto:peter.renaud@canterbury.ac.nz)

## Preface to Version 2

In this version, a number of mistakes have been corrected and hopefully, a smaller number introduced. As well, another chapter has been added; this one dealing with the energy spectrum of the hydrogen atom.

It seems clear to the author, that if physicists are to be persuaded to look more closely at the role that Clifford algebras should play, then more effort should be put into using Clifford algebras to solve, or at least analyse, problems which they regard as important. Some of these (such as the Stern-Gerlach experiment), are straightforward but their relative simplicity can also be a disadvantage - if the conventional solution is simple, why look for another method? On the other hand, problems such as finding the energy levels of hydrogen in the relativistic case, are decidedly tricky. It cannot realistically be expected that a Clifford algebra approach will be very much simpler, but we can hope that the approach sheds more light on what is usually a tedious, algebraic exercise. In particular, the role that the various spin operators (usually denoted by  $J$ ,  $S$  and  $K$ ) seems clearer since the Clifford approach explains quite easily why only the spin-orbit operator  $K$  is needed. But that perhaps is a personal view only.

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# Chapter 1

## Clifford algebra - the basic theory

### 1.1 What is a Clifford algebra?

Clifford algebras are defined on vector spaces on which a “metric”<sup>1</sup>, i.e. a non-degenerate quadratic form  $Q$  has been chosen. Different quadratic forms will generally lead to different algebras. The most important property of the quadratic form is its *signature*: the numbers of positive and negative eigenvalues.

This is for real spaces. For complex vector spaces, it turns out that signature is unimportant and that the Clifford algebra depends only on the dimension of the space. For this reason, complex Clifford algebras are of much less importance in physics and will only rarely be discussed here. We will be concerned only with *real* Clifford algebras. Although complex spaces are the traditional settings for many areas in modern physics, we will see that many of the important Clifford algebras have either an inherent complex structure (even though they are real algebras), or can be “complexified” in a trivial fashion. An example of “complexification” is  $M(4, R)$ , the algebra of all  $4 \times 4$  real matrices. We will see below that this is a Clifford algebra and it has the trivial complexification  $M(4, C)$  where  $C$  is now the field of

---

<sup>1</sup>Mathematicians would call this a ‘pseudo-metric’ as we do not usually assume that the length of a vector is positive. But the term is standard in relativity theory and we abide by this usage.

complex numbers. This algebra is important in the Dirac theory.

In any case, it is the geometrical ideas behind Clifford algebras which make them so appealing and it is these ideas which will be emphasized in these notes. Geometry is much harder to find in complex spaces.

All (real) Clifford algebras are defined on the underlying real vector space,  $R^n$ , the space of vectors with  $n$  real components. Vectors should be regarded as column vectors

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}$$

though sometimes for typographical reasons we will write them in the usual way one writes the coordinates of a point, i.e. as  $(x_1, x_2, \dots, x_n)$ . (Of course we can always write  $x = [x_1, x_2, \dots, x_n]^T$  where  $x^T$  is the transpose of  $x$ .)

Clifford algebras are often described in terms of tensor-product spaces, modulo a quadratic form ([1]). Mathematically this has the appeal of providing a basis free definition. But from a practical viewpoint (and our approach will always be a practical one), this approach is unnecessarily complicated and obscure, as well as disguising much of the geometry which is inherent in Clifford algebras and which makes their study so rewarding. (It also often makes algebraic manipulations more cumbersome, which is hardly a useful feature.) Our approach instead is to define Clifford algebras in terms of basis vectors in  $R^n$  always bearing in mind that the choice of basis is immaterial and that if pushed, one can always introduce these ideas in a basis-free way.

So what are Clifford algebras?

Begin with the vector space  $R^n$  on which we are given a non-degenerate quadratic form  $Q$ . Such a form can always be written as  $Q(x) = x^T Q x$  where with a small abuse of notation, we also use  $Q$  to denote a symmetric  $n \times n$  matrix. Since  $Q$  is symmetric, its eigenvalues are real and the fact that  $Q$  is non-degenerate means that  $Q$  has  $p$  positive and  $q$  negative eigenvalues with  $p + q = n$ . This pair  $(p, q)$  is called the *signature* of  $Q$  and is the only

important property of  $Q$  in defining the associated Clifford algebra. This algebra will be written as  $Cl(p, q)$  or sometimes as  $Cl(Q)$ .

Choose an orthonormal basis  $e_1, e_2, \dots, e_n$  in  $R^n$ . Here orthonormal means *relative to  $Q$*  so that we require

$$e_i^T Q e_j = 0, \quad i \neq j \quad (1.1)$$

$$e_i^T Q e_i = \begin{cases} +1, & i = 1 \dots p \\ -1, & i = p + 1 \dots n. \end{cases} \quad (1.2)$$

The inner product  $x^T Q y$  will also be written as  $\langle x, y \rangle$ .

Define multiplication to be an *associative* operation which satisfies the two conditions

$$1 \quad e_i^2 = \begin{cases} +1, & i = 1 \dots p \\ -1, & i = p + 1 \dots n \end{cases} \quad (1.3)$$

$$2 \quad e_i e_j = -e_j e_i, \quad \text{if } i \neq j.$$

These two rules<sup>2</sup> allow us to define unambiguously any product  $e_{i_1} e_{i_2} \dots e_{i_r}$  where  $A = \{i_1, i_2, \dots, i_r\}$  is an (ordered) subset of  $\{1, 2, \dots, n\}$ . Further we can always assume that  $i_1 < i_2 < \dots < i_r$  because by condition 2 above we can always reorder the indices. For example

$$e_3 e_2 e_1 = (e_3 e_2) e_1 = -(e_2 e_3) e_1 = -e_2 (e_3 e_1) = +e_2 (e_1 e_3) = (e_2 e_1) e_3 = -e_1 e_2 e_3$$

We write  $e_A$  for the product  $e_{i_1} e_{i_2} \dots e_{i_r}$  (this includes the case where the indices are not necessarily ordered and may even be repeated.)

**Example 1.1** In  $Cl(0, 2)$  (so that  $e_1^2 = e_2^2 = -1$ ),

$$e_{121} = e_1 e_2 e_1 = -e_1 e_1 e_2 = -(-1) e_2 = e_2.$$

---

<sup>2</sup>together with associativity

The Clifford algebra  $Cl(p, q)$  is the vector space spanned by the products  $e_A$  where  $A$  is an arbitrary subset of  $\{1, 2, \dots, n\}$  whose elements can always be written in increasing order.<sup>3</sup>

It follows from this that  $Cl(p, q)$  has dimension  $2^n$ .

A point of notation. In special relativity we work in  $R^4$  and it is conventional in that case to write the coordinates of a vector as  $(x^0, x^1, x^2, x^3)$ . So we will write a basis for  $R^4$  as  $e_0, e_1, e_2, e_3$ . In the case where the metric is denoted by physicists as  $+- - -$  the Clifford algebra will be  $Cl(1, 3)$  so that  $e_0^2 = +1, e_1^2 = e_2^2 = e_3^2 = -1$ . The other case is where the metric is written as  $- + + +$  and the Clifford algebra will then be  $Cl(3, 1)$  so that  $e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = +1$ . As we will see later, the two Clifford algebras  $Cl(1, 3)$  and  $Cl(3, 1)$  are *not* isomorphic and this has led some (albeit few) physicists to suggest that there might be some physics in this.<sup>4</sup>

**Exercise 1.1** a. Show that  $e_{12}^2 = -1$  in both  $Cl(2, 0)$  and  $Cl(0, 2)$ .

b. Show that  $e_{123}^2 = -1$  in  $Cl(3, 0)$  while  $e_{123}^2 = +1$  in  $Cl(0, 3)$ . This has led some authors to preferring to work with  $Cl(3, 0)$  as  $e_{123}$  can then be used as a substitute for the imaginary  $i$ . See e.g. [7].

c. Show further that in either algebra,  $e_{123}$  is in the centre of the algebra, meaning that it commutes with all elements of the algebra. (Hint: It suffices to show that  $e_{123}$  commutes with each of  $e_1, e_2, e_3$ .)

## 1.2 Examples of Clifford Algebras

We will look at examples in the cases  $n = 1, 2, 3$  and 4. Higher dimensional algebras do crop up from time to time in physics but are of lesser importance for our purposes. The cases  $n = 6$  is useful however in shedding some

<sup>3</sup>By convention when  $\phi$  is the null set, we define  $e_\phi = 1$ .

<sup>4</sup>There is no consistent view as to which metric should be chosen. The choice varies with authors and times. In any case physicists are pretty well united in the view that the choice doesn't really matter meaning that there is no physics in the choice. But this view is not universal. We will return to this point later.

geometric light on the groups  $SU(3)$  and  $SU(4)$ .<sup>5</sup>

### 1.2.1 $\mathbf{n} = 1$

There are two Clifford algebras defined on  $R^1$ :  $Cl(1,0)$  and  $Cl(0,1)$ . These both have dimension  $2^1 = 2$  and so are useful in representing points in the plane.

1.  $Cl(1,0)$  has the single basis vector  $e_1$  with  $e_1^2 = 1$ . This is not a division algebra because there are non-zero elements such as  $1 \pm e_1$  which do not have inverses.

**Exercise 1.2** Show that  $(1 + e_1)(1 - e_1) = 0$  and deduce that  $1 \pm e_1$  have no inverses.

If we make the correspondence

$$e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then  $Cl(1,0)$  is isomorphic to the algebra of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ . This algebra is of limited importance.

2.  $Cl(0,1)$  again has the single basis vector  $e_1$  this time with  $e_1^2 = -1$ . So clearly  $Cl(0,1)$  is isomorphic to the algebra of complex numbers with  $a + be_1 \longleftrightarrow a + bi$ . But for reasons we will discuss in more detail later, this is not a particularly useful idea (basically because  $a + be_1$  is the sum of a scalar and a vector).

### 1.2.2 $\mathbf{n} = 2$

Here there are three Clifford algebras;  $Cl(2,0)$ ,  $Cl(1,1)$  and  $Cl(0,2)$ . Their dimensions are all  $2^2 = 4$ .

---

<sup>5</sup>Basically because  $SU(4)$  and the two-fold covering group of  $SO(6)$  are isomorphic. This is an interesting result we will look at later.



1. Consider firstly  $Cl(2,0)$ . Take as basis  $e_1, e_2$  with  $e_1^2 = e_2^2 = 1$ . Note that

$$(e_{12})^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -e_1^2 e_2^2 = -1.$$

We can represent  $Cl(2,0)$  by the algebra  $M(2, R)$  of all  $2 \times 2$  real matrices by taking

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (1.4)$$

This gives an isomorphism between  $Cl(2,0)$  and  $M(2, R)$ .

2. For  $Cl(1,1)$  we will have the basis  $e_1, e_2$  with  $e_1^2 = +1, e_2^2 = -1$ . Working as above we now find that  $e_{12}^2 = +1$ . This time we can represent the basis elements by

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1.5)$$

and we have immediately the same result as in 1. above: namely that  $Cl(1,1)$  is isomorphic to  $M(2, R)$ .

So  $Cl(2,0)$  and  $Cl(1,1)$  are isomorphic algebras. But in fact this is merely an *accidental* isomorphism and has no real significance. This is an important point to note because it shows the *algebraic* properties of a Clifford algebra are not enough to describe the algebra fully. In these last two examples, the Clifford algebras are defined by two quite different metrics. One is positive definite, i.e. describes Euclidean geometry, while the other is hyperbolic. They could hardly be more different. But the algebras they define are isomorphic. So, and this is the vital point to understand, the algebraic properties of a Clifford algebra give only half the picture. The other half is the underlying geometry.

3. Before considering  $Cl(0,2)$ , let us briefly remind ourselves of an important division algebra,<sup>6</sup> the quaternions  $H$ .<sup>7</sup>

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<sup>6</sup>There are in fact only three associative division algebras over the reals: the reals themselves, the complex numbers and the quaternions.

<sup>7</sup>Discovered independently by William Hamilton and Olinde Rodrigues. Hamilton, an Irish mathematician and amateur bridge carver, is usually credited with their discovery but the French mathematician Rodrigues had described many of their properties some years earlier.

To define the quaternions, start with  $H = R^4$  with a basis written  $1, i, j, k$  and define an associative multiplication which also satisfies the properties

$$i^2 = j^2 = k^2 = -1, \tag{1.6}$$

$$ij = k, jk = i, ki = j. \tag{1.7}$$

This algebra is non-commutative since for example

$$ji = (ki)i = ki^2 = -k = -ij \text{ and similarly } kj = -jk, ik = -ki.$$

Returning now to  $Cl(0,2)$ , choose a basis  $e_1, e_2$  with  $e_1^2 = e_2^2 = -1$ . It follows (as above) that  $e_{12}^2 = -1$ . It is now trivial to verify that if we make the correspondence  $e_1 \leftrightarrow i, e_2 \leftrightarrow j, e_{12} \leftrightarrow k$ , then  $Cl(0,2)$  is isomorphic to the quaternions  $H$ .

**Exercise 1.3** Draw up and compare, the multiplication tables in each case, i.e. for  $H$  and  $Cl(0,2)$ .

However, this is not a particularly interesting result since while  $e_1$  and  $e_2$  are vectors in  $R^2$ ,  $e_{12}$  is not. This means that  $i, j, k$  in a sense have different standings with  $k$  singled out as the different one. There is nothing like this in the quaternions.

We will see a much better interpretation of quaternions in Clifford algebras shortly.

### 1.2.3 $n = 3$

There are two important algebras here,  $Cl(3,0)$  and  $Cl(0,3)$ .<sup>8</sup> Either algebra can be usefully applied to describe geometrical ideas in  $R^3$  such as rotations and reflections.

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<sup>8</sup>The other two algebras are based on hyperbolic geometry and play a lesser role in physics.

1. Consider  $Cl(3,0)$  with basis  $e_1, e_2, e_3$  satisfying  $e_1^2 = e_2^2 = e_3^2 = +1$ . The Pauli spin matrices  $\sigma_i, i = 1, 2, 3$  also have this property and we can use the correspondence

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad e_3 = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.8)$$

It follows that we need to define

$$e_{12} = e_1 e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and similarly

$$e_{23} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_{31} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and finally

$$e_{123} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

It is clear that taking real linear combinations of these eight matrices, we obtain all  $2 \times 2$  complex matrices. It follows that  $Cl(3,0)$  is isomorphic to  $M(2, \mathbb{C})$ . Because of the connection between the basis elements and the Pauli matrices,  $Cl(3,0)$  is often called the *Pauli algebra*.

Note that  $e_{123}^2 = -1$ , a result which has led some to prefer this algebra to its rival  $Cl(0,3)$  because it gives us another possible representation for  $i$ . But there is a problem with this which we will discuss in the next chapter.

Matrices in  $M(2, \mathbb{C})$  and therefore too, elements of  $Cl(3,0)$  are useful when describing rotations in  $R^3$ . To see this, we write a point  $(x^1, x^2, x^3)$  in  $R^3$  as a matrix

$$(x^1, x^2, x^3) = x^1 e_1 + x^2 e_2 + x^3 e_3 \longleftrightarrow \begin{bmatrix} x^3 & x^1 - x^2 i \\ x^1 + x^2 i & -x^3 \end{bmatrix}$$

i.e. as a trace-zero anti-Hermitian matrix.

This representation has the disadvantage that the  $x^3$  component of a point is singled out for (unfair) special consideration. We will see shortly that  $Cl(0,3)$  does not suffer from this problem.

2. For  $Cl(0,3)$  the basis vectors  $e_1, e_2, e_3$  satisfy  $e_1^2 = e_2^2 = e_3^2 = -1$ . This immediately brings to mind the defining properties of the quaternions, namely  $i^2 = j^2 = k^2 = -1$ . We cannot just identify the basis vectors with  $i, j, k$  since the quaternion algebra is four dimensional while  $Cl(0,3)$  has dimension  $2^3 = 8$ . Instead we can take the representation to be

$$e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$$

in which case

$$e_{23} = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

and similarly

$$e_{31} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad e_{12} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

and finally

$$e_{123} = e_{12}e_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

By considering linear combinations of the eight basis vectors for  $Cl(0,3)$  we find that the algebra is isomorphic to the algebra of  $2 \times 2$  quaternions of the form

$$\begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

so that  $Cl(0,3)$  is simply  $H \oplus H$  (see also the following exercise).

In this representation, there is no preferred direction as there is in the case of  $Cl(3,0)$ .

**Exercise 1.4** *The matrix*

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

*describes a rotation through  $\frac{\pi}{2}$  in  $R^2$ .*

*Show that if  $e'_A = U^T e_A U$  (with  $A$  a subset of  $\{1,2,3\}$ ) then every  $e'_A$  is in diagonal form. (It suffices to show this for  $e_i$ ,  $i = 1,2,3$ .) This shows more clearly that  $Cl(0,3)$  is isomorphic with  $H \oplus H$ , the direct sum of two copies of the quaternions, with  $e_1 \leftrightarrow (i, -i)$  etc.*

In the next chapter we will discuss how quaternions are natural objects for describing rotations in  $R^3$ . This in turn means that  $Cl(0,3)$  is arguably a more suitable algebra than  $Cl(3,0)$  at least as far as rotations (and later, spin) is concerned.

### 1.2.4 $n = 4$

This is the dimension we need to consider when discussing the role Clifford algebras might play in special relativity. The two metrics  $-+++$  and  $----$  lead to two different algebras  $Cl(3,1)$  and  $Cl(1,3)$ . That they are not isomorphic has led to some speculation that there might be some interesting physics in the choice of metric [8].

We will also briefly look at  $Cl(0,4)$  and  $Cl(4,0)$  since  $R^4$  plays a role in describing Pauli spinors in real space. This will be dealt with more thoroughly later.

1. We discuss  $Cl(3,1)$  first since it can be represented by  $4 \times 4$  matrices and this provides a link with the Dirac equation and the Dirac algebra - a matter we will discuss later. Because of this, it is the Clifford algebra preferred by some physicists.

Here we need a basis  $e_0, e_1, e_2, e_3$  which satisfies the conditions

$$e_0^2 = -1, e_1^2 = +1, e_2^2 = +1, e_3^2 = +1.$$

Before continuing, it is worthwhile asking ourselves what type of representation we might be after. Since  $Cl(3,1)$  obviously contains  $Cl(3,0)$  as a sub-algebra and  $Cl(3,0)$  is generated by the Pauli matrices, it would be highly desirable if  $Cl(3,1)$  also contained a copy of the Pauli matrices. After all, Pauli matrices generate rotations in  $R^3$  and rotations are important examples of Lorentz transformations.

It is easy to show that there is no non-trivial  $2 \times 2$  matrix  $e_0$  which anti-commutes with the Pauli matrices. So we would need to increase the dimensions of the matrices. One way to proceed is to start with the  $2 \times 2$  Pauli

matrices  $\sigma_i$  and then define the generators  $e_i$  ( $i = 1, 2, 3$ ) by

$$e_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}.$$

The last generator  $e_0$  can then be found using the conditions  $e_0 e_i = -e_i e_0$  as well as  $e_0^2 = -1$ .

**Exercise 1.5** Show that we can take  $e_0 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}$  where  $I = I_2$  is the  $2 \times 2$  identity matrix.

The problem with this approach is that we end up with a matrix representation for  $Cl(3, 1)$  where it is difficult to identify important geometric concepts. For example it is not at all obvious how Lorentz transformations should be described. Even rotations, which are generated by the Pauli matrices, are now generated by the  $e_i$ s ( $i = 1, 2, 3$ ) and each  $e_i$  contains two *different* versions of  $\sigma_i$ . What does that mean?

Finally the matrix algebra itself is not particularly attractive. The matrices are not all real but clearly (on dimensional grounds) not all complex matrices are included either. Describing it is not straightforward or particularly satisfying.

There is another approach which yields better results. Start with the Clifford algebra  $Cl(2, 0)$  which we already know to be isomorphic to the algebra  $M(2, R)$  of all  $2 \times 2$  real matrices. Take the generators to be

$$f_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now define  $e_\mu, \mu = 0, 1, 2, 3$  via

$$e_1 = \begin{bmatrix} f_1 & 0 \\ 0 & -f_1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} f_2 & 0 \\ 0 & -f_2 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix},$$

$$e_0 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}.$$

It is now easy to verify that the  $e_\mu$ s all anti-commute and that  $e_0^2 = -1, e_i^2 = +1, i = 1, 2, 3$ . A little more calculation shows that the 16 matrices  $e_A, A \subset \{1, 2, 3\}$  in fact span all of  $M(4, R)$ . So we have a representation of  $Cl(3, 1)$  as the algebra of all  $4 \times 4$  real matrices.

**Exercise 1.6** *The reasoning above can be generalized to give the following result (which however we will not need).*

*$Cl(p + 1, q + 1)$  is isomorphic to the algebra of all  $2 \times 2$  matrices whose elements are the elements of  $Cl(p, q)$ .*

The proof is not difficult and may be found in [1], Ch 16.8.

This representation has some advantages over the previous one. For a start, it is easier to describe. Furthermore it bears a direct relationship with the Dirac algebra: the algebra of all  $4 \times 4$  *complex* matrices. (This is a good example of a *real* Clifford algebra having an obvious ‘complexification’.) But the price to pay when we complexify, is that any geometric structure in the real algebra becomes obscured or even lost.

Another disadvantage with this representation is that the Pauli matrices have largely disappeared (or at least, have become lost). On the positive side however, is that Lorentz transformations (being  $4 \times 4$  matrices) can be nicely represented in this algebra.

As we will now see, the Clifford algebra  $Cl(1,3)$  avoids these problems and seems the more appropriate algebra when dealing with ideas in special relativity.

2. The  $+---$  metric gives the Clifford algebra  $Cl(1,3)$  which in many respects appears superior to  $Cl(3,1)$  for the simple reason that it describes rotations and Lorentz transformations in a more natural and geometric manner. This aspect will be explored in the following chapter. For the present, let us look at how this algebra can be described (as usual) as a matrix algebra.

$Cl(1,3)$  contains  $Cl(0,3)$  in an obvious way and the latter algebra will prove highly useful in describing rotations and spin. So we seek a representation for  $Cl(1,3)$  which extends the representation we already have for  $Cl(0,3)$ . By this we mean that we will take the (quaternion) matrix representations for  $e_1, e_2, e_3$  which are already present. As for  $e_0$ , we need only choose a matrix which anti-commutes with each  $e_i$ ,  $i = 1, 2, 3$  and which squares to 1. This then leads to the following representation for  $Cl(1,3)$ .

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (1.9)$$

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}, \quad (1.10)$$

$$e_{23} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad e_{31} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad e_{12} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \quad (1.11)$$

$$e_{01} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad e_{02} = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix}, \quad e_{03} = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}, \quad (1.12)$$

$$e_{023} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_{031} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad e_{012} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}, \quad e_{123} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad (1.13)$$

$$e = e_{0123} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (1.14)$$

It is clear now that these 16 matrices provide a basis for  $M(2, H)$ , the algebra of all  $2 \times 2$  matrices with quaternion components.

**Exercise 1.7** Show that another representation can be obtained, starting



with

$$e_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad e_3 = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}$$

and using these to define the other basis terms  $e_A$ . Which representation to use, is largely a matter of personal choice. It doesn't really matter.

3. Finally let us consider the Clifford algebras which arise from a euclidean metric. Start with  $Cl(4,0)$ . It would be nice if we could extend the representation of the Pauli algebra  $Cl(3,0)$  which we know to be isomorphic to  $M(2,C)$ , perhaps to something like  $M(n,C)$ . But it is clear on dimensional grounds that this is impossible. Quaternions however seem to give better hope and we can start with something like

$$e_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}.$$

To find  $e_4$ , the properties that  $e_4^2 = +1$  and that  $e_4$  anti-commutes with  $e_1, e_2, e_3$  is enough to show that we can choose

$$e_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

**Exercise 1.8** Now calculate the other basis vectors and deduce that  $Cl(4,0)$  is isomorphic to  $M(2,H)$ .

4. Finally the case of  $Cl(0,4)$  can be treated in a similar way. We can start with

$$e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$$

and find  $e_4$  by a similar process to be (for example)

$$e_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

**Exercise 1.9** Show that in this case too,  $Cl(0,4)$  is isomorphic to  $M(2,H)$ .

So three of the cases studied when  $n = 4$ , give the same representation for the Clifford algebra, namely  $M(2,H)$ . Nothing should be read into this result. The underlying geometries are quite different and it just goes to show, yet again, that algebra is only half the story. Still, it illustrates why, if you think algebraically alone, it is easy to be led astray.

### 1.3 Gradings in a Clifford algebra

Since the basis elements of the Clifford Algebra  $Cl(p, q)$ ,  $p + q = n$  are the elements  $e_A$  where  $A$  is a subset of  $\{1, 2, \dots, n\}$ , the algebra splits naturally as a direct sum  $Cl(p, q) = \bigoplus Cl^r(p, q)$  of subspaces spanned by the elements  $e_A$  where  $A$  has  $r$  elements. This is the  $Z_n$ -grading of  $Cl(p, q)$ .<sup>9</sup> For example,

1.  $Cl^0(p, q)$  is the 1-dimensional subspace of real scalars (i.e. spanned by 1 or  $e_\phi$ .)
2.  $Cl^1(p, q)$  is the  $n$ -dimensional subspace spanned by the basis vectors  $e_i$ ,  $i = 1, 2, \dots, n$ . So  $Cl^1(p, q) = R^n$ .
3.  $Cl^2(p, q)$  is the  $\binom{n}{2}$ -dimensional subspace spanned by the basis vectors  $e_{ij}$ ,  $1 \leq i < j \leq n$ . This is the space of *bivectors* in the algebra.
4. Generally  $Cl^r(p, q)$  is the  $\binom{n}{r}$ -dimensional subspace spanned by the basis vectors  $e_{i_1 i_2 \dots i_r}$ ,  $1 \leq i_1 < \dots < i_r \leq n$ .
5. Finally  $Cl^n(p, q)$  is the 1-dimensional subspace spanned by  $e = e_{12 \dots n}$ . This is the subspace of *pseudoscalars*.  $e = e_{12 \dots n}$  is often called the (unit) pseudoscalar.<sup>10</sup>

The elements in  $Cl^1(p, q)$  are sometimes called the 1- vectors to distinguish them from elements in  $Cl^r(p, q)$  for higher values of  $r$ .<sup>11</sup>

Similarly elements of  $Cl^2(p, q)$  are called 2-vectors or *bivectors*. Likewise we have 3- vectors (trivectors) and so on.

For many applications, the  $Z_n$ -grading is too fine and we are mostly interested in whether the number of elements in  $A$  is either even or odd.

<sup>9</sup>The symbol  $Z$  is traditionally used to denote the set of integers.

<sup>10</sup>In relativity where we use the algebras  $Cl(1,3)$  or  $Cl(3,1)$  this would be the pseudoscalar  $e = e_{0123}$ .

<sup>11</sup>Relativists would call them 4-vectors but this slight ambiguity of notation should not cause problems.

This leads to the  $Z_2$  grading where  $Cl(p, q)$  is decomposed into the sum of even and odd terms,

$$Cl(p, q) = Cl^+(p, q) \oplus Cl^-(p, q) \quad (1.15)$$

where

$$Cl^+(p, q) = \bigoplus_r Cl^r(p, q), \quad r \text{ is even} \quad (1.16)$$

and

$$Cl^-(p, q) = \bigoplus_r Cl^r(p, q), \quad r \text{ is odd} \quad (1.17)$$

Clearly too

$$\dim Cl^+(p, q) = \dim Cl^-(p, q) = 2^{n-1}. \quad (1.18)$$

Elements in  $Cl^+(p, q)$  are called the *even* elements, while those in  $Cl^-(p, q)$  are called odd. Note that  $Cl^+(p, q)$  is a *sub-algebra* of  $Cl(p, q)$ . We will see later that although in general  $Cl(p, q)$  and  $Cl(q, p)$  are *different* algebras, i.e. non-isomorphic,  $Cl^+(p, q)$  and  $Cl^+(q, p)$  are isomorphic. This has important consequences if we wish to argue for one choice of metric over another, as for example the choice of which Minkowski metric to use in special relativity.

### 1.3.1 Involutions

A Clifford algebra  $Cl(p, q)$ , or in this case, more conveniently written as  $Cl(Q)$ , has what is known as the *universal property* [1, ch 13.4]. Briefly this means that any linear map  $f : R^n \rightarrow R^n$  which preserves the quadratic form, i.e.  $Q(f(x)) = Q(x)$  for all  $x \in R^n$ , can be extended to an automorphism or anti-automorphism from  $Cl(Q) \rightarrow Cl(Q)$  (which with mild abuse of notation we still write as  $f$ ). This may seem a complicated idea but in practice it is very simple as there are only three important examples.

Consider the two maps  $f(x) = x$  and  $f(x) = -x$  from  $R^n \rightarrow R^n$ . Since  $Q(-x) = Q(x)$ , each of these maps preserves  $Q$ . So we are led to four cases. (All other cases are combinations of one of these with an isometry on  $R^n$ ).

1.  $f(x) = x, x \in R^n$  and we extend  $f$  to an automorphism. This is just the identity map from  $Cl(Q) \rightarrow Cl(Q)$ . This case is of no importance.
2.  $f(x) = x, x \in R^n$  and we extend  $f$  to an *anti*-automorphism. Now this case is of considerable importance and we usually write  $f(v)$  as  $\tilde{v}, (v \in Cl(Q))$ . This map is called the *reversion* map. Using the anti-automorphism property  $\widetilde{uv} = \tilde{v}\tilde{u}$ , we have

- (a)  $\tilde{v} = v$  for  $v \in Cl^0(Q)$  or  $Cl^1(Q)$ .
- (b)  $\tilde{v} = -v$  for  $v \in Cl^2(Q)$  or  $Cl^3(Q)$ .
- (c)  $\tilde{v} = v$  for  $v \in Cl^4(Q)$  etc.

**Exercise 1.10** Show that if  $v \in Cl^r(Q)$ ,

$$\tilde{v} = (-1)^{\frac{r(r-1)}{2}} v.$$

3.  $f(x) = -x, x \in R^n$  and we extend  $f$  to an automorphism. Here we write  $f(v)$  as  $\hat{v}$  for  $v \in Cl(Q)$  and we call this map the *grade involution* map. Using the automorphism property  $\widehat{uv} = \hat{u}\hat{v}$ , we have
- (a)  $\hat{v} = v$  for  $v \in Cl^0(Q)$  or  $Cl^2(Q)$ ,
  - (b)  $\hat{v} = -v$  for  $v \in Cl^1(Q)$  or  $Cl^3(Q)$  etc.

We call this map *grade involution* because it depends only on the  $Z_2$  grading.

$$\hat{v} = \begin{cases} v, & \text{if } v \in Cl^+(Q), \\ -v, & \text{if } v \in Cl^-(Q). \end{cases}$$

**Exercise 1.11** Derive this result by showing that if  $v \in Cl^r(Q)$ ,

$$\hat{v} = (-1)^r v.$$

4.  $f(x) = -x, x \in R^n$  and we extend  $f$  to an anti-automorphism. In this case we write  $f(v)$  as  $\bar{v}$  for  $v \in Cl(Q)$  and we call this map the *Clifford conjugation* map. Using the anti-automorphism property we have
- (a)  $\bar{v} = v$  for  $v \in Cl^0(Q)$  or  $Cl^3(Q)$ ,
  - (b)  $\bar{v} = -v$  for  $v \in Cl^1(Q)$  or  $Cl^2(Q)$  etc.

**Exercise 1.12** Show that if  $v \in Cl^r(Q)$ ,

$$\bar{v} = (-1)^{\frac{r(r+1)}{2}} v.$$

**Exercise 1.13** *These three involutions are not all independent. Clifford conjugation is in fact the composite map of grade involution followed by reversion.*

If we define a *paravector* to be the sum of a scalar and a 1–vector, then Clifford conjugation mimics many of the properties of complex conjugation. Paravectors are an important field of research in Clifford analysis but will not be discussed in this work.<sup>12</sup>

### 1.3.2 The inner product on $Cl(p, q)$

Using the algebraic structure in a Clifford algebra, the inner product  $\langle \cdot, \cdot \rangle$  on  $R^n$  can be expressed in terms of the basis vectors  $e_i$  as

$$\langle e_i, e_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ e_i^2, & \text{if } i = j. \end{cases}$$

**Exercise 1.14** *Let  $x, y \in R^n$  show that*

1.  $\langle x, y \rangle = \frac{xy + yx}{2}$ .
2.  $\langle x, x \rangle = x^2$ .

This last example shows that in the cases  $p \neq 0, q \neq 0$  there exist vectors in  $R^{p,q}$  with  $\langle x, x \rangle = 0$ . Since then  $x^2 = 0$ , such vectors cannot have inverses.<sup>13</sup> All other vectors  $x$  will have inverses with

$$x^{-1} = \frac{x}{\langle x, x \rangle}.$$

When written in the above algebraic way, it becomes clear how we might extend the inner product to all of  $Cl(p, q)$ . Since a basis for  $Cl(p, q)$  consists of the elements  $e_A$  with  $A \subset \{1, \dots, n\}$ , it suffices to define

$$\langle e_A, e_B \rangle = \begin{cases} 0, & \text{if } A \neq B, \\ e_A^2, & \text{if } A = B. \end{cases}$$

---

<sup>12</sup>The name was coined by J. G. Maks in his 1989 doctoral thesis.

<sup>13</sup>Such vectors are appropriately called null vectors in relativity.

It follows from this definition, that in the  $Z_n$ -grading, all the subspaces  $Cl^r(p, q)$  are orthogonal and therefore too, the subspaces  $Cl^+(p, q)$  and  $Cl^-(p, q)$ .

Appealing as this definition may be, it does lead to problems. One of these will be discussed more fully when we come to consider how Pauli spinors might be represented in a (real) vector space. Very briefly here<sup>14</sup>, that space will be taken to be  $Cl^-(0, 3)$ . This space has the basis  $e_1, e_2, e_3, e_{123}$  and since  $e_i^2 = -1, i = 1, 2, 3$  while  $e_{123}^2 = +1$ , the inner product as defined above, leads to a hyperbolic metric. This is difficult to reconcile with the known properties of spinors as it would lead to an unphysical geometric picture.

Instead, we will take the lead from linear algebra, where the inner product of (real) matrices is usually taken to be

$$\langle A, B \rangle = \text{trace}(A^T B).$$

If we now replace the transpose of a matrix by the *reversion* of an element in the Clifford algebra, and trace by the real or scalar part (denoted by  $RI(\cdot)$ ), we are led to an alternative definition of the inner product on  $Cl(Q)$ .

**Definition 1.1** *If  $x, y \in Cl(Q)$ , define*

$$\langle x, y \rangle = RI(\tilde{x}y). \tag{1.19}$$

**Exercise 1.15** *Show that in the  $Z_n$ -grading, all the subspaces  $Cl^r(p, q)$  are again orthogonal and therefore too, the subspaces  $Cl^+(p, q)$  and  $Cl^-(p, q)$ .*

**Exercise 1.16** *Show that in  $Cl(0, 3)$  this leads to a negative-definite metric on  $Cl^-(0, 3)$ .*

When dealing with spinors later, and also the group  $SO(4)$ , this will be the correct inner product to use.

Although this inner product is *negative*-definite, we can hardly complain. It is entirely consistent with our deliberate choice of using  $Cl(0, 3)$  rather than  $Cl(3, 0)$ .<sup>15</sup>

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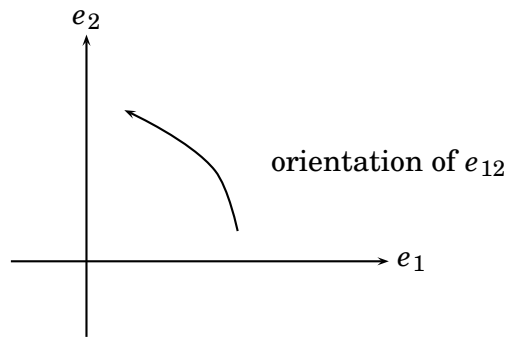
<sup>14</sup>for reasons which will be discussed at length in a later chapter,

<sup>15</sup>Inner products on Clifford algebras are discussed fully by Dorst in [9].

### 1.3.3 What are bivectors, trivectors etc geometrically?

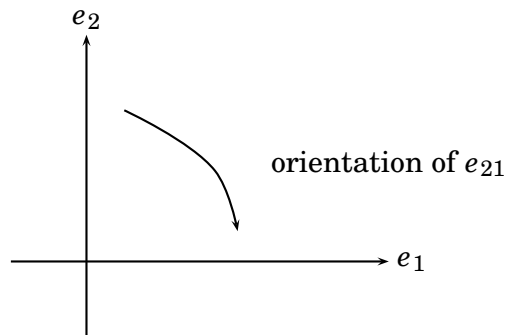
Just as vectors have the geometric meaning of a scalar length and a direction, so we can give a geometric picture to a bivector, at least in simple cases.

Consider first, two basis vectors  $e_1, e_2$  in  $Cl(Q)$  (as usual we use this notation when signature is not important). These define a plane in  $R^n$  and this plane can be given an orientation, i.e. an (imaginary) rotation from  $e_1$  to  $e_2$ .



$e_{12}$  is then described geometrically as a scalar (1 in this case, being the area of the square defined by  $e_1$  and  $e_2$ ) and an orientation from  $e_1$  to  $e_2$ . In other words whereas a vector has a scalar length and a direction, so in this case  $e_{12}$  has a scalar area and an *orientation*.

The corresponding diagram for  $e_{21}$  would be



The orientation for  $e_{21}$  (a rotation from  $e_2$  to  $e_1$ ) is now equal and opposite to that of  $e_{12}$ . An immediate consequence of this geometric interpretation is the reason behind the assumption that  $e_1e_2 = -e_2e_1$ .

What about the more general case when we multiply two arbitrary vectors  $x, y$  in  $R^n$ ?

The first thing to notice is that we no longer have a bivector. For example let  $x = ae_1 + be_2$  and  $y = ce_1 + de_2$  with  $a, b, c, d$  scalars. Then

$$\begin{aligned} xy &= (ae_1 + be_2)(ce_1 + de_2) = (ac \langle e_1, e_1 \rangle + bd \langle e_2, e_2 \rangle) + (ad - bc)e_{12} \\ &= \langle x, y \rangle + (ad - bc)e_{12}. \end{aligned}$$

So in general the product of two vectors  $x$  and  $y$  will be the sum of a scalar and a bivector. It is only a pure bivector when the vectors are orthogonal. The scalar part of  $xy$  is the inner-product  $\langle x, y \rangle$  of the vectors and the bivector part consists of the area of the parallelogram formed by  $x$  and  $y$ , together with the orientation of  $x$  to  $y$ . (It is sometimes described as the oriented area of the parallelogram formed by the vectors and in the plane that they span.)

**Exercise 1.17** Show that for any two vectors  $x, y$  in  $Cl(Q)$

$$xy = \left( \frac{xy + yx}{2} \right) + \left( \frac{xy - yx}{2} \right)$$

gives the decomposition of the product  $xy$  as a sum of a scalar (the inner-product of  $x$  and  $y$ ) and a bivector.

Can we give a similar geometric description for an arbitrary element of  $Cl^2(p, q)$ ?

In some cases the answer will be yes: for example the bivector  $u = e_{12} + e_{13}$  can be written as  $u = e_1(e_2 + e_3)$  which expresses  $u$  as the product of two orthogonal vectors. This is possible since the two components of  $u$  have  $e_1$  in common. But in general, the geometric picture is more complicated and probably not worth pursuing. For example in  $R^4$  what is the geometric meaning of  $v = e_{12} + e_{34}$ ? Here we are asked to give meaning to

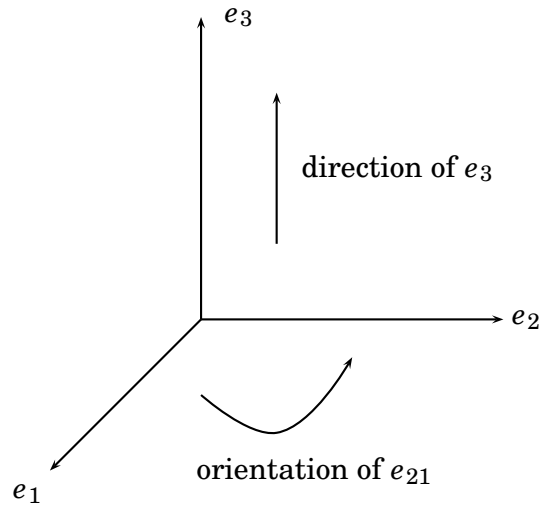


the ‘sum’ of two orientations. There does not seem to be any obvious answer. Unlike directions which can be added, orientations cannot.

As an aside, it is worth pointing out that if the geometric difference between  $u$  and  $v$  (in the paragraph above) is fairly clear, algebraically too, the two bivectors have different properties. For any two vectors  $x, y$ , we can define the matrix  $A = xy^T - yx^T$  which has rank 2 (assuming that  $x$  and  $y$  are linearly independent). So in the example above,  $u$  corresponds to a rank 2 matrix. In the same way we can find a matrix which represents  $v$  and this matrix is the sum of two rank 2 matrices. In this case it is a matrix of rank 4. It is this algebraic difference between  $u$  and  $v$  which lies behind their different geometric properties.

In any case, we can only really make sense of the direction of the sum of two vectors if the vectors are connected in some way, for example having the ends meeting in a common point. It is the same with orientations. If they are not connected in some way, they cannot be added.

Trivectors too can be given a geometric interpretation though it is necessarily more complicated. As before, start with the three orthonormal vectors  $e_1, e_2, e_3$  described in the following figure. The first two,  $e_1, e_2$  define an orientation and the right hand rule determines the direction of  $e_3$ . So for trivectors, the same idea now really requires that of *handedness* as shown below.



Finally, what of 4–vectors in Clifford algebras on  $R^4$ , i.e. psudoscalsars? These arise in relativity and it would be useful to have some idea of their properties, especially any property which distinguishes them from ordinary scalars. An important example which we will discuss later is that of *helicity*: the projection of the spin of a particle in the direction of its momentum.

Since we are now in  $R^4$  it is unrealistic to give any geometric meaning to the idea of an oriented 4–volume. However geometry still plays an important role when we consider possible invariance under certain discrete symmetry groups such as parity (space inversion):  $(x, y, z) \rightarrow (-x, -y, -z)$  or time inversion:  $t \rightarrow -t$ .

## 1.4 Groups in Clifford algebras

Clifford algebras contain a number of groups which have important applications. These groups are central to the study of the two-fold covering groups for some of the classical groups such as rotation or Lorentz groups. Here we will give just a few examples since more groups will be studied further in Chapter 3 where we will look at specific examples which are of special

importance in geometry and physics.

1. The *Clifford group*  $\Gamma(p, q)$ .<sup>16</sup> This group can be defined in a number of equivalent ways and we choose the definition which gives us its important properties most easily.

**Definition 1.2**  $\Gamma(p, q)$  is the (multiplicative) group generated by invertible 1-vectors in  $R^{p,q}$ .

Since a product of 1-vectors must be either an even or an odd element of  $Cl(p, q)$  we have

$$\Gamma(p, q) = Cl^+(p, q) \cup Cl^-(p, q).$$

(So that  $\Gamma(p, q)$  cannot contain linear combinations of even and odd elements.)

**Exercise 1.18** If  $g$  is an invertible 1-vector, the map

$$g : x \rightarrow gxg^{-1}$$

is a map from  $R^{p,q} \rightarrow R^{p,q}$ , i.e. maps 1-vectors to 1-vectors.

(One way to show this is to write  $g^{-1}$  in terms of  $\tilde{g}$  and deduce that if  $y = gxg^{-1}$  then  $\tilde{y} = y$ . This means that  $y$  cannot contain a 3-component.)

2. The *Pin group*  $\text{Pin}(p, q)$ . (This is the first of the important covering groups.)

$$\text{Pin}(p, q) = \{g \in \Gamma(p, q) : g\tilde{g} = \pm 1\}.$$

So if  $g \in \text{Pin}(p, q)$ ,  $\tilde{g}$  is a scalar multiple of  $g^{-1}$ . This is not true for arbitrary elements of  $\Gamma(p, q)$ .

**Exercise 1.19** Let  $g = 1 + e_{0123}$  in  $Cl(1, 3)$ . Show that  $g^{-1}$  exists but is not a scalar multiple of  $\tilde{g}$  (and so,  $g$  is not in  $\text{Pin}(1, 3)$ .)

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<sup>16</sup>Actually first studied by R. O. S. Lipschitz in the 1880s and sometimes called the Clifford-Lipschitz group.

**Example 1.2** Consider  $\text{Pin}(0,1)$ . Let  $e_1$  be the basis vector for  $R^{0,1}$  so that  $e_1^2 = -1$ . Then if  $g$  is in  $\text{Pin}(0,1)$  we have  $\tilde{g} = g$  (check this!) so that if  $g$  is odd then  $g = \pm e_1$  while if  $g$  is even,  $g = \pm 1$ . This means that  $\text{Pin}(0,1)$  is the cyclic group  $Z_4 = \{1, e_1, e_1^2 = -1, e_1^3 = -e_1\}$ .

**Example 1.3** Consider instead,  $\text{Pin}(1,0)$ . Now if  $e_1$  is the basis vector for  $R^{1,0}$  then  $e_1^2 = +1$  and a similar argument shows that  $\text{Pin}(1,0)$  is now the direct product of the two cyclic groups  $Z_2 = \{\pm 1\}$  with  $\{1, e_1\}$  (which of course is also cyclic of order 2).

So the two Pin groups  $\text{Pin}(0,1)$  and  $\text{Pin}(1,0)$  are non-isomorphic. In general this is true for all Pin groups: the choice of metric is important.

**Example 1.4** Consider now  $\text{Pin}(0,2)$ . Let  $e_1, e_2$  be a basis for  $R^{0,2}$  so that  $e_1^2 = e_2^2 = -1$ . Suppose that  $g \in \text{Pin}(0,2)$ . Then  $g\tilde{g} = \pm 1$  and  $g$  is either even or odd. If  $g$  is even then  $g = a + be_{12}$  in which case  $\tilde{g} = a - be_{12}$ . Then  $g\tilde{g} = a^2 + b^2$  i.e.  $g$  is of the form  $g = \cos\theta + e_{12}\sin\theta$  for some  $\theta$  and  $g$  is an element of the group  $SO(2)$ .

If now  $g$  is odd, then a similar argument shows that  $g$  is of the form  $g = \cos\theta e_1 + \sin\theta e_2$ . From this we can deduce that  $\text{Pin}(0,2)$  is the semi-direct product of  $SO(2)$  with the cyclic group  $Z_2$ .

3. The Spin group  $\text{Spin}(p, q)$ . This is the subgroup of  $\text{Pin}(p, q)$  consisting of even elements only<sup>17</sup>, i.e.

$$\text{Spin}(p, q) = \text{Pin}(p, q) \cap Cl^+(p, q).$$

**Example 1.5**  $\text{Spin}(0,1) = \text{Spin}(1,0) = Z_2 = \{\pm 1\}$ . This follows easily from examples 2 and 3.

4. Finally the Spin group  $\text{Spin}(p, q)$  has the further subgroup

$$\text{Spin}^\uparrow(p, q) = \{g \in \text{Spin}(p, q) : g\tilde{g} = +1\}.$$
<sup>18</sup>

<sup>17</sup>So we could write  $\text{Spin}(p, q) = \text{Pin}^+(p, q)$

<sup>18</sup>Some authors write  $\text{Spin}^+(p, q)$  instead. This can be confusing since the + sign is generally used to emphasize that we are working in  $Cl^+(p, q)$ . The  $\uparrow$  symbol seems more appropriate since in the case of  $Cl(1,3)$ , it means that these are the elements of  $\text{Spin}(1,3)$  which preserve the arrow of time as relativists would put it.

For Euclidean metrics, this subgroup has no significance since  $\text{Spin}^\dagger(n, 0) = \text{Spin}(n, 0)$ . But for non-Euclidean metrics, the subgroup is generally a proper subgroup. (See Lounesto [1].)

The connection between the Pin and Spin groups and the orthogonal groups are as follows.

$\text{Pin}(p, q)$ ,  $\text{Spin}(p, q)$  and  $\text{Spin}^\dagger(p, q)$  are respectively the two-fold covering groups of  $O(p, q)$ ,  $SO(p, q)$  and  $SO^\dagger(p, q)$  (where  $SO^\dagger(p, q)$  is the connected component of  $SO(p, q)$ ).

More complicated (and interesting) examples of groups which lie naturally in Clifford algebras, will be seen later.

## 1.5 Notes and references

In this author's view, the classic book on Clifford algebras is

1. Lounesto, Pertti (2001), Clifford algebras and spinors, Cambridge University Press.

It contains a wealth of ideas and applications to physics.

There is a number of excellent texts of which a random selection is 2 - 7 below.

2. Porteous, Ian R. (1995), Clifford algebras and the classical groups, Cambridge University Press.

3. Hestenes, David. Sobczyk, Garret (1984), Clifford Algebra to Geometric Calculus; A Unified Language for Mathematics and Physics; Series: Fundamental Theories of Physics, Vol. 5, Springer.

4. Doran, Chris, Lasenby, Anthony (2015), Geometric Algebra for Physicists, Cambridge University Press.

5. Snygg, John (1997) Clifford Algebra: A Computational Tool for Physicists, Oxford University Press.

6. Vaz, Jayme, da Rocha, Roldao (2016), An Introduction to Clifford Algebras and Spinors, Oxford University Press.

7. Baylis, W E (2004), Relativity in introductory physics, Canadian Journal of Physics. 82 (11): 853–873.

(In this approach, space-time vectors are defined as scalar +  $R^3$  vectors, i.e. as *paravectors*.)

8. Berg, M, DeWitt-Morette, C, Gwo, S, Kramer, E (2001), The Pin groups in physics, Reviews in Mathematical Physics Vol. 13, No. 08, pp. 953-1034.

9. Dorst, L, (2013) The inner products of geometric algebra, in Applications of Geometric Algebra in Computer Science and Engineering, 35-46, Springer-Verlag.

There is also a wealth of articles devoted to Clifford algebras and applications to sundry fields such as physics, geometry and computer science. The following is a small selection which the author has found most useful.

Todorov, Ivan, (2011) Clifford algebras and spinors, at arxiv:1106.3197v2

Gallier, Jean, (2014) Clifford algebras, Clifford groups and a generalization of the quaternions: the Pin and Spin groups, at arxiv:0805.0311v3

Dutailly, Jean Claude, (2018), Clifford algebras - new results, at <https://hal.archives-ouvertes.fr/hal-01826553>

# Chapter 2

## Rotations and reflections

### 2.1 Introduction

We now have a good basic understanding of how a Clifford algebra is built up starting with the space  $R^n$  equipped with a (usually indefinite) metric  $Q$  with signature  $(p, q)$ . The algebra resembles a stack of vector spaces which are graded by the way we define their bases. The bottom stack is the reals, followed by the vectors in  $R^n$ , then the bivectors etc. Each of these spaces is equipped with an inner product, such that the spaces are all orthogonal to one another.

At this stage we are in the position of describing some of the most important ideas concerning the geometry of transformations and how these ideas can be expressed through Clifford algebra. These ideas are most easily seen in two or three dimensions (and this is where most of the applications will be found), but in general there is little problem in generalising them to arbitrary dimensions.

Many of the most important ideas in geometry, are defined in terms of symmetries, usually transformations in  $R^2$  or  $R^3$ , which leave certain properties invariant or unchanged. For example, translations and rotations leave objects looking pretty well the same. In physics these symmetries have the additional, critically important, properties, that they give rise to conservation laws. It is these concepts we now want to consider, especially the way Clifford algebras include them in a most natural way.

Two of the most basic and important conservation laws are those of momentum and angular momentum. There are others of course such as energy and charge but we will deal with the first two to start with because they can be derived from symmetries in two or three dimensional space.

As we will soon see, these two symmetries are each derivable in term of reflections so it is with this idea that we will start.

## 2.2 Reflections

Consider first, reflections<sup>1</sup> in  $R^2$ . The simplest example is a reflection in a line  $l$  through the origin.

Assume too for simplicity that the metric on  $R^2$  is the usual Euclidean metric so that the Clifford algebra is  $Cl(2,0)$ .

Consider the following figure where a point  $P$  is reflected to the point  $P'$  in a line  $l$  through the origin.

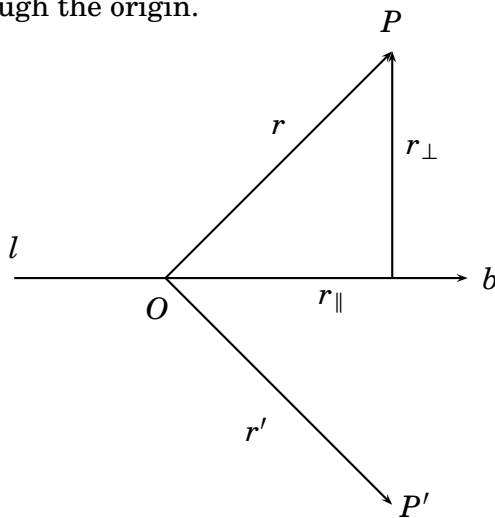


Figure 2.1

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<sup>1</sup>See e.g. Lounesto [1], ch 1.12.



Let  $b$  be a direction vector of the line through the origin ( $b$  need not be normalized). The projection  $r_{\parallel}$  of  $r$  onto  $b$  is

$$r_{\parallel} = \langle r, \frac{b}{\|b\|} \rangle \frac{b}{\|b\|} = \langle r, b \rangle \frac{b}{\|b\|^2} = \langle r, b \rangle b^{-1} \quad (2.1)$$

(since  $b^2 = \|b\|^2$ ).

Also

$$r_{\perp} = r - r_{\parallel} = (rb - \langle r, b \rangle)b^{-1} \quad (2.2)$$

so that the reflection of  $r$  in the line  $l$  is

$$r' = r_{\parallel} - r_{\perp} = (2\langle r, b \rangle - rb)b^{-1}. \quad (2.3)$$

Using the fact that

$$\langle r, b \rangle = \frac{rb + br}{2},$$

i.e.

$$2\langle r, b \rangle - rb = br$$

the reflection of  $r$  in  $l$  is then given by

$$r' = brb^{-1}. \quad (2.4)$$

If we want to generalize this result to reflections in higher dimensions, it is more appropriate to describe  $r'$  in terms of a vector *perpendicular* to the line, rather than parallel. Consider the following figure.

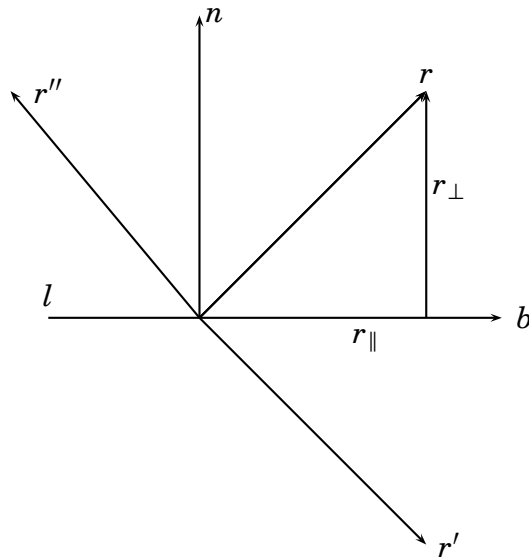


Figure 2.2

Here  $n$  is a normal vector to the line and  $r''$  is the reflection through  $n$ . Since  $r' = -r''$  we have

$$r' = -r'' = -nrn^{-1}. \quad (2.5)$$

**Exercise 2.1** Show that the reflection  $r'$  of a point  $r$ , through the line  $\langle r, n \rangle = d$  is

$$r' = -nrn^{-1} + 2dn^{-1}. \quad (2.6)$$

**Exercise 2.2** Show too that formula (2.4) holds in the algebra  $Cl(0,2)$ . (Basically because all we need to use is the definition  $x^2 = \langle x, x \rangle$  for any 1-vector  $x$ .)

More generally, a point  $r$  in  $R^n$  can be reflected through a *hyperplane* (of dimension  $n - 1$ .) In the case of a hyperplane through the origin given by the equation  $\langle r, n \rangle = 0$ , the equation (2.5) still holds so that the reflected point is again  $r' = -nrn^{-1}$ .

Similarly equation (2.6) holds for a reflection of a point through a general hyperplane.

## 2.3 Translations

A translation in  $R^n$  can be written as the product of two *parallel* reflections. Consider the following figure in  $R^2$ . Here we have two lines with equations  $l_1: \langle r, n \rangle = d_1$  and  $l_2: \langle r, n \rangle = d_2$ . Assume that  $n$  is a unit vector and  $d_2 > d_1$ . Put  $d = \frac{d_2 - d_1}{2}$ .

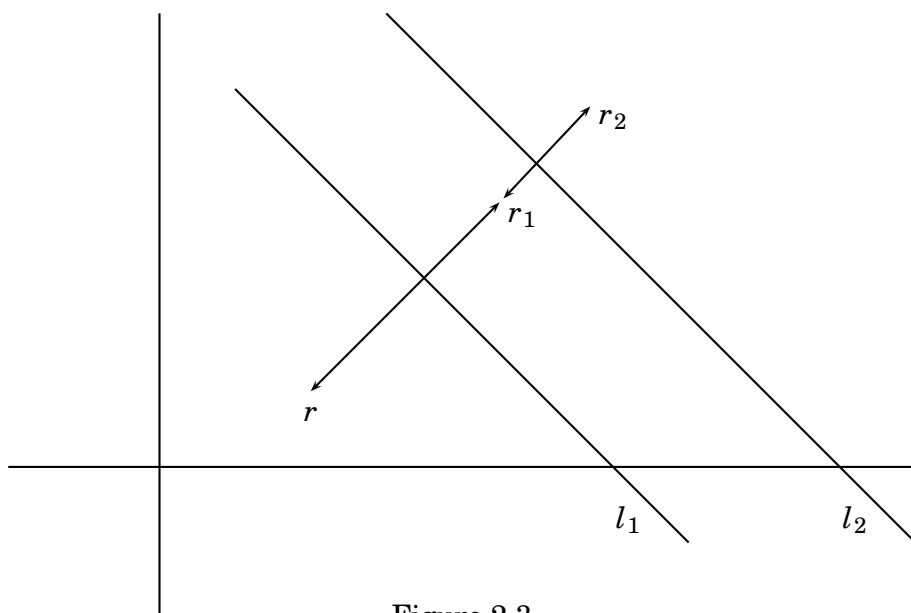


Figure 2.3

Again work in either  $Cl(2,0)$  or  $Cl(0,2)$ . From Exercise 2.1 the reflection through line  $l_1$  maps

$$r \rightarrow r_1 = -nrn^{-1} + 2d_1n^{-1}.$$

Similarly  $r_1$  is reflected to  $r_2$  by the line  $l_2$  so that

$$\begin{aligned} r_2 &= -nr_1n^{-1} + 2d_2n^{-1} \\ &= -n(-nrn^{-1} + 2d_1n^{-1})n^{-1} + 2d_2n^{-1} \\ &= \dots \\ &= r + 2(d_2 - d_1)n \quad \text{since } n \text{ is a unit vector} \\ &= r + dn \end{aligned}$$

i.e.  $r$  is translated to  $r + dn$ .

So a translation can be expressed as the product of two successive reflections in parallel lines.

## 2.4 Generalized reflections - inversions

Consider a reflection  $R$  in  $R^n$  about a hyperplane through the origin. From Fig 2.1 above, we see that a point  $r$  is decomposed as the sum  $r = r_{\parallel} + r_{\perp}$  and the reflection of  $r$  is then

$$Rr = r' = r_{\parallel} - r_{\perp} \quad (2.7)$$

The projection operator  $P$  associated with the hyperplane subspace is

$$Pr = r_{\parallel} \quad (2.8)$$

so that

$$Rr = Pr - (r - r_{\parallel}) = Pr - (r - Pr) = 2Pr - r \quad (2.9)$$

or

$$R = 2P - I. \quad (2.10)$$

Although for a reflection, the hyperplane is a subspace of dimension  $n - 1$ , equation (2.10) can be used to *define* a generalized reflection through *any* subspace.

**Exercise 2.3** *Show the following:*

1. *If  $R$  is a (generalized) reflection, then  $R$  can be written as a symmetric matrix with  $R^2 = I$ .*
2. *Conversely if  $R$  is symmetric with  $R^2 = I$ , then  $P = \frac{R + I}{2}$  is a projection operator and  $R$  is the reflection about the subspace associated with  $P$  (i.e. the range of  $P$ ).*
3. *If  $R$  is a reflection about  $P$  (i.e. about the subspace associated with  $P$ ), then  $-R$  is the reflection associated with  $P^{\perp} = I - P$ . In particular in  $R^3$ ,  $P = 0$  gives the space inversion map,  $Rx = -x$ .*

There are two important examples of reflections in  $R^{1,3}$  which play a fundamental role in physics. They are time reversal and space inversion.

In  $R^{1,3}$ , the time reversal map is the map

$$T(x^0, x^1, x^2, x^3) = (-x^0, x^1, x^2, x^3). \quad (2.11)$$

It is straightforward to check that this operator can be described in the Clifford algebra  $Cl(1,3)$  by

$$Tx = -e_0 x e_0^{-1} \quad (2.12)$$

which has the same form as (2.5) and is a genuine reflection (i.e. not a generalized one). It is the reflection about the hyperplane  $\{x : x^0 = 0\}$ .

The space inversion map in  $R^{1,3}$  is defined by

$$S(x^0, x^1, x^2, x^3) = (x^0, -x^1, -x^2, -x^3) \quad (2.13)$$

and we can also easily verify that it is given by the Clifford algebra map

$$Sx = -e_{123} x e_{123}^{-1} \quad (2.14)$$

so that in  $R^{1,3}$ , this is a generalized reflection (as  $e_{123}$  is not a 1-vector). The associated subspace is the 1-dimensional subspace spanned by  $e_0$ .

**Exercise 2.4** 1. *Show that space inversion can be written as the product of three (genuine) reflections.*

2. *Show too that equations (2.12) and (2.14) also hold in  $Cl(3,1)$ . So the choice of signature is not important here.*

## 2.5 Rotations

A rotation is always in a plane. Although we often talk about a rotation ‘about an axis’, this is because we live in  $R^3$  and given a plane and a point in the plane, there is a line through that point and perpendicular to the plane. This is the axis of rotation. In higher dimensional spaces, there is no longer any such axis.

Warning! Rotations in  $R^n$  preserve the (Euclidean) metric and therefore are elements of  $SO(n)$  - the group of isometries on  $R^n$  with determinant 1. But there are isometries which are *not* rotations. Some authors however equate matrices in  $SO(n)$  with rotations. This does not make good geometrical sense and it is always preferable to define rotations as acting in a plane.

For simplicity, we will only consider rotations about the origin. More general rotations (about a point in a plane) can be reduced to this case by an appropriate translation.

In this section we will show how rotations can be expressed in very simple terms, using the ideas which allowed us to write reflections in Clifford algebra notation.

We start with the case of rotations in  $R^2$ . Here the plane is given to us so that the only variable is the angle of rotation.

### 2.5.1 Rotations in $R^2$

There are two ways we can use Clifford algebras to describe rotations in  $R^2$ . However only the second way can be generalized to the general  $R^n$  case. So let us start with the way which *cannot*, just to see what the problem is.

This approach is simply another way of describing rotations in the complex plane. Here a rotation through an angle  $\theta$  is expressed algebraically by the function  $z \rightarrow e^{i\theta} z$ .

We work in  $Cl(0,2)$ : the case  $Cl(2,0)$  is entirely similar and is left as an exercise. Choose an orthonormal basis  $e_1, e_2$ . Since rotations map 1-vectors to 1-vectors,  $e^{i\theta} = \cos\theta + i\sin\theta$  must be replaced by an even vector, of norm 1. As  $e_{12}^2 = -1$ , the obvious approach is to try a map of the form

$$x \rightarrow qx = (\cos\theta + e_{12}\sin\theta)x \quad (2.15)$$

A simple calculation gives

$$qe_1 = (\cos\theta + e_{12}\sin\theta)e_1 = \cos\theta e_1 + \sin\theta e_2 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad (2.16)$$

and

$$qe_2 = (\cos\theta + e_{12}\sin\theta)e_2 = -\sin\theta e_1 + \cos\theta e_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \quad (2.17)$$

So left multiplication by  $q$  corresponds to matrix multiplication by the matrix

$$R_q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad (2.18)$$

and this correctly describes a rotation in  $R^2$ .

**Exercise 2.5** *Show that in  $Cl(2,0)$ , the map  $R_q$  above, describes a rotation through the angle  $-\theta$ .*

Alternately instead of (2.15), we could have tried right multiplication

$$x \rightarrow xq^{-1} = x(\cos\theta - e_{12}\sin\theta) \quad (2.19)$$

The reason why we choose  $q^{-1}$  rather than  $q$  here, is because right multiplication by  $q^{-1}$  gives the same matrix as left multiplication by  $q$ , i.e. the matrix  $R_q$ .

**Exercise 2.6** *Verify this last statement.*

How could we extend this idea to higher dimensions? First note that formulae like (2.15) or (2.19) do not work in  $R^3$ . The reason is simple: the vector  $x$  now has an  $e_3$  component and then  $qx$  will have an  $e_{123}$  term, i.e. it will no longer be a 1-vector. Furthermore we cannot get around this problem by defining  $q$  more generally. If we try to write  $q = \cos\theta + \sin\theta p$  where  $p$  is a bivector satisfying  $p^2 = -1$  then it is not hard to show that if  $x = pe_{123}$  then  $x$  is a 1-vector and  $qx$  will again have a 3-vector component.

**Exercise 2.7** *Show this!*

Another reason why this idea doesn't extend to  $R^3$  is the fact that no axis of rotation will exist. For if  $x$  is a point on this axis then, no matter how we define  $q$ ,  $qx = x$ . But unless  $q = 1$ , which is the trivial case,  $q - 1$  has an inverse so that  $x = 0$ . So there can be no axis of rotation.

There are two pointers to a way around this problem. The first is to notice that equations (2.15) and (2.19) show that left multiplication by  $q$  and right multiplication by  $q^{-1}$  each give the same rotation in  $R^2$ . Since there is no reason why we should prefer one to the other, we might combine them to give a composite rotation (though we will have to go to half angles for  $q$  to give the whole angle rotation). So we could try

$$x \rightarrow q x q^{-1} \tag{2.20}$$

where

$$q = \cos \frac{\theta}{2} + e_{12} \sin \frac{\theta}{2}.$$

From the results above, we know that this represents a rotation through an angle  $\theta$ .

The second clue comes from looking at reflections. As is well known in Euclidean geometry, if we have two lines (in  $R^2$  still) intersecting at a point, then the two successive reflections in the lines produce a rotation about the point of intersection, through an angle, twice the angle between the lines.

Figure 2.4 below shows a double reflection. First the point  $P$  is reflected in the line  $l_1$  to a point  $P'$ . Then  $P'$  is reflected in  $l_2$  to  $P''$ . The net result  $P \rightarrow P''$  is a reflection through an angle  $2\theta$  where  $\theta$  is the angle between  $l_1$  and  $l_2$ .



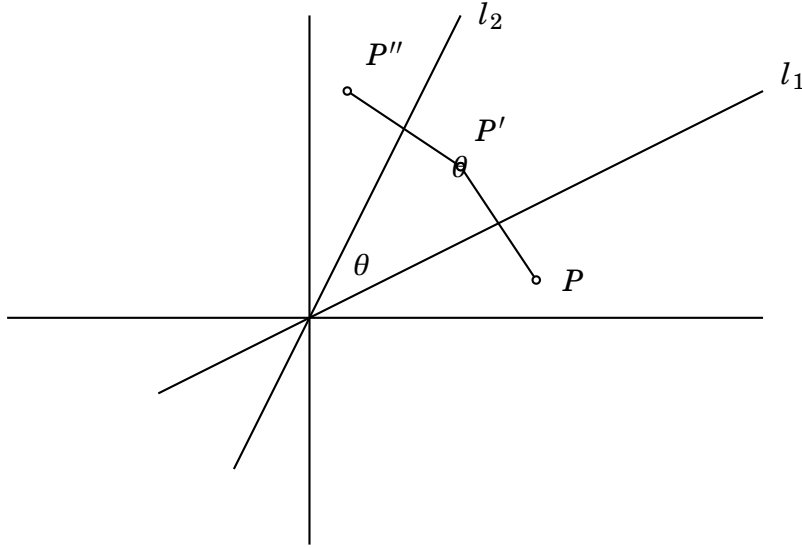


Figure 2.4

If the lines  $l_1$  and  $l_2$  intersect at the origin and have equations  $\langle a, x \rangle = 0$  and  $\langle b, x \rangle = 0$  respectively, the composition of a reflection about  $l_1$  followed by a reflection about  $l_2$  is

$$x \rightarrow (ba)x(ba)^{-1} = qxq^{-1} \quad (2.21)$$

where  $q = ba$  is a  $0 + 2$  vector. Note that we can always assume that  $q$  is a unit bivector (a unit quaternion in this case). So if the angle between the lines is  $\frac{\theta}{2}$  and we write  $q = \cos \frac{\theta}{2} + e_{12} \sin \frac{\theta}{2}$  we obtain a rotation through  $\theta$ .

In summary then, a rotation in  $R^2$  through  $\theta$  can be written as

$$R_\theta x = qxq^{-1} \quad (2.22)$$

where  $q = \cos \frac{\theta}{2} + e_{12} \sin \frac{\theta}{2}$ .

## 2.5.2 Rotations in $R^3$

When describing rotations in  $R^2$ , we worked with  $Cl(0, 2)$  though as exercise 2.5 showed, this choice is not important and we could just as well have chosen  $Cl(2, 0)$ . So for rotations in  $R^3$  we will use the algebra  $Cl(0, 3)$  and leave

it as an exercise to show that  $Cl(3,0)$  also works perfectly well.

Let  $e_1, e_2, e_3$  be an orthonormal basis for  $R^3$ . Bearing in mind equation (2.21), we look at transformations of the form  $x \rightarrow qxq^{-1}$  where  $q = ab$  is the product of two 1- vectors. Then as before,  $q$  is the sum of a scalar term and a bivector, i.e.  $q$  is a quaternion. Furthermore the expression  $qxq^{-1}$  is unchanged when  $q$  is multiplied by a non-zero scalar and so we may assume that  $q$  is a *unit* quaternion, i.e. if  $q = q_0 + q_1e_{23} + q_2e_{31} + q_3e_{12}$  then  $|q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2 = 1$ .

Therefore we can write

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} p$$

where  $p$  is a unit, (pure) bivector such that  $p^2 = -1$ . (We sometimes call a bivector pure if we want to emphasize that it has no scalar part.)  $p$  takes the place of  $i$  in the complex plane. But in fact  $p$  contains much more information as it gives us the axis of rotation in  $R^3$ .

To show that we have a rotation, we need also to show that if  $x$  is a 1- vector then so is  $R_\theta x = qxq^{-1}$ . This can be done by calculating the coefficients of  $R_\theta x$  in full, thus verifying that there is no 3- component. But there is another much neater way, variations of which we will later use.

Recall the inversion operation  $x \rightarrow \tilde{x}$  in  $Cl(0,3)$ . It is an anti-automorphism such that  $\tilde{\tilde{x}} = x$  for  $x$  a 1- vector. It follows that  $\tilde{\tilde{x}} = -x$  for a 3- vector  $x$ . Noting that if  $q$  is a unit quaternion  $q^{-1} = \tilde{q}$ , then if

$$y = R_\theta x = qxq^{-1} = qx\tilde{q}$$

$$\tilde{y} = \widetilde{(qx\tilde{q})} = \tilde{\tilde{q}}\tilde{\tilde{x}}\tilde{q} = qx\tilde{q} = y$$

so that  $y$  has no 3- vector term, i.e.  $y$  is a 1- vector.

As an example, take  $p = e_{23}$  so that  $q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} e_{23}$ . Then a straightforward calculation gives (in vector format)

$$R_\theta x = R_\theta \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

i.e.  $R_\theta$  is a rotation about the  $e_1$  axis as expected.

More generally, to find the axis of rotation we need to find the invariant vector  $x$ , i.e. to solve the equation  $qxq^{-1} = x$  i.e. solve the equation  $qx = xq$ .

Write  $q = q_0 + q_1p$  with  $p$  a bivector. Since  $e_{123}$  commutes with all elements of  $Cl(0,3)$ , the equation  $qx e_{123} = xq e_{123}$  implies that  $(pe_{123})x = x(pe_{123})$  (we may assume that  $q \neq 1$  in which case  $q_1 \neq 0$ .) But  $pe_{123}$  is a 1-vector which commutes with  $x$  and therefore it must be a multiple of  $x$ . So the axis of rotation is the line defined by the 1-vector  $pe_{123}$  or (better) the line defined by the axial bivector  $p$ .

Finally it is worth noting that in  $R^3$  every matrix in  $SO(3)$  is a rotation. (This is *not* true in  $R^n$  when  $n > 3$ .)

To show this, suppose that  $A \in SO(3)$ . Then since  $A^T = A^{-1}$ , the eigenvalues of  $A$  have modulus 1. Likewise the characteristic polynomial of  $A$  has real components which means that the eigenvalues come in conjugate pairs, i.e. if  $e^{i\alpha}$  is an eigenvalue, so is  $e^{-i\alpha}$ . Since there are 3 eigenvalues, one must be real and so equal to  $\pm 1$ . The eigenvalues must then be  $\pm 1, e^{i\alpha}, e^{-i\alpha}$ . Finally since the determinant of  $A$  is  $+1$ , the eigenvalues must be  $1, e^{i\alpha}, e^{-i\alpha}$ . So  $A$  is a rotation with axis of rotation the line of eigenvectors corresponding to the eigenvalue 1.

It is perhaps worth emphasizing too, that describing a rotation in terms of (unit) quaternions is very much simpler than messing about with Euler angles. The quaternion  $q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} p$  contains all the information we need to describe a rotation.

**Exercise 2.8** Show that in  $R^3$ , the affine map

$$x \rightarrow Rx + a \quad (\text{where } R \text{ is a rotation and } a \in R^3)$$

can be written as the product of two reflections. (This exercise will be extended to  $R^{1,3}$  in the next chapter when we look at the Poincaré group.)

Note that writing a rotation (about the origin) as a product of two reflections, requires knowing two vectors in the plane of rotation. For example if

$R$  is a rotation in the  $y, z$ -plane,  $R$  is given by the map

$$x \rightarrow qxq^{-1} \quad \text{where } q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} e_{23}$$

We can then take  $a = e_2$  and  $b = -\cos \frac{\theta}{2} e_2 + \sin \frac{\theta}{2} e_3$ . Then  $q = ab$ . But other choices of  $a$  and  $b$  are certainly possible.

### 2.5.3 Rotations in $R^4$

In general we will not be interested in rotations in  $R^n$  when  $n > 3$ , since geometric ideas are more difficult to see. However the case  $n = 4$  does have some interest for at least two reasons. Firstly, although the Euclidean metric on  $R^4$  has quite different properties to the Minkowski metric, at least the dimensions are the same and this does lead to some similarities (largely because both cases contain ordinary 3-space.)

Secondly it turns out to be possible to describe general isometries in  $SO(4)$  in terms of quaternion maps, just as it is in  $SO(3)$ . This is due to the fact that we can take as a model for  $R^4$ , the space  $Cl^-(0, 3)$ .<sup>2</sup> In other words we can take a basis to be  $e_1, e_2, e_3, e_{123}$ . The action of quaternions on  $R^3$  then extends in a very natural way to an action on  $R^4$ .

An extra bonus with this approach is that we derive geometric properties which will resurface when we later come to look at the geometry underlying spin 1/2 particles.

The central idea is that of a *double rotation* in  $R^4$ .<sup>3</sup> This means the following. Suppose that  $M$  is a 2-dimensional subspace of  $R^4$  and  $M^\perp$  its (also 2-dimensional) ortho-complement. If we have rotations in each of these subspaces, they combine to give an isometry of  $R^4$ . If we choose a basis for each of these subspaces, the isometry can be written in matrix form as

$$\begin{bmatrix} \cos \alpha & -\sin \alpha & & \\ \sin \alpha & \cos \alpha & & \\ & & \cos \beta & -\sin \beta \\ & & \sin \beta & \cos \beta \end{bmatrix}.$$

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<sup>2</sup>We will also see later that this space is an excellent candidate for the space of Pauli spinors.

<sup>3</sup>A full account of the theory of double rotations can be found, e.g. in [2].

This is a double rotation in the orthogonal planes  $M$  and  $M^\perp$ .

An interesting special case of this occurs when  $\alpha = \pm\beta$ . Then the two rotations are called *isoclinic*. In this case we can also ask whether the two rotations are in the same ‘sense’ or in the opposite ‘sense’. If  $u$  and  $v$  are two vectors in the plane, ‘sense’ means ‘from  $u$  to  $v$ ’ or vice versa. This is just the idea of an orientation in the plane, we discussed when describing bivectors. We therefore need to have an order on the basis  $\{e_1, e_2, e_3, e_{123}\}$ . If we take  $e_{123}$  as the last basis vector, this essentially means an orientation of  $e_1, e_2, e_3$  which is just a matter of choosing either the right hand rule or the left hand rule. We will choose the right hand rule so that the order of the basis is simply  $\{e_1, e_2, e_3, e_{123}\}$ .

Now suppose that

$$q = q_0 + q_1 e_{23} + q_2 e_{31} + q_3 e_{12} = q_0 + q_1 i + q_2 j + q_3 k$$

is a quaternion. If  $x \in R^4$ , write  $x = x^1 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_{123}$ . A simple calculation shows that left multiplication by  $q$  is given by the matrix  $L_q$  where

$$L_q = \begin{bmatrix} q_0 & -q_3 & q_2 & -q_1 \\ q_3 & q_0 & -q_1 & -q_2 \\ -q_2 & q_1 & q_0 & -q_3 \\ q_1 & q_2 & q_3 & q_0 \end{bmatrix}.$$

This means that the isoclinic matrix (in the case  $\beta = \alpha$ )

$$\begin{bmatrix} \cos \alpha & -\sin \alpha & & \\ \sin \alpha & \cos \alpha & & \\ & & \cos \alpha & -\sin \alpha \\ & & \sin \alpha & \cos \alpha \end{bmatrix}$$

is just left multiplication by  $L_q$  with  $q = \cos \alpha + \sin \alpha k$ .

Similarly right multiplication by  $q$  corresponds to the matrix

$$R_q = \begin{bmatrix} q_0 & q_3 & -q_2 & -q_1 \\ -q_3 & q_0 & q_1 & -q_2 \\ q_2 & -q_1 & q_0 & -q_3 \\ q_1 & q_2 & q_3 & q_0 \end{bmatrix}$$

which means that the isoclinic matrix (in the case  $\beta = -\alpha$ )

$$\begin{bmatrix} \cos \alpha & -\sin \alpha & & \\ \sin \alpha & \cos \alpha & & \\ & & \cos \alpha & \sin \alpha \\ & & -\sin \alpha & \cos \alpha \end{bmatrix}$$

is now right multiplication by  $q^{-1}$  (where again  $q = \cos \alpha + \sin \alpha k$ ).

**Exercise 2.9** Show that all left and right multiplication matrices above ( $L_{q_1}$  and  $R_{q_2}$ ) commute with each other. Can you see why without calculating the matrix products?

There is a general result, that any isometry in  $SO(n)$  can be written as the product of at most  $\frac{n(n-1)}{2}$  simple rotations in one of the coordinate planes (the so-called Givens rotations.<sup>4</sup> In  $R^4$  such a rotation in the  $e_1, e_2$  plane is of the form

$$A = \begin{bmatrix} \cos \theta & -\sin \theta & & \\ \sin \theta & \cos \theta & & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix}$$

In this case we can write

$$A = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & & \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & & \\ & & \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ & & \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & & \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & & \\ & & \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ & & -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \\ = LR$$

where  $L$  (resp.  $R$ ) is a left (resp. right) multiplication matrix by a quaternion. This leads to the basic result.

**Theorem 2.1** All matrices  $X \in SO(4)$  are of the form  $X = L_{q_1} R_{q_2}^{-1}$  where  $q_1, q_2$  are unit quaternions.

<sup>4</sup>See [3], chapter 5 for an efficient way to obtain these Givens rotations.

Alternately we can say that every 4–dimensional rotation is the product of a left isoclinic and a right isoclinic rotation.

In a sense this result is to be expected. The group  $SO(n)$  is an  $\frac{n(n-1)}{2}$ –parameter family so that  $SO(4)$  is a 6–parameter group. The group  $H^1$  of unit quaternions is a 3–parameter group so that the direct sum  $H^1 \oplus H^1$  is a 6–parameter group describing rotations in  $R^4$  by left and right multiplication as described above. The total number of parameters being equal for the two groups, the groups must be the same (though actually  $H^1 \oplus H^1$  is the two-fold covering group of  $SO(4)$  and so should be written as  $\text{Spin}(4)$ .) This is not an entirely watertight argument but can be made so with a few additional topological arguments.

Having looked at the cases where we had a Euclidean metric on  $R^4$ , let us turn to the (more useful) case of the Minkowski metric.

## 2.6 Lorentz transformations in $R^{1,3}$

(See e.g. Lounesto, [1], chapter 9.)

First recall some of the basic ideas and notation in  $R^{1,3}$ . Take a basis  $e_0, e_1, e_2, e_3$  for  $R^{1,3}$  so that in the Clifford algebra  $Cl(1,3)$

$$e_0^2 = 1, e_1^2 = e_2^2 = e_3^2 = -1$$

A vector in  $R^{1,3}$  is written as

$$x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \quad (2.23)$$

We will sometimes write  $ct$  for  $x^0$  or even just  $t$  since we usually assume that  $c = 1$ .

The scalar product of two vectors in  $R^{1,3}$  is given by

$$\langle x, y \rangle = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

so that

$$\langle x, x \rangle = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

In physics we introduce the quadratic form, i.e. the metric tensor

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2.24)$$

in which case

$$\langle x, y \rangle = x^T g y = \sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu y^\nu. \quad (2.25)$$

Suppose that we have two inertial systems. An event described by the vector  $x$  in the first system corresponds to a vector  $x'$  in the second. We can write

$$x' = Lx + a \quad \text{or} \quad x'^\mu = \sum L^\mu_\nu x^\nu + a^\mu \quad (2.26)$$

where  $a$  is a constant vector and  $L$  is a linear transformation  $L : R^{1,3} \rightarrow R^{1,3}$  which preserves the inner product. That is

$$\langle Lx, Ly \rangle = \langle x, y \rangle.$$

This last condition is equivalent to the condition:

$$L^T g L = g.$$

Such a transformation  $L$  is called a Lorentz transformation.

### 2.6.1 Some examples of Lorentz transformations

Lorentz transformations fall into one of four sets, depending on whether the direction of time,  $e_0$ , is preserved (or not) and whether the orientation of the 4-volume is preserved or not. We will discuss this at some further length in the next chapter but for the time being let us consider just those transformations which preserve the  $e_0$  direction (so that the matrix element  $L^0_0 > 0$ ) and which have determinant 1, i.e. in the usual notation, transformations in  $SO(1,3)$ . As in  $R^4$ , these transformations are basically rotations but there are two cases. If the plane of rotation is in physical space  $R^3$ , i.e. the subspace spanned by  $e_1, e_2, e_3$ , these are ordinary rotations, while in the second



case, the plane contains  $e_0$ . Then the transformations are sometimes called ‘imaginary’ rotations, but more commonly they are called boosts. We look at each case in turn.

### 1. (Space) rotations

Space rotations are in the subspace of  $R^{1,3}$  spanned by  $e_1, e_2, e_3$ . So they will be written as  $4 \times 4$  matrices but their properties are just those of rotations in  $R^3$ .

So for example, the rotation the  $e_2 - e_3$  plane, i.e. about the  $e_1$  or  $x$ -axis, through an angle  $\theta$  is given by the matrix

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (2.27)$$

which is conventionally written in exponential form as

$$R(\theta) = e^{J_1\theta} \quad (2.28)$$

where

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.29)$$

We say that  $J_1$  is the (infinitesimal) generator of  $R(\theta)$ .

Similarly for the other two planes,  $e_3 - e_1$  and  $e_1 - e_2$  the generators are

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.30)$$

A general rotation in the plane with (unit) normal  $n$ , i.e. about the axis given by  $n$  and through an angle  $\theta$  is given by

$$R_n(\theta) = e^{J \cdot n\theta}$$

where  $J$  is the vector  $(J_1, J_2, J_3)$ .

Just as  $R^3$  rotations can be expressed in the form  $x \rightarrow qx\tilde{q}$ , so rotations in  $R^{1,3}$  can also be written in this way but since  $x$  is no longer a vector in  $R^3$  a little care must be taken. We will look at this in detail in the following chapter.

## 2. Boosts

In most physics texts, a boost in the  $e_1$  direction is written as

$$\begin{aligned}x'^0 &= \gamma(x^0 - \beta x^1) \\x'^1 &= \gamma(-\beta x^0 + x^1) \\x'^2 &= \gamma x^2 \\x'^3 &= \gamma x^3\end{aligned}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \quad \beta = \frac{v}{c}. \quad (2.31)$$

Since  $\gamma^2 - \gamma^2\beta^2 = 1$ , we may put

$$\gamma = \cosh \phi, \quad \gamma\beta = \sinh \phi \quad (2.32)$$

and we then have

$$\begin{bmatrix}x'^0 \\x'^1 \\x'^2 \\x'^3\end{bmatrix} = \begin{bmatrix}\gamma & -\gamma\beta & 0 & 0 \\-\gamma\beta & \gamma & 0 & 0 \\0 & 0 & 1 & 0 \\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}x^0 \\x^1 \\x^2 \\x^3\end{bmatrix} = \begin{bmatrix}\cosh \phi & -\sinh \phi & 0 & 0 \\-\sinh \phi & \cosh \phi & 0 & 0 \\0 & 0 & 1 & 0 \\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}x^0 \\x^1 \\x^2 \\x^3\end{bmatrix} \quad (2.33)$$

In terms of the parameter  $\phi$ , a boost in the  $x$ -direction can also be written in exponential (generator) form as

$$B_1(\phi) = e^{-K_1\phi} \quad \text{where} \quad (2.34)$$

$$K_1 = \begin{bmatrix}0 & 1 & 0 & 0 \\1 & 0 & 0 & 0 \\0 & 0 & 0 & 0 \\0 & 0 & 0 & 0\end{bmatrix}. \quad (2.35)$$

In the same way, for boosts in the  $y$  and  $z$  directions, the generators are

$$K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (2.36)$$

A general boost in the direction of the unit vector  $n$ , with parameter  $\phi$  is then given by

$$B_n(\phi) = e^{-K \cdot n \theta} \quad (2.37)$$

where  $K$  is the vector  $(K_1, K_2, K_3)$ .

The matrices  $J_i, K_j$ , define the Lorentz Lie algebra and satisfy the commutation relations

- $[J_1, J_2] = J_3$  and cyclically,
- $[K_1, K_2] = -J_3$  and cyclically,
- $[J_i, K_i] = 0$  etc,
- $[J_1, K_2] = K_3$  and cyclically.

How should we express boosts in terms of the Clifford algebra  $Cl(1,3)$ ?

There are a number of possible approaches but one which looks appealing is to look at boosts as ‘imaginary’ rotations and then follow the ideas which led to rotations being described by quaternions. Let us see how this approach works. (Note however that this is not the standard way of doing things! So it also has the advantage of novelty.)

A boost is normally defined as a movement in one direction. For simplicity let us suppose that we have a boost in the  $e_1$  direction. Of course the  $e_0$  component also changes so what we have is movement in the  $e_0 - e_1$  plane (this is why boosts are called imaginary rotations.) Since (space) rotations are the products of reflections, boosts may also be considered in this way.

So let us consider again reflections in the plane but this time in the  $e_0 - e_1$  plane, so that the metric is now the Minkowski metric, i.e. we are working in

the Clifford algebra  $Cl(1,1)$ . If  $l$  is the line  $\langle n, x \rangle = 0$ <sup>5</sup>, with  $n$  a unit vector, then it is straightforward to verify that the reflection of a point  $r$  in the line  $l$  is again given by the equation

$$r' = -nrn^{-1}. \quad (2.38)$$

Note however that, since the metric is no longer Euclidean, there is a restriction on the line  $l$  because its (normal) vector  $n$  *must* have an inverse. If  $n = n^0 e_0 + n^1 e_1$  this means that

$$n^2 = \|n\|^2 = (n^0)^2 - (n^1)^2 \neq 0. \quad (2.39)$$

Then if  $n^2 \neq 0$  we have  $n^{-1} = \frac{n}{\|n\|^2}$  as before.

So we are not to consider reflections through lines on the light cone. This makes physical sense as otherwise we could reflect from inside the light cone to its exterior and it is difficult to make much physical sense of this.

Since a boost is to be considered as an ‘imaginary’ rotation in the  $e_0 - e_1$  plane and a rotation is the product of two reflections, consider two lines

$$l_1 : \langle a, x \rangle = 0, \quad l_2 : \langle b, x \rangle = 0 \quad (2.40)$$

with  $a$  and  $b$  their unit normals respectively. Then as with ordinary reflections (see equation (2.21) above) we should consider the product of the reflections in  $l_1$  followed by  $l_2$ . This gives the transformation

$$Bx = (ba)x(ba)^{-1} = u x u^{-1} \quad (2.41)$$

where

$$u = ba = (b^0 e_0 + b^1 e_1)(a^0 e_0 + a^1 e_1) \quad (2.42)$$

$$= (b^0 a^0 - b^1 a^1) + (b^0 a^1 - b^1 a^0) e_{01}. \quad (2.43)$$

Note that  $u$  is *not* a quaternion because of the presence of the  $e_{01}$  term.

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<sup>5</sup>The inner product  $\langle \cdot, \cdot \rangle$  is therefore the inner product inherited from  $Cl(1,3)$  or alternately, from  $g$ .

To clarify these ideas a bit more, let us write this transformation in matrix form. First, since  $a$  is a unit vector we have  $(a^0)^2 - (a^1)^2 = 1$  so that we can write

$$a = \cosh \alpha e_0 + \sinh \alpha e_1$$

and similarly for  $b$ , also a unit vector

$$b = \cosh \beta e_0 + \sinh \beta e_1.$$

(For reasons which become clearer in the next chapter, we will take the + sign for both cosh terms.)

Then

$$\begin{aligned} u &= ba = (\cosh \alpha \cosh \beta - \sinh \alpha \sinh \beta) \\ &\quad + (\sinh \alpha \cosh \beta - \cosh \alpha \sinh \beta)e_{01} \\ &= \cosh \theta + \sinh \theta e_{01} \end{aligned}$$

where  $\theta = \alpha - \beta$  is the ‘angle’ between the two lines (or vectors  $a, b$ .)

A simple calculation gives

$$ue_0 = (\cosh \theta + \sinh \theta e_{01})e_0 = \cosh \theta e_0 - \sinh \theta e_1 = \begin{bmatrix} \cosh \theta \\ -\sinh \theta \end{bmatrix} = e_0 u^{-1} \quad (2.44)$$

and similarly

$$ue_1 = e_1 u^{-1} = \begin{bmatrix} -\sinh \theta \\ \cosh \theta \end{bmatrix}. \quad (2.45)$$

Combining these last two equations we find that

$$Bx = uxu^{-1} = \begin{bmatrix} \cosh 2\theta & -\sinh 2\theta \\ -\sinh 2\theta & \cosh 2\theta \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}. \quad (2.46)$$

Finally since the boost  $B$  is in the  $e_0 - e_1$  plane, it can be written as a  $4 \times 4$  matrix acting on  $R^{1,3}$  as

$$B = \begin{bmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.47)$$

(where we have written  $\phi = 2\theta$ ). which is the Lorentz boost given in equation (2.33).

Alternately in Clifford algebra notation, we can write

$$Bx = uxu^{-1} \quad \text{where } u = \cosh \frac{\phi}{2} + \sinh \frac{\phi}{2} e_{01}. \quad (2.48)$$

A general boost has a (unit) direction vector  $u = u^1 e_1 + u^2 e_2 + u^3 e_3$  and can again be written as

$$Bx = uxu^{-1} \quad (2.49)$$

with

$$u = \cosh \frac{\phi}{2} + \sinh \frac{\phi}{2} e_0 u. \quad (2.50)$$

A useful point to keep in mind, is that in Clifford algebra notation, boosts are in the subspace spanned by  $1, e_{01}, e_{02}, e_{03}$ , while rotations are in the subspace spanned by  $1, e_{23}, e_{31}, e_{12}$ .

**Exercise 2.10** *Show that if  $B$  is a boost and  $R$  a rotation, then  $RBR^{-1}$  is a boost. What is its direction?*

(Algebraically this is a straightforward consequence of the previous remark. Geometrically this is also easy to understand as  $RBR^{-1}$  is simply  $B$  when viewed in the new coordinate system defined by the rotation  $R$ .)

**Exercise 2.11** *If  $B$  is a boost and  $R$  a rotation then there exists another boost  $B'$  such that  $BR = RB'$ .*

**The product of two boosts.** The subspace spanned by  $1, e_{01}, e_{02}, e_{03}$  is not closed under multiplication. Whereas the product of two rotations is another rotation (because the subspace spanned by  $1, e_{23}, e_{31}, e_{12}$  is closed under multiplication), the product of two boosts will involve terms in this second subspace, i.e. terms which behave like rotations. So the following result seems reasonable.

**Theorem 2.2** *If  $B_1, B_2$  are boosts then there is a boost  $B$  and a rotation<sup>6</sup>  $R$  such that*

$$B_1 B_2 = BR$$

*Further, the axis of rotation of  $R$  is orthogonal to the plane determined by the direction vectors of  $B_1$  and  $B_2$ .*

(Note that from exercise 11 we could just as well have asked for a decomposition  $B_1 B_2 = RB'$  - same  $R$  but different  $B$ ).

*Proof:* Most proofs of this important result are technically quite challenging as they use  $4 \times 4$  matrix calculations. These tend to obscure the basic ideas.

However, armed with Clifford algebra ideas, we will see that the derivation is really quite simple. The end result is of course the same and necessarily tricky when it comes to describing  $R$  and  $B$  but this is unavoidable. What is avoidable is an obscure proof.

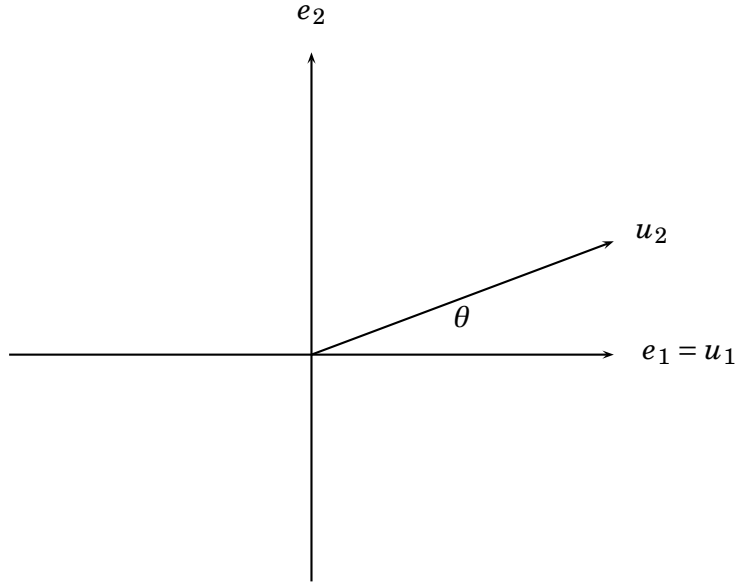
The reason for this simplicity is that, for this proof, Clifford algebra allows us to describe Lorentz transformations as  $2 \times 2$  *real* matrices. We will need the general matrix result - the polar decomposition theorem - but in the  $2 \times 2$  case this is trivial to derive.

Now let us go (fairly gently) through the main ideas.

The first thing to note is that this is essentially a (real) 2-dimensional problem. The reason is the following. Consider two boosts  $B_1, B_2$  in the (unit) directions,  $u_1, u_2$ . These vectors span a plane and we can take as basis vectors for the plane,  $e_1 = u_1$  and  $e_2 \perp e_1$  so that we have the following picture.

---

<sup>6</sup> $R$  is called the Thomas rotation or precession, or sometimes the Thomas-Wigner rotation or more economically (and meanly) the Wigner rotation. Its existence still causes some controversy with some calling it a paradox. See e.g [4], pp 38-43.



$$u_2 = \cos\theta e_1 + \sin\theta e_2.$$

So we don't need all of  $Cl(1,3)$  to describe the two boosts; just  $Cl(1,2)$  will do. Furthermore since boosts (and rotations) are *even* elements of the algebra,  $Cl^+(1,2)$  is in fact the right setting. This algebra is spanned by  $e_{01}, e_{02}, e_{12}$  (as well as 1).

It is easy to verify that  $Cl^+(1,2)$  is just  $M(2,R)$  - all  $2 \times 2$  real matrices. This can be seen by writing

$$e_{01} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{02} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (2.51)$$

which together with  $I = I_2$  span all of  $M(2,R)$ .

We can then write the two boosts as

$$B_1 = \cosh \frac{s}{2} + \sinh \frac{s}{2} e_{01} \quad (2.52)$$

$$B_2 = \cosh \frac{t}{2} + \sinh \frac{t}{2} \cos\theta e_{01} + \sinh \frac{t}{2} \sin\theta e_{02} \quad (2.53)$$

so that in the representation given by (2.51),

$$B_1 = \begin{bmatrix} \cosh \frac{s}{2} + \sinh \frac{s}{2} & 0 \\ 0 & \cosh \frac{s}{2} - \sinh \frac{s}{2} \end{bmatrix} = \begin{bmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{bmatrix} \quad (2.54)$$



and

$$B_2 = \begin{bmatrix} \cosh \frac{t}{2} + \sinh \frac{t}{2} \cos \theta & \sinh \frac{t}{2} \sin \theta \\ \sinh \frac{t}{2} \sin \theta & \cosh \frac{t}{2} - \sinh \frac{t}{2} \cos \theta \end{bmatrix}. \quad (2.55)$$

It is important to see that the matrix expression for any boost in the plane is a *symmetric* matrix as there is no  $e_{12}$  term.

So

$$A = B_1 B_2 = \begin{bmatrix} e^{\frac{s}{2}} (\cosh \frac{t}{2} + \sinh \frac{t}{2} \cos \theta) & e^{\frac{s}{2}} \sinh \frac{t}{2} \sin \theta \\ e^{-\frac{s}{2}} \sinh \frac{t}{2} \sin \theta & e^{-\frac{s}{2}} (\cosh \frac{t}{2} - \sinh \frac{t}{2} \cos \theta) \end{bmatrix} \quad (2.56)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{say.} \quad (2.57)$$

Note that  $B_1 B_2$  has determinant 1 and in general is not symmetric, which immediately shows that  $B_1 B_2$  is not a pure boost.

At this stage we could invoke a standard result in matrix theory, which states that if  $A$  is a square matrix with positive determinant, then there exists an orthogonal matrix  $R$  (with determinant 1), and a positive definite (symmetric) matrix  $H$  such that  $A = HR$ . This is the polar decomposition of  $A$  and the decomposition is unique (see e.g. [4], chapter 4.2).<sup>7</sup>

This result is not very hard to prove but in the  $2 \times 2$  case, the decomposition is actually very simple to construct. It can be obtained as follows.

If  $A = HR$  as in the decomposition above, then  $AR^T = H$  is symmetric and this condition is enough to find  $R$  and  $H$ . Since  $R^T$  is a rotation (being orthogonal, with determinant 1), it must be of the form

$$R^T = \begin{bmatrix} u & v \\ -v & u \end{bmatrix} \quad \text{with } u^2 + v^2 = 1.$$

Then

$$\begin{aligned} H = AR^T &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u & v \\ -v & u \end{bmatrix} \\ &= \begin{bmatrix} au - bv & av + bu \\ cu - dv & cv + du \end{bmatrix} \end{aligned}$$

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<sup>7</sup>The result is actually wider than that, but this version is all we need.

and since  $H$  is symmetric

$$av + bu = cu - dv, \text{ i.e. } (a + d)v = (c - b)u$$

so that we can take

$$u = k(a + d), \quad v = k(c - b) \quad (2.58)$$

where  $k$  is a normalizing constant given by

$$k = \frac{1}{\sqrt{(a + d)^2 + (b - c)^2}} \quad (2.59)$$

(as  $u^2 + v^2 = 1$ ).

So we now have the rotation component  $R$  of  $B_1 B_2$

$$R = \frac{1}{\sqrt{(a + d)^2 + (b - c)^2}} \begin{bmatrix} a + d & -(c - b) \\ c - b & a + d \end{bmatrix}. \quad (2.60)$$

This is the Thomas-Wigner rotation.

As for the symmetric (positive definite part),

$$\begin{aligned} H = AR^T &= k \begin{bmatrix} a(a + d) - b(c - b) & a(c - b) + b(a + d) \\ c(a + d) - d(c - b) & c(c - b) + d(a + d) \end{bmatrix} \\ &= \frac{1}{\sqrt{(a + d)^2 + (b - c)^2}} \begin{bmatrix} a^2 + b^2 + 1 & ac + bd \\ ac + bd & c^2 + d^2 + 1 \end{bmatrix}. \end{aligned}$$

Since  $H$  is symmetric with the determinant 1,  $H$  is a pure boost and the decomposition

$$B_1 B_2 = HR \quad (2.61)$$

is the required decomposition of  $B_1 B_2$ .

We have of course, still to write this decomposition (2.61) in terms of the original parameters for  $B_1$  and  $B_2$  but from (2.56) and (2.57) we have

$$a = e^{\frac{s}{2}} \left( \cosh \frac{t}{2} + \sinh \frac{t}{2} \cos \theta \right), \quad (2.62)$$

$$b = e^{\frac{s}{2}} \sinh \frac{t}{2} \sin \theta, \quad (2.63)$$

$$c = e^{-\frac{s}{2}} \sinh \frac{t}{2} \sin \theta, \quad (2.64)$$

$$d = e^{-\frac{s}{2}} \left( \cosh \frac{t}{2} - \sinh \frac{t}{2} \cos \theta \right). \quad (2.65)$$

. These substitutions give  $R$  and  $B$ . For example, if we write

$$R = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

where now  $\phi$  is the Thomas-Wigner angle, then (see (2.60)),

$$\begin{aligned} \tan \phi &= \frac{c - b}{a + d} \\ &= \frac{-\sinh \frac{s}{2} \sinh \frac{t}{2} \sin \theta}{\cosh \frac{s}{2} \cosh \frac{t}{2} + \sinh \frac{s}{2} \sinh \frac{t}{2} \cos \theta} \end{aligned}$$

gives an expression for  $\phi$ .

The expression for the boost  $B$  is similar, but messy and is left to the brave reader.

## 2.7 Notes and References

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# Chapter 3

## Groups

### 3.1 Introduction<sup>1</sup>

Group theory is of fundamental importance in modern physics. Among many examples, it is the language of the standard model for particle physics. It is also basic to the description of diverse areas from crystallography to the symmetries observed in the physical world and consequent conservation laws. Physicists just cannot do without groups.

Groups may be finite or infinite. Finite groups find applications in describing discrete symmetries such as parity or time reversal. Permutation groups are basic to the quark model and through their Young diagrams, are essential in describing the irreducible representations of unitary groups.

Infinite groups may be discrete or continuous and it is with continuous groups that we are mostly concerned.<sup>2</sup> The most important continuous groups in physics are the Lie groups. Lie groups have the great advantage that they can be *linearized*. This involves imposing a Lie multiplication on the vector space, tangent to the group at the identity. Studying the properties of Lie groups can then often be replaced by studying the resultant Lie algebra and this is now an area in linear algebra which is generally regarded

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<sup>1</sup>There is any number of excellent books on group theory with applications to physics; for example [1] and [2].

<sup>2</sup>There are discrete infinite groups such as the group of integers under addition, but they will not be of interest here.

as an easier field of study.

Examples of Lie groups of interest to physicists are the following.

1. The unitary group  $U(n)$  of all  $n \times n$  complex, unitary matrices, i.e. matrices  $A$  such that  $A^{-1} = A^H$ , where  $A^H = \overline{A}^T$  is the conjugate transpose of  $A$ . These are precisely the matrices which preserve the usual, conjugate-linear, inner-product on  $C^n$ .  $U(n)$  contains the subgroup  $SU(n)$  (the special unitary group) of matrices with determinant 1.
2. The orthogonal group  $O(n)$  and its special subgroup  $SO(n)$ . This is the equivalent of the unitary group for real spaces. So  $O(n)$  consists of all matrices  $A$  such that  $A^{-1} = A^T$ . As discussed in the previous chapter,  $SO(n)$  contains the rotations in  $R^n$  and furthermore every element of  $SO(n)$  is a product of rotations.  $O(n)$  includes reflections as well.
3. For more general metrics which define the space  $R^{p,q}$  and the associated Clifford algebra  $Cl(p,q)$ , there are the more general orthogonal groups  $O(p,q)$  and  $SO(p,q)$ . Of these, the case of  $R^{1,3}$  is fundamental as the groups now contain the Lorentz transformations. By adding translations, we arrive at the Poincaré groups.
4. If  $n$  is even we can add a symplectic structure on  $R^n$ . Then we can look for matrices which preserve this structure. If we restrict ourselves to orthogonal matrices, we obtain the (compact) symplectic group on  $R^n$ . In the case  $n = 6$ , this is the group of transformations which preserve Hamilton's equations.
5. Finally we have the conformal group. In the case of  $R^{1,3}$ , this is the largest group which preserves the structure of Maxwell's equations and so is basic to electromagnetic theory.

In each of these cases there is a corresponding Lie algebra and this algebra may be used to obtain the required results for the group.

However, in the low-dimensional cases of interest to us, each group sits inside a Clifford algebra and this algebra, being associative, is much easier

to handle than its Lie counterpart.<sup>3</sup>

This chapter then is concerned with the study of the most important examples from each of these families - important in the sense that they have immediate applications to physics. In each case we will see that the group of interest lies naturally inside a Clifford algebra of similar dimension to that of the group. This algebra can be given a matrix representation and it is this representation which provides us most easily with the group properties we need.

## 3.2 Spin(3) - the covering group of $SO(3)$ <sup>4</sup>

Since the two Clifford algebras  $Cl^+(p, q)$  and  $Cl^+(q, p)$  are isomorphic so too are the two groups  $Spin(n, 0)$  and  $Spin(0, n)$ . So we will write them more simply as  $Spin(n)$ .  $Spin(n)$  is the two-fold covering group of  $SO(n)$ .

The conventional way of representing  $Spin(3)$  is as  $SU(2)$ . The reason is as follows. Firstly, identify  $R^3$  as  $2 \times 2$  Hermitian, trace-zero matrices<sup>5</sup>, e.g. by

$$x = (x^1, x^2, x^3) \leftrightarrow A = \begin{bmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{bmatrix}.$$

Then a rotation in  $R^3$  can be implemented by a matrix  $U \in SU(2)$  and the rotation is given by

$$A \rightarrow U^H A U$$

Since  $U$  is determined only up to a  $\pm$  sign, this gives the two-fold representation of  $SO(3)$  by  $SU(2)$ .

There are some drawbacks with this approach. For a start, there is a loss of symmetry in  $R^3$  as the  $x^3$  or  $z$  direction has been singled out as the diagonal element. Of course any component would do just as well but the fact remains that one component has to be chosen.

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<sup>3</sup>This is hardly surprising since in many cases the Lie product is simply the commutator product inside an associative algebra. So the Lie product does little more than complicate the issue.

<sup>4</sup>See e.g. Lounesto, op.cit., chapter 19. for a general overview of the Spin and Pin groups.

<sup>5</sup>See chapter 1. The Pauli spin matrices now span this space.

The second issue is more problematic and illustrates quite well what happens when you stray too far from a basis-free picture. Given a rotation in one of the coordinate planes, it is easy to construct the relevant unitary matrix (up to a sign of course). But given a rotation about an arbitrary axis, the solution is more tricky. Even worse still is the problem of starting with an arbitrary  $SU(2)$  matrix and deriving the actual rotation in  $R^3$  which it represents. This means finding the axis and angle of rotation. Of course this can be done but it involves finding the eigenvalues and eigenvectors of the matrix (and it is here that you are glad that you are only working with  $2 \times 2$  matrices).

By the way, if you prefer the positive euclidean metric on  $R^3$ , i.e. you are happy to work in the Clifford algebra  $Cl(3,0)$ , the above approach (and its problems) is hard to avoid. As we saw in chapter 1, to find the natural representation of  $Cl(3,0)$  you start with the Pauli matrices<sup>6</sup> to give a basis for the algebra and are then led inexorably to the isomorphism between  $Cl(3,0)$  and  $M(2, C)$  - the algebra of  $2 \times 2$  complex matrices. The above representations of rotations by  $SU(2)$  matrices now follows.

Suppose instead that we work in  $Cl(0,3)$ . We saw in Chapter 2 that every rotation in  $R^3$  is the product of two reflections, so that we can write a rotation  $R_\theta$  as

$$R_\theta : x \rightarrow (ab)x(ab)^{-1} \quad (3.1)$$

with  $a, b \in R^3$ .

Then  $q = ab$  is a bivector in  $Cl(0,3)$ , i.e. a quaternion. Since  $R_\theta$  is unchanged when we scale  $q$  by  $q \rightarrow \lambda q$ , with  $\lambda > 0$  we can assume that  $q$  is a unit quaternion ( $q\tilde{q} = 1$ ) so that  $q^{-1} = \tilde{q}$ . So  $R_\theta$  is given by

$$R_\theta : x \rightarrow qx\tilde{q} \quad (3.2)$$

with  $q \in H^1$ , the set of unit quaternions.  $H^1$  forms a group under multiplication and is therefore another candidate for the covering group  $\text{Spin}(3)$ . Of course the two groups  $H^1$  and  $SU(2)$  are isomorphic and the real question is: which one is the most natural, or simplest, representation for describing rotations.

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<sup>6</sup> $Cl(3,0)$  is often referred to as the Pauli algebra.



Our contention is that  $H^1$  is the better. The problems with  $SU(2)$  described above, no longer exist for  $H^1$ . The reason is that the way a (unit) quaternion is written, *automatically* provides a description of the axis and rotation angle of the rotation. For if we write<sup>7</sup>

$$q = \cos \frac{\theta}{2} + p \sin \frac{\theta}{2} \tag{3.3}$$

where  $p$  is a unit, pure quaternion then  $\theta$  is the angle of rotation and

$$p = p^1 e_{23} + p^2 e_{31} + p^3 e_{12}$$

gives the axis of rotation. Note too that  $p$  is a bivector, i.e. is an *axial* vector in physicists' language, and not an ordinary polar or 1–vector. This is just as it should be.

Finally we no longer have to write the vector  $x \in R^3$  as a Hermitian, trace-zero matrix as we had to with  $SU(2)$ . We simply write  $x = x^1 e_1 + x^2 e_2 + x^3 e_3$ . in the term  $qx\tilde{q}$ . What could be more natural?

(This is another hint that choosing the negative metric on  $R^3$  might have unintended benefit.)

### 3.3 Spin(4) - the covering group of $SO(4)$

The group  $SO(4)$  does not arise very often in physics, compared to its more favoured relatives,  $SO(3)$ ,  $SU(2)$ ,  $SU(3)$  and even  $SO(6)$ . This is largely because when working in  $R^4$  it is almost invariably the case that we impose the Minkowski metric on the space. However, its covering group Spin(4) is usually written as  $SU(2) \times SU(2)$  and as seen above, this implies a close connection with rotations in  $R^3$ . So it seems worthwhile to look a little more at the group Spin(4).

Despite these somewhat negative comments,  $R^4$  does have a role to play when describing rotations. We will see this in more detail in the next chapter but there are two immediate clues why this should be so.

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<sup>7</sup>We need half angles here as  $q$  appears twice in equation (3.2).

Firstly  $SU(2)$  acts on  $C^2$ , the two (complex) dimensional space of Pauli spinors. This space can be regarded as ‘equivalent’ to  $R^4$  and furthermore, relativity issues which impose the Minkowski metric, do not arise. Secondly (and this is just another way of looking at  $C^2$ ), there is a natural real 4-dimensional space associated with  $R^3$ , namely  $Cl^-(0,3)$  - the space of *odd* vectors in  $Cl(0,3)$ . If  $e_1, e_2, e_3$  is a (orthonormal as ever) basis for  $R^3$  then  $e_1, e_2, e_3, e_{123}$  is a natural basis for  $Cl^-(0,3)$ . Note too that with our choice of inner-product on  $Cl(0,3)$  (see Definition 1.1, above), we have a *negative-definite*, (i.e. *not hyperbolic*) metric on  $Cl^-(0,3)$ . So  $Cl^-(0,3)$  is essentially  $R^4$ . It follows that  $Spin(4)$  does have a role to play in ordinary space.

As we saw in chapter 2, every orthogonal map in  $SO(4)$  can be expressed geometrically as the product of isoclinic rotations and in turn each such rotation can be written as either left or right multiplication by a quaternion  $q$  in  $H^1$  acting on  $R^4$  with the basis  $e_1, e_2, e_3, e_{123}$ . Since left and right multiplication commute we have the result that  $Spin(4)$  is isomorphic to the direct sum  $H^1 \oplus H^1$ .

To summarize: a rotation  $R$  in  $R^4$  can be expressed as

$$Rx = q_1 x q_2^{-1}$$

where  $x \in R^4$  is written as  $x = x^1 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_{123}$  and  $q_1, q_2$  are unit quaternions.

### 3.4 The Lorentz groups<sup>8</sup>

Lorentz transformations on  $R^{1,3}$  were defined in chapter 2. The group of all Lorentz transformations (called the full Lorentz group) is  $O(1,3)$ . We have already seen the important examples of rotations and boosts, but there are two other important ones.

1. Time reversal  $T$ : this is the Lorentz transformation

$$T: \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \begin{bmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

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<sup>8</sup>For an extensive and detailed analysis of these groups - as well as the Poincaré group - see [3].

2. Space inversion (parity)  $P$ : similarly  $P$  is the Lorentz transformation

$$P : \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \begin{bmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{bmatrix}.$$

These two transformations do not belong to the connected component of the Lorentz group (for example because the determinants of their matrix representations are  $-1$ ).

Let us now consider the full Lorentz group  $O(1,3)$  and some of its important subsets. Note first that if  $L$  is a Lorentz transformation, its determinant is  $\pm 1$ . (This follows immediately from the defining relation  $g = L^T g L$  where  $g$  is the Minkowski metric on  $R^{1,3}$ ). So we have the first classification of Lorentz transformations into two types

- *Proper transformations*, where  $\det(L) = +1$ . These include rotations and boosts.
- *Improper transformations* where  $\det L = -1$ . These include space and time inversion.

Lorentz transformations can also be classified according to whether or not they reverse the ‘arrow of time’, i.e. according to the sign of  $L_0^0 = \langle e_0, L e_0 \rangle$  of the transformation  $L$ . So we have

- *Orthochronous transformations*, where  $L_0^0 \geq 1$ . Examples of these are rotations, boosts and space inversion.
- *Anti-orthochronous transformations* where  $L_0^0 \leq -1$ . These include time reversal.

So the Lorentz group  $O(1,3)$  can be partitioned into four disjoint components

$$L_+^\uparrow, L_-^\uparrow, L_+^\downarrow, L_-^\downarrow$$

where the symbols  $\uparrow, \downarrow, +, -$  indicate respectively that

$$L_0^0 \geq 1, L_0^0 \leq -1, \det(L) = +1, \det(L) = -1.$$

Note that  $L_-^\uparrow = P L_+^\uparrow$  and  $L_+^\downarrow = P T L_+^\uparrow$  (where  $P$  is space inversion and  $T$  is time reversal).

### 3.4.1 Subgroups of the Lorentz group

Of the four above components of the Lorentz group, only  $L_+^\uparrow$  is a subgroup (it is the component which contains the identity). Other subgroups can be made up by joining various components. This leads us to the following classification of the Lorentz groups.

- *The orthochronous proper Lorentz group*

$$SO^\uparrow(1,3) = L_+^\uparrow = \{L \in SO(1,3) : L_0^0 \geq 1\}.$$

- *The proper Lorentz group*

$$\begin{aligned} SO(1,3) = L_+ &= L_+^\uparrow \cup L_+^\downarrow \\ &= L_+^\uparrow \cup PTL_+^\uparrow. \end{aligned}$$

- *The orthochronous Lorentz group*

$$\begin{aligned} O^\uparrow(1,3) = L^\uparrow &= L_+^\uparrow \cup L_-^\uparrow \\ &= L_+^\uparrow \cup PL_+^\uparrow \\ &= \{L \in O(1,3) : L_0^0 \geq 1\}. \end{aligned}$$

- *The (full) Lorentz group*

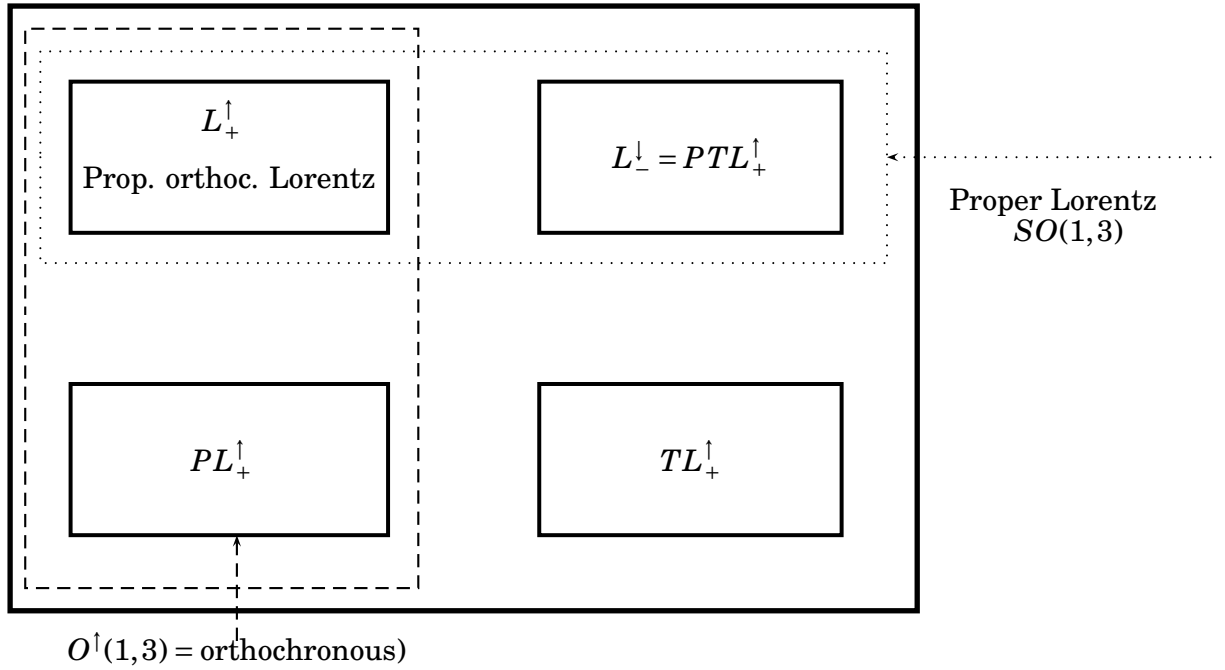
$$\begin{aligned} O(1,3) &= L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow \\ &= L_+^\uparrow \cup PTL_+^\uparrow \cup PL_+^\uparrow \cup TL_+^\uparrow. \end{aligned}$$

### 3.4.2 Representations of the Lorentz (and covering) groups by quaternion matrices

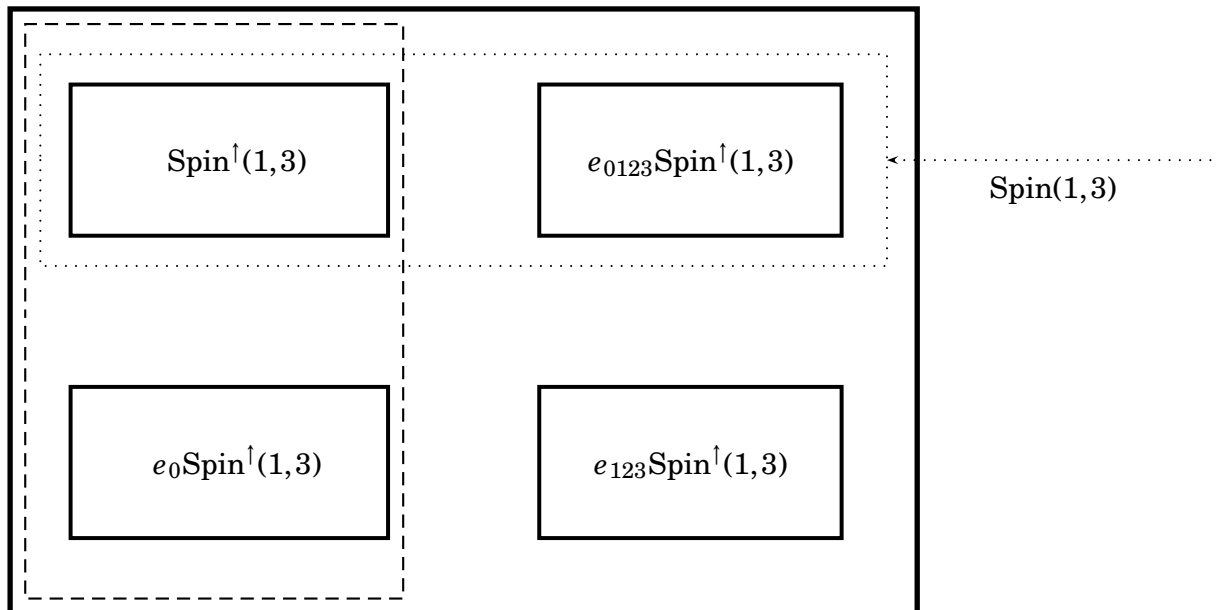
The principal Lorentz groups of interest have been listed above and we know from chapter 1 that their two-fold covering groups are the following.

$\text{Pin}(1,3)$  is the covering group for the full group  $O(1,3)$ ;  $\text{Spin}(1,3)$  for  $SO(1,3)$  and  $\text{Spin}^\uparrow(1,3)$  for the orthochronous group  $SO^\uparrow(1,3)$ . As suggested earlier, Clifford algebra makes it easy to deal with these covering groups.

**Lorentz Group  $O(1,3)$**



**Covering Group  $\text{Pin}(1,3)$**



The Lie algebra only really handles  $SO^{\uparrow}(1,3)$ . On the other hand, the Clifford algebra  $Cl(1,3)$  can be used to describe the entire Lorentz group.  $Cl(1,3)$  is the algebra of  $2 \times 2$  quaternion matrices and it is with these matrices that we will represent the groups.

Because all the Lorentz groups are subgroups of the Clifford group  $\Gamma(1,3)$ , we start with a description of that group.

### 3.4.3 The Clifford group $\Gamma(1,3)$

Recall from chapter 1 that an element of  $\Gamma(1,3)$  is a product of (invertible) vectors in  $R^{1,3}$  and so (depending on the number of vectors in the product) will be either even or odd. That is we can write

$$\Gamma(1,3) = \Gamma^+(1,3) \cup \Gamma^-(1,3) \quad (3.4)$$

and this means that we only need to describe  $\Gamma^+(1,3)$  because for example,  $\Gamma^-(1,3) = e_0\Gamma^+(1,3)$ .

Begin with an element  $g \in \Gamma^+(1,3)$ . Since  $g \in Cl^+(1,3)$  we can write

$$g = \alpha + \beta e = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad (3.5)$$

with  $\alpha, \beta$  quaternions. We can distinguish two cases.

*Case 1:*  $\alpha = 0$ . Then  $g = \beta e = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$  with  $\beta \neq 0$ . Note that in this case

$$\tilde{g} = \tilde{e}\tilde{\beta} = \tilde{\beta}e$$

(since  $\tilde{e} = e$  and  $e$  commutes with all quaternions).

We need to check that  $g \in \Gamma(1,3)$ , i.e. that the map  $x \rightarrow gxg^{-1}$  is a map from  $R^{1,3} \rightarrow R^{1,3}$ . Since the map is unchanged under scaling, we may assume that  $|\beta| = 1$ . Then  $g^{-1} = -\tilde{\beta}e$  and if  $x \in R^{1,3}$ ,

$$gxg^{-1} = (\beta e)x(-\tilde{\beta}e) = \dots = -\beta x \tilde{\beta} \quad \text{which is in } R^{1,3}.$$

---

<sup>9</sup>because  $e$  commutes with quaternions and also  $ex = -xe$ .

So  $g$  does belong to  $\Gamma(1,3)$  and furthermore the map consists of a space inversion together with a rotation in  $R^3$ . So in particular, this map is in  $O^\uparrow(1,3)$  but not in  $SO^\uparrow(1,3)$ .

*Case 2:  $\alpha \neq 0$ .* To simplify the argument, write

$$g = \alpha + \beta e = (1 + \gamma e)\alpha \quad (3.6)$$

where  $\gamma = \beta\alpha^{-1}$ .  $\alpha \in \Gamma(1,3)$  (since the map  $x \rightarrow \alpha x \alpha^{-1}$  is a rotation in  $R^3$ ). So we may assume that  $\alpha = 1$  and we need only find conditions on  $\gamma$  for  $g = 1 + \gamma e$  to be in  $\Gamma(1,3)$ .

Note too that

$$(1 + \gamma e)(1 - \gamma e) = 1 + \gamma^2$$

from which we deduce that if  $\gamma^2 = -1$  then  $g$  is not invertible, while if  $\gamma^2 \neq -1$  then  $g$  is invertible with

$$g^{-1} = (1 + \gamma e)^{-1} = (1 + \gamma^2)^{-1}(1 - \gamma e). \quad (3.7)$$

So now the important question is: what are the conditions on  $\gamma$  for the map  $x \rightarrow g x g^{-1}$  to map  $R^{1,3} \rightarrow R^{1,3}$ .

Take the case  $x = e_0$  and to make the algebra clearer, let us write the terms in quaternion representation form. Then

$$g e_0 g^{-1} = (1 + \gamma^2)^{-1} \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ -\gamma & 1 \end{bmatrix} \quad (3.8)$$

$$= (1 + \gamma^2)^{-1} \begin{bmatrix} 1 - \gamma^2 & 2\gamma \\ 2\gamma & \gamma^2 - 1 \end{bmatrix} \quad (3.9)$$

which must be of the form  $\begin{bmatrix} t & p \\ p & -t \end{bmatrix}$  with  $t$  real and  $p$  pure.

This implies that

$$\gamma^2 = \frac{1-t}{1+t} \quad (3.10)$$

i.e.  $\gamma^2$  is real and therefore  $\gamma$  must be pure.

That this condition is then sufficient for  $g$  to be in  $\Gamma(1,3)$  is left as part of the following exercise.

**Exercise 3.1** Show that if  $\gamma$  is pure,  $\gamma^2 \neq -1$ , then the map  $x \rightarrow gxg^{-1}$  maps  $R^{1,3}$  to  $R^{1,3}$ .

So we have the following description of  $\Gamma^+(1,3)$ .

$\Gamma^+(1,3)$  consists of elements of the form (with  $\alpha, \beta$  quaternions).

1.  $\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} = \beta e$  or
2.  $\begin{bmatrix} \alpha & -p\alpha \\ p\alpha & \alpha \end{bmatrix} = (1+pe)\alpha$  where  $p$  is pure and  $p^2 \neq -1$ .

*Corollary* Since  $\Gamma^-(1,3) = e_0\Gamma^+(1,3)$ ,  $\Gamma^-(1,3)$  consists of elements of the form

1.  $\begin{bmatrix} 0 & -\beta \\ -\beta & 0 \end{bmatrix} = e_{123}\beta$  or
2.  $\begin{bmatrix} \alpha & -p\alpha \\ -p\alpha & -\alpha \end{bmatrix}$  where  $p$  is pure with  $p^2 \neq -1$ .

### 3.4.4 The spin group $\text{Spin}(1,3)$

Recall that  $g \in \text{Spin}(1,3)$  means that  $g \in \Gamma^+(1,3)$  with  $g\tilde{g} = \pm 1$ . From the above there are two cases.

Case 1:  $g = \beta e$ . Then  $\tilde{g} = \bar{\beta}e$  so that

$$g\tilde{g} = |\beta|^2 e^2 = -|\beta|^2.$$

So  $|\beta| = 1$  i.e  $\beta = u$  is a *unit* quaternion and  $g = ue$ .

**Exercise 3.2** Show that in this case

$$ge_0g^{-1} = -e_0.$$

*Deduce that the Lorentz transformation defined by  $g$  is the composition of a rotation in  $R^3$  and the time reversal operator. So  $g \notin \text{Spin}^+(1,3)$ .*



Case 2: In this case  $g = \alpha + \beta e$  with  $\gamma = \beta\alpha^{-1}$  pure. Then

$$\begin{aligned} g\tilde{g} &= (\alpha + \beta e)(\bar{\alpha} + \bar{\beta}e) \\ &= (|\alpha|^2 - |\beta|^2) + (\alpha\bar{\beta} + \beta\bar{\alpha})e \end{aligned}$$

from which we deduce that  $|\alpha|^2 - |\beta|^2 = \pm 1$  and that  $\alpha\bar{\beta}$  is pure. It follows that  $g$  must be in one of the two forms

$$1. \quad g = \begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ \sinh \frac{\phi}{2} p u & \cosh \frac{\phi}{2} u \end{bmatrix} \text{ or}$$

$$2. \quad g = \begin{bmatrix} \sinh \frac{\phi}{2} p u & -\cosh \frac{\phi}{2} p u \\ \cosh \frac{\phi}{2} p u & \sinh \frac{\phi}{2} p u \end{bmatrix}.$$

where  $u$  is a unit quaternion and  $p$  is now a pure, unit quaternion.

Note that case 1 is in fact subsumed in case 2 so that  $\text{Spin}(1,3)$  consists exactly of the matrices in case 2.

### 3.4.5 The spin group $\text{Spin}^\uparrow(1,3)$

The two sets of matrices which make up  $\text{Spin}(1,3)$  are disconnected and only elements of the first set can be continuously reached from the identity. It is for this reason especially that we can deduce that  $\text{Spin}^\uparrow(1,3)$  consists of matrices of the form

$$\begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ \sinh \frac{\phi}{2} p u & \cosh \frac{\phi}{2} u \end{bmatrix}.$$

Note that an element in  $\text{Spin}^\uparrow(1,3)$  can be decomposed in the obvious way as

$$\begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ \sinh \frac{\phi}{2} p u & \cosh \frac{\phi}{2} u \end{bmatrix} = \begin{bmatrix} \cosh \frac{\phi}{2} & -\sinh \frac{\phi}{2} p \\ \sinh \frac{\phi}{2} p & \cosh \frac{\phi}{2} \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} = BR \quad (3.11)$$

where  $B = \begin{bmatrix} \cosh \frac{\phi}{2} & -\sinh \frac{\phi}{2} p \\ \sinh \frac{\phi}{2} p & \cosh \frac{\phi}{2} \end{bmatrix}$  represents a boost, with parameter  $\phi$  in the direction given by  $p$ , necessarily a bivector (axial vector) as boosts are regarded as imaginary rotations, and  $R = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$  is a rotation. This is possibly as simple a way there is, of showing that every (proper, orthochronous)

Lorentz transformation can be expressed as the product of a boost and a rotation.

**Exercise 3.3** (*The Pin(1,3) group*).

*Show that the elements of the Pin group fall into one of four classes.*

1.  $\begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ \sinh \frac{\phi}{2} p u & \cosh \frac{\phi}{2} u \end{bmatrix},$
2.  $\begin{bmatrix} \sinh \frac{\phi}{2} p u & -\cosh \frac{\phi}{2} p u \\ \cosh \frac{\phi}{2} p u & \sinh \frac{\phi}{2} p u \end{bmatrix},$
3.  $\begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ -\sinh \frac{\phi}{2} p u & -\cosh \frac{\phi}{2} u \end{bmatrix},$
4.  $\begin{bmatrix} \sinh \frac{\phi}{2} p u & -\cosh \frac{\phi}{2} p u \\ -\cosh \frac{\phi}{2} p u & -\sinh \frac{\phi}{2} p u \end{bmatrix}.$

## 3.5 The Poincaré group

The (full) Lorentz group consists of all linear transformations on  $R^{1,3}$  which preserve the Minkowski metric. If we ask instead for *affine* transformations which preserve the metric, we obtain the (full) Poincaré group. It is also known as the *inhomogeneous* Lorentz group and may be written as  $IO(1,3)$ . This latter notation allows us to specify certain components of the group, using the notation for the corresponding part of the Lorentz group. For example we can write  $ISO^\uparrow(1,3)$  for those affine transformations in the Poincaré group whose corresponding Lorentz transformation is in  $SO^\uparrow(1,3)$ . Likewise, the various Spin and Pin groups can be enlarged to their affine version, written ISpin, IPin respectively.

**Definition 3.1** *If  $L$  is a Lorentz transformation and  $a \in R^{1,3}$ , define the Poincaré transformation  $(L,a)$  on  $R^{1,3}$  by*

$$(L,a) : x \rightarrow Lx + a. \tag{3.12}$$

It follows easily that the Poincaré transformations form a group with identity element  $(I, 0)$  and with

$$(L, a)^{-1} = (L^{-1}, -L^{-1}a) \quad (3.13)$$

The Poincaré group is a 10-parameter, non-compact group.

Note that if we want to consider elements in the covering groups<sup>10</sup> we would replace  $L$  by its  $2 \times 2$  quaternion matrix, e.g. if  $L \in \text{Spin}^{\uparrow}(1, 3)$  then  $L$  is replaced by

$$\begin{bmatrix} \cosh \frac{\phi}{2} u & -\sinh \frac{\phi}{2} p u \\ \sinh \frac{\phi}{2} p u & \cosh \frac{\phi}{2} u \end{bmatrix}$$

and  $a$  by

$$\begin{bmatrix} a^0 & a^1 e_1 + a^2 e_2 + a^3 e_3 \\ a^1 e_1 + a^2 e_2 + a^3 e_3 & -a^0 \end{bmatrix}.$$

**Exercise 3.4** Show that the subgroup of translations, i.e.  $R^{1,3}$ , is a normal subgroup of the Poincaré group. This is an important result when we want to construct unitary representations of the Poincaré group. It is the first step in showing that the Poincaré group is the semidirect product of the additive group  $R^{1,3}$  with the Lorentz group.

**Exercise 3.5** Consider the operation given by equation (3.12) which defines the group operation for the Poincaré group. Show that this idea can be extended to provide a more general multiplicative structure on  $Cl^+(1, 3) \times Cl^-(0, 3)$ .

The importance of the Poincaré group lies in the fact that all inertial frames connected by translation, boosts and rotations, i.e. by all (proper, orthochronous) Poincaré transformations, are equivalent so that (at least locally) the laws of physics are form-invariant under these transformations.

---

<sup>10</sup>It is often much more convenient to do this as the quaternion representation of Lorentz transformations simplifies the mathematics.

### 3.6 The conformal group in $R^{1,3}$

Conformal transformations are maps on a space, which preserve angles. Together they form the conformal group on a vector space. Their importance can be seen from the fact that the conformal group in  $R^{1,3}$  is the largest group which leaves Maxwell's equations structurally invariant ([4,5]).

The study of the conformal group in  $R^{p,q}$  usually involves the conformal compactification of the space because this allows the transformations to be represented by linear transformations in the new space which normally has dimension 2 more than the original space. (See e.g. [6,7,8].) So for example the conformal group of Minkowski space,  $R^{1,3}$  leads to its isomorphism with  $SO(2,4)$ . But this embedding into a higher dimensional space comes at the expense of the geometric properties of the transformations. This is particularly a problem in  $R^{1,3}$  where we might well prefer to keep the geometric nature of the various types of transformations in sight.

However, we will show that this linearization procedure can be achieved with no loss of geometric insight if, instead of using compactification, we let the conformal transformations act on two copies of the associated Clifford algebra. Although we will be only concerned with the conformal group of Minkowski space (where the geometry is clearest and we are not burdened with extra notation), generalization to the general case is straightforward.

Note that there is not complete unanimity as to what constitutes the conformal group. Most authors restrict it to a connected component. One advantage of a Clifford algebra approach, is that it naturally leads to a description of the covering group and even allows the inclusion of operators such as inversions, which are not normally included in the conformal group, even though they are conformal transformations.

Our aim then is to show that the conformal group (more properly, the covering group of the conformal group) can be realized by the action of  $Cl(1,3)$  on the *vector space*  $Cl(1,3) \oplus Cl(1,3)$ . Although this larger space can be viewed as the vector space of the Clifford algebra  $Cl(2,3)$  (which is the approach taken by Lounesto and Latvamaa, [9]), this is unnecessary. Imposing an algebraic structure tends to obscure the more important geometric ideas and also raises problems of interpretation - e.g., what does the extra gener-

ator represent physically or geometrically?

This section is taken from Gresnigt and Renaud [10].

### 3.6.1 Conformal transformations

The 10-parameter Poincaré group is the semi-direct product of the 6-parameter Lorentz group with the 4-parameter group of space-time translations. The Poincaré group may then be enlarged to the conformal group by adding dilatations

$$x \rightarrow \rho x \quad (\rho > 0)$$

as well as special *conformal* transformations

$$x \rightarrow \frac{x + \langle x, x \rangle a}{\sigma(x)}, \quad \text{where } \sigma(x) = 1 + 2\langle a, x \rangle + \langle a, a \rangle \langle x, x \rangle$$

which correspond to local scale changes.

**Exercise 3.6** *The special conformal transformations may also be obtained as the product of an inversion*

$$I : x \rightarrow x^{-1} = \frac{x}{\langle x, x \rangle}$$

*followed by a translation and another inversion.*

The generators of the (identity component of the) conformal group may be realized as differential operators acting on Minkowski space. The operators corresponding to Lorentz transformations ( $M_{\mu\nu}$ ), translations ( $P_\mu$ ), dilatations ( $D$ ) and special conformal transformations ( $K_\mu$ ) satisfy the following commutation relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\sigma\rho}] &= g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} \\
[P_\lambda, M_{\mu\nu}] &= g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu \\
[D, M_{\mu\nu}] &= 0 \\
[K_\lambda, M_{\mu\nu}] &= g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu \\
[P_\mu, P_\nu] &= 0 \\
[D, P_\mu] &= -P_\mu \\
[P_\mu, K_\nu] &= 2(M_{\mu\nu} - g_{\mu\nu}D) \\
[D, K_\mu] &= K_\mu \\
[K_\mu, K_\nu] &= 0
\end{aligned}$$

Note that there is some divergence between authors. Some require that the generators be Hermitian in which case the imaginary number  $i$  makes an occasional appearance in these equations. Since we are dealing with Lie, i.e. anti-symmetric, products it is perhaps more logical to define these generators to be skew-Hermitian. This has the added bonus that only real algebras ever have to be used. In this context, we follow the definitions in Barut and Raczka [7] and Lounesto [8].

### 3.6.2 The Clifford algebra representations

We regard  $Cl(1,3)$  as acting on the vector space  $Cl(1,3) \oplus Cl(1,3)$  by left multiplication. It is then straightforward to verify that the above commutation relations are satisfied by the following operators.

$$\begin{aligned}
M_{\mu\nu}(x, y) &= \frac{1}{2}(e_{\mu\nu}x, e_{\mu\nu}y) \\
P_\mu(x, y) &= (e_\mu y, 0) \\
K_\mu(x, y) &= (0, e_\mu x) \\
D(x, y) &= \frac{1}{2}(-x, y)
\end{aligned}$$

where  $x, y \in Cl(1,3)$ .

Although the inversion operator is not in the identity component, it too has a very natural representation in this context as

$$I(x, y) = (y, x).$$

It then follows that  $K_\mu = IP_\mu I$ .

Since these operators are defined through the action of the associative algebra  $Cl(1,3)$ , they can be expected to have extra algebraic properties.<sup>11</sup> As an example, the translation generators  $P_\mu$  satisfy the property

$$P_\mu P_\nu = 0$$

(which trivially implies that  $[P_\mu, P_\nu] = 0$ .) As is shown below in this section, this property is important when we want to show that  $P_\mu$  generates a translation in the direction  $e_\mu$ .

The transformations that arise from these generators are now easy to describe geometrically.

### 3.6.3 Lorentz transformations

It is well known that the elements  $M_{\mu\nu}$  generate (the proper orthochronous) Lorentz transformations on  $R^{1,3}$ . In the Clifford algebra setting, where we could define  $M_{\mu\nu} = \frac{e_{\mu\nu}}{2}$ , this is particularly simple. For example, a boost in the direction  $(n_1, n_2, n_3)$ , with velocity  $v = \tanh \phi$  (remember that we are using units with  $c = 1$ ), may be represented as

$$x \rightarrow x' = axa^{-1}$$

where  $a = \exp(\phi n)$  and  $n = n_1 e_{01} + n_2 e_{02} + n_3 e_{03}$ . Similarly spatial rotations through an angle  $\theta$ , are of the same form but now  $a = \exp(\theta ne)$  with  $n$  describing the axis of rotation and  $e = e_{0123}$ .

### 3.6.4 Translations

Consider now the generator  $P_\mu$  defined by

$$P_\mu(x, y) = (e_\mu y, 0)$$

Clearly  $P_\mu^2 = 0$  so that

$$\exp(tP_\mu)(x, y) = (x + te_\mu y, y)$$

---

<sup>11</sup>As a general rule, if a Lie algebra structure is imposed on an associative algebra via  $[A, B] = AB - BA$ , some properties of  $AB$  may be lost.

More generally, if  $a = (a^\mu e_\mu)$  is a vector in  $R^{1,3}$ , then

$$\exp(ta^\mu P_\mu)(x, y) = (x + ta^\mu e_\mu y, y)$$

and in particular

$$\exp(ta^\mu P_\mu)(x, 1) = (x + ta, 1)$$

so that  $P_a = a^\mu P_\mu$  generates the translation operation  $T_a : x \rightarrow x + a$  in  $R^{1,3}$  when this space is identified with the hyperplane  $R^{1,3} \oplus (1)$  in  $Cl(1, 3) \oplus Cl(1, 3)$ , (see figure 3.1).

It might also be worth pointing out, that there is no *algebraic* reason why we should only consider translations in the direction of a 1-vector. If  $u$  is *any* element of  $Cl(1, 3)$ , we can define an operator  $P_u$  by

$$P_u(x, y) = (uy, 0)$$

and this generates a translation in  $Cl(1, 3) \oplus Cl(1, 3)$ . This then leads to a generalization of the conformal group and it would be interesting to characterize this extended group further.

### 3.6.5 Special conformal transformations

The generator  $K_\mu$  behaves much like  $P_\mu$ , but on the second component space of  $Cl(1, 3) \oplus Cl(1, 3)$ . It too generates a translation  $U_b : y \rightarrow y + b$  in  $R^{1,3}$  when the space is identified with the hyperplane  $(1) \oplus R^{1,3}$ , this time of the form

$$\exp(ta^\mu K_\mu)(1, y) = (1, y + ta)$$

This illustrates an advantage of our approach. The special conformal transformations act in an entirely similar way to translations, but on the second component subspace rather than the first. In that sense, they are no more non-linear than translations.

Again we can generalize special conformal transformations to operators  $K_u$  defined by

$$K_u(x, y) = (0, ux)$$



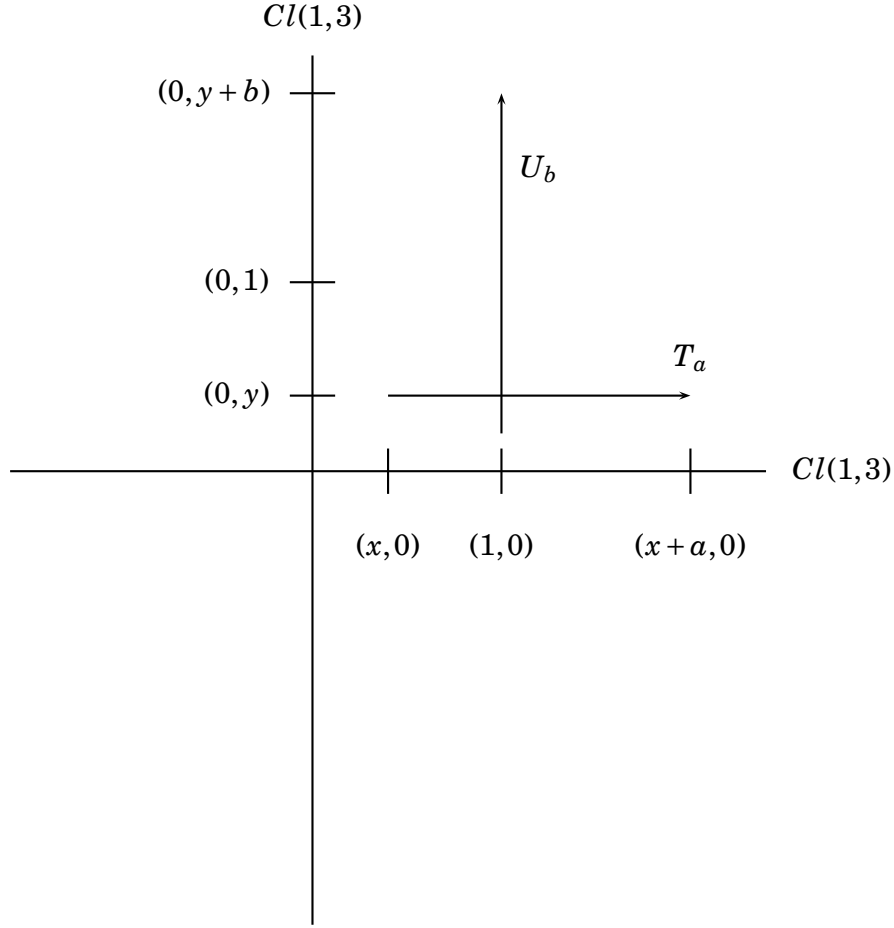


Figure 3.1: Geometric representation of the translations and special conformal transformations on the space  $Cl(1,3) \oplus Cl(1,3)$ . The generators  $T_a$  and  $U_b$  act on the hyperspaces  $R^{1,3} \oplus (1)$  and  $(1) \oplus R^{1,3}$  respectively.

### 3.6.6 Dilatations

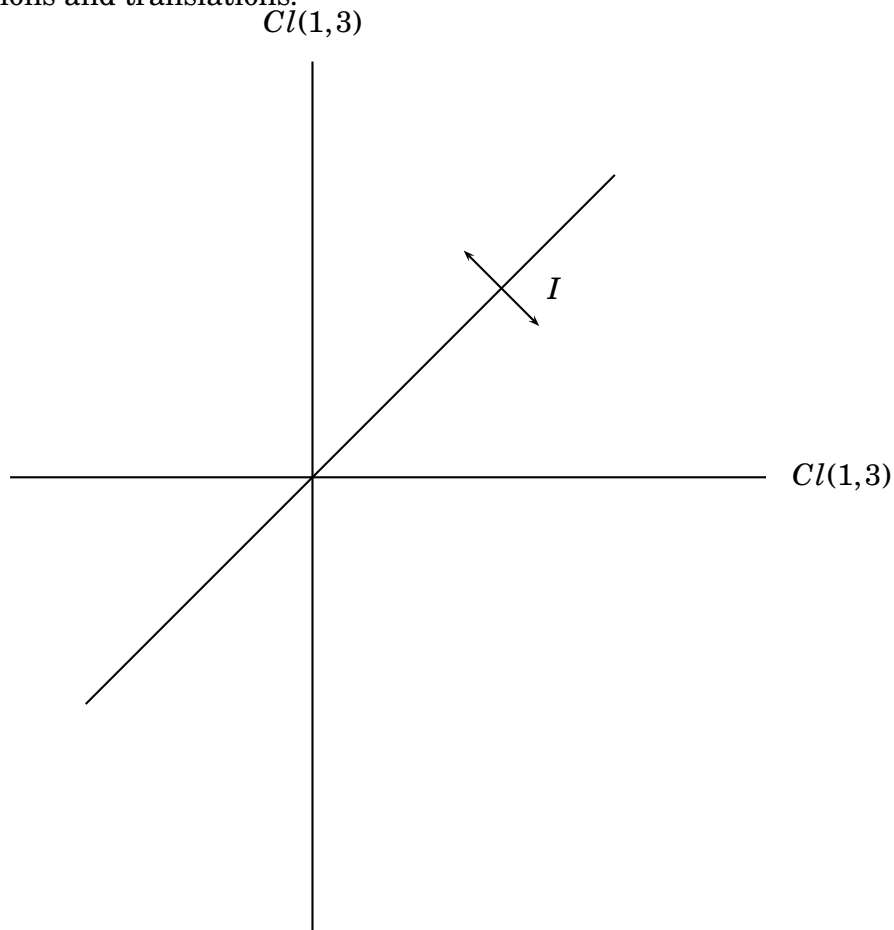
The operator  $D$  defined by  $D(x, y) = \frac{1}{2}(x, y)$  generates the transformations

$$\exp(tD)(x, y) = \left( e^{-\frac{1}{2}t}x, e^{\frac{1}{2}t}y \right)$$

which, in the special cases where either  $x$  or  $y$  is 0, can represent dilatations of 1-vectors in  $R^{1,3}$ . Again, there is no algebraic reason why dilatations cannot be considered on all of  $Cl(1,3)$  or in fact, on all of  $Cl(1,3) \oplus Cl(1,3)$ .

### 3.6.7 Inversions

Although inversions are not part of the connected component of the conformal group (and hence do not appear in the conformal Lie algebra), they are conformal transformations which in our picture, interchange the two component subspaces and thus provide a link between special conformal transformations and translations.



**To summarize:** This provides an alternative linearization procedure for the conformal group of  $R^{p,q}$ . To achieve linearization, we let the conformal transformations act on two copies of the associated Clifford algebra instead of the standard procedure which involves compactifying  $R^{p,q}$ , (where the conformal transformations are then represented by linear transformations in  $R^{p+1,q+1}$ ).

In particular when we consider the conformal transformations of Minkowski space  $R^{1,3}$ , it highlights the geometrical advantages provided by this Clifford algebra approach. Representing the conformal algebra in  $R^{1,3}$  in terms of  $Cl(1,3)$  rather than some larger Clifford algebra, preserves and in fact emphasizes the geometric nature of conformal transformations.

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An extensive review of this topic can be found in:

Kastrup, H. A. On the Advancements of Conformal Transformations and their Associated Symmetries in Geometry and Theoretical Physics arXiv:0808.2730v1.

# Chapter 4

## Spin and group representations

### 4.1 Introduction

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The conventional approach used to describe spin  $\frac{1}{2}$  particles is based on describing the space of Pauli spinors as a complex, two-dimensional, space on which the Pauli spin matrices act as the spin operators. Higher level spin,  $s$ , is similarly defined on complex spaces of dimension  $2s + 1$ . This theory meshes very elegantly with the theory of irreducible representations of the covering group  $\text{spin}(3) = SU(2)$  of the rotation group  $SO(3)$ .

This approach has been extraordinarily successful but, based as it is on complex spaces, lacks any obvious geometric interpretation. An alternate approach to describe spin 1, favoured by a number of authors, uses a Clifford algebra, based on the space  $R^3$ . This idea will be expanded on in the next section. The aim is to develop a *real* theory based on real vector spaces. Such an approach is immediately more geometric and therefore easier to interpret. It has a number of useful consequences, one of which is that it establishes the role of quaternions to describe general rotations in  $R^3$ . The scalar part of the quaternion gives the angle of rotation while the pure part describes the axis and therefore too, the plane of rotation. There is little doubt that such a description is much easier to interpret and handle than the more cumbersome use of matrices based on Euler angles. All in all this

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<sup>1</sup>Group representations - on which the theory of spin is based - is extensively covered in [1]. It is not a book for the fainthearted.

means that the geometry of spin 1 particles is quite easy to describe.

Recall that the groups  $SU(2)$ ,  $Spin(3)$  and  $H^1$  - the group of unit quaternions, are all isomorphic. When working with the Clifford algebra arising from  $R^3$ ,  $H^1$  is the easiest to use.

When it comes to spin  $\frac{1}{2}$ , the situation is not so clear. The Clifford algebra approach to the Dirac equation as proposed by a number of authors, uses  $Cl^+(1,3)$ , the even part of the Clifford algebra  $Cl(1,3)$ , to describe the Dirac spinors. An obvious conclusion from this is that this space should also contain the Pauli spinors. So we have to find a four-dimensional subspace of  $Cl^+(1,3)$  and the only clear candidate is the space of quaternions. The problem with this approach is that quaternions are objects which *describe* rotations. It is not at all obvious that they also describe the objects which are themselves being rotated since these latter objects should at the very least include ordinary (polar) vectors and these are not in  $Cl^+(1,3)$ . In fact it is generally accepted that the pure part of a quaternion should be interpreted as an *axial* vector. (See e.g. [2], chapter 5.4.)

In this chapter we will be especially interested in describing spin  $\frac{1}{2}$  and its relationship with spin 1. The case of spin 0 is relatively trivial and higher spins can usually be understood better in terms of the composition of lower spin particles. We will see this in the case of spin 1 below.

The choice of signature of the metric in  $R^3$  is still a matter of debate and in any case is not that important for  $R^3$ . Since the spacetime signature is usually <sup>2</sup> taken to be  $+, -, -, -$ , we will follow this convention when dealing with  $R^3$  and work with the Clifford algebra  $Cl(0,3)$ .

## 4.2 Description of spin 1

The spin 1 representation of  $SO(3)$  is the usual representation of rotations by  $3 \times 3$  real matrices. As described in Chapter 2, a rotation in  $R^3$  may be considered as equivalent to two successive reflections. Since a reflection is

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<sup>2</sup>Well at least by a reasonable number of authors.

described by

$$x \longrightarrow -axa^{-1}$$

where  $x$  and  $a$  are vectors in  $R^3$ , a rotation can be described by

$$x \longrightarrow (ba)x(ba)^{-1} = qxq^{-1} \quad (4.1)$$

where  $q = ba$  is the product of two vectors and so is of the form  $q = q_0 + q_1e_{23} + q_2e_{31} + q_3e_{12}$ , i.e. a quaternion.

Clearly we can assume that  $q$  has unit norm in which case  $q^{-1} = \tilde{q}$  and the map can be written as  $x \longrightarrow qx\tilde{q}$ . We can write

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} p$$

where  $\theta$  is the angle of rotation and  $p$  is a unit, pure quaternion which describes the plane of rotation.

So geometrically, rotations are described by quaternions, which act on the space  $R^3 = \langle e_1, e_2, e_3 \rangle$  in  $Cl(0, 3)$  according to equation (4.1).

Note that we can trivially extend  $R^3$  to  $Cl^-(0, 3) = \{e_1, e_2, e_3, e_{123}\}$ , the space of all odd elements of  $Cl(0, 3)$ . The map (4.1) extends to a map from  $Cl^-(0, 3) \longrightarrow Cl^-(0, 3)$ . Because  $e_{123}$  commutes with all elements of  $Cl^-(0, 3)$ , under this map  $Cl^-(0, 3)$  splits into two subspaces;  $R^3$  and the one dimensional subspace spanned by  $e_{123}$ . On  $R^3$  the map describes a rotation while on the one-dimensional subspace, the map is the identity. This will become quite relevant when we deal with spin  $\frac{1}{2}$ .

### 4.3 Description of spin $\frac{1}{2}$

Normally to describe the spin  $s$  representation of  $SU(2)$ , we need a complex representation space of dimension  $2s + 1$ . In the case of *integer* spin, a basis for this space can always be chosen so that all the matrices in the representation are *real*. In this case, the space can be regarded as a real vector space of dimension  $2s + 1$ . (So for example, in the case  $s = 1$  above, the representation space can be chosen to be  $R^3$  rather than  $C^3$ .)

This idea does not carry over to half-integer spin  $s$ . In this case, the complex space must be made “twice as large” to become real. So we need a real representation space of dimension  $2 \times (2s + 1)$ . In particular for  $s = \frac{1}{2}$ , the representation space must be four-dimensional real space. What should that space be in the Clifford algebra setting?

We are guided by two ideas. Firstly, two spin  $\frac{1}{2}$  representations can be combined to give a spin 1 representation according to the well known formula

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1. \quad (4.2)$$

This suggests that the spin  $\frac{1}{2}$  representation should also act on vectors, i.e. the representation space for spin  $\frac{1}{2}$  should be a four dimensional space which contains  $R^3$ .

Secondly, in the spin 1 equation,  $x \rightarrow qx\tilde{q}$ ,  $q$  appears twice. Looking at equation (4.2), this indicates that the spin  $\frac{1}{2}$  map should be something like  $x \rightarrow qx$  or  $x \rightarrow x\tilde{q}$ .

If we accept these heuristic arguments, the spin  $\frac{1}{2}$  map should be

$$x \rightarrow qx \quad (\text{or} \quad x \rightarrow x\tilde{q})$$

where  $x$  is in some 4-dimensional real space which contains  $R^3$ . Since  $q$  is an even vector in  $Cl(0,3)$ , the only possible candidate for this space is  $Cl^-(0,3)$ .

So we will take the spin  $\frac{1}{2}$  representation of the group of unit quaternions,  $H^1$ , to be the map

$$\pi_q : x \rightarrow qx, \quad x \in Cl^-(0,3). \quad (4.3)$$

**Exercise 4.1** *Show that this representation is equivalent to the map*

$$\tilde{\pi}_q : x \rightarrow x\tilde{q} \quad (4.4)$$

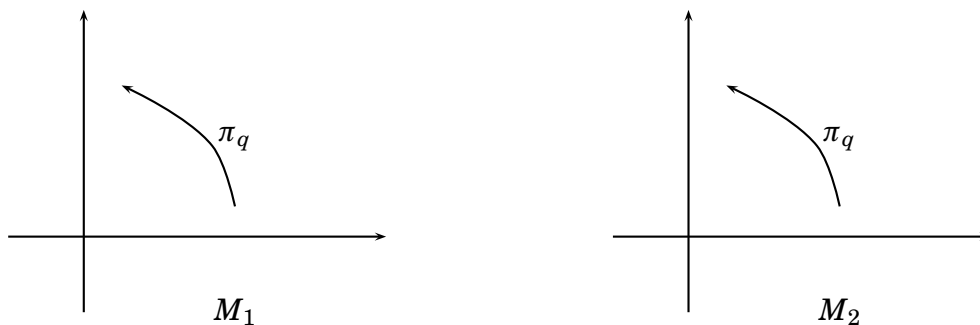
*under the reversion map  $u \rightarrow \tilde{u}$  in  $Cl(0,3)$ . So the choice of representations is not important at this stage (but see the final comment in this section).*

Let us consider some of the geometry of the map  $\pi_q$  given by (4.3).

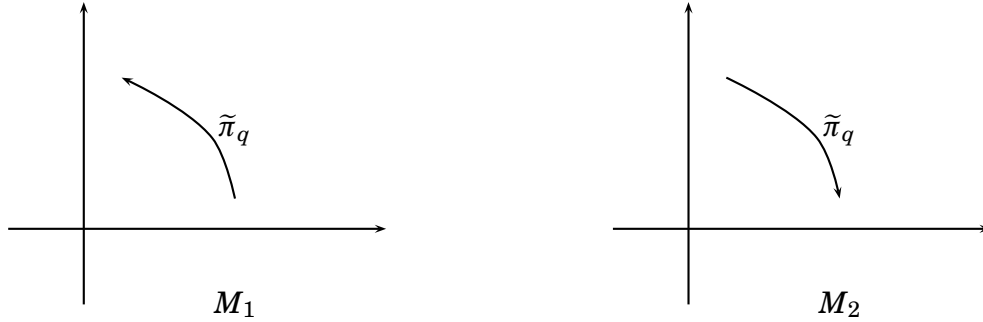


Write again  $q = \cos \frac{\theta}{2} + p \sin \frac{\theta}{2}$  with  $p$  a unit, pure quaternion. Under  $\pi_q$ ,  $Cl^-(0,3)$  splits into two 2-dimensional subspaces. The first,  $M_1$ , is the plane of the rotation (in  $R^3$ ). This plane has a normal vector determined by  $p$ . On  $M_1$ ,  $\pi_q$  acts as a rotation through an angle  $\frac{\theta}{2}$ .

The second subspace  $M_2$ , is the orthogonal complement of  $M_1$  in  $Cl^-(0,3)$ . It is the 2-dimensional subspace spanned by  $pe_{123}$  and  $e_{123}$ . On  $M_2$ ,  $\pi_q$  also acts as a rotation through an angle  $\frac{\theta}{2}$ .



Now consider the map  $\tilde{\pi}_q : x \rightarrow x\tilde{q}$ . A straightforward calculation shows that  $Cl^-(0,3)$  splits into the same two subspaces  $M_1$  and  $M_2$  but  $\tilde{\pi}_q$  acts on these in a slightly different way. On  $M_1$ ,  $\tilde{\pi}_q$  and  $\pi_q$  are the same. But on  $M_2$ ,  $\tilde{\pi}_q$  is the *inverse* rotation to  $\pi_q$  i.e. it rotates vectors in  $M_2$  in the *opposite* direction.



All this makes it easy to interpret the formula

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

geometrically.

The map  $\frac{1}{2} \otimes \frac{1}{2}$  corresponds to the composite map  $\pi_q \tilde{\pi}_q$  (or  $\tilde{\pi}_q \pi_q$ ) :  $x \rightarrow qx\tilde{q}$ . On  $M_1$  the two rotations add to give a net rotation through an angle  $\theta$ . However on  $M_2$ , the rotations cancel to give the identity map. In other words  $\pi_q \tilde{\pi}_q$  splits into the spin 1 map on  $R^3$  plus the identity (spin 0) map on the subspace spanned by  $e_{123}$ .

It is also of some interest to note that the two  $\frac{1}{2}$  maps in (4.2) are slightly different. One refers to  $\pi_q$  and the other to  $\tilde{\pi}_q$ . Perhaps physicists should be writing

$$\frac{1}{2} \otimes \tilde{\frac{1}{2}} = 0 \oplus 1.$$

## 4.4 Higher spin values

We will not be overly interested in higher values of the spin- $s$  representations of  $\text{Spin}(3) = H^1$ . This is largely because the dimension of the spin space very quickly becomes too high for any real geometric ideas to shine through. However we will briefly look at the next two cases  $s = 2$  and  $s = 3/2$ . (We treat them in this order since the real dimensions of their representation

spaces are 5 and 8 respectively.)

Probably the simplest way, for us, to construct high dimensional spin representations is to use the equation<sup>3</sup>

$$s_1 \otimes s_2 = \bigoplus_{s_k=|s_1-s_2|}^{s_1+s_2} s_k \quad (4.5)$$

which describes how the tensor product of two representations can be decomposed into its irreducible components. (Strictly speaking this equation is for complex representations. For real representations, extra copies of the  $s_k$  component sometimes appears.)

#### 4.4.1 Case $s = 2$

Here our starting point is the equation obtained from (4.5)

$$1 \otimes 1 = 0 \oplus 1 \oplus 2 \quad (4.6)$$

which means that the spin 2 representation will be found as one of the components of the tensor product of two spin 1 representations. Since tensor products can profitably be defined in terms of bilinear functionals, this means that we should start with bilinear functionals on the spin 1 representation space, i.e. on  $R^3$ .<sup>4</sup> Furthermore bilinear functionals may themselves be expressed in terms of matrices and some geometric ideas are more easily expressed there (e.g. eigenvalues and determinants).

So we take for the representation space of  $1 \otimes 1$ , the space  $M(3, R)$  of  $3 \times 3$  real matrices. Then equation (4.6) gives for the dimensions of the component subspaces

$$9 = 1 + 3 + 5 \quad (4.7)$$

and we need to identify each of these subspaces in matrix terms.

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<sup>3</sup>See e.g. [3], chapter 4.

<sup>4</sup>Using bilinear functionals avoids the messy construction of tensors based on the tensor products  $e_1 \otimes e_j$  of basis elements.

Note that the tensor product representation in the space  $1 \otimes 1$  (of bilinear functionals at this stage) can be expressed as

$$(\pi_q)(x, y) = f(qx\tilde{q}, qy\tilde{q}) \quad (4.8)$$

$$= f(R_q x, R_q y) \quad (4.9)$$

where  $R_q(x) = qx\tilde{q}$  is the rotation induced by  $q$ .

Writing  $f$  in terms of a matrix  $A$ ,  $f(x, y) = x^T A y$ , we can write the representation  $\pi_q$  on  $M(3, R)$  as

$$\pi_q A = R_q^T A R_q \quad (4.10)$$

It is now easy to describe the three component subspaces in  $1 \otimes 1$ . In matrix terms they are given by

- 0  $\leftrightarrow$  multiples of the identity  $I$
- 1  $\leftrightarrow$  skew matrices
- 2  $\leftrightarrow$  symmetric matrices with trace 0.

**Exercise 4.2** *Show that each of these three subspaces has the correct dimension and are invariant under the group action  $\pi_q$ .*

#### 4.4.2 Case $s = \frac{3}{2}$

For this case we require an 8-dimensional representation for  $H^1$ .

As in the previous case, we can describe this representation, this time starting with the decomposition result

$$1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}. \quad (4.11)$$

So we begin with the tensor product space  $R^3 \otimes Cl^-(0, 3)$ , the carrier space for the representation  $1 \otimes \frac{1}{2}$ . This is the 12-dimensional space of bilinear functionals

$$f : R^3 \times Cl^-(0, 3) \rightarrow R. \quad (4.12)$$

The representation of  $H^1$  is found by combining the two representations on  $R^3$  and  $Cl^-(0, 3)$  respectively. To keep the notation tidy we call all these

representations  $\pi_q$  leaving it to the context to identify which is which. This should not cause problems.

Define therefore

$$(\pi_q f)(x, y) = f(qx\tilde{q}, qy) \quad (4.13)$$

with  $x \in R^3$  and  $y \in Cl^-(0, 3)$ .

From equation (4.11) we see that this representation splits into the sum of two irreducible representations; one ( $s = \frac{1}{2}$ ) will have dimension 4 while the other  $s = \frac{3}{2}$  has dimension 8. This is the one we want to describe.

To do that, we have to find a 4-dimensional subspace of  $R^3 \otimes Cl^-(0, 3)$ , invariant under  $\pi_q$ . This is straightforward. Define  $V$  to be the subspace of bilinear functionals of the form

$$f_\alpha(x, y) = \langle x, y\tilde{\alpha} \rangle, \quad \alpha \in H. \quad (4.14)$$

(Note that both  $x$  and  $y\tilde{\alpha}$  are in  $Cl^-(0, 3)$  so that the inner product is well defined.)

Then a short calculation gives

$$\pi_q f_\alpha = f_{\tilde{q}\alpha} \quad (4.15)$$

so that  $V$  is an invariant subspace corresponding to the (sub)representation  $s = \frac{1}{2}$ .

The orthogonal complement  $V^\perp$  of  $V$  will then serve as the 8-dimensional  $\frac{3}{2}$  representation space for  $H^1$ .

Let us now describe this representation. As in the spin 2 case, it is easier to describe if we replace bilinear functionals by matrices, i.e. we write

$$f(x, y) = x^T A y \quad (4.16)$$

where now  $x \in R^3$ ,  $y \in Cl^-(0, 3)$  and so,  $A$  is a  $3 \times 4$  matrix.

Next, note that in the case above where  $f_\alpha(x, y) = \langle x, y\tilde{\alpha} \rangle$ , we have the following result

**Exercise 4.3**  $f_\alpha(x, y) = x^T A_\alpha y$  where

$$A_\alpha = \begin{bmatrix} \alpha^1 & \alpha^0 & \alpha^3 & -\alpha^2 \\ -\alpha^2 & -\alpha^3 & \alpha^0 & \alpha^1 \\ -\alpha^3 & \alpha^2 & -\alpha^1 & \alpha^0 \end{bmatrix} \quad (4.17)$$

and  $\alpha = \alpha^0 + \alpha^1 i + \alpha^2 j + \alpha^3 k$ .

Also when we regard  $R^3 \otimes Cl^-(0, 3)$  as a space of matrices, the inner product on that space becomes

$$\langle A, B \rangle = \text{Trace}(A^T B)$$

as can easily (though a trifle messily) be verified. So the matrices in the representation space  $V^\perp$  are  $3 \times 4$  matrices  $A$  such that  $A \perp A_\alpha$  for all  $\alpha \in H$ .

Another straightforward exercise shows that these matrices are therefore of the form

$$A = [u | B] \quad (4.18)$$

where  $B = [b_{ij}]$  is  $3 \times 3$  with trace 0, and  $u = [b_{23} - b_{32}, b_{31} - b_{13}, b_{12} - b_{21}]^T$ .

**A final remark:** It is tempting to see into this characterisation of  $3 \times 4$  matrices (given by (4.18)), the idea that the only important matrix is the trace-zero matrix  $B$ , since  $A$  can be derived from  $B$ . These matrices  $B$  constitute an 8-dimensional space and the temptation is to use this space as our representation space.

But this idea should be suppressed since the actual role of this space is in the spin 2 representation as described previously. Furthermore, the group action on the matrices  $B$  is wrong. The correct action on matrices  $A = [u | B]$  is the action

$$\pi_q A = {}_q O_q^T A L_q \quad (4.19)$$

where  $O_q$  is the rotation in  $R^3$  induced by  $q$  and  $L_q$  is left multiplication by  $q$  in  $Cl^-(0, 3)$ .

## 4.5 The spin $\frac{1}{2}$ representation of the Lorentz group

The Dirac equation which we will be looking at in chapter 6, is the fundamental equation which describes relativistic spin  $\frac{1}{2}$  particles. For a proper understanding of the equation, it is important to first understand the irreducible spin  $\frac{1}{2}$  representation of the Lorentz group. Conventionally there are two such representations, denoted as  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  (see e.g. Ryder [3], chapter 2.3). These are representations over complex spaces and as such, are *inequivalent*. However we will show that, regarded as *real* representations, they are now equivalent, and so we are justified in calling the representation, the (real) spin  $\frac{1}{2}$  representation of the Lorentz group.

In this section we will look in some detail at this representation and in particular the roles Pauli and Dirac spinors play. It turns out that if we only want the representation of the (sub)-Lorentz group  $SO(1,3)$  then we need only use Pauli spinors for the representation space. But if we want the representation of the *full* Lorentz group, this space is too small and we have to introduce Dirac spinors. This approach illustrates the fundamental fact that Dirac spinors are composed of two Pauli spinors.

### 4.5.1 The spin $\frac{1}{2}$ representation of $Cl^+(1,3)$

We start by discussing briefly the technique by which the irreducible representations of the Lorentz group are traditionally obtained. To begin with, we should note that these representations are defined in terms of the Lie algebra of the group. The consequence is that only the connected component of the Lorentz group is relevant and extensions of its representations to the whole Lorentz group must be done separately. As we will see, the Clifford algebra approach avoids this (admittedly small) problem.

In the following we will restrict ourselves to the spin  $\frac{1}{2}$  representation as this is where the bulk of the new ideas occur (that is, the ideas of developing a theory which includes both rotations and boosts).

The traditional method is as follows. Start by regarding Lorentz transformations as being made up of rotations and boosts. Rotations as we have seen in chapter 3 form the group  $SO(3)$  (not  $O(3)$  since we want to remain in the connected component of the Lorentz group). This rotation group has the covering group  $SU(2)$ .

Next, boosts are regarded as *imaginary* rotations. Though they do not form a group, their generators may be combined with the generators of the rotation group, in a way as to form two copies of  $SU(2)$ . This of course is just an algebraic trick whose sole virtue (admittedly a large one) is to simplify the mathematical development.<sup>5</sup>

This leads to the two spin  $\frac{1}{2}$  representations of the restricted Lorentz group through its two-fold covering group  $SL(2, C)$ . This representation still acts on  $C^2$ .

However it does not provide a representation for the whole Lorentz group  $O(1, 3)$  since, for example, time reversal is not included. In order to accommodate this transformation, an extension of the representations to  $C^4 = C^2 \oplus C^2$  is necessary.

The representation theory as expressed in Clifford algebra, is different. We still start with  $\text{Spin}(3) = SU(2)$  but this time expressed as  $H^1$ . This acts as we saw in chapter 2, on  $Cl^-(0, 3)$  by left multiplication. It would then be possible to extend this action (still on  $Cl^-(0, 3)$ ) to give a representation of the restricted Lorentz group (see exercise 4.3) but there is another, more natural procedure.<sup>6</sup> This is to define a representation of  $Cl^+(1, 3)$  on the direct sum space  $Cl^-(0, 3) \oplus Cl^-(0, 3)$ . When we come to the study of the Dirac equation, this space will be seen as the natural space of Dirac spinors.

To extend the representation  $\pi_q$  of  $H^1$  on  $Cl^-(0, 3)$ , we need to define the group action  $\pi_e$  (where as usual  $e = e_{0123}$ ). We can do this as follows.

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<sup>5</sup>So we find statements such as “the Lorentz group is essentially  $SU(2) \times SU(2)$ ” which is not merely wrong but misleading - since for example, one group is compact while the other is not. Try explaining this to a bright student.

<sup>6</sup>Natural in the sense that it is coordinate-free.



**Definition 4.1** For  $q_1 + q_2 e \in Cl^+(1, 3)$ , define

$$\pi_{q_1 + q_2 e} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} q_1 \alpha - q_2 \beta \\ q_2 \alpha + q_1 \beta \end{bmatrix} \quad (4.20)$$

with  $\alpha, \beta \in Cl^-(0, 3)$ .

More simply, we extend the definition of  $\pi$  above by defining

$$\pi_e \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}. \quad (4.21)$$

**Exercise 4.4** Show that this definition does lead to a representation of  $Cl^+(1, 3)$ .

Show further that this representation is faithful (i.e. 1-1) by showing that  $\pi_u = I \implies u = 1$ .

Note that restricting  $\pi$  to  $\text{Spin}(1, 3)$ , we obtain a representation of the (covering group of the) restricted Lorentz group  $SO(1, 3)$ .

Before continuing to a representation of the full Lorentz group,  $O(1, 3)$ , let us consider the results so far and contrast them with the conventional representation of the restricted Lorentz group.

The conventional treatment gives *two* spin  $\frac{1}{2}$  representations of  $SL(2, C)$ , denoted by  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  and these representations are *not equivalent*. These lead to two types of spinors (sometimes called “dotted” and “undotted” spinors). In the Clifford algebra setting these correspond to two choices for  $\pi_e$ , namely  $\pi_e = \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . An important deviation from the conventional treatment is that these two representations are now *equivalent*.

For if we define

$$\pi_e^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \pi_e^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

then defining  $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  it is trivial to show that  $U\pi^1 = \pi^2 U$ .

So this distinction between two classes of spinors disappears. This is one of the more notable differences between the two approaches. (If we look back

at chapter 1.2.4,  $U$  is nothing more than time reversal as might have been expected.)

## 4.5.2 The spin $\frac{1}{2}$ representation of $Cl(1,3)$

We can now consider the spin  $\frac{1}{2}$  representation of the full Lorentz group. As in the restricted case, there is just one representation. Again, the easiest path is via a representation of the full Clifford algebra  $Cl(1,3)$ .

The importance of the full Lorentz group is that it contains the Lorentz transformations of time reversal and space inversion. We know from chapter 2 that these transformations are generated respectively by the elements  $e_0$  and  $e_{123}$  acting as generalised inversions. Since  $e_{123} = e_0 e$ , it suffices to extend the representation  $\pi$  to  $\pi_{e_0}$ . But this is easy (again using the idea of time reversal).

**Theorem 4.1** *Let  $\pi$  be the spin  $\frac{1}{2}$  representation of  $Cl^+(1,3)$  on  $Cl^-(0,3) \oplus Cl^-(0,3)$ <sup>7</sup> as defined above. Extend  $\pi$  to all of  $Cl(1,3)$  by defining*

$$\pi_{e_0} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}. \quad (4.22)$$

*Then  $\pi$  is a representation of  $Cl(0,3)$  on  $Cl^-(0,3) \oplus Cl^-(0,3)$ .*

The proof is straightforward. All that needs showing is that  $\pi_{e_0}^2 = I$  and that  $\pi_{e_0} \pi_{e_i} = -\pi_{e_i} \pi_{e_0}$  for  $i = 1, 2, 3$ .

So we now have a representation of the Clifford algebra  $Cl(1,3)$ . Since the full Lorentz group is contained in the algebra, it provides a representation of the group itself.

It is clear now why we gave  $Cl(1,3)$  the matrix representation in chapter 1. This representation is the most natural one for dealing with spin  $\frac{1}{2}$

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<sup>7</sup>It is tempting to identify  $Cl^-(0,3) \oplus Cl^-(0,3)$  with  $Cl^-(1,3)$  which of course can be done as the dimensions agree. But this is misleading.  $Cl^-(0,3)$  contains the Pauli spinors and these are not relativistic objects. Combining them in pairs is still a non-relativistic construction.

objects, in other words, for describing the Dirac equation. This will be seen later in chapter 6.

## 4.6 One-parameter groups

In this section, we discuss some of the 1-parameter groups which are generated by transformations in coordinate space. These groups in turn, induce other 1-parameter groups, this time defined on function spaces. The fact that the first groups are defined on spaces such as  $R^3$  or  $R^{1,3}$  means that results and ideas from Clifford algebra can be used to describe them more geometrically. As is well known, 1-parameter groups are of fundamental importance in physics since invariance under the group action leads to conservation laws.

### 4.6.1 Rotation groups

We will start with one of the simplest examples, rotations in  $R^3$ . For a fixed axis of rotation (or equivalently, a plane of rotation), given by a pure, unit quaternion  $p$  (i.e. an axial vector), rotations in that plane through an angle  $\theta$  can be written in the form (see chapter 2).

$$R_\theta x = qx\tilde{q} \quad (4.23)$$

where  $q$  is the quaternion  $q = \cos \frac{\theta}{2} + p \sin \frac{\theta}{2}$ .

This is a 1-parameter group since  $R_\theta R_\phi = R_{\theta+\phi}$ . We can write this in exponential form since  $q = \exp\left(\frac{\theta p}{2}\right)$ . Then (4.23) becomes

$$R_\theta x = \exp\left(\frac{\theta p}{2}\right) x \exp\left(\frac{-\theta p}{2}\right). \quad (4.24)$$

The generator for this group can be found as follows. For small  $\theta$  we have to order  $O(\theta^2)$

$$\begin{aligned} R_\theta x &= \left(1 + \frac{\theta p}{2}\right) x \left(1 - \frac{\theta p}{2}\right) \\ &= x + \frac{\theta}{2}(px - xp) \end{aligned}$$

so that

$$R'_0(x) = \left. \frac{d}{d\theta} R_\theta x \right|_{\theta=0} = \frac{1}{2}(px - xp).$$

$R'_0$  the generator of the group Using more common notation and recalling that the axis of rotation is part of its definition, we write  $J_p = R'_0$ . It follows that

$$R_\theta x = \exp(\theta J_p) x. \quad (4.25)$$

**Exercise 4.5** Derive (4.25) by showing that  $(R_\theta)$  satisfies the differential equation

$$R'_\theta = R'_0 R_\theta.$$

**Exercise 4.6** In the case of rotations in the  $x^1 - x^2$  plane, choose  $p = e_{12}$  and verify that in this case

$$R'_0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and that

$$R_\theta = \exp(\theta R'_0).$$

(Of course the more usual notation would be to write the generator as  $J_3$  since it generates rotations about the  $x^3$  axis.)

Note that we do not write generators in the form  $iA$  with  $A$  Hermitian, but instead write them as anti-symmetric operators. This is a reminder that there is no need to introduce complex numbers to describe rotations.

Operators on coordinate space induce operators on spaces of functions as shown by the following diagram

$$\begin{array}{ccc} R^3 & \xrightarrow{R} & R^3 \\ & \searrow R^* f & \downarrow f \\ & & Y \end{array}$$

$R^* f$  is the operator induced by  $R$ .

So the 1-parameter group  $(R_\theta)$  induces another 1-parameter group, written  $(U_\theta)$  defined on functions via

$$(U_\theta f)(x) = f(R_\theta x)$$

**Exercise 4.7** Verify the group property  $U_{\theta+\phi} = U_\theta U_\phi$ .

As in the case of the group  $(R_\theta)$  we are interested in the generator for the group  $(U_\theta)$  as well as its corresponding exponential form.

To find the generator, again consider  $U_\theta$  for small values of  $\theta$ . Then to order  $O(\theta^2)$  and using the notation of (4.25),

$$\begin{aligned}(U_\theta f)(x) &= f(R_\theta x) = f(x + \theta J_p x) \\ &= f(x) + \theta J_p x \cdot \nabla f\end{aligned}$$

giving the generator  $A$  of  $(U_\theta)$  as

$$(Af)(x) = J_p x \cdot \nabla f \quad (4.26)$$

As before we then have the exponential form

$$U_\theta = \exp(\theta A) \quad (4.27)$$

**Exercise 4.8** (*Keeping in touch with reality!*) In the case where  $R_\theta$  is a rotation about the  $x^3$  axis we saw above that the generator for  $(R_\theta)$  is

$$R'_0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using equation (4.26) show that the generator for  $(U_\theta)$  is given by

$$(Af)(x) = x^1 \frac{\partial f}{\partial x^2} - x^2 \frac{\partial f}{\partial x^1}.$$

(The more usual notation would be to write  $Af$  as  $J_3 f$  with

$$J_3 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$$

and where the complex  $i$  has been omitted.)

## 4.6.2 Boosts in a specified direction

This next example consists of boosts  $B = B_t$  in a fixed direction. From chapter 2 we know that these can be written as  $B_t : x \rightarrow L_t x L_t^{-1}$  with

$$L_t = \begin{bmatrix} \cosh \frac{t}{2} & -\sinh \frac{t}{2} p \\ \sinh \frac{t}{2} p & \cosh \frac{t}{2} \end{bmatrix}$$

which describes a boost with velocity  $v$  given by

$$\cosh t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and in the direction given by the pure, unit quaternion  $p$  (actually as is seen below, the direction is more accurately specified by  $pe$ ). It is easily verified that this is indeed a 1-parameter group acting on  $R^{1,3}$ .

As in the first example ( $L_t$ ) induces a 1-parameter group (again written as ( $U_t$ )) on the space of functions on  $R^{1,3}$  according to the diagram

$$\begin{array}{ccc} R^{1,3} & \xrightarrow{B_t} & R^{1,3} \\ & \searrow U_t f & \downarrow f \\ & & Y \end{array}$$

Let us now look for the generators and exponential forms of these two groups.

Start with ( $B_t$ ) and work as before. First we have for small  $t$

$$L_t = I + \frac{t}{2} \begin{bmatrix} 0 & -p \\ p & 0 \end{bmatrix} + O(t^2)$$

which gives

$$L'_0 = \frac{1}{2} \begin{bmatrix} 0 & -p \\ p & 0 \end{bmatrix} = \frac{pe}{2}$$

for the generator of ( $L_t$ ). Similarly a small calculation gives

$$B_t x = \left(1 + \frac{t}{2} pe\right) x \left(1 - \frac{1}{2} pe\right) + O(t^2) \quad (4.28)$$

$$= x + \frac{te}{2}(px + xp) + O(t^2) \quad (4.29)$$

(recalling that  $ex = -xe$ )

so that

$$B'_0 x = \frac{e}{2}(px + xp)$$

and  $B'_0$  is the generator for  $(B_t)$ . As in the first example we also have the exponential forms

$$L_t x = \exp\left(\frac{tpe}{2}\right)x \text{ and} \quad (4.30)$$

$$B_t x = \exp\left(\frac{tpe}{2}\right)x \exp\left(-\frac{tpe}{2}\right). \quad (4.31)$$

Again the more common notation would be to write  $K_p = B'_0$ .

Now consider the group  $(U_t)$  on a space of functions. Again for small values of  $t$  we have

$$\begin{aligned} (U_t f)(x) &= f(B_t x) = f(x + tK_p x) + O(t^2) \\ &= f(x) + tK_p x \cdot \nabla f + O(t^2) \end{aligned}$$

giving the generator of  $(U_t)$  as

$$(Af)(x) = K_p x \cdot \nabla f$$

(Note the clear similarity with (4.26) in the case of the rotation group.)

**Exercise 4.9** Consider the case of a boost in the  $x^1$  direction (so that we choose  $p = e_{23}$ ). Show that

$$B'_0 x = x^0 e_1 + x^1 e_2$$

or writing  $B'_0$  as a  $4 \times 4$  matrix

$$B_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Deduce that the generator for  $(U_t)$  can then be written as

$$K_1 f = x^0 \frac{\partial f}{\partial x^1} + x^1 \frac{\partial f}{\partial x^0}.$$

### 4.6.3 Translation groups

For a fixed vector  $a \in R^{1,3}$ , we can define the translation 1-parameter group  $(T_t)$  by

$$T_t : x \rightarrow x + ta.$$

These are affine rather than linear transformations so will not have an exponential form.

These in turn define a translation group on functions as

$$(U_t f)(x) = f(x + ta) \quad (4.32)$$

and since  $f(x + ta) = f(x) + ta \cdot \nabla f + O(t^2)$ , the generator for  $(U_t)$  is given by

$$P f = a \cdot \nabla f. \quad (4.33)$$

The exponential form is therefore

$$U_t = \exp t(a \cdot \nabla). \quad (4.34)$$

**Summary:** to summarize the groups looked at so far; there are three symmetry 1-parameter families of groups defined on function spaces. Their generators may be written (in the more usual notation) as

*Rotations:*

$$J_{23} = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2}, \quad J_{31} = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3}, \quad J_{12} = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$$

*Boosts:*

$$K_{01} = x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0}, \quad K_{02} = x^0 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^0}, \quad K_{03} = x^0 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^0},$$

*Translation:*

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3.$$

Note that the  $P$  and  $J$  generators commute with the energy operator  $H = P_0$  and so have a special place in quantum mechanics. These commutation properties all follow from Clifford algebra ones. For example  $J_{23}$  commutes with  $P_0$  because  $e_{23}e_0 = e_0e_{23}$ .



#### 4.6.4 Two other examples

There are two further examples of 1-parameter groups which are not so commonly studied but which still merit some attention. The first is the *dilatation* group

$$D_t x = e^t x \quad x \in R^{1,3} \quad (4.35)$$

and its associated group on functions

$$(U_t f)(x) = f(e^t x). \quad (4.36)$$

It is straightforward to show that the generator of  $(U_t)$  is

$$(Df)(x) = x \cdot \nabla f(x)$$

and so,  $U_t = e^D$ .

**Exercise 4.10** Show that

$$[D, P_\mu] = -P_\mu.$$

The second example can be considered as a refinement of the rotation group.

Define  $R_\theta$ , this time on the space  $Cl^-(0, 3)$  by

$$R_\theta x = qx \quad \text{where } q = \cos \frac{\theta}{2} + p \sin \frac{\theta}{2} \quad (4.37)$$

for some fixed pure, unit quaternion  $p$ .

(We have already seen in chapter 2 that  $R_\theta$  defines the spin  $\frac{1}{2}$  representation of  $\text{Spin}(3)$ .)

We can write this immediately in exponential form

$$R_\theta x = \exp\left(\frac{\theta p}{2}\right) x$$

giving the group generator as  $\frac{px}{2}$ .

The induced 1-parameter group on functions is

$$(U_\theta f)(x) = f(R\theta x) = f(x) + \theta px \cdot \nabla f + O(\theta^2) \quad (4.38)$$

so that the generator is  $(Af)(x) = \frac{px}{2} \cdot \nabla f$ .

(Recall that when working in  $Cl^-(0,3)$ , and with the usual notation,

$$x \cdot \nabla = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}.$$

**Exercise 4.11** (*Rotations in the  $x^2 - x^3$  plane*). Take  $p = e_{23}$ . Show that in this case

$$\frac{px}{2} \cdot \nabla f = \frac{1}{2} \left( x^2 \frac{\partial f}{\partial x^3} - x^3 \frac{\partial f}{\partial x^2} \right) + \frac{1}{2} \left( x^1 \frac{\partial f}{\partial x^4} - x^4 \frac{\partial f}{\partial x^1} \right).$$

We could just as well have defined  $R_\theta$  by

$$R_\theta x = x\tilde{q} = x \exp - \left( \frac{\theta p}{2} \right) x.$$

In this case the generator for  $(U_\theta)$  would be  $(Af)(x) = -\frac{xp}{2} \cdot \nabla f$ .

**Exercise 4.12** Repeat exercise 4.8 for this case and notice that combining the two rotations (i.e. to obtain  $x \rightarrow qx\tilde{q}$ ) gives the right generator for the rotation group of section 4.6.1.

## 4.7 Representations of the Lorentz and Poincaré group

### 4.7.1 Representations of the Lorentz group

<sup>8</sup> The fundamental problem with finding representations of the Lorentz group, is that the group is not compact. This means that all the nice results for compact groups, such as irreducible representations are all finite dimensional, do not always apply. In particular there are irreducible, infinite dimensional,

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<sup>8</sup>For a fairly comprehensive treatment of the Lorentz group, see [4].

representations. Let us look a little more closely at this problem.

The irreducible representations in fact fall into two categories. First, there are finite dimensional representations but they are not unitary. Secondly, there are unitary representations but they are infinite dimensional. These two categories are quite different and have to be studied separately.

**Some finite dimensional cases.** We will start with the first category because here the irreducible representations are straightforward to describe. Conventionally this is because one uses the “Weyl unitary trick” which goes as follows.

The rotation generators ( $J_i$ ) and the boost generators ( $K_i$ ) (see 4.6.1 and 4.6.2 above) can be combined to define two new sets of generators ( $A_i$ ) and ( $B_i$ ) by

$$A_i = \frac{1}{2}(J_i + iK_i)$$
$$B_i = \frac{1}{2}(J_i - iK_i)$$

and both the families ( $A_i$ ) and ( $B_i$ ) now generate  $SU(2)$  and furthermore, the families *commute*. This means that the Lie algebra of the Lorentz group can be identified with that of  $SU(2) \oplus SU(2)$ . Since the irreducible representations of  $SU(2)$  are indexed by the spin number  $j = 0, \frac{1}{2}, 1, \dots$ , the irreducible representations of the Lorentz group are now indexed by the pairs  $(j, j')$  with  $j, j' = 0, \frac{1}{2}, 1, \dots$ . Just as for  $SU(2)$ , the low dimensional representations can be nicely described in Clifford algebra terms, so too we can describe the corresponding ones for the Lorentz group.

Two important examples are the following.

1.  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  which describe right and left Weyl spinors and
2.  $(\frac{1}{2}, \frac{1}{2})$  - the four vector representation.

Recall that the *real* dimension of the spin  $j$  representation of  $SU(2)$  is

$$\begin{aligned} 2j + 1, & \quad \text{if } j = 0, 1, 2, \dots, \\ 2(2j + 1), & \quad \text{if } j = \frac{1}{2}, \frac{3}{2}, \dots \end{aligned}$$

We therefore see that both the  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  are 4-dimensional while the  $(\frac{1}{2}, \frac{1}{2})$  representation is 8-dimensional.

These representations are easy to describe in Clifford algebra form since the  $j = \frac{1}{2}$  representation of  $H^1 = SU(2)$  is simply left (or right) multiplication.

**The  $(\frac{1}{2}, 0)$  representation.** In the same way that the  $SU(2)$  representation on  $C^2$  can be extended to a representation of  $SL(2)$  still on  $C^2$ , so we can start with the left multiplication representation of  $H^1$  on  $Cl^-(0, 3)$  defined by left multiplication

$$\pi_q : \alpha \rightarrow q\alpha$$

and extend it to a map (still denoted by  $\pi$ )

$$\pi_{q_1+q_2e} : Cl^-(0, 3) \rightarrow Cl^-(0, 3).$$

The question is now, how to define  $\pi_e$ ? Since  $e$  commutes with all elements in  $Cl^+(0, 3)$  (in which lies the Lorentz group  $L_+$ ), we need an operator on  $Cl^-(0, 3)$  which commutes with all operations of left multiplication by quaternions. The only candidate is *right* multiplication and since  $e^2 = -1$ , this must be right multiplication by a *unit, pure* quaternion  $p$ . This leads to the following extension of  $\pi$  to all of  $Cl^+(0, 3)$

$$\pi_{q_1+q_2e} : \alpha \rightarrow q_1\alpha + q_2\alpha p.$$

**Exercise 4.13** *Show that  $\pi$  is a representation.*

2. *Show that this representation too is faithful.*

3. *Show that two different choices of  $p$  lead to equivalent representations. To do this, choose  $p_1, p_2$  two unit, pure quaternions. Call their corresponding representations as defined above,  $\pi^1$  and  $\pi^2$ .*

Since  $p_1, p_2$  correspond to unit vectors in  $R^3$ , either can be rotated to the other, i.e. there exists a unit quaternion  $q$  such that  $qp_1\tilde{q} = p_2$ . Then define the orthogonal map on  $Cl^-(0, 3)$  by  $Ux = x\tilde{q}$  and show that  $U\pi^1 = \pi^2U$ .

The  $(0, \frac{1}{2})$  representation is essentially the same but this time using right multiplication to give firstly the representation of  $H^1$  on  $Cl^-(0, 3)$  via

$$\pi_q : \alpha \rightarrow \alpha\tilde{q}.$$

Again if we fix a pure, unit quaternion  $p$ , then  $\pi$  extends to give the  $(0, \frac{1}{2})$  representation representation of  $Cl^+(1, 3)$  as

$$\pi_{q_1+q_2e} \alpha = \alpha\tilde{q}_1 - p\alpha\tilde{q}_2.$$

(The reason for the negative sign will appear in the theorem below.) As in the previous case, different choices of  $p$  give equivalent representations.

There is a divergence here between the conventional treatment and the Clifford algebra one. In the usual treatment, the two spin 1/2 representations are *not* equivalent. But in the Clifford algebra setting, they *are* equivalent and the isometry which provides this equivalence is the reversion map. More exactly we have the following theorem (whose proof is left to the reader)

**Theorem 4.2** *Let  $p$  be a unit, pure quaternion and write the two representations as*

$$\pi_{q_1+q_2e}^l \alpha = q_1\alpha + q_2\alpha p \tag{4.39}$$

$$\pi_{q_1+q_2e}^r \alpha = \alpha\tilde{q}_1 - p\alpha\tilde{q}_2. \tag{4.40}$$

*Then if  $C$  is the reversion map on  $Cl(0, 3)$ ,*

$$C\pi_{q_1+q_2e}^l = \pi_{q_1+q_2e}^r C. \tag{4.41}$$

Why is there this divergence, i.e. why are the two representations. *inequivalent* in the usual treatment but equivalent in the Clifford algebra one?

The reason is that in the complex case, we require an intertwining operator  $U$  such that  $U\pi^l = \pi^rU$  to be a unitary map and this requires  $U$  to be *complex* linear. In the Clifford setting, we only require it to be real linear.

This is a weaker requirement.

**The  $(\frac{1}{2}, \frac{1}{2})$  representation** This 8-dimensional representation of the Clifford algebra  $Cl^+(1, 3)$  is just the representation of the algebra as  $2 \times 2$  quaternion matrices, as defined in chapter 1. Being  $2 \times 2$  matrices, elements act on vectors

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

with  $\alpha, \beta$  in  $Cl^-(0, 3)$ . Its usefulness will become more apparent when we come to study the Dirac equation.

**Infinite-dimensional cases** (See e.g. [1], chapter 17.)

Turn now to infinite-dimensional representations of the Lorentz group (now written as  $\mathcal{L}$ ). These are usually obtained as *induced* representations of a smaller subgroup - typically the stabiliser subgroup of an element from the set that  $\mathcal{L}$  acts on. The basic ideas are as follows.

Suppose that  $\mathcal{L}$  acts on a set  $X$ . Usually this will be  $R^{1,3}$ . Start with the following two definitions.

**Definition 4.2** Let  $x_0 \in X = R^{1,3}$ . The orbit of  $x_0$  is

$$orb(x_0) = \{Lx_0 : L \in \mathcal{L}\} \quad (4.42)$$

*i.e. all vectors in  $R^{1,3}$  which can be reached from  $x_0$  by a Lorentz transformation.*

**Definition 4.3** The stabilizer of  $x_0$  is the set of group elements which fixes  $x_0$ , *i.e.*

$$\mathcal{L}_{x_0} = \text{Stabilizer}(x_0) = \{L \in \mathcal{L} : Lx_0 = x_0\} \quad (4.43)$$

**Exercise 4.14** The following result is very useful. Suppose that a group  $G$  acts on a space  $X$ , If two elements of  $X$  are in the same orbit, then their stability groups are the same. Show this by showing that if  $gx_0 = x_1$  then  $g^{-1}G_{x_1}g = G_{x_0}$ .

Suppose still that a group  $G$  acts on  $X$ , the action being written as  $x \rightarrow gx$ . Let  $x_0 \in X$  and write  $K$  for the stabilizer subgroup of  $x_0$ . If we define a map

$$\phi : G/K \rightarrow \text{Orb}(x_0), \quad gK \rightarrow gx_0$$

then it is easy to check that  $\phi$  is well defined ( $g_1K = g_2K \implies g_1x_0 = g_2x_0$ ), 1-1 and onto. Furthermore the following diagram is commutative which means that we can identify the group action on  $X$  with the group action on  $G/K$ .

$$\begin{array}{ccc} G/K & \xrightarrow{g} & G/K \\ \downarrow \phi & & \downarrow \phi \\ X & \xrightarrow{g} & X \end{array}$$

So to classify all the stability subgroups of  $\mathcal{L}$ , it suffices to find all the orbits and the stability group of any member of an orbit.

There are six types of orbits (here  $m \geq 0$ ).

- (i)  $O_m^+ = \{x : x^2 = m^2, m > 0, x^0 > 0\}$ ,
- (ii)  $O_m^- = \{x : x^2 = m^2, m > 0, x^0 < 0\}$ ,
- (iii)  $O_{em} = \{x : x^2 = -m^2, m > 0\}$ ,
- (iv)  $O_0^+ = \{x : x^2 = 0, x^0 > 0\}$ ,
- (v)  $O_0^- = \{x : x^2 = 0, x^0 < 0\}$ ,
- (vi)  $O_0^0 = \{0\}$ .

We can consider representations which correspond to mass shells as orbits (examples (i) to (iii) above), or where the orbits are part of the light cone (examples (iv) to (vi)).

To develop the general idea of an induced representation, it suffices to look at case (i).

Fix  $m > 0$  and consider the point  $x_0 = me_0 = (m, 0, 0, 0) \in R^{1,3}$ . Any Lorentz transformation which fixes  $x_0$  must be a rotation in  $R^3$  and so  $\text{Orb}(x_0)$  can be taken to be  $\text{Spin}(3) = H^1$ . Though this is not a very important point, we will often work with the covering groups rather than (in this case) the rotation group  $SO(3)$ . It is quite easy to skip from one to the other and we will do this without further explanation. So elements of  $\mathcal{L}$  as well as points in  $R^{1,3}$  will be written in  $2 \times 2$  quaternion form and the action  $x \rightarrow gx$ , as  $x \rightarrow LxL^{-1}$ .

Finally, again for simplicity, we will only consider the connected component  $\mathcal{L}_+^\uparrow$  of  $\mathcal{L}$ . This keeps the basic ideas simple.

We need what is sometimes called a Borel section, which we can take to be a function  $c$  as defined below.

Suppose that  $G$  acts on  $X$ . Write the stabiliser subgroup of a point  $x_0$  as  $K$ . For each  $x \in X$ , choose  $c(x) \in G$  in such a way that  $c(x)x_0 = x$  and  $c(x_0) = e$  (group identity). Let  $\pi$  be a representation of  $K$  on a vector space  $V$ . Take a space of functions  $\phi : X \rightarrow V$  and define the operator  $U_g$  ( $g \in G$ ) on these functions by

$$(U_g f)(x) = \pi(c(x)^{-1} g c(g^{-1}x)) f(g^{-1}x). \quad (4.44)$$

Then  $g \rightarrow U_g$  is a representation of  $G$ .

(Much has been omitted here! We need a measure on  $X$  and a more precise definition of the space of functions, so as to make it a Hilbert space in which case the representation is now unitary. See [1] for details.)

Now consider the case  $G = \mathcal{L}$ ,  $X = \text{Orb}(me_0)$  and  $K = H^1$ . From the definition of  $X = O_m^+$ , an arbitrary element of  $X$  can be written (uniquely) as

$$x = m \begin{bmatrix} \cosh \phi & \sinh \phi p \\ \sinh \phi p & -\cosh \phi \end{bmatrix}$$

(where  $p$  is a unit pure quaternion and we can assume that  $\phi \geq 0$ .)

So there is a 1 – 1 map between elements of  $X$  and pure boosts  $B_x$  where

$$B_x = \begin{bmatrix} \cosh \frac{\phi}{2} & -\sinh \frac{\phi}{2} p \\ \sinh \frac{\phi}{2} p & \cosh \frac{\phi}{2} \end{bmatrix}$$



and furthermore it is easy to show that

$$B_x x_0 B_x^{-1} = x.$$

We can now define the Borel section function  $c$  defined above by

$$c : x = m \begin{bmatrix} \cosh \phi & \sinh \phi p \\ \sinh \phi p & -\cosh \phi \end{bmatrix} \rightarrow B_x = \begin{bmatrix} \cosh \frac{\phi}{2} & -\sinh \frac{\phi}{2} p \\ \sinh \frac{\phi}{2} p & \cosh \frac{\phi}{2} \end{bmatrix}. \quad (4.45)$$

Having found  $c$ , we now look at equation (4.44) for the induced representation  $g \rightarrow U_g$  of  $\mathcal{L}$ . Note that it suffices to consider the case where  $g$  is a *boost*  $B$ , since every Lorentz transformation  $L$  is of the form  $L = BU$  with  $U$  a rotation, and our assumption is that we already have a representation for the rotation group. From equation (4.44), we see that we first have to calculate  $c(B^{-1}x)$  where  $B$  is a boost.

So let  $B$  be a boost and write  $B' = c(B^{-1}x)$ . Then

$$\begin{aligned} B'x_0 &= B^{-1}x \\ \text{i.e. } BB'x_0 &= x = B_x x_0 \\ \therefore B_x^{-1}BB' &= U \text{ for some } U \in H^1. \end{aligned}$$

So

$$B^{-1}B_x = B'U^{-1}. \quad (4.46)$$

This equation shows that  $U$  arises from writing the product of two boosts as the product of a boost and a (Thomas-Wigner) rotation (as discussed in chapter 2).

It is now straightforward to describe the representation given by equation (4.44). We have

$$\begin{aligned} c(x)^{-1}Bc(B^{-1}x) &= c(x)^{-1}BB' \\ &= c(x)^{-1}B(B^{-1}B_xU) \quad \text{from (4.46)} \\ &= c(x)^{-1}B_xU \\ &= U \quad \text{since } c(x) = B_x. \end{aligned}$$

Since  $U$  corresponds to a unit quaternion  $q$ , we can finally write (4.44) in the form

$$(U_B f)(x) = \pi_q f(B^{-1}x) \quad (4.47)$$

with  $q$  arising as the Wigner angle defined by  $B$  and  $x$ .

For example, consider the spin  $\frac{1}{2}$  representation of  $H^1$  given by left multiplication by (unit) quaternions. Then the induced representation of  $\mathcal{L}$  is given by

$$(U_B f)(x) = qf(B^{-1}x) \quad (4.48)$$

## 4.7.2 The Poincaré group

(See e.g. [1], chapter 17.2.)

In chapter 3.5 we looked at the Poincaré group as it might be represented in the Clifford algebra  $Cl(1, 3)$ , following on from the observation that Lorentz transformations can be expressed in that algebra. Here we will look very briefly at the way group representations can be built up from those of the underlying Lorentz group  $\mathcal{L}$ .

We will restrict the discussion to the proper, orthochronous Lorentz group  $\mathcal{L}$ . This extension of Lorentz representations is actually quite easy as the Poincaré group is the semi-direct product of the additive group  $R^{1,3}$  with  $\mathcal{L}$ . So the only real problem facing us is how to write the irreducible representations of  $R^{1,3}$  in Clifford algebra form.

Classically these representations are of the form

$$a \rightarrow \exp(i \langle a, t \rangle) \quad (4.49)$$

for some fixed  $t \in R^{1,3}$ . Further since we can write an element  $(L, a)$  in the Poincaré group  $\mathcal{P}$  as

$$(L, a) = (I, a)(L, 0)$$

It follows that a representation of  $\mathcal{P}$  will be of the form

$$\pi_{(L,a)}\phi(x) = \exp(i \langle a, t \rangle)\pi_L\phi(x). \quad (4.50)$$

To write this in Clifford algebra form we therefore only need to replace the imaginary  $i$ . Since it is clear that we need an even element which commutes

with all other even elements and which squares to  $-1$ , the only candidate is the pseudoscalar  $e = e_{0123}$ . Apart from that the theory reads exactly as does the conventional theory.

## 4.8 Notes and references

1. Barut, A. O. and Raczka, R. Theory of group representations and applications (1986), World Scientific.
2. Lounesto, P. op. cit.
3. Ryder, L. op. cit.
4. Sibel Bařkal, Young S Kim and Marilyn E Noz, Physics of the Lorentz group, IOP Concise Physics (2015).

# Chapter 5

## Maxwell's equations

### 5.1 Introduction

We are now in a position to consider some of the basic equations in relativistic quantum physics, with a particular interest in the way these shed light on the Dirac equation which will be studied in more detail in the following chapter. Of interest here are Maxwell's equations and the allied Klein-Gordon equation and the Proca equations. Each of these equations or sets of equations, can be expressed very simply in Clifford algebra form and it is this very simplicity which highlights the connections (and differences) between them.

As usual we choose natural units with  $c = \hbar = 1$ .

### 5.2 The Klein-Gordon equation

This is the simplest of the three as it is concerned with real-valued functions only. (The others have vector or spinor arguments.)

First in  $R^{1,3}$ , write as usual the space-time derivatives as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right), \quad (5.1)$$

$$\text{and } \partial^\mu = \left( \frac{\partial}{\partial t}, -\nabla \right). \quad (5.2)$$

With respect to a fixed (orthonormal) basis  $(e_\mu)$  in  $R^{1,3}$ , the four-gradient or Dirac operator is then defined by

$$d = e_0\partial^0 + e_1\partial^1 + e_2\partial^2 + e_3\partial^3 = e_0\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3. \quad (5.3)$$

With the obvious notation it follows that

$$d^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \square \quad \text{the D'Alembertian operator.}$$

The Klein-Gordon equation for a particle of mass  $m$  can then be written as

$$(d^2 + m^2)\psi = 0.$$

**Exercise 5.1** Find conditions on the 1-vector  $k$  in  $R^{1,3}$  for the function

$$\psi(x) = \cos(k \cdot x + \phi)$$

to satisfy the Klein-Gordon equation.

There are problems however with the Klein-Gordon equation when trying to describe single particles. These problems arise as the equation is a *scalar* equation. They are largely overcome by using the Dirac equation as we will see.

## 5.3 Maxwell's equations

There is a number of ways by which Maxwell's equations can be written.<sup>1</sup> In the simplest cases they can be expressed in the form<sup>2</sup>

$$\begin{aligned}\operatorname{div} \underline{E} &= \rho \\ \operatorname{div} \underline{B} &= 0 \\ \operatorname{curl} \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \operatorname{curl} \underline{B} - \frac{\partial \underline{E}}{\partial t} &= \underline{J}.\end{aligned}$$

Here  $\underline{E}$  is the electric field,  $\underline{B}$  the magnetic field and  $\rho, \underline{J}$  are the charge and current densities respectively. The second equation implies the absence of magnetic monopoles.

Ignoring topological issues<sup>3</sup>, the second and third equations imply that there exists a vector potential

$$A = (A^\mu) = (A^0, \underline{A}) \quad (5.4)$$

such that

$$\underline{B} = \operatorname{curl} \underline{A} \quad (5.5)$$

$$\text{and } \underline{E} = -\left(\frac{\partial \underline{A}}{\partial t} + \nabla A^0\right). \quad (5.6)$$

However  $(A^\mu)$  is not uniquely determined by the second and third equations. If  $\chi$  is any scalar function then we can define the *gauge transformation*

$$\underline{A}' = \underline{A} - \nabla \chi, \quad (5.7)$$

$$A^{0'} = A^0 + \frac{\partial \chi}{\partial t} \quad (5.8)$$

---

<sup>1</sup>Maxwell's equations can be concisely expressed in the language of differential forms. (See e.g. [1], chapter 4.6.) In many respects this is very similar to the Clifford algebra technique which we describe here.

<sup>2</sup>Here and in much of what follows, we write vectors in underlined form when it seems more usual and appropriate - even when there is strictly no necessary reason, e.g. if they are considered as elements of a Clifford algebra.

<sup>3</sup>See e.g. [1], chapter 3, if you don't want to ignore them.

and this transformation leaves  $\underline{E}$  and  $\underline{B}$  invariant.

Different choices for  $\chi$  lead to different *gauge* conditions. From the Clifford algebra point of view, the most interesting is the Lorenz condition

$$\partial_0 A^0 + \text{div} \underline{A} = 0. \quad (5.9)$$

The reason for this will appear shortly.

Let us now recast Maxwell's equations in Clifford algebra form.<sup>4</sup>

Begin by defining the electromagnetic field tensor  $F$  as a bivector in  $Cl(1,3)$  as

$$F = E^1 e_{01} + E^2 e_{02} + E^3 e_{03} + B^1 e_{23} + B^2 e_{31} + B^3 e_{12}. \quad (5.10)$$

Then

$$\begin{aligned} dF &= (e_0 \partial_0 - e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3) (E^1 e_{01} + E^2 e_{02} + E^3 e_{03} + B^1 e_{23} + B^2 e_{31} + B^3 e_{12}) \\ &= (-\text{div} \underline{E}) e_0 \\ &\quad + (\partial_0 E^1 - \partial_2 B^3 + \partial_3 B^2) e_1 + \text{cyclic terms} \\ &\quad + (-\text{div} \underline{B}) e_{123} \\ &\quad + (\partial_0 B^1 + \partial_2 E^3 - \partial_3 E^2) e_{023} + \text{cyclic terms.} \end{aligned}$$

Comparing these terms with those in Maxwell's equations leads us to define the (source) 1-vector  $J$

$$J = (J^\mu) = (\rho e_0 + J^1 e_1 + J^2 e_2 + J^3 e_3)$$

and Maxwell's equations become

$$dF = -J. \quad (5.11)$$

Note now that the absence of monopoles ( $\text{div} \underline{B} = 0$ ) follows directly from the assumption that  $J$  or  $dF$  is a 1-vector. (In general  $dF$ , the derivative of a bivector, could have been a 1 + 3 vector.)

---

<sup>4</sup>See Lounesto, [2], chapter 8.

As before, we use the fact that the 3-component parts of  $dF$  are zero, to define the scalar and vector potentials. This means that we should introduce a 1-vector  $A = (A^\mu)$  in which case

$$\begin{aligned} dA &= (e_0\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3)(A^0e_0 + A^1e_1 + A^2e_2 + A^3e_3) \\ &= \partial_0A^0 + \text{div } \underline{A} \\ &\quad + (\partial_0\underline{A} + \nabla A^0)_1e_{01} + \cdots \\ &\quad - (\text{curl } \underline{A})_1e_{23} + \cdots \end{aligned}$$

or by (5.5), (5.6)

$$dA = L - F \quad (5.12)$$

where

$$L = \partial_0A^0 + \text{div } \underline{A}. \quad (5.13)$$

We see now that the Lorenz condition on  $(A^\mu)$  (see (5.9)) is equivalent to the requirement that  $dA$  is a bivector, i.e. has no scalar part.

Assuming the Lorenz condition, Maxwell's equations become

$$F = -dA \quad (5.14)$$

$$dF = -J. \quad (5.15)$$

Since  $F$  is determined by the potential  $A$ , we can write Maxwell's equations in terms of  $A$  (and the source  $J$ ). By equation (5.12),

$$\square A = d^2A = d(dA) = dL - dF = d(\partial^0\phi + \text{div } \underline{A}) + J. \quad (5.16)$$

This shows the usefulness of applying the Lorenz condition on  $(\phi, \underline{A})$  since with this condition, (5.16) now simplifies to

$$\square A = J. \quad (5.17)$$

Notice too how gauge invariance is now trivial to verify. If  $A' = A + d\psi$  for some arbitrary function  $\psi$  then

$$\begin{aligned} dA' &= dA + \square\psi \\ &= (L + \square\psi) - F \end{aligned}$$

which expresses  $dA'$  as a 0 + 2 vector and therefore  $F$  is unchanged.



### 5.3.1 Proca equations

Maxwell's equations describe massless particles (photons) of spin 1. To describe massive, spin 1 particles (e.g. the  $W^\pm$  bosons of weak interaction), we need to use instead, the Proca equations. These are a modification of Maxwell's equations with a mass term added. They are usually expressed as follows.<sup>5</sup>

Begin with a potential 1-vector  $A = (A^\mu)$ . The Proca equations are then

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (5.18)$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \quad (5.19)$$

(5.18) defines the anti-symmetric field tensor  $F$  which as in the Maxwell case is obtained from a 1-potential.

From (5.19) it follows that

$$m^2 \partial_\nu A^\nu = 0 \quad (5.20)$$

and if  $m \neq 0$  this means that  $\partial_\nu A^\nu = 0$ .

Let us now put all this into Clifford algebra form.

Start with a 1-vector  $A$ . Then as in (5.12) we can write

$$F = L - dA \quad (5.21)$$

with  $F$  the tensor field (now a bivector) given by (5.18) and

$$L = \partial_0 A^0 + \text{div} \underline{A}$$

as before.

So now (5.20) tells us that  $L = 0$  in which case the Proca equations become

$$F = -dA \quad (5.22)$$

$$(d^2 + m^2)A = (\square + m^2)A = 0. \quad (5.23)$$

---

<sup>5</sup>See Ryder [3], chapter 2.8.

Note how the presence of the mass term destroys gauge invariance. Unlike Maxwell's equations which are gauge invariant, Proca's equations are not.

(This is to be expected on the alternative argument that  $A$  should have only 3 degrees of freedom as befits a particle of spin 1.)

### 5.3.2 Energy conservation laws

The energy (per unit volume) contained in an electromagnetic field is

$$u = \frac{1}{2}(\underline{E}^2 + \underline{B}^2). \quad (5.24)$$

A theorem due to Poynting gives a relationship between the change in the energy stored in the field and the work which has to be done on charges by the field and energy flow. In differential form it can be expressed as **Poynting's theorem**

$$\frac{\partial}{\partial t} \left( \frac{\underline{E}^2 + \underline{B}^2}{2} \right) + \operatorname{div}(\underline{E} \times \underline{B}) + \underline{J} \cdot \underline{E} = 0. \quad (5.25)$$

The proof is a straightforward application of Maxwell's equations. (See [4], chapter 8.)

Let us see now how a neat proof can be obtained in Clifford algebra. This method also brings out two interesting ideas.

Begin with Maxwell's equations in the form

$$dF = -J.$$

Multiplying by  $F$ ,

$$FdF = -FJ. \quad (5.26)$$

Now consider this equation in component form. A small calculation gives

$$\begin{aligned}
FdF &= \left( -\underline{E} \cdot \frac{\partial \underline{E}}{\partial t} - \underline{B} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{E} \cdot \text{curl} \underline{B} - \underline{B} \cdot \text{curl} \underline{E} \right) e_0 \\
&+ ((\text{div} \underline{E})E_1 + (\text{div} \underline{B})B_1) e_1 + \cdots \\
&+ \left( -\underline{E} \cdot \left( \frac{\partial \underline{B}}{\partial t} + \text{curl} \underline{E} \right) + \underline{B} \cdot \left( \frac{\partial \underline{E}}{\partial t} - \text{curl} \underline{B} \right) \right) e_{123} \\
&+ (\text{div} \underline{B})E_1 - \text{div} \underline{E})B_1) e_{123} \\
&+ ((\text{div} \underline{B})E_1 - (\text{div} \underline{E})B_1) e_{023} + \cdots
\end{aligned}$$

while

$$\begin{aligned}
FJ &= (-\underline{E} \cdot \underline{J}) e_0 \\
&+ (\underline{B} \times \underline{J})_1 e_1 + \cdots \\
&+ \underline{B} \cdot \underline{J} e_{123} \\
&+ (\underline{E} \times \underline{J} + \rho \underline{B})_1 e_{023} + \cdots
\end{aligned}$$

Comparing the  $e_0$  terms gives

$$-\frac{1}{2} \frac{\partial}{\partial t} (\underline{E}^2 + \underline{B}^2) + \underline{E} \cdot \text{curl} \underline{B} - \underline{B} \cdot \text{curl} \underline{E} = \underline{E} \cdot \underline{J} \quad (5.27)$$

and from the vector identity

$$\text{div}(\underline{E} \times \underline{B}) = -\underline{E} \cdot \text{curl} \underline{B} + \underline{B} \cdot \text{curl} \underline{E}$$

we obtain Poynting's theorem, (5.25).

So the Clifford algebra form for Maxwell's equations gives us the energy conservation law very directly. But it gives us more since that law only requires equating the  $e_0$  terms in  $FdF$  and  $-FJ$ . For example equating the other scalar term, i.e. the  $e_{123}$  term, gives

$$\underline{E} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{B} \cdot \frac{\partial \underline{E}}{\partial t} + \underline{E} \cdot \text{curl} \underline{E} + \underline{B} \cdot \text{curl} \underline{B} = \underline{B} \cdot \underline{J} \quad (5.28)$$

although it is not clear just what physical conservation law this describes.

**Exercise 5.2** *In a similar vein, consider the 1-vector and 3-vector terms and find two more "Poynting type" results.*

So this is one advantage that a Clifford algebra approach has: it not only provides a simple algebraic proof of Poynting's theorem but also gives us extra results.

Furthermore, *essentially the same procedure* can be applied to obtain other conservation laws in different circumstances. We shall see this in the next chapter when we consider the Dirac equation. The Dirac equation also has a conservation law, conservation of probability current, and we will show how to obtain this result in an identical manner.<sup>6</sup>

A question arises: can we deduce Maxwell's equations ( $dF = -J$ ) from these conservation laws ( $FdF = -FJ$ )? In other words, is multiplication by  $F$  a reversible process? (For aesthetic reasons it would be nice to deduce field equations from conservation laws.)

When  $F$  is thought of as a tensor, this question does not make immediate sense. Although tensor products can be defined, tensors are not usually interpreted as operators on some vector space (though of course they act that way through interior products) which can be multiplied, i.e. composed, in the way that operators can.

However in Clifford algebra terms,  $F$  is an element of the algebra  $Cl(1,3)$  and the question of whether or not  $F$  has an inverse  $F^{-1}$  (i.e. satisfying  $FF^{-1} = F^{-1}F = I$ ) is perfectly reasonable. It turns out that invertibility or not of the electromagnetic field  $F$  is a very natural property of the field.

Being an element of  $Cl(1,3)$ , we can write the bivector  $F$  as a  $2 \times 2$  quaternion matrix. This simplifies matters considerably as the question now becomes one of matrix inversion. We can then use well-known results in matrix theory which tell us just when a  $2 \times 2$  block matrix has an inverse.

Since  $\underline{E}$  and  $\underline{B}$  are to be represented as (pure) quaternions, we will simplify the notation a little by writing

$$E = E^1 e_{01} + E^2 e_{02} + E^3 e_{03} = -(E^1 e_{23} + E^2 e_{31} + E^3 e_{12})e = -p_E e, \quad (e = e_{0123})$$

---

<sup>6</sup>A common procedure is by way of the adjoint Dirac equation. But this is not needed and is really an unnecessary complication.

and similarly  $B = p_B$ .

Then

$$F = -p_E e + p_B = \begin{bmatrix} p_B & p_E \\ -p_E & p_B \end{bmatrix} \quad (5.29)$$

where  $p_E, p_B$  are pure quaternions.

Assume that  $F \neq 0$ .

Suppose first that  $F$  is singular. We can consider two cases.

Case 1:  $p_E = 0$ . Then  $F = p_B$  and by assumption  $p_B \neq 0$ . But then  $F^{-1} = p_B^{-1} = -\frac{p_B}{p_B^2}$ . So this case cannot arise.

Case 2:  $p_E \neq 0$ . A standard result in matrix theory<sup>7</sup> is that if

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a matrix in block form with  $A$  invertible, then

$$X \text{ is invertible} \iff D - CA^{-1}B \text{ is invertible.}$$

So since  $F$  is singular, we must have

$$p_B + p_E p_B^{-1} p_E = 0 \quad (5.30)$$

so that

$$(p_E p_B^{-1})^2 = -1. \quad (5.31)$$

This means that

$$p_E p_B^{-1} = p \quad (5.32)$$

for some *pure, unit* quaternion  $p$ .

Since the norm on quaternions is multiplicative, this means that

$$\|p_E\| = \|p_B\|$$

---

<sup>7</sup>See e.g. [5], p.44

and since too, we can rescale  $F$  (we are only assuming that  $F$  is non-singular) we can assume that

$$\|p_E\| = \|p_B\| = 1.$$

Since  $p_E, p_B$  are pure, this implies that

$$p_E^2 = p_B^2 = -1.$$

(Note that in terms of  $\underline{E}$  and  $\underline{B}$  this means that  $\underline{E}^2 - \underline{B}^2 = 0$ .)

From equation (5.32), we then have

$$-p_E p_B = p \tag{5.33}$$

and applying reversion,

$$-\widetilde{p}_B \widetilde{p}_E = \widetilde{p} \tag{5.34}$$

or

$$p_B p_E = p. \tag{5.35}$$

Subtracting (5.33) and (5.35) gives

$$p_E p_B + p_B p_E = 0$$

in other words,  $\langle p_E, p_B \rangle = 0$ , or  $\underline{E} \cdot \underline{B} = 0$ .

To summarize: if  $F$  is singular then  $\underline{E}^2 - \underline{B}^2 = 0$ , and  $\underline{E} \cdot \underline{B} = 0$ .

The converse is also true, for if  $\underline{E}^2 - \underline{B}^2 = \underline{E} \cdot \underline{B} = 0$  then

$$F^2 = (-\underline{E}e + \underline{B})^2 = (\underline{E}^2 - \underline{B}^2) - 2\underline{E} \cdot \underline{B} = 0.$$

and so  $F$  cannot have an inverse, i.e.  $F$  is singular.

### 5.3.3 Plane wave solutions of Maxwell's equations

Start with the source-free ( $J = 0$ ) Maxwell equations in the form

$$dF = 0.$$

Then

$$\square F = d^2 F = 0 \quad (5.36)$$

and so  $F$  (i.e. all its components) satisfy the wave equation.

We look for basic plane wave equations, i.e. solutions of the form

$$F = F_0 \cos(k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 + \phi) = F_0 \cos(\omega t - \underline{k} \cdot \underline{x} + \phi) \quad (5.37)$$

where  $F_0, \underline{k}, \omega = k^0, \phi$  are constants, with  $\underline{k}^2 = \omega^2$ , i.e.  $k$  is a null vector in  $R^{1,3}$ .

(When working in Clifford algebra it is often advisable *not* to use an exponential form unless it is clear just what the imaginary  $i$  should be. The reason for this caution is that not only do we need  $i^2 = -1$  but also (preferably at least) that  $i$  should have nice commutation properties with the other elements of interest, which are often bivectors. Furthermore since we will also be interested in the polarization of waves, using the real cosine form is more geometrically natural than the “complex” exponential form.)

From (5.37) we see that

$$\partial_0 F = -\omega F_0 \sin(\omega x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 + \phi)$$

and for  $i = 1, 2, 3$

$$\partial_i F = k_i F_0 \sin(\omega x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 + \phi)$$

which gives

$$dF = -k \sin(\omega x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 + \phi). \quad (5.38)$$

Since  $dF = 0$ , it follows that

$$kF_0 = 0 \quad (5.39)$$

and because  $k$  is null, we can expect non-trivial solutions.

Expanding equation (5.39) and writing

$$F_0 = E_0^1 e_{01} + \cdots + B_0^1 e_{23} + \cdots$$

gives the system of equations

$$\begin{aligned}\underline{k} \cdot \underline{E}_0 &= 0 \\ k_0 \underline{E}_0 + \underline{k} \times \underline{B}_0 &= \underline{0} \\ \underline{k} \cdot \underline{B}_0 &= 0 \\ k_0 \underline{B}_0 - \underline{k} \times \underline{E}_0 &= \underline{0}.\end{aligned}$$

Assuming that  $k \neq 0$ , it then follows that  $\underline{k}, \underline{E}_0, \underline{B}_0$  form an orthogonal system with  $\|\underline{E}_0\| = \|\underline{B}_0\|$  which is right-handed if  $k_0 > 0$  and left-handed if  $k_0 < 0$ .

**Exercise 5.3** *Conversely, suppose that  $\underline{E}$  and  $\underline{B}$  are orthogonal vectors of equal lengths in  $R^3$ . Choose  $\underline{k}$  orthogonal to  $\underline{E}$  and  $\underline{B}$ . Define  $k_0 = \pm \|\underline{k}\|$  according as the system  $\underline{k}, \underline{E}, \underline{B}$  is right- or left-handed. Show that  $kF = 0$ .*

## 5.4 Polarization

From equation (5.39) we see that a single electromagnetic wave is a transverse wave with the electric and magnetic fields orthogonal to each other and to the direction of the wave, determined by  $\underline{k}$ . Such waves are called *plane* or *linearly* polarized. Conventionally we choose the plane of polarization to be the plane determined by  $\underline{E}$  and  $\underline{k}$  (or if we use the term linearly polarized, by the line containing  $\underline{E}$ ).

The physics of polarization becomes more interesting when it comes to dealing with the superposition of two or more waves. For simplicity we will restrict ourselves to the case of two waves, and where the waves are moving in the same direction which we choose to be  $e_3$ , so that we can write  $\underline{k}$  as  $(0, 0, k, \omega)$  in each case.

Since for a single wave its magnetic component is determined by its electric component (because  $k_0 \underline{B}_0 = \underline{k} \times \underline{E}_0$ ), we need only consider how the electric fields of two waves interact.



Consider then two waves with electric fields

$$\underline{E}^1 = E_0^1 \cos(\omega t - kz + \phi^1) \underline{u}^1 \quad (5.40)$$

$$\text{and } \underline{E}^2 = E_0^2 \cos(\omega t - kz + \phi^2) \underline{u}^2 \quad (5.41)$$

where  $E_0^1, E_0^2$  are constants and  $\underline{u}^1, \underline{u}^2$  are (unit) vectors in the  $e_1 - e_2$  plane. A simple calculation shows that in the case  $\underline{u}^2 = \underline{u}^1$ , then  $\underline{E}^1 + \underline{E}^2$  can be written in the form

$$\underline{E}^1 + \underline{E}^2 = A \cos(\omega t - kz + \phi') \underline{u}^1$$

so that the sum of two collinear waves is again linearly polarized.

Assume now that  $\underline{u}^2 \perp \underline{u}^1$ . Furthermore since only relative phases are of importance, we can also assume that  $\phi^1 = 0$  and the waves are now

$$\underline{E}^1 = E_0^1 \cos(\omega t - kz) \underline{u}^1 \quad (5.42)$$

$$\text{and } \underline{E}^2 = E_0^2 \cos(\omega t - kz + \phi) \underline{u}^2, \quad \phi \in [0, 2\pi). \quad (5.43)$$

Conventionally we distinguish three cases.

**Case 1: Linear polarization.** Suppose first that  $\phi = 0$ . In this case

$$\begin{aligned} \underline{E} &= \underline{E}^1 + \underline{E}^2 = (E_0^1 \underline{u}^1 + E_0^2 \underline{u}^2) \cos(\omega t - kz) \\ &= A \cos(\omega t - kz) \underline{v} \end{aligned}$$

for some constant  $A$  and (unit) vector  $\underline{v}$  orthogonal to  $e_3$ . So the sum  $\underline{E}^1 + \underline{E}^2$  is again linearly polarised. The two waves are said to be in phase.

A similar argument occurs when  $\phi = \pi$ . Again the sum is linearly polarized but now we say that the two waves are out of phase.

In each of these two cases, the direction of the electric vector  $\underline{E}$  is fixed.

**Case 2: Circular polarization.** Suppose now that  $\phi = \pi/2$  and that  $E_0^1 = E_0^2$  ( $=E_0$  say.) Then

$$\underline{E} = \underline{E}^1 + \underline{E}^2 = E_0 (\cos(\omega t - kz)\underline{u}^1 - \sin(\omega t - kz)\underline{u}^2)$$

The magnitude of  $\underline{E}$  is constant with  $|\underline{E}| = E_0$  but the direction now varies with time.  $\underline{E}$  now rotates in a circle and we therefore call the polarization *circular polarization*.

**Case 2: Elliptical polarization.** The two cases above are special cases of the general polarization state of the sum of the two waves. The general case is called elliptical polarization and in this case the vector  $\underline{E}$  is a function of time  $t$  and describes an ellipse. Details can be found in [4], chapter 9.

### 5.4.1 Jones vectors and the Clifford algebra $Cl(0,3)$

A different way to represent polarized states due to Jones (see e.g. [6]) is interesting for a number of reasons, one of which is its similarity with the way spin  $\frac{1}{2}$  quantum states can be written. Sakurai [7] discusses this in detail and uses this very similarity to argue why complex spaces are needed for a mathematical basis for quantum mechanics. We will discuss this later and at this stage will confine ourselves to giving a brief outline of how Jones vectors are used to describe polarization states (especially those above), and what these vectors correspond to in a Clifford algebra. Explanations of how and why we make this correspondence, is deferred to chapter 7 where it will be discussed in more detail (though in the context of quantum states).

#### Jones vectors

Consider again a monochromatic plane wave travelling in the  $z$  direction. The electric field of the wave can be written in complex exponential form as

$$\begin{aligned} \underline{E} &= \begin{bmatrix} E_1(t) \\ E_2(t) \end{bmatrix} = \begin{bmatrix} E_{10}e^{i(\omega t - kz + \phi_1)} \\ E_{20}e^{i(\omega t - kz + \phi_2)} \end{bmatrix} \\ &= \begin{bmatrix} E_{10}e^{i\phi_1} \\ E_{20}e^{i\phi_2} \end{bmatrix} e^{i(\omega t - kz)}. \end{aligned}$$

The Jones vector for this state is then defined to be

$$\mathbf{J} = \begin{bmatrix} E_{10}e^{i\phi_1} \\ E_{20}e^{i\phi_2} \end{bmatrix} \quad (5.44)$$

and provides a concise description of the amplitude and phase of  $\underline{E}$  in the  $x$  and  $y$  directions. Because there is some leeway in the way we can describe these properties, the usual method is to assume that  $\mathbf{J}$  is a unit vector in  $C^2$  whose first component is real.

Typical examples of Jones vectors are:

1. Linear polarization in the  $x$  (horizontal) direction.

$$\mathbf{J}_H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (5.45)$$

2. Linear polarization in the  $y$  (vertical) direction.

$$\mathbf{J}_V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (5.46)$$

3. Linear polarization at an angle  $\pm\frac{\pi}{4}$  to the  $x$  axis (diagonal direction).

$$\mathbf{J}_{D\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \quad (5.47)$$

4. Circular polarization.

$$\mathbf{J}_{C\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}. \quad (5.48)$$

(The sign depending on whether we have left or right circular polarization.)

There are also “Jones matrices” which are projection maps in  $C^2$  onto Jones vectors. Since they are wholly determined by the corresponding vector, they do not add very much to the theory. However, very importantly, they remind us that in cases such as these, where vectors can be rescaled and where the (absolute) phase is not important, there are advantages in replacing the vectors with their corresponding 1-dimensional projection in  $C^2$  (or more simply

identifying vectors with lines.)

With this idea in mind, we replace the 2-(complex)-dimensional space  $C^2$  with the four (real) dimensional space  $Cl^-(0,3)$  with orthogonal basis  $\{e_1, e_2, e_3, e_{123}\}$ . Each of the Jones vectors, or more accurately the corresponding Jones (projection) matrix, is then described by a two-dimensional projection onto a subspace of  $Cl^-(0,3)$ . In some cases the projection may look a bit arbitrary but there is method to this and a deeper explanation will be given when we come to describe the Stern-Gerlach experiment later. Suffice to say at this stage that we follow Sakurai [7] where he uses polarization vectors to provide a justification for his description of the Stern-Gerlach spin states.

The Jones vectors as given above, correspond to the following projections (which for the sake of comparison, are given the same symbol.)

1.  $J_H$  is the projection onto the subspace  $\langle e_1, e_2 \rangle$  (i.e. spanned by  $e_1, e_2$ ).

$$J_H = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \quad (5.49)$$

2.  $J_V$  is the projection onto the subspace  $\langle e_3, e_{123} \rangle$ .

$$J_V = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \quad (5.50)$$

3.  $J_{D+}$  is the projection onto the subspace  $\langle \frac{e_1+e_3}{\sqrt{2}}, \frac{e_2+e_{123}}{\sqrt{2}} \rangle$ .

$$J_{D+} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \quad (5.51)$$

4.  $J_{D-}$  is the projection onto the subspace  $\langle \frac{e_1 - e_3}{\sqrt{2}}, \frac{e_2 - e_{123}}{\sqrt{2}} \rangle$ .

$$J_{D-} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \quad (5.52)$$

5.  $J_{C+}$  is the projection onto the subspace  $\langle \frac{e_2 - e_3}{\sqrt{2}}, \frac{e_1 + e_{123}}{\sqrt{2}} \rangle$ .

$$J_{C+} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \quad (5.53)$$

6.  $J_{C-}$  is the projection onto the subspace  $\langle \frac{e_2 + e_3}{\sqrt{2}}, \frac{e_1 - e_{123}}{\sqrt{2}} \rangle$ .

$$J_{C-} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \quad (5.54)$$

## 5.5 Lagrangians

<sup>8</sup> When working with relativistic fields, it is often convenient to start with the Lagrangian function  $L$  which defines the field and then use  $L$  to obtain necessary results. For example, the Euler-Lagrange equations applied to  $L$  yield the defining equations for the field (such as Maxwell's equations for the electromagnetic field). Another important use of the Lagrangian is to verify that local gauge invariance holds. In this case, we need only look at the invariance of  $L$  as all else will follow.

Lagrangians are functions of the fields  $\phi_i$  defined on  $R^{1,3}$ , and their first order partial derivatives  $\partial_\mu \phi_i = \frac{\partial \phi_i}{\partial x^\mu}$ .

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<sup>8</sup>See [8] for a much deeper analysis of Lagrangians in Clifford algebras.

In this section we will confine ourselves to the Klein-Gordon and Maxwell Lagrangians. These are enough to give you an idea of how useful Clifford algebra terminology can be, and especially how compact the ideas are expressed. In the next chapter we will look at the Lagrangian for the Dirac field, where ideas become more interesting.

### 5.5.1 The Klein-Gordon Lagrangian<sup>9</sup>

This is the Lagrangian for a spin 0 field and so there is just one scalar field  $\phi$  and the Lagrangian is a function  $L(\phi, \frac{\partial\phi}{\partial x^\mu})$ . The Euler Lagrange equation is then

$$\partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \phi)} \right) = \frac{\partial L}{\partial \phi}.$$

This case is particularly transparent when we use the notation of Clifford algebras. For a start, since we want  $L$  to be relativistically invariant, the dependence of  $L$  on the partial derivatives of  $\phi$  must in fact be a dependence on  $d\phi$ , So  $L$  must be of the form  $L = L(\phi, d\phi)$ .

The Euler-Lagrange equation now becomes much simpler

$$d \left( \frac{\partial L}{\partial(d\phi)} \right) = \frac{\partial L}{\partial \phi}. \quad (5.55)$$

Since the Klein-Gordon equation is linear, we expect the Lagrangian to be quadratic (in  $\phi$  and  $d\phi$ ). So we are led to the definition of the Klein-Gordon Lagrangian as

$$L = L_{KG}(\phi, d\phi) = \frac{1}{2}(d\phi)^2 - \frac{1}{2}m^2\phi^2. \quad (5.56)$$

(As usual, the factor  $\frac{1}{2}$  is purely conventional and may be omitted if you prefer.)

Then from equation (5.55)

$$d \left( \frac{\partial}{\partial(d\phi)} \left( \frac{1}{2}(d\phi)^2 - \frac{1}{2}m^2\phi^2 \right) \right) = \frac{\partial}{\partial \phi} \left( \frac{1}{2}(d\phi)^2 - \frac{1}{2}m^2\phi^2 \right)$$

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<sup>9</sup>See e.g. [4], chapter 10 for a more detailed introduction to this case.

i.e.

$$d(d\phi) = -m^2\phi$$

giving  $\square\phi + m^2\phi = 0$  and we have the Klein-Gordon equation.

Note how simple the Lagrangian looks when expressed in terms of  $d\phi$  instead of the customary  $\frac{\partial\phi}{\partial x^\mu}$ .

### 5.5.2 The Maxwell Lagrangian

Recall from (5.14) and (5.15) that we can write Maxwell's equations (in vacuum) as

$$F = -dA, \quad dF = -J.$$

Finding the Lagrangian for Maxwell's equations is therefore just a little more tricky as we have two (but dependent) fields involved, namely  $F$  and  $A$ . So it seems sensible to look for a Lagrangian just in terms of  $A$  (whose components at least are independent) and the partial derivatives  $\partial_\mu A^\nu$ .

Again because we want to end up with a linear system, we look for a Lagrangian which is no more than quadratic in  $A$  and  $dA$ . This leads us to try

$$L = L_{EM} = \frac{1}{2}(dA)^2 + \langle J, A \rangle. \quad (5.57)$$

As in the previous case, we have

$$\frac{\partial L}{\partial(dA)} = dA = -F$$

so that

$$d\left(\frac{\partial L}{\partial(dA)}\right) = -dF$$

while

$$\frac{\partial L}{\partial A} = J$$

(see the following exercise.)

So from the Euler-Lagrange equation we obtain  $-dF = J$  which is Maxwell's equation.

**Exercise 5.4** *To verify that  $\frac{\partial L}{\partial A} = J$ , show that (in the more usual notation) if  $f(A) = \langle J, A \rangle$  then  $\nabla f = J$ .*

## 5.6 Notes and references

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# Chapter 6

## The Dirac Equation

### 6.1 Introduction

It is no exaggeration to say that the Dirac equation which describes spin  $\frac{1}{2}$  particles, is the very cornerstone of relativistic quantum mechanics. It is however a *complex* theory with particles described by Dirac spinors which are four-component, complex vectors.

Against that we have the view that, to describe a theory *geometrically*, it should be a real theory, that is, one based on *real* vector spaces. As we saw in Chapter 3, a real theory which describes the spin  $\frac{1}{2}$  ideas can be achieved, based on using the space  $Cl^-(0, 3)$  rather than  $C^2$  to describe Pauli spinors. So it is not unreasonable to hope for an extension of these ideas to describe Dirac spinors.

In a search for a more geometric theory, the pioneering works of Hestenes [1] and others (see e.g. [2]), have demonstrated the important role that Clifford algebras should play in the geometric interpretation of Dirac's basic ideas and results. In particular the algebra  $Cl(1, 3)$  plays a central role.

In his reformulation of the Dirac equation, Hestenes had to find a replacement for the complex number  $i$ . He found this in the bivector element  $e_{12}$  (in our notation;  $\gamma_{12}$  in his) and was then able to construct a more geometric theory of Dirac particles in which complex Dirac spinors were replaced by elements of the *real* 8-dimensional subspace  $Cl^+(1, 3)$  of the Clif-

ford algebra  $Cl(1,3)$ .

Let us firstly look at the traditional physicists' viewpoint and then at Hestenes' approach and some of the more immediate consequences which stem from his approach.

## 6.2 The traditional physicists' viewpoint

In co-ordinate space, the (free particle) Dirac equation may be written as

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad (6.1)$$

(see e.g. Ryder [3]). Here as usual,

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \quad (6.2)$$

and the  $\gamma^\mu$ s are  $4 \times 4$  complex matrices which satisfy the conditions

$$(\gamma^0)^2 = 1, (\gamma^i)^2 = -1, i = 1, 2, 3 \quad (6.3)$$

and the commutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0, \mu \neq \nu. \quad (6.4)$$

The vector  $\psi \in C^4$  is called a Dirac spinor.

## 6.3 Hestenes' (real) Dirac equation

Hestenes [1] reformulated the Dirac equation as a *real* equation in the Clifford algebra  $Cl(1,3)$  (which he calls the space-time algebra, STA). In his notation, the basis vectors are again written as  $\gamma^\mu$ , so as to closely mirror Dirac's notation. Being the basis vectors for  $Cl(1,3)$ , the  $\gamma^\mu$ s will satisfy equations (6.3) and (6.4). The Dirac equation is now written as

$$\gamma^\mu \partial_\mu \psi + im\psi \gamma^0 \quad (6.5)$$

where  $i = \gamma^2 \gamma^1$  and  $\psi \in Cl^+(1,3)$ .

So the 4-complex-dimensional Dirac spinor space is replaced by the *real* 8-dimensional space  $Cl^+(1,3)$ .

Two consequences arise from this approach.

1. The operator  $\gamma^\mu \partial_\mu$  behaves as a 1-vector which means that  $\gamma^\mu \partial_\mu \psi$  is now an *odd* vector and so is no longer a Dirac spinor. (This is why the extra term  $\gamma^0$  appears on the right in equation (6.5)). This is in contrast with the original Dirac theory where  $\gamma^\mu \partial_\mu \psi$  is *still* a spinor (equal to  $-im\psi$ ).

2. A more basic question is: why choose  $Cl^+(1,3)$  to represent spinors? Certainly it is (real) 8-dimensional as required and has the attraction of being closed under multiplication (as opposed to  $Cl^-(1,3)$ ) but this is not a sound *geometric* argument. In fact,  $Cl^-(1,3)$  contains the 1-vectors and it would be expected that any geometric interpretation would most likely start with these vectors.

To understand this basic problem of what geometric meaning we can give to Dirac spinors, it is useful to look at the connection between Dirac and Pauli spinors. As we have already seen, Pauli spinors have an immediate geometric interpretation and this connection might help in the Dirac case.

## 6.4 Dirac and Pauli spinors

The conventional relationship between Pauli and Dirac spinors is elegantly presented in Ryder, [3]. It is based on the representation of the Lorentz group and briefly goes as follows.

Consider first, the restricted Lorentz group; this is the component of the Lorentz group connected to the identity. Alternately it can be defined as the subgroup of transformations which are *orthochronous* (preserve the direction of time) and *proper* (preserve orientation). Its two-fold covering group is  $SL(2, C)$  which therefore acts naturally on Pauli spinors (when regarded as elements of  $C^2$ ).

However, if we include the parity transformation which on  $R^{1,3}$  would be

$$P = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix},$$

this representation space is too small and has to be expanded to  $C^4 = C^2 \oplus C^2$ , i.e. to pairs  $\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  of Pauli spinors.

In a Clifford algebra setting, we have already seen in chapter 3, that  $M = Cl^-(0,3)$  is (geometrically) an appealing choice for the space of Pauli spinors. This means that we should choose  $\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  with  $\xi, \eta \in M$ . Alternately we can write  $\psi = \xi + \eta e$  which is a general element of  $Cl^-(1,3)$ . So Dirac spinors are precisely the odd elements of the Clifford algebra  $Cl(1,3)$ .

It is worth pointing out here that the parity transformation  $P$  given above is now represented as a generalised reflection (see Chapter 2), by the element  $e_{123}$  (or  $\gamma_{123}$  if you prefer.) For if we consider the map

$$P : Cl^-(1,3) \rightarrow Cl^-(1,3), \quad P : x \rightarrow -e_{123}x(e_{123})^{-1}$$

then

$$P(e_0) = e_0, \quad P(e_i) = -e_i, \quad i = 1, 2, 3$$

as required.

In the same way, the map  $T$  which reverses the arrow of time is the Lorentz transformation

$$T = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

which can be represented as a reflection by  $e_0$ .

## 6.5 The Dirac equation

Let us go back to the first issue regarding Hestenes' equation (6.5), which was raised in section 6.3 above. This was that the operator

$$d = e_0\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3$$

interchanges odd and even elements so that its action on a Dirac spinor is no longer a Dirac spinor. Hestenes 'solved' this problem by introducing the extra term  $\gamma_0 = e_0$  in equation (6.5), but there is another, more natural, way around the problem.

Recall from chapter 1, the following representation for  $Cl(1,3)$ .

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (6.6)$$

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}, \quad (6.7)$$

$$e_{23} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad e_{31} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad e_{12} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \quad (6.8)$$

$$e_{01} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad e_{02} = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix}, \quad e_{03} = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}, \quad (6.9)$$

$$e_{023} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_{031} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad e_{012} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}, \quad (6.10)$$

$$e_{123} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad (6.11)$$

$$e_{0123} = e = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (6.12)$$

To this end we will use this representation of  $Cl(1,3)$  as  $2 \times 2$  quaternion matrices and instead define the Dirac  $d$  operator to be

$$\begin{aligned} d &= \partial_0 e_0 - \partial_1 e_1 - \partial_2 e_2 - \partial_3 e_3 \\ &= \partial_0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \partial_1 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \partial_2 \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} - \partial_3 \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \end{aligned}$$

(where again  $\partial_0 = \frac{1}{c} \frac{\partial}{\partial t}$  though we will usually assume that  $c = \hbar = 1$ .)

(If you prefer, write  $\gamma_\mu$  for  $e_\mu$  etc. This will now accord more with conventional notation. On the downside however, it replaces the traditional  $e_\mu$  notation in Clifford algebras, with Dirac's gamma matrix notation. The point is not important; choose whichever notation you prefer.)

In terms of components where we write  $\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  this becomes

$$d\psi = d \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \partial_0 \begin{bmatrix} \xi \\ -\eta \end{bmatrix} - \partial_1 \begin{bmatrix} i\eta \\ i\xi \end{bmatrix} - \partial_2 \begin{bmatrix} j\eta \\ j\xi \end{bmatrix} - \partial_3 \begin{bmatrix} k\eta \\ k\xi \end{bmatrix}. \quad (6.13)$$

Note that  $d^2$  is the d'Alembertian operator

$$d^2 = \square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

It is important to note as well that with this definition of  $d$ , we no longer have an associative law of the form

$$d(\psi\phi) = (d\psi)\phi \quad (\text{where } \phi \text{ is a constant in } Cl(1,3)).$$

In fact it is easily seen that  $d(\psi e) = -(d\psi)e$ . But we still have

$$d(\psi q) = (d\psi)q$$

when  $q$  is a *quaternion*. (Essentially this is because  $q$  commutes with  $e$ .)

**Exercise 6.1** *Derive the above two properties of the  $d$  operator.*

So, for spin  $\frac{1}{2}$  particles, we will take the Dirac equation to be

$$d\psi + m\psi u = 0 \quad (6.14)$$

where  $d$  is defined by (6.13),  $m \geq 0$  and  $u$  is a fixed *unit, pure* quaternion, so that  $u^2 = -1$ . Unless specifically mentioned, we normally assume that  $m \neq 0$ .

Note that as in the classical setting, the spinor  $\psi$  (and its components  $\xi$  and  $\eta$ ) satisfy the Klein-Gordon equation, because

$$\begin{aligned} d^2\psi &= d(d\psi) = -d(m\psi u) = -m(d\psi)u = m^2\psi u^2 = -m^2\psi \\ \text{i.e. } \square\psi + m^2\psi &= 0. \end{aligned}$$

To summarize, the Dirac equation is formulated as

$$d\psi + m\psi u = 0 \tag{6.15}$$

where  $\psi \in Cl^-(1,3)$  and  $u$  is a constant unit pure quaternion. Comparing this with the classical Dirac equation, we see that  $u$  takes the place of  $i$ . Note however that  $u$  does not commute with  $\psi$ . This is an interesting departure from the classical viewpoint.

How important is the choice of  $u$ ? Apart from being a unit, pure, quaternion, the choice is not critical. For if  $u'$  is another unit pure quaternion, there is a *rotation*  $\Lambda$  (which can therefore be regarded as a quaternion) such that

$$\Lambda u \Lambda^{-1} = u'.$$

(Think in terms of vectors. Given any two unit vectors in  $R^3$ , one can always be rotated to the other.)

If we write  $\psi' = \psi \Lambda^{-1}$  then an easy calculation shows that

$$d\psi' + m\psi' u' = 0. \tag{6.16}$$

This means the the choice of  $u$  is to a large extent arbitrary; it is up to individual observers to each choose their own  $u$  in the same way that observers choose their own orthonormal bases. We will return to this point later.

**Exercise 6.2** *Verify equation (6.16).*

Let us now look at a few simple cases.

### 6.5.1 A Dirac particle at rest<sup>1</sup>

Recall that in the standard theory (and with the usual notation), the Dirac particle at rest obeys the equation

$$\gamma^0 p_0 \psi = m \psi$$

where  $\gamma^0$  (in the standard representation) is  $\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$  and  $p_0 = i\partial_t$ . This has the solution

$$\psi = \begin{bmatrix} \psi_P e^{-imt} \\ \psi_N e^{+imt} \end{bmatrix}$$

with  $\psi_P, \psi_N$  constant vectors in  $C^2$ .

So we say that there are two positive and two negative energy solutions. Since  $\psi_P, \psi_N \in C^2$  they describe spin  $\frac{1}{2}$  particles.

In our formulation, start with  $\psi = \psi(t)$  satisfying

$$d\psi + m\psi u = 0, \quad u^2 = -1. \quad (6.17)$$

Since  $(\square + m^2)\psi = 0$  and  $\square = \frac{\partial^2}{\partial t^2}$ , this gives

$$\psi = \psi_0 e^{-umt} + \psi_1 e^{umt} \quad (6.18)$$

with  $\psi_0 = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}$  and  $\psi_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}$  (constant) elements of  $Cl^-(1,3)$ .

Then from (6.18)

$$m\psi u = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} m u e^{-umt} + \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} m u e^{umt}$$

while

$$d\psi = - \begin{bmatrix} \xi_0 \\ -\eta_0 \end{bmatrix} m u e^{-umt} + \begin{bmatrix} \xi_1 \\ -\eta_1 \end{bmatrix} m u e^{umt}.$$

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<sup>1</sup>See e.g. [3], chapter 2.5



Substituting in (6.17) now gives

$$\eta_0 = 0, \quad \xi_1 = 0.$$

So  $\psi$  is of the form

$$\psi = \xi e^{-umt} + (\eta e) e^{umt}$$

where  $\xi, \eta$  are constant elements in the Pauli spinor space  $M = Cl^-(0, 3)$ .

There are therefore two basic solutions of the form

$$\psi_P = \xi \exp(-umt); \quad \psi_N = (\eta e) \exp(umt) \quad (6.19)$$

with  $\xi, \eta \in M$ . These correspond to the classical positive and negative energy solutions respectively. As expected, to interpret spin geometrically, the underlying space  $C^2$  has been replaced by  $M$ .

Note that  $\psi_P$  and  $\psi_N$  belong to *different* subspaces since from (6.19)  $\psi_P \in M$  while  $\psi_N \in Me$ . This distinction is not so clear in the conventional formulation.

The spin  $\frac{1}{2}$  nature of these solutions should be clear. For a fixed choice of  $u$ , the solutions are invariant under left multiplication by elements of  $H^1$  (the unit quaternion group), and so, form a (real) 4-dimensional space invariant under the group action of  $H^1$ . As we remarked in chapter 3, this provides the natural geometric setting for describing the spin  $\frac{1}{2}$  representation of  $SO(3)$ .

## 6.5.2 Plane wave solutions

Let us recall the basic features of a (traveling) plane wave.

A plane wave in  $R^{1,3}$  is described by a function of the form

$$\phi(x) = f(k_0x^0 - k_1x^1 - k_2x^2 - k_3x^3) = f(k_0t - \underline{k} \cdot \underline{x})$$

where

$$x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = x^0 e_0 + \underline{x} \in R^{1,3}$$

and  $f : R \rightarrow Cl(1,3)$  is a function of one variable only.

The vector  $\underline{k}$  gives the direction of propagation (and so is assumed non-zero);  $k_0$  is the (angular) frequency so that  $k_0 = \frac{2\pi}{T}$  where  $T$  is the period. (For that reason we always assume that  $k > 0$ ).  $\|\underline{k}\|$  is the (angular) wave number so that  $\|\underline{k}\| = \frac{2\pi}{\lambda}$  where  $\lambda$  is the wavelength. Finally the speed of the wave is  $\frac{k_0}{\|\underline{k}\|}$ . The profile of the wave is determined by the function  $f$ .

Here we are interested in plane wave solutions of the Dirac equation.

Start with the Dirac equation  $d\psi + m\psi u = 0$  and let

$$k = k_0 e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3 \in R^{1,3}$$

so that  $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$ .

Let  $f : R \rightarrow Cl^-(1,3)$  and look for solutions of the form

$$\psi(x) = f(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3) = f(k_0 t - \underline{k} \cdot \underline{x}).$$

Note that

$$\square\psi = k^2 f''$$

so that  $k^2 = 0$  only gives the trivial solution. Assuming that  $k^2 \neq 0$ , we obtain from the Klein-Gordon equation

$$f'' + \left(\frac{m^2}{k^2}\right) f = 0. \quad (6.20)$$

Now  $k^2 < 0$  gives unbounded solutions so that we may assume that  $k^2 > 0$ , i.e. that  $k$  is time-like. Then from (6.20),  $f$  is a linear combination of the functions  $\exp\left(\pm \frac{um}{\|\underline{k}\|}(k_0 t - \underline{k} \cdot \underline{x})\right)$  which is more conventionally (and familiarly) written as

$$f(t) = \exp \pm u(Et - \underline{p} \cdot \underline{x})$$

where  $E = \frac{mk_0}{\|\underline{k}\|}$  is the energy of the wave and  $\underline{p} = \frac{m\underline{k}}{\|\underline{k}\|} = p^1 e_1 + p^2 e_2 + p^3 e_3$  its momentum. Note that

$$E^2 - \|\underline{p}\|^2 = m^2$$

as required.

Consider first solutions of the form

$$\psi = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} \exp -u(\underline{E}t - \underline{p} \cdot \underline{x})$$

where  $\psi_0 = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} = \xi_0 + \eta_0 e$  is a constant element in  $Cl^{-}(1, 3)$ . (These are the so called “positive” energy solutions for  $u$ . Replacing  $u$  by  $-u$  we obtain the “negative” energy solutions for  $u$ .)

Substituting in the Dirac equation leads to the following linear system which provides a relationship between  $\xi_0$  and  $\eta_0$ .

$$\begin{bmatrix} E - m & p \\ -p & E + m \end{bmatrix} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.21)$$

where we now write (with a slight abuse of notation)  $p = p_1 e_{23} + p_2 e_{31} + p_3 e_{12}$ .

**Exercise 6.3** *Show that the Dirac equation leads to equation (6.21) and verify that this equation has non-trivial solutions by showing that the coefficient matrix has zero determinant.*

Solving (6.21) gives

$$p\xi_0 = (E + m)\eta_0 \text{ or equivalently } p\eta_0 = (m - E)\xi_0 \quad (6.22)$$

so that the general solution can be expressed as

$$\psi = \begin{bmatrix} E + m \\ p \end{bmatrix} \xi \exp -u(\underline{E}t - \underline{p} \cdot \underline{x})$$

with  $\xi \in Cl^{-}(0, 3)$ .

(The fraction  $\frac{p}{E + m}$  should be compared with the term  $\frac{\sigma \cdot p}{E + m}$  which appears in the conventional treatments. See, e.g. Ryder, [3]).

Similarly, the “negative” energy solutions are

$$\psi = \begin{bmatrix} E + m \\ p \end{bmatrix} \xi e \exp +u(Et - \underline{p} \cdot \underline{x})$$

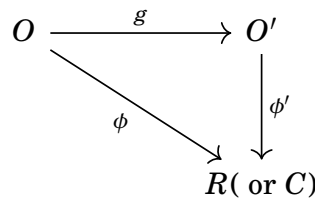
again with  $\xi \in Cl^-(0,3)$ .

Note that in the small momentum limit  $p \rightarrow 0$  we obtain again the solutions (6.19) for a particle at rest.

## 6.6 Invariance of the Dirac Equation

It is a fundamental belief, held by all physicists, that the important equations in physics must be invariant, in the sense that equivalent observers describe *exactly* the same equation (though in their own coordinate system). This belief is regarded as so basic and intuitive that in many instances (perhaps the large majority even) this is rarely even mentioned. The main exception to this, is in books whose main emphasis lies in the physical application of group theory (or symmetries). Such groups include groups of transformations such as Galilean, Lorentz or Poincaré which are necessary for the description of equivalence between observers.

One explanation for this apparent lack of obvious interest, is that in many cases invariance is actually almost trivial to establish. The reason for this is that the unknowns of interest are functions (of space, time or space-time) whose values are *scalars*. In such cases the following diagram shows that invariance is little more than applying the chain rule of differentiation.



In this diagram we have observers  $O$  and  $O'$  whose coordinate frames  $x^\mu$  and  $x'^\nu$  are connected by a transformation  $g$ . The observer  $O$  wants to solve for  $\phi$  (given in terms of a differential equation, say) while  $O'$  is similarly interested in solving for the *same* function, denoted in the primed coordinate system as  $\phi'$ . Equivalence of the two observers requires that  $\phi = \phi' \circ g$ .

Showing the equivalence of the two differential equations for  $\phi$  and  $\phi'$  usually amounts to little more than using the chain rule

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial \phi'}{\partial x'^\nu} \quad (6.23)$$

and noting that  $\frac{\partial x'^\nu}{\partial x^\mu}$  is given by  $g$ .

Equations such as the Klein-Gordon equation and Schrodinger's equation are all instances of this case.

But when it comes to the Dirac equation, the situation is complicated by the fact that spinors are not scalar valued. The appropriate diagram now looks like this.

$$\begin{array}{ccc} R^{1,3} & \xrightarrow{L} & R^{1,3} \\ \downarrow \psi & & \downarrow \psi' \\ V & \xrightarrow{S(L)} & V \end{array}$$

This diagram illustrates the situation where an observer  $O$  is looking for a solution  $\psi$  to the Dirac equation and similarly an observer  $O'$  looks for a corresponding solution  $\psi'$ . Both  $\psi$  and  $\psi'$  are spinors meaning that they are functions defined on  $R^{1,3}$  with values in spinor space denoted (temporarily) by  $V$ .  $L$  is the Lorentz transformation which relates the two observers and their coordinate systems  $x^\mu, x'^\nu$ .

The complication is that now we have to find a map  $S(L) : V \rightarrow V$  to make this diagram commutative, i.e. such that

$$S(L)(\psi) = \psi'(L)$$

and that furthermore with this choice of  $S(L)$ ,  $\psi'$  again satisfies the Dirac equation.

(Though we do not need this result, it is easy to see that the map  $L \rightarrow S(L)$  must define a representation of the Lorentz group on  $V$ .)

In the conventional setting for the Dirac equation, spinors are 4-dimensional complex vectors so we must find a representation of the Lorentz group on

$C^4$ . This can be done, though the construction is not obvious since there is no clear link between the (real) space  $R^{1,3}$  and the (complex) space  $C^4$ . For this reason many authors who discuss the invariance of the Dirac equation, avoid describing this representation and are content with pointing the reader elsewhere for details.

Finally there is one more point where authors are not unanimous. For the observers  $O$  and  $O'$ , the Dirac equation is conventionally written as (with  $\hbar = c = 1$ )

$$i\gamma^\mu \partial_\mu \phi - m\phi = 0, \quad i\gamma'^\mu \partial_\mu \phi' - m\phi' = 0 \quad (6.24)$$

Since all observers should be free to choose their Dirac  $\gamma$  matrices independently, it might be thought that we should not assume that  $\gamma'^\mu = \gamma^\mu$ . However, all systems of  $\gamma$  matrices are unitarily equivalent, i.e. there is a unitary matrix  $U$  such that

$$U^H \gamma^\mu U = \gamma'^\mu \quad \text{for all } \mu.$$

So we will assume that  $\gamma'^\mu = \gamma^\mu$  as  $U$  can be absorbed into the definition of  $S(L)$ .

As we will now see, in the Clifford algebra setting for the Dirac equation this extra complication is entirely absent. The reason for this is that Dirac spinors take values in  $Cl^-(1,3)$  and any Lorentz transformation  $L$  on  $R^{1,3}$  *naturally* induces the transformation  $S(L)$  on  $Cl^-(1,3)$  as follows. Since  $L$  can be written in the form

$$Lx = gxg^{-1}, \quad g \in M_2(H)$$

(see chapter 2), we can choose  $S(L) = g$  (by which we mean left multiplication by  $g$ ). The diagram now becomes

$$\begin{array}{ccc} R^{1,3} & \xrightarrow{L} & R^{1,3} \\ \downarrow \psi & & \downarrow \psi' \\ Cl^-(1,3) & \xrightarrow{g} & Cl^-(1,3) \end{array}$$

We can now verify that the Dirac equation is the same for both observers, i.e. we now show that if

$$\left( e_0 \frac{\partial \psi}{\partial x^0} - e_1 \frac{\partial \psi}{\partial x^1} - e_2 \frac{\partial \psi}{\partial x^2} - e_3 \frac{\partial \psi}{\partial x^3} \right) + m\psi u = 0 \quad (6.25)$$

then

$$\left( e_0 \frac{\partial \psi'}{\partial x'^0} - e_1 \frac{\partial \psi'}{\partial x'^1} - e_2 \frac{\partial \psi'}{\partial x'^2} - e_3 \frac{\partial \psi'}{\partial x'^3} \right) + m \psi' u = 0 \quad (6.26)$$

(Note that in equation (6.26) we do not need a different  $u'$  because we can use the same basis vectors  $e_\mu$  for each observer as discussed above.)

Because there are two sets of indices,  $x^\mu$  and  $x'^\nu$ , we will show this in some detail.

Start with equation (6.25). Write  $x' = Lx$ . Multiplying by  $g$  gives

$$\begin{aligned} -mg\psi u &= (ge_0g^{-1}) \left( \frac{\partial g\psi}{\partial x^0} \right) - (ge_1g^{-1}) \left( \frac{\partial g\psi}{\partial x^1} \right) - \dots \\ \text{or } -m\psi'(x')u &= Le_0 \frac{\partial \psi'}{\partial x^0} - Le_1 \frac{\partial \psi'}{\partial x^1} - \dots \end{aligned}$$

since  $(g\psi)(x) = \psi'(Lx) = \psi'(x')$ .

From the chain rule

$$\frac{\partial \psi'}{\partial x^\mu} = \sum_\nu \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial \psi'}{\partial x'^\nu} = \sum_\nu L_{\nu\mu} \frac{\partial \psi'}{\partial x'^\nu}$$

so that

$$\begin{aligned} -m\psi'(x')u &= Le_0 \sum_\nu L_{\nu 0} \frac{\partial \psi'}{\partial x'^\nu} - Le_1 \sum_\nu L_{\nu 1} \frac{\partial \psi'}{\partial x'^\nu} - \dots \\ &= \sum_\nu (L_{\nu 0}Le_0 - L_{\nu 1}Le_1 - \dots) \frac{\partial \psi'}{\partial x'^\nu}. \end{aligned}$$

Comparing this with equation (6.26), we see that we need to show that

$$L_{00}Le_0 - L_{01}Le_1 - \dots = e_0$$

and for  $i = 1, 2, 3$  that

$$L_{i0}Le_0 - L_{i1}Le_1 - \dots = -e_i.$$

Writing  $Le_\nu$  as a column vector, this is the same as showing that

$$[Le_0 | Le_1 | Le_2 | Le_3] \begin{bmatrix} L_{00} & L_{01} & \dots & \dots \\ -L_{01} & -L_{11} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = [e_0 | -e_1 | -e_2 | -e_3]$$

or more compactly, writing  $\eta = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$  for the Minkowski metric

$$L\eta L^T = \eta$$

and this result holds because  $L^T$  is also a Lorentz transformation.

## 6.7 The prediction of Dirac antiparticles

The classical Dirac equation describes both particles and antiparticles. We see this in the example of a particle at rest, where we have both positive energy ( $e^{-imt}$ ) and negative energy ( $e^{+imt}$ ) solutions. Negative energy particles, initially regarded as an embarrassment, were brilliantly interpreted by Dirac as positive energy, anti-particles.

The Clifford algebra approach gives a different perspective to this idea. When we go from  $-umt$  to  $+umt$ , it is not  $m$  that changes to  $-m$  but  $u$  that changes sign. Geometrically,  $u$  represents an oriented plane in which case  $-u$  represents the same plane but with opposite orientation. One choice (arbitrary no doubt) represents a particle and the other, its anti-particle. There is no need to introduce the idea of negative energy which is confusing at best.

Suppose for example, that we have a solution  $\psi$  of the Dirac equation  $d\psi + m\psi u = 0$ . Then recalling that  $d(\psi e) = -(d\psi)e$ , we have

$$\begin{aligned} d(\psi e) &= -(d\psi)e = -(-m\psi u)e \\ &= m\psi u e = m(\psi e)u \end{aligned}$$

or

$$d(\psi e) + m(\psi e)(-u) = 0$$

so that  $\psi e$  is a solution of the Dirac equation where  $u$  has been replaced by  $-u$ .

One interpretation then, is that the map  $\psi \rightarrow \psi e$  might be seen to convert particles to anti-particles in the conventional setting. (See e.g. Ryder



[3], chapter 11.2.) It would be interesting to follow this up although there are some problems with this idea. Still it's a good example of how Clifford algebra brings up some questions very naturally.

## 6.8 The particle probability density function

Recall that from the classical Dirac equation

$$\gamma^\mu \partial_\mu \psi + im\psi = 0$$

we can define a “current” vector

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

such that the conservation law

$$\partial_\mu j^\mu = 0$$

holds. In particular

$$j^0 = |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2$$

is positive and serves as the probability density for the particle in question.

We will now derive a corresponding current vector and conservation law. This will be expressed in a way which invites comparison with the electromagnetic energy conservation law which results from Maxwell's equations.

Recall that this conservation law can be written (in the source-free case) as

$$\frac{\partial}{\partial t} \left( \frac{\underline{E}^2 + \underline{B}^2}{2} \right) + \text{div} (\underline{E} \times \underline{B}) = 0$$

with  $\frac{\underline{E}^2 + \underline{B}^2}{2}$  the electromagnetic energy density and  $\underline{E} \times \underline{B}$  the electromagnetic flux (Poynting's vector).

Return now to the Dirac equation

$$d\psi + m\psi u = 0, \quad \psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Then multiplying throughout by  $\tilde{\psi}$  (recall that  $\tilde{\psi}$  is the reversion of  $\psi$  as defined in chapter 1)

$$\tilde{\psi}d\psi + m\tilde{\psi}\psi u = 0.$$

Now  $\tilde{\psi}\psi$  is even, invariant under reversion and therefore has no bivector part (i.e. has scalar and pseudoscalar components only). Consequently  $\tilde{\psi}\psi u$  is a pure bivectors so that  $\text{Rl}(m\tilde{\psi}\psi u) = 0$ , and therefore

$$\text{Rl}(\tilde{\psi}d\psi) = 0.$$

Since  $\psi = \xi + \eta e$ , ( $\xi, \eta \in M = Cl^-(0, 3)$ ),

$$\tilde{\psi} = \tilde{\xi} + \tilde{e}\tilde{\eta} = \tilde{\xi} + e\tilde{\eta} = \tilde{\xi} - \tilde{\eta}e.$$

Then

$$\begin{aligned}\tilde{\psi}d\psi &= (\tilde{\xi} - \tilde{\eta}e)(\partial_0(\xi - \eta e) - e_{23}\partial_1(\eta + \xi e) - e_{31}\partial_2(\eta + \xi e) - e_{12}\partial_3(\eta + \xi e)) \\ &= (\tilde{\xi}\partial_0\xi + \tilde{\eta}\partial_0\eta) \\ &\quad + (\tilde{\eta}\partial_0\xi - \tilde{\xi}\partial_0\eta)e \\ &\quad + (\tilde{\eta}e_{23}\partial_1\xi - \tilde{\xi}e_{23}\partial_1\eta) + \text{cyclic terms} \\ &\quad - (\tilde{\xi}e_{23}\partial_1\xi + \tilde{\eta}e_{23}\partial_1\eta)e - \text{cyclic terms}.\end{aligned}$$

The second and fourth terms have no real parts so that

$$\text{Rl}(\tilde{\psi}d\psi) = \text{Rl}((\tilde{\xi}\partial_0\xi + \tilde{\eta}\partial_0\eta) + (\tilde{\eta}e_{23}\partial_1\xi - \tilde{\xi}e_{23}\partial_1\eta) + \text{cyclic terms}).$$

Now write

$$\xi = \xi_0 e_{123} + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 = \xi_0 e_{123} + \underline{\xi}$$

so that

$$\tilde{\xi} = -\xi_0 e_{123} + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 = -\xi_0 e_{123} + \underline{\xi}$$

and similarly for  $\eta$ . Then after a little algebra we obtain

$$\text{Rl}(\tilde{\psi}d\psi) = -\left(\partial_0\left(\frac{\|\xi\|^2 + \|\eta\|^2}{2}\right) + \text{div}\left(\underline{\xi} \times \underline{\eta} + \eta_0 \underline{\xi} - \xi_0 \underline{\eta}\right)\right)$$

where  $\|\xi\|^2 = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2$  and similarly for  $\|\eta\|^2$ .

So we have the conservation law for the Dirac equation in a form reminiscent of Maxwell's energy conservation law

$$\partial_0 \left( \frac{\|\xi\|^2 + \|\eta\|^2}{2} \right) + \operatorname{div} \left( \underline{\xi} \times \underline{\eta} + \eta_0 \underline{\xi} - \xi_0 \underline{\eta} \right) = 0.$$

This also illustrates that the term  $\|\xi\|^2 + \|\eta\|^2$  corresponds to  $j^0 = |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2$  in the usual formulation.

As in the case of the energy conservation law for Maxwell's equations, we can ask whether the Dirac equation can be re-obtained from the conservation law

$$\tilde{\psi} d\psi + m\tilde{\psi}\psi u = 0$$

which is equivalent to asking whether left multiplication by  $\tilde{\psi}$  is invertible. Again we explore this idea using the representation of  $Cl(1, 3)$  as  $2 \times 2$  quaternion matrices, given in chapter 1 (equations (1.9) - (1.15)).

This representation gives the quaternion matrices

$$\tilde{\xi} = -\xi_0 e_{123} + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 = \begin{bmatrix} 0 & q_1 \\ q_1 & 0 \end{bmatrix}$$

where  $q_1 = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$ .

Similarly

$$\tilde{\eta} e = \begin{bmatrix} q_2 & 0 \\ 0 & -q_2 \end{bmatrix}$$

where  $q_2 = \eta_0 + \eta_1 i + \eta_2 j + \eta_3 k$ .

So the question becomes whether the matrix

$$X = \begin{bmatrix} q_2 & q_1 \\ q_1 & -q_2 \end{bmatrix}$$

is invertible. (At this stage, the reader should compare the problem with the corresponding one for Maxwell's equations. The similarity is striking.)

Assume that  $X$  is singular, i.e. not invertible. There are two case.

**Case 1:**  $q_2 = 0$ . Then  $X = \begin{bmatrix} 0 & q_1 \\ q_1 & 0 \end{bmatrix}$  and if  $q_1 \neq 0$ , we have

$X^{-1} = \begin{bmatrix} 0 & q_1^{-1} \\ q_1^{-1} & 0 \end{bmatrix}$  which contradicts the assumption that  $X$  is singular.

So  $q_1 = 0$  which means that  $\psi = 0$ .

Since we may assume that  $\psi \neq 0$ , case 1 cannot occur.

**Case 2:**  $q_2 \neq 0$ . Recall (see chapter 5.3) that if

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with  $A$  invertible, then  $X$  is invertible iff  $D - CA^{-1}B$  is invertible.

So here  $X$  is singular iff  $q_2 + q_1 q_2^{-1} q_1 = 0$ .

This condition is equivalent to the requirement that  $(q_1^{-1} q_2)^2 = -1$ , which means that

$$q_1^{-1} q_2 = p$$

for some *unit, pure* quaternion  $p$ . From this we deduce that  $\|q_1\| = \|q_2\|$  which we may also assume by normalization, to be 1. Likewise we also have  $RI(\widetilde{q_1} q_2) = 0$ . In terms now of  $\xi$  and  $\eta$  we can summarize this argument by saying the following.

Let  $\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ . The operation of left multiplication by  $\widetilde{\psi}$  is *not* invertible precisely when

$$\|\xi\| = \|\eta\| \quad \text{and} \quad \xi \perp \eta.$$

So we have another striking similarity (the first was the energy conservation law) between the Dirac field and the electromagnetic one. In electromagnetism, the field tensor  $F = (\underline{E}, \underline{B})$  is null if  $\underline{E}^2 = \underline{B}^2$  and  $\underline{E} \cdot \underline{B} = 0$ . Similarly in the Dirac field we can call a Dirac spinor  $\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  *null* if  $\|\xi\| = \|\eta\|$  and  $\xi \perp \eta$ . Clearly this condition is Lorentz invariant, i.e. all observers will agree on whether the field is null or not.

## 6.9 The massless case - Weyl equations

Consider the Dirac equation with  $m = 0$ . We look for solutions corresponding to Weyl spinors in  $C^2$ , i.e. with four real degrees of freedom. As we will see, these solutions are in the Pauli space  $M = Cl^-(0,3)$  and describe Weyl fermions with different “handedness”.

We start with the equation

$$d\psi = 0 \tag{6.27}$$

where  $\psi = \xi + \eta e$ ,  $\xi, \eta \in M$ . Note that  $u$  no longer appears in the equation.

Then

$$0 = \partial_0 \begin{bmatrix} \xi \\ -\eta \end{bmatrix} - (e_{23}\partial_1 - e_{31}\partial_2 - e_{12}\partial_3) \begin{bmatrix} \eta \\ \xi \end{bmatrix}.$$

Define  $\partial_s$  (the space derivative) to be

$$\partial_s = e_{23}\partial_1 + e_{31}\partial_2 + e_{12}\partial_3$$

so that

$$\partial_0\xi - \partial_s\eta = 0 \tag{6.28}$$

$$\partial_0\eta + \partial_s\xi = 0. \tag{6.29}$$

Let  $u$  be a unit pure quaternion (but note that  $u$  is no longer connected to the Dirac equation -  $u$  is chosen by the observer) and define the operators  $\partial_{\pm}$  by

$$\partial_{\pm}\phi = \partial_0\phi \pm \partial_s\phi u.$$

Let

$$\psi_+ = \xi + \eta u, \quad \psi_- = \xi - \eta u.$$

Then it follows from (6.28), (6.29) that

$$\begin{aligned} \partial_+\psi_+ &= \partial_0(\xi + \eta u) + \partial_s(\xi + \eta u)u \\ &= (\partial_0\xi - \partial_s\eta) + (\partial_0\eta + \partial_s\xi)u \\ &= 0 \end{aligned}$$

and similarly

$$\begin{aligned}\partial_- \psi_- &= \partial_0(\xi - \eta u) - \partial_s(\xi - \eta u)u \\ &= (\partial_0 \xi - \partial_s \eta) - (\partial_0 \eta + \partial_s \xi)u \\ &= 0.\end{aligned}$$

We can call these equations,

$$\partial_+ \psi_+ = 0, \quad \partial_- \psi_- = 0 \quad (6.30)$$

the Weyl equations and  $\psi_{\pm}$  the corresponding Weyl spinor solutions. Note that  $\psi_{\pm} \in M$ . They behave like Pauli spinors.

*Note:* 1.  $u$  is an arbitrary, pure, unit quaternion. It doesn't (at this stage) need to be related to  $u$  in the Dirac equation (which in any case doesn't appear in the Dirac equation when  $m = 0$ .)

2. Under the map  $u \rightarrow \tilde{u} = -u$ , the Weyl spinors  $\psi_{\pm}$  are exchanged.

3. Consider the first Weyl equation  $\partial_+ \psi_+ = 0$ . If we give the (energy - momentum operator) interpretation

$$p_0 = u\partial_0, \quad p_i = -u\partial_i, \quad i = 1, 2, 3$$

the Weyl equation becomes

$$(p_0 - p_1 e_{23} - \dots)\psi_+ = 0$$

which means that  $\psi_+$  can be interpreted as a right handed spinor (see Ryder, [3], equation 2.95). Similarly  $\psi_-$  is a left handed spinor.

**Exercise 6.4** *The derivation of the Weyl equations (6.24) and (6.25) can be generalized to the general Dirac equation and gives the so-called Weyl form of the Dirac equation.*

$$\partial_+ \psi_+ = -m\psi_- u \quad (6.31)$$

$$\partial_- \psi_- = -m\psi_+ u \quad (6.32)$$

so that  $\psi_+$  and  $\psi_-$  are coupled only through their mass terms.

### 6.9.1 Plane wave solutions of the Weyl equations

Consider the massless Dirac (or Weyl) equation  $d\psi = 0$ . We look for solutions of the form

$$\psi(x) = \psi_0 f(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3) \quad (6.33)$$

with  $\psi_0$  a constant (Dirac) spinor.

Since  $\square\phi = 0$ , we must have  $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2 = 0$   
i.e.  $k = k_0 e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3$  is a null vector.

Choose  $u$ , a unit, pure quaternion (again different observers will choose  $u$  differently). Let us look for basic solutions of the form

$$\psi(x) = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} \exp^{-u(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3)}$$

with  $k^2 = 0$ . A short calculation gives

$$\begin{aligned} 0 = d\psi = & \begin{bmatrix} \xi_0 \\ -\eta_0 \end{bmatrix} (-k_0) \exp^{-u(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3)} \\ & - e_{23} \begin{bmatrix} \eta_0 \\ \xi_0 \end{bmatrix} (k_1) \exp^{-u(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3)} - \dots \end{aligned}$$

which leads to the equations

$$k_0 \xi_0 + (k_1 e_{23} + \dots) \eta_0 = 0 \quad (6.34)$$

$$k_0 \eta_0 - (k_1 e_{23} + \dots) \xi_0 = 0 \quad (6.35)$$

(though these equations are not independent since  $k$  is null).

Writing  $p = k_1 e_{23} + k_2 e_{31} + k_3 e_{12}$ , a pure quaternion, gives (after a simplifying normalisation)

$$\psi(x) = \begin{bmatrix} k_0 \xi_0 \\ p \xi_0 \end{bmatrix} \exp^{-u(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3)} \quad (6.36)$$

$$= (k_0 - p e) \xi_0 \exp^{-u(k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3)} \quad (6.37)$$

where  $\xi_0$  is an arbitrary constant odd vector in  $M$ .

Note that for a fixed  $k$ , the solutions are determined by a real 4-dimensional subspace  $(k_0 + pe)\xi_0$  with  $\xi_0 \in M$ . So Weyl spinors, like Pauli spinors, are determined by the 4-dimensional space  $M = Cl^-(0,3)$  (in the conventional treatment, this is  $C^2$ ).

From (6.30), we find that the corresponding solutions of the Weyl equations are

$$\psi_{\pm}(x) = (k_0\xi_0 \pm p\xi_0u) \exp^{-u(k_0x_0 - k_1x_1 - k_2x_2 - k_3x_3)}. \quad (6.38)$$

These Weyl spinors *are* in the Pauli spin space  $M = Cl^-(0,3)$ .

## 6.10 The Dirac Lagrangian

Let us begin by recalling the Dirac Lagrangian in the conventional treatment of the Dirac equation given as (see e.g. [4], chapter 10.2)

$$\gamma^\mu \partial_\mu \psi + im\psi = 0.$$

The Lagrangian is defined to be

$$L = \bar{\psi} (\gamma^\mu \partial_\mu \psi + im\psi) \quad (6.39)$$

where  $\psi, \bar{\psi}$  are treated as independent field variables. We then obtain trivially,

$$\frac{\partial L}{\partial(\partial_\mu \bar{\psi})} = 0, \quad \frac{\partial L}{\partial \bar{\psi}} = \gamma^\mu \partial_\mu \psi + im\psi.$$

Applying the Euler-Lagrange equation (to  $\bar{\psi}$ ), i.e.

$$\partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \right) = \frac{\partial L}{\partial \bar{\psi}}$$

gives

$$\gamma^\mu \partial_\mu \psi + im\psi = 0$$

i.e. the Dirac equation.

All this is rather trivial (and hardly seems worth the effort of defining the Lagrangian) except that, applying the Euler-Lagrange equation to  $\psi$  gives the adjoint Dirac equation which in conventional form, is

$$\partial_\mu \bar{\psi} \gamma^\mu - im\bar{\psi} = 0.$$



In Clifford terms we can define the Dirac Lagrangian as

$$L = (d\psi + m\psi u)\tilde{\psi}. \quad (6.40)$$

(Note that there is no guarantee that  $L$  is real-valued. All we can say is that it is an even element in  $Cl^+(1,3)$ . Of course (6.39) also defines, in general, a complex Lagrangian so this may not be an important issue.)

The order of the terms in (6.40) is important as these may not commute (we will see shortly why we chose this particular order). As above, we regard  $\psi$  and  $\tilde{\psi}$  as independent variables.

Applying the Euler-Lagrange equation to  $\tilde{\psi}$ , in the form we used in (5.55) in the case of Maxwell's equations,

$$d\left(\frac{\partial L}{\partial(d\tilde{\psi})}\right) = \frac{\partial L}{\partial\tilde{\psi}} \quad (6.41)$$

gives

$$0 = d\psi + m\psi u$$

i.e. the Dirac equation as expected.

### 6.10.1 Gauge invariance of the Lagrangian

Consider the *global* phase transformation

$$\psi \rightarrow \psi' = \psi e^{-\theta u} \quad (6.42)$$

where  $\theta$  is a real constant. Then  $d\psi' = (d\psi)e^{-\theta u}$  and defining the transformed Lagrangian as

$$L' = (d\psi' + m\psi' u)\tilde{\psi}'$$

we find

$$L' = \left((d\psi)e^{-\theta u} + m\psi e^{-\theta u} u\right) e^{+\theta u} \tilde{\psi} = L. \quad (6.43)$$

So  $L$  is invariant under this (global) phase transform.

(It is here that we see why the order of terms in (6.40) was important.)

Now consider *local* phase transformations

$$\psi' = \psi e^{-\theta(x)u}, \quad x \in R^{1,3}.$$

Let us first look at the conventional way a local phase transformation introduces an extra field if we want the Lagrangian to remain invariant. We will follow the treatment given by Griffiths, ([4], chapter 10.3).

Here the local gauge transformation on  $\psi$  is

$$\psi' = \psi e^{-i\theta(x)}, \quad x \in R^{1,3}$$

which after differentiating, gives

$$\partial_\mu \psi' = \partial_\mu (\psi e^{-i\theta}) = e^{-i\theta} \partial_\mu \psi - i(\partial_\mu \theta) e^{-i\theta} \psi \quad (6.44)$$

and this extra term means that the Dirac Lagrangian (defined in (6.39)) changes since

$$\begin{aligned} L \rightarrow L' &= \bar{\psi}' (\gamma^\mu \partial_\mu \psi' + im\psi') \\ &= e^{i\theta} \bar{\psi} \left( \gamma^\mu e^{-i\theta} \partial_\mu \psi - i e^{-i\theta} \gamma^\mu (\partial_\mu \theta) + im e^{-i\theta} \psi \right) \\ &= \bar{\psi} (\gamma^\mu \partial_\mu \psi + im\psi - i \gamma^\mu (\partial_\mu \theta) \psi) \\ &= L - i \bar{\psi} \gamma^\mu (\partial_\mu \theta) \psi. \end{aligned}$$

So a problem arises if we require that  $L$  should be invariant under a local phase transformation. To obtain invariance we need to add an extra term to  $L$ . This leads to the introduction of the 1-vector  $A = (A_\mu)$  and a new Lagrangian

$$L = \bar{\psi} (\gamma^\mu \partial_\mu \psi + im\psi) - iq(\bar{\psi} \gamma^\mu \psi) A_\mu \quad (6.45)$$

where  $A_\mu$  is the necessary new field and changes under the local phase transformation to

$$A'_\mu = A_\mu - \frac{1}{q} \partial_\mu \theta. \quad (6.46)$$

However even this Lagrangian doesn't quite suffice since having to include  $(A_\mu)$  it must also include a 'free' term for  $(A_\mu)$  itself. This leads to the complete Lagrangian

$$L = \bar{\psi} (\gamma^\mu \partial_\mu \psi + im\psi) - i \left( q(\bar{\psi} \gamma^\mu \psi) A_\mu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad (6.47)$$

where as for Maxwell's equations,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

How do we see all this in Clifford algebra terms?

From equation (6.40), we start with  $L = (d\psi + m\psi u)\tilde{\psi}$ . Then putting  $\psi' = \psi e^{-\theta(x)u}$ , we first need to calculate  $d\psi'$ . We will show this in detail as it is not obvious at first sight what  $d\theta$  should be (as it has to be in matrix form).

Write

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \text{ so that } \psi' = \begin{bmatrix} \xi e^{-\theta u} \\ \eta e^{-\theta u} \end{bmatrix}.$$

Then

$$\begin{aligned} d\psi' &= e_0 \partial_0 \psi' - e_1 \partial_1 \psi' - \dots \\ &= \partial_0 \begin{bmatrix} \xi e^{-\theta u} \\ -\eta e^{-\theta u} \end{bmatrix} - i \partial_1 \begin{bmatrix} \eta e^{-\theta u} \\ \xi e^{-\theta u} \end{bmatrix} - \dots \\ &= \begin{bmatrix} \partial_0 \xi \\ -\partial_0 \eta \end{bmatrix} e^{-\theta u} + \begin{bmatrix} \xi e^{-\theta u} (-\partial_0 \theta) u \\ -\eta e^{-\theta u} (-\partial_0 \theta) u \end{bmatrix} \\ &\quad - i \begin{bmatrix} \partial_1 \eta \\ \partial_1 \xi \end{bmatrix} e^{-\theta u} - \dots - i \begin{bmatrix} \eta e^{-\theta u} (-\partial_1 \theta) u \\ \xi e^{-\theta u} (-\partial_1 \theta) u \end{bmatrix} - \dots \\ &= \begin{bmatrix} \partial_0 \xi - i \partial_1 \eta - \dots \\ -\partial_0 \eta - i \partial_1 \xi - \dots \end{bmatrix} e^{-\theta u} \\ &\quad + \begin{bmatrix} \xi \\ -\eta \end{bmatrix} (-\partial_0 \theta) e^{-\theta u} u - i \begin{bmatrix} \eta \\ \xi \end{bmatrix} (-\partial_1 \theta) e^{-\theta u} u - \dots \\ &= (d\psi) e^{-\theta u} - \partial_0 \theta \begin{bmatrix} \xi \\ -\eta \end{bmatrix} u e^{-\theta u} + i \partial_1 \theta \begin{bmatrix} \eta \\ \xi \end{bmatrix} u e^{-\theta u} - \dots \\ &= (d\psi) e^{-\theta u} - \begin{bmatrix} \partial_0 \theta & -i \partial_1 \theta - \dots \\ -i \partial_1 \theta - \dots & -\partial_0 \theta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} u e^{-\theta u} \\ &= (d\psi) e^{-\theta u} - (d\theta) \psi u e^{-\theta u} \end{aligned}$$

where we have defined  $d\theta$  to be the  $2 \times 2$  matrix

$$d\theta = \begin{bmatrix} \partial_0 \theta & -i \partial_1 \theta - \dots \\ -i \partial_1 \theta - \dots & -\partial_0 \theta \end{bmatrix}.$$

(It is important to get the order of terms right here!)

Note that  $d\theta$  is a 1-vector, expressed as a  $2 \times 2$  matrix according to the representation we are using for  $Cl(1,3)$ .

How does the Lagrangian defined by (6.40) change?

$$\begin{aligned}
L' &= (d\psi' + m\psi'u)\widetilde{\psi}' \\
&= \left( (d\psi)e^{-\theta u} - (d\theta)\psi u e^{-\theta u} + m\psi u e^{-\theta u} \right) e^{\theta u} \widetilde{\psi} \\
&= (d\psi - (d\theta)\psi u + m\psi u) \widetilde{\psi} \\
&= L - (d\theta)\psi u \widetilde{\psi}.
\end{aligned}$$

So again, to obtain invariance, we need to introduce a 1-vector  $A$  and a new Lagrangian

$$L = (d\psi + m\psi u - qA\psi u)\widetilde{\psi} \quad (6.48)$$

where under the local phase transformation,

$$A \rightarrow A' = A - \frac{1}{q} d\theta.$$

With this new Lagrangian (6.48), we find

$$\begin{aligned}
L' &= (d\psi' + m\psi'u - qA'\psi'u)\widetilde{\psi}' \\
&= (d\psi' + m\psi'u)\widetilde{\psi}' - qA'\psi'u\widetilde{\psi}' \\
&= (d\psi - (d\theta)\psi u + m\psi u)\widetilde{\psi} - qA'\psi'u\widetilde{\psi}' \\
&= (d\psi - (d\theta)\psi u + m\psi u)\widetilde{\psi} - q\left(A - \frac{1}{q}d\theta\right)\psi e^{-\theta u} u e^{\theta u} \widetilde{\psi} \\
&= (d\psi - (d\theta)\psi u + m\psi u)\widetilde{\psi} - qA\psi u\widetilde{\psi} + (d\theta)\psi u\widetilde{\psi} \\
&= (d\psi + m\psi u - qA\psi u)\widetilde{\psi} \\
&= L.
\end{aligned}$$

So  $L$  as in (6.48) is now invariant.

As in the usual treatment, we also want  $L$  to include a 'free' term for  $A$ . This leads to the complete Dirac-Maxwell Lagrangian

$$L = L_{DM} = (d\psi + m\psi u - qA\psi u)\widetilde{\psi} - \frac{u}{2}(dA)^2. \quad (6.49)$$

**Exercise 6.5** *Following the convention treatment, define the covariant derivative*

$$D\psi = d\psi - qA\psi u \quad (6.50)$$

*and the transformed derivative,*

$$D'\psi' = d\psi' - qA'\psi' u. \quad (6.51)$$

*Show that*

$$D'\psi' = (D'\psi)e^{-\theta u}.$$

## 6.11 Notes and references

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# Chapter 7

## Selected topics

In this chapter we look in more detail at three examples which are of basic importance in physics and where the geometry of Clifford algebras might well be thought to give a helpful hand in understanding their role and how they might best be regarded.

The first consists of the Stern-Gerlach experiment. This experiment which is conceptually quite simple to describe, has no classical counterpart, and so is regarded as of fundamental importance in showing why classical mechanics has to give way to quantum mechanics. Some authors (such as Sakurai) treat this experimental result as leading to an irremedial breach with classical mechanics and for that reason, analyse the ideas involved in detail. They are then led to the conclusion that complex vector spaces are necessary for its description. It is this last point especially which we will consider here.

The second example concerns the problem of the choice of signature metric, i.e. the choice of which Clifford algebra,  $Cl(1,3)$  or  $Cl(3,1)$ , one should use. The algebras are not isomorphic and this is relevant as their Pin group are also non-isomorphic. However in many (if not most) applications, it is only their even sub-algebras which are used and these *are* isomorphic (which means that the Spin groups are also isomorphic). So the difference is probably minor and this explains why most physicists would say that the choice of metric is unimportant. Mathematicians, wedded to the differing algebraic properties of the two algebras would disagree.

In this section we will show that if we take a more restricted view, the *actions* of the algebras are the same. In this view, we do not consider the whole algebras but only the actions of the even parts on the space of odd elements. We argue (perhaps not totally convincingly) that when it comes to applications in physics, this is all that is needed.

Whether or not this view is justified is really a matter of applications in physics. So the jury is still out.

The third example concerns the group  $SU(3)$ . Unlike  $SU(2)$  which is isomorphic to  $Spin(3)$  and which therefore can be described via the Clifford algebra  $Cl(0,3)$ ,  $SU(3)$  does not arise naturally in any Clifford algebra. Since it is a group of fundamental importance in elementary particle physics - it is central to the Standard Model - it would be interesting, to say the least, to find a geometric setting for the group.

Fortunately, the larger group  $SU(4)$  *does* have a nice Clifford algebra setting. This is because  $SU(4)$  is isomorphic to  $Spin(6)$  and therefore can be described by  $Cl(0,6)$ . It would seem that the underlying space  $R^6$  is too high a dimensional space for much geometry to peep through, but the analysis of  $SU(4)$  as  $Spin(6)$  shows that in this context,  $R^6$  splits naturally as  $R^3 \oplus R^3$  and now we have more hope of geometric results.

In this section we look at the isomorphism between  $SU(4)$  and  $Spin(6)$ . This treatment is fairly standard (though some authors do a good job in hiding the geometric ideas involved).  $SU(3)$  can then be obtained by fixing a basis vector in  $C^4$  and the problem then becomes one of characterising the corresponding subgroup of  $Spin(6)$ . Once this is done, it is useful to see how the Gell-Mann matrices (which generate  $SU(3)$ ) look geometrically.

Finally we look at the commutation relations for the Poisson product in  $R^3$ . Phase space on  $R^3$  is  $R^3 \oplus R^3$  and again we see how  $R^6$  is better regarded in this context, as  $R^3 \oplus R^3$ .

## 7.1 The Stern-Gerlach experiment

### 7.1.1 Introduction

The Stern-Gerlach experiment is often given as a basic argument why complex vector spaces are necessary to describe spin, and so quantum mechanics generally (see e.g. [1]). The idea in this section will be to replace the standard 2-dimensional complex space  $C^2$  with the 4-dimensional real space  $Cl^-(0,3)$  with the usual basis  $e_1, e_2, e_3, e_{123}$ . The rationale is, that with a real vector space, the ideas behind the Stern-Gerlach experiment will have a clearer geometric meaning.

Instead of describing the experiment using Pauli spinors and spin matrices, we will look at a way of presenting the underlying ideas of the experiment using the spin  $\frac{1}{2}$  representation of the group of unit quaternions,  $H^1$  rather than the more conventional group  $SU(2)$ <sup>1</sup>. The Pauli matrices are replaced by elements of  $Cl^+(0,3)$  while the eigenstates (conventionally given as two-component complex vectors) become two-dimensional real subspaces of  $Cl^-(0,3)$ .

### 7.1.2 The conventional mathematical treatment

Let us briefly recall the mathematical ideas involved, as they are usually given. A system such as a silver atom, in the presence of an inhomogeneous magnetic field (e.g. in the  $z$ -direction) falls into one of two spin states. Following Sakurai's notation, these are written as  $|S_z; + \rangle$  and  $|S_z; - \rangle$ . A similar notation holds for fields in the  $x$ - and  $y$ -directions. Since (pure) states such as these are usually expressed as unit vectors in some space, the problem is to obtain vector space expressions for these states in such a way that the transition probabilities between states  $\phi, \psi$  are correctly obtained as  $|\langle \phi, \psi \rangle|^2$ . If we start with the two possible states  $|S_z; + \rangle$  and  $|S_z; - \rangle$  which we may write more simply as  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  respectively, then Sakurai shows rigorously that the vector space cannot be  $R^2$  as it is not large enough to also contain the states given by the  $x$  and  $y$  directions. However the space may be taken to be  $C^2$  in which case these other states can then be described. So (the argument goes), complex vector spaces are necessary

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<sup>1</sup>Recall that  $H^1$  is contained in  $Cl^+(0,3)$  and is generated by  $1, i = e_{23}, j = e_{31}, k = e_{12}$ .



for a description of spin  $\frac{1}{2}$  systems.

But, as we have already seen above, spin  $\frac{1}{2}$  can be described more geometrically using the Clifford algebra  $Cl(0,3)$ . So our aim here will be to show that the results of the Stern-Gerlach experiment can also be described in terms of this real algebra. Before we do that however, we need to look a little closer at how we represent states and (spin) operators.

States such as eigenstates of an operator are typically represented by unit vectors in some complex vector space. Since a complex multiple (of modulus 1) of this vector represents the same state, we may replace this state vector by the one-dimensional subspace it generates or better still, by the projection onto that subspace. This approach (carefully presented by Jauch [2]) has the advantage that it allows us to describe *mixed* states as *density operators*, i.e by operators  $W$  of the form<sup>2</sup>

$$Wx = \sum_j p_j \langle \psi_j, x \rangle \psi_j \quad (7.1)$$

or

$$W = \sum_j p_j \psi_j \psi_j^T \quad (7.2)$$

where  $p_j \geq 0$ ,  $\sum_j p_j = 1$  and  $(\psi_j)$  is an orthonormal set of vectors (written as column vectors).  $W$  is a pure state when  $W = \psi\psi^T$  for some (unit) vector  $\psi$  or alternately when  $W$  is a 1-dimensional projection.

If a system is in state  $W$  as above and  $A$  is an observable (i.e. a Hermitian operator) then the expected (average) value of measuring  $A$  is

$$\langle A \rangle = \sum_j p_j \langle \psi_j, A \psi_j \rangle = \text{trace}(WA). \quad (7.3)$$

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<sup>2</sup>At least for the finite dimensional case. The more general case is slightly more complicated.

### 7.1.3 An unconventional mathematical treatment

For our purposes we can identify  $C^2$  with  $Cl^-(0,3) \equiv R^4$  via the following map

$$C^2 \leftrightarrow R^4, \quad \begin{bmatrix} x+iy \\ z+it \end{bmatrix} \leftrightarrow xe_1 + ye_2 + ze_3 + te_{123}. \quad (7.4)$$

Consider now the state  $|S_z; + \rangle$  which is usually represented in  $C^2$  by the vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  (here ‘usually’ means ‘as in Sakurai’s book’.) Since states are unchanged when multiplied by a unit vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} e^{i\theta} \\ 0 \end{bmatrix} \quad (7.5)$$

$$\leftrightarrow \cos\theta e_1 + \sin\theta e_2. \quad (7.6)$$

So  $|S_z; + \rangle$  can be represented by a unit circle in the  $e_1, e_2$  plane or better, by the projection  $P_{\langle e_1, e_2 \rangle}$  operator onto the plane.

Similarly the state  $|S_z; - \rangle$  can be represented by the projection  $P_{\langle e_1, e_{123} \rangle}$  operator onto the  $e_1, e_{123}$  plane.

What about states in other coordinate directions?

Consider as another example  $|S_x; + \rangle$  usually represented as  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ . The above ideas lead to the representation of this state as the projection onto the 2-dimensional subspace spanned by  $\frac{e_1+e_3}{\sqrt{2}}, \frac{e_2+e_{123}}{\sqrt{2}}$ . Similarly

$$|S_x; - \rangle \equiv \text{Proj onto } \left\{ \frac{e_1-e_3}{\sqrt{2}}, \frac{e_2-e_{123}}{\sqrt{2}} \right\} \quad (7.7)$$

and finally

$$|S_y; \pm \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \pm \frac{i}{\sqrt{2}} \end{bmatrix} \equiv \text{Proj onto } \left\{ \frac{e_2 \mp e_3}{\sqrt{2}}, \frac{e_1 \pm e_{123}}{\sqrt{2}} \right\}. \quad (7.8)$$

In general if we have the state  $S = \begin{bmatrix} a+ib \\ c+id \end{bmatrix}$  with  $a^2 + b^2 + c^2 + d^2 = 1$ , then  $S$

corresponds to the projection onto the subspace spanned by  $f_1, f_2$  where

$$f_1 = ae_1 + be_2 + ce_3 + de_{123}, \quad (7.9)$$

$$f_2 = -be_1 + ae_2 - de_3 + ce_{123}. \quad (7.10)$$

(Clearly  $f_1, f_2$  form an orthonormal pair.)

The matrix form for this general projection is easily found to be

$$P = \begin{pmatrix} a^2 + b^2 & 0 & ac + bd & ad - bc \\ 0 & a^2 + b^2 & bc - ad & ac + bd \\ ac + bd & bc - ad & c^2 + d^2 & 0 \\ ad - bc & ac + bd & 0 & c^2 + d^2 \end{pmatrix}. \quad (7.11)$$

Note (as a check) that the trace of  $P$  is 2 which is to be expected as  $P$  projects onto a 2-dimensional subspace.

As a special case, consider the state conventionally written as

$$\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} = \cos \alpha |S_z; + \rangle + \sin \alpha |S_z; - \rangle \quad (7.12)$$

which corresponds in the Stern-Gerlach experiment to a polarizer placed at an angle  $\alpha$  to the positive  $z$  direction. The corresponding subspace has a basis

$$\cos \alpha e_1 + \sin \alpha e_3, \cos \alpha e_2 + \sin \alpha e_{123}. \quad (7.13)$$

(We will use this result later).

So generally if the system is in a state such as  $|S_z; + \rangle$  then this complex vector (i.e. 1-dimensional subspace in  $C^2$ ) corresponds to a 2-dimensional subspace in  $Cl^-(0, 3)$ . Note that if  $P$  is the projection onto the subspace, the associated state is  $W = \frac{P}{2}$ .

The problem is now the following. Suppose that the system is in a certain state  $W$  (e.g. it might be in state  $|S_z; + \rangle$ ). Perform the experiment again, (e.g. change the field to the  $x$  direction). Let this new state correspond to a projection  $Q$  on  $Cl^-(0, 3)$ . We then have to calculate trace  $(WQ)$  and verify that it gives the correct value for the transition probability.

To keep the notation simple, so as to have the ideas more firmly in mind, let us consider first how we would calculate the probability of the transition  $|S_z; + \rangle$  to  $|S_x; + \rangle$ .

In this case, the system is originally in the state

$$W = \frac{1}{2}P_{\{e_1, e_2\}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the projection  $Q$  corresponding to the new state  $|S_x; + \rangle$  is the projection onto the subspace spanned by  $\frac{e_1+e_3}{\sqrt{2}}, \frac{e_2+e_{123}}{\sqrt{2}}$ . An easy exercise shows that

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

As a check, note that  $\text{trace}(WQ) = \frac{1}{2}$  as expected.

The general case is not much harder though tedious to calculate. Start with an initial state  $\phi$  in  $C^2$  and a final state  $\psi$ . Write

$$\phi = \begin{bmatrix} a + ib \\ c + id \end{bmatrix}, \quad \psi = \begin{bmatrix} x + iy \\ z + it \end{bmatrix} \quad (7.14)$$

and use equation (7.11) to define the two corresponding projections  $P$  and  $Q$ . Again let the initial state be  $W = \frac{1}{2}P$ . We can calculate  $\text{trace}(WQ)$ .

**Exercise 7.1** Verify that this is the same as  $|\langle \phi, \psi \rangle|^2$ .

So this ‘real space’ setting for the Stern-Gerlach experiment is obtained by substituting  $R^4$  for  $C^2$  and then using  $Cl^-(0, 3)$  to represent  $R^4$ . I.e. the setting is obtained to some extent using the Clifford algebra  $Cl(0, 3)$ . But actually, few of the algebra properties are used; in fact the only one is the existence of the basis  $e_1, e_2, e_3, e_{123}$  for  $Cl^-(0, 3)$ .

A better approach is the following. Again start with an initial state  $\phi$  in  $C^2$  which corresponds to a projection  $P$  on a two-dimensional subspace  $M$  of  $Cl^-(0,3)$ . Define

$$R = 2P - I$$

so that  $R$  is the reflection corresponding to the subspace  $M$ . This means that

$$Rx = \begin{cases} x, & x \in M, \\ -x, & x \in M^\perp. \end{cases} \quad (7.15)$$

Note that  $R^2 = I$ .

We can also regard  $R$  as a rotation through an angle  $\pi$  in the plane  $M^\perp$ . We have seen<sup>3</sup> that a rotation  $R$  can be written in terms of left and right multiplication by quaternions as

$$Rx = q_1 x \tilde{q}_2$$

where  $x \in Cl^-(0,3)$  and  $q_1, q_2$  are unit quaternions. ( $\tilde{q}$  is the conjugate of  $q$ ). What are  $q_1$  and  $q_2$ ?

Since  $R^2 = I$  we have  $q_i^2 = \pm 1, i = 1, 2$ . The cases  $q_i^2 = 1$  can be ruled out since these cases do not give (non-trivial) rotations in a plane. So  $q_i$  must be pure and to emphasize that we write

$$Rx = p_1 x \tilde{p}_2, \quad (p_1, p_2 \text{ unit, pure quaternions}) \quad (7.16)$$

So now we have to find  $p_1$  and  $p_2$ .

Choose an orthonormal basis  $u_1, u_2$  for  $M$ . Here, because we are dealing with rotations in  $R^4$ , we use the Euclidean inner-product<sup>4</sup>

$$\langle u, v \rangle = u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3 \quad (7.17)$$

where  $u = u^1 e_1 + u^2 e_2 + u^3 e_3 + u^0 e_{123} \in Cl^-(0,3)$  and similarly for  $v$ . Alternately we can write

$$\langle u, v \rangle = -Rl(\tilde{u}v). \quad (7.18)$$

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<sup>3</sup>See chapter 2.5.3

<sup>4</sup>For simplicity we have chosen a positive-definite inner-product. This is just the negative of the inner-product inherited on  $Cl(0,3)$ .

Consider first the action of  $R$  on  $M$ .

Since  $Ru_1 = u_1$  and  $Ru_2 = u_2$  we obtain

$$u_1 = p_1 u_1 \widetilde{p}_2 \quad \text{or} \quad p_2 = u_1^{-1} p_1 u_1. \quad (7.19)$$

Similarly for  $u_2$ ,

$$u_2 = p_1 u_2 \widetilde{p}_2 \quad \text{or} \quad p_2 = u_2^{-1} p_1 u_2. \quad (7.20)$$

Combining these equations gives

$$p_2 = u_1^{-1} p_1 u_1 = u_2^{-1} p_1 u_2 \quad (7.21)$$

i.e.

$$p_1 = (u_1 u_2^{-1}) p_1 (u_1 u_2^{-1})^{-1} \quad (7.22)$$

and so

$$p_1 (u_1 u_2^{-1}) = (u_1 u_2^{-1}) p_1. \quad (7.23)$$

Since  $u_1 u_2^{-1} = -u_1 \widetilde{u}_2$  and has no real part (as  $u_1, u_2$  are orthogonal), then  $u_1 u_2^{-1}$  is *pure* and because  $u_1$  and  $u_2$  are orthonormal, then  $u_1 u_2^{-1}$  is a pure *unit* quaternion. Since it commutes with  $p_1$  (itself a pure unit quaternion) then we have

$$p_1 = \pm u_1 \widetilde{u}_2 \quad (7.24)$$

which gives  $p_1$  in terms of the basis vectors of  $M$ . The choice of sign is unimportant as it can be absorbed into the  $p_2$  term for  $R$  and so we can choose  $p_1 = u_1 \widetilde{u}_2$ .

We can now find  $p_2$ . From (7.21),

$$p_2 = u_1^{-1} p_1 u_1 = u_1^{-1} (u_1 \widetilde{u}_2) u_1 = \widetilde{u}_2 u_1. \quad (7.25)$$

To summarize:

Let  $P$  be a two-dimensional projection onto  $M$  and let  $u_1, u_2$  be an orthonormal basis for  $M$ . Put  $R = 2I - P$ , the associated reflection. Then  $Rx = p_1 x \widetilde{p}_2$  where  $p_1 = u_1 \widetilde{u}_2$  and  $p_2 = u_2 \widetilde{u}_1$ .

**Exercise 7.2** Show that if  $u_1, u_2$  are orthonormal then  $q = u_1 u_2$  is a unit quaternion.

(Hint: Work with the quaternions  $q_i = u_i e_{123}$  and the fact that the norm on the algebra of quaternions is multiplicative.)

Recall that a bivector in  $R^3$  of the form  $p = uv$  where  $u, v$  are vectors in  $R^3$ , has the geometric meaning of giving an orientation to the plane determined by  $u$  and  $v$ . It is worth noting that this idea extends to planes in  $Cl^-(0, 3)$  in the following sense.

If  $M$  is a plane in  $Cl^-(0, 3)$ , there is a *unit* bivector  $p$  associated with  $M$ .  $p$  is unique apart from sign which determines an orientation of  $M$ .

To find  $p$ , choose an orthonormal basis  $u_1, u_2$  for  $M$  as above. Define  $p = u_1 \widetilde{u}_2$ . Then as we have seen,  $p$  is a unit bivector. To show uniqueness (apart from sign), let  $v_1, v_2$  be another orthonormal basis for  $M$ . Then (interchanging  $v_1$  and  $v_2$  if necessary) we can write

$$\begin{aligned} v_1 &= \cos\theta u_1 + \sin\theta u_2 \\ v_2 &= -\sin\theta u_1 + \cos\theta u_2 \end{aligned}$$

and a simple calculation shows that  $v_1 \widetilde{v}_2 = u_1 \widetilde{u}_2$ . The ambiguity in sign stems from the fact that we might have to interchange basis elements, i.e. change the orientation of  $M$ .

Note too that we have to be careful about the way we use the basis vectors. Given a basis  $u, v$ , then  $p$  is the product of  $u$  and  $\widetilde{v}$ . The order of  $u$  and  $v$  is not important (though it will change the sign on  $p$  as we would expect). But we cannot take  $\widetilde{u}v$ . This would be the bivector associated with the plane spanned by  $\widetilde{u}, \widetilde{v}$  which might well be different from  $M$ .

Now we can return to the description of the geometry underlying the Stern-Gerlach experiment. Start with a subspace  $M_1$  spanned by the orthonormal basis  $u_1, v_1$  corresponding in the classical treatment to a vector  $\phi \in C^2$ . Suppose too that we also have a final state which we take to be another two-dimensional subspace  $M_2$  corresponding to the state vector  $\psi \in C^2$ .

Let  $u_2, v_2$  be an orthonormal basis for  $M_2$ . Let  $P_i$  be the projection onto  $M_i$ ,  $i = 1, 2$ .

Put  $W = \frac{P_1}{2}$  for the initial state. The transition probability of ending in the state corresponding to  $P_2$  is then  $\text{trace}(WP_2)$ .

Let  $R_1, R_2$  be the reflections associated with  $M_1$  and  $M_2$  respectively. We know that  $P_i = \frac{I + R_i}{2}$  so that

$$\text{trace}(WP_2) = \frac{1}{2} \text{Trace} \left( \frac{I + R_1}{2} \frac{I + R_2}{2} \right) \quad (7.26)$$

$$= \frac{1}{8} \text{Trace} (I + R_1 + R_2 + R_1R_2) \quad (7.27)$$

Now  $\text{Trace}(I) = 4$  and  $\text{Trace}(R_i) = 0$  since  $R_i$  being a two-dimensional reflection has eigenvalues  $1, 1, -1, -1$ . So

$$\text{trace}(WP_2) = \frac{1}{2} + \frac{1}{8} \text{Trace}(R_1R_2). \quad (7.28)$$

Write as before

$$R_1x = p_1xp'_1, \quad R_2x = p_2xp'_2$$

where

$$p_1 = u_1\widetilde{v}_1, \quad p'_1 = \widetilde{v}_1u_1, \quad p_2 = u_2\widetilde{v}_2, \quad p'_2 = \widetilde{v}_2u_2.$$

Then

$$R_1R_2x = q_1x\widetilde{q}_2$$

where

$$q_1 = p_1p_2, \quad q_2 = p'_1p'_2$$

In other words,  $R_1R_2 = L_{q_1}R_{q_2}$  where  $L_{q_1}$  is left multiplication by  $q_1$  and similarly  $R_{q_2}$  is right multiplication by  $\widetilde{q}_2$ . It is straightforward to show that

$$\text{Trace}(R_1R_2) = 4Rl(q_1)Rl(q_2).$$

Furthermore if we write  $q_1 = \cos\theta_1 + \sin\theta_1\widehat{q}_1$  for the decomposition of  $q_1$  into its real and pure parts and similarly for  $q_2$ , we have

$$\text{Trace}(R_1R_2) = 4Rl(q_1)Rl(q_2) = 4\cos\theta_1\cos\theta_2.$$



This gives the transition probability as

$$\text{Trace}(WP_2) = \frac{1}{2} + \frac{1}{8}\text{Trace}(R_1R_2) \quad (7.29)$$

$$= \frac{1 + \cos\theta_1 \cos\theta_2}{2} \quad (7.30)$$

$$= \frac{1 + \cos\theta_{M_1M_2}}{2} \quad (7.31)$$

where  $\cos\theta_{M_1M_2} = \cos\theta_1 \cos\theta_2$  gives the so-called *total angle* between the subspaces  $M_1, M_2$ .

So the transition probability can be neatly expressed as

$$\text{Trace}(WP_2) = \cos^2\left(\frac{\theta_{M_1M_2}}{2}\right). \quad (7.32)$$

(In general two  $r$ -dimensional subspaces in  $R^n$  have  $r$  *principal angles* defined through pairs of *singular vectors*. The product of the cosines of these angles gives the total angle between the subspaces. See, e.g. [3] for a geometric description of these angles from a Clifford algebra perspective.)

## 7.2 On the signature of metrics

### 7.2.1 Introduction

An interesting question - though realistically it may turn out to be unimportant - is whether the choice of signature in  $R^3$  or  $R^{1,3}$  has any consequences in physics. This is a matter of some argument though the prevailing thought (especially among physicists - mathematicians might well disagree) is that it does not.

However even among physicists this view is not shared universally. The question has been analysed in depth by Berg et al [4], who claim that in a neutrinoless double beta decay, the neutrino emitted and reabsorbed in the course of the interaction can only be described in terms of  $\text{Pin}(3,1)$  (see Chapter 3 above for the definition of the Pin groups.)

On the other hand, the choice of signature will have an effect on geometry and therefore perhaps on physics. This is due to the connection between signature and 'handedness'. This was dealt with by Butler and McAven ([5]) with the connection described as follows.

Consider  $R^3$  with a negative metric, i.e. consider the Clifford algebra  $Cl(0,3)$ . Choose a right-handed orthonormal basis  $e_1, e_2, e_3$ . Righthandedness implies a certain orientation so that e.g.  $(e_1, e_2) \rightarrow e_3$  etc. while  $(e_2, e_1) \rightarrow -e_3$ . Now consider the (axial or bi-) vectors  $i = e_{23}, j = e_{31}, k = e_{12}$ . Note that the order  $1 \rightarrow 2 \rightarrow 3$  described for the  $e_i$ s carries over to  $i, j, k$  this time expressed also by multiplication, e.g.  $e_{23}e_{31} = e_{12}$ . *The fact that both the vectors and bivectors have the same direction or orientation is a direct consequence of the metric.* If instead we choose the positive metric so that  $e_i^2 = +1$  then the vectors and bivectors given above would now have *different* orientations.

**Exercise 7.3** *Show this.*

The most appropriate choice of signature in relativistic physics is still a matter of some argument. The signature of a metric is derived from a quadratic form on a real vector space and whether we choose the associated

symmetric matrix to be  $Q$  or  $-Q$  is a choice which many physicists would regard as unimportant and claim that either choice leads to the same physics, in the sense that either choice leads to identical predictions.

On the other hand, mathematicians would argue that the Clifford algebras defined by the metrics are non-isomorphic and that this might very well have physical consequences (as suggested by Berg et al.) In particular, the two Pin groups are different even if the two Spin groups are isomorphic. Is it possible that these might lead to different results?

At this stage, it is fair to say that the point is unresolved. What evidence there is, seems currently to give physicists the upper hand.

As we know, Clifford algebras split into a direct sum of two sub-spaces, the even vectors  $Cl^+$  and the odd vectors  $Cl^-$ . These spaces seem to play fundamentally different roles in physics. The first contains *operators* such as rotations, boosts, etc while the second contains vector and spinor type quantities. This means that the elements of  $Cl^+$  act on the elements of  $Cl^-$  (in the sense e.g. that rotations act on vectors) but that addition of an even and an odd vector may be physically meaningless. In other words, the full algebraic operations, while permissible mathematically, may be meaningless *physically*. If Clifford algebra is to be regarded as underpinning a physical theory, then concepts which are important in mathematics, should likewise be important in the physics it wishes to describe. By the same token (and usually ignored), concepts which are *meaningless* in the physical theory, should be expunged or at the very least, ignored, in the mathematical description of the theory.

In particular, if the physical theory does not give a meaning to the sum of an even vector with an odd vector, then mathematically we should refrain from any such use. This means that we are *not permitted* to use the full algebraic structure of the Clifford algebra.

It is this idea that we pursue here.

## 7.2.2 A map between $Cl^+(p, q)$ and $Cl^+(q, p)$

(While the only physically important metrics are those of space or space-time, the general case is no more difficult and saves us the problem of describing one case or the other.)

As usual, denote by  $Cl(p, q)$  the Clifford algebra on  $R^{p+q}$  with an orthonormal basis  $e_i$  satisfying

$$e_i^2 = \begin{cases} +1, & i = 1 \dots p, \\ -1, & i = p+1 \dots p+q. \end{cases} \quad (7.33)$$

Similarly for  $Cl(q, p)$  we can choose a basis  $f_i$  such that

$$f_i^2 = \begin{cases} -1, & i = 1 \dots p, \\ +1, & i = p+1 \dots p+q. \end{cases} \quad (7.34)$$

Note that with this choice  $e_i^2 = -f_i^2$  for all  $i = 1 \dots n = p+q$ .

These choices of basis vectors will make it easier to describe the isomorphism between  $Cl^+(p, q)$  and  $Cl^+(q, p)$  given below.

Recall that the *reversion* operator on a Clifford algebra is an *anti-automorphism*  $x \rightarrow \tilde{x}$  satisfying  $\tilde{\tilde{x}} = x$  for any 1-vector  $x$ . It follows from these two properties that

$$\tilde{e}_{ij} = -e_{ij} \quad (7.35)$$

and more generally<sup>5</sup> we have for any set  $A$  of distinct indices

$$\tilde{e}_A = \sigma(|A|)e_A \quad (7.36)$$

where  $|A|$  is the number of elements of  $A$  and

$$\sigma(n) = (-1)^{\frac{n(n-1)}{2}} = \begin{cases} +1, & n \equiv 0, 1 \pmod{4}, \\ -1, & n \equiv 2, 3 \pmod{4}. \end{cases} \quad (7.37)$$

Using the anti-automorphism property of reversion, we have

$$e_A \tilde{e}_A = \prod_{i \in A} e_i^2 \quad (7.38)$$

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<sup>5</sup>See chapter 1.3.1 where the reversion operation is dealt with in more detail.

and it then follows from (7.36) that for any orthonormal basis  $e_i$ ,

$$(e_A)^2 = \sigma(|A|) \prod_{i \in A} e_i^2. \quad (7.39)$$

Now suppose that  $(e_i)$  and  $(f_i)$  are bases for  $Cl(p, q)$  and  $Cl(q, p)$  respectively satisfying (7.33) and (7.34).

Define  $U : e_i \rightarrow f_i$  and extend  $U$  in the obvious way to the map (still denoted by  $U$ ) from  $Cl(p, q) \rightarrow Cl(q, p)$ ,  $U : e_A \rightarrow f_A$ .

Now define  $\pi : Cl(p, q) \rightarrow Cl(q, p)$  by

$$\pi x = \widetilde{(Ux)}. \quad (7.40)$$

Looking at the basis vectors in  $Cl(p, q)$  we see that for 1-vectors  $\pi e_i = f_i$ , while for 2-vectors,  $\pi e_{ij} = -f_{ij}$ . More generally (by (7.36)), for any set  $A$  of (distinct) indices

$$\pi e_A = \sigma(|A|) f_A. \quad (7.41)$$

In particular if  $e_A \in Cl^+(p, q)$ , i.e.  $A$  has an even number of indices,

$$\pi e_A = (-1)^{\frac{|A|}{2}} f_A. \quad (7.42)$$

We have the following result.

*Lemma:* If  $A$  is a set of distinct indices, then

$$(f_A)^2 = (-1)^{|A|} (e_A)^2. \quad (7.43)$$

*Proof:* From (7.39)

$$(e_A)^2 = \sigma(|A|) \prod_{i=1}^{|A|} e_i^2$$

and therefore for  $(f_A)$ ,

$$\begin{aligned} (f_A)^2 &= \sigma(|A|) \prod_{i=1}^{|A|} f_i^2 \\ &= \sigma(|A|) (-1)^{|A|} \prod_{i=1}^{|A|} e_i^2 \quad (\text{since } f_i^2 = -e_i^2 \text{ for each } i) \\ &= (-1)^{|A|} (e_A)^2. \end{aligned}$$

Now we can derive the required isomorphism.

**Theorem 7.1**  $\pi$  is an isomorphism between  $Cl^+(p, q)$  and  $Cl^+(q, p)$ .

*Proof:* Clearly  $\pi$  is a bijection between  $Cl^+(p, q)$  and  $Cl^+(q, p)$ . So it suffices to show that

$$\pi e_A \pi e_B = \pi e_{AB} \text{ for index sets } A \text{ and } B \text{ (necessarily even).}$$

Letting  $H = A \cap B$ , we can write without loss of generality,  $e_A = e_J e_H$  and  $e_B = e_H e_K$  for disjoint  $J$  and  $K$ . Then

$$\begin{aligned} e_A e_B &= e_J e_H e_H e_K \\ &= (e_H)^2 e_J e_K \end{aligned}$$

so that by (7.42) and noting that  $|J| + |K|$  must be even (as  $|A|$  and  $|B|$  are even),

$$\pi e_A e_B = (e_H)^2 (-1)^{\frac{|J|+|K|}{2}} f_{JK}. \quad (7.44)$$

Also (again from (7.42))

$$\pi e_A = (-1)^{\frac{|J|+|H|}{2}} f_{JH}, \quad \pi e_B = (-1)^{\frac{|H|+|K|}{2}} f_{HK}$$

and therefore

$$\begin{aligned} \pi e_A \pi e_B &= (-1)^{|H|} (-1)^{\frac{|J|+|K|}{2}} f_{JH} f_{HK} \\ &= (-1)^{|H|} (-1)^{\frac{|J|+|K|}{2}} (f_H)^2 f_{JK} \\ &= (-1)^{|H|} (-1)^{\frac{|J|+|K|}{2}} (-1)^{|H|} (e_H)^2 f_{JK} \quad \text{by (7.43)} \\ &= (-1)^{\frac{|J|+|K|}{2}} (e_H)^2 f_{JK} = \pi e_A e_B. \end{aligned}$$

So reversion has the remarkable property of switching signature. Another equally interesting property is given below.

### 7.2.3 The action of $Cl^+(p, q)$ on $Cl^-(p, q)$

Elements of  $Cl^+(p, q)$  act multiplicatively on  $Cl^-(p, q)$  via either

$$\rho_u^l(x) = ux \quad \text{or} \quad \rho_u^r(x) = x\tilde{u} \quad (7.45)$$

where  $u \in Cl^+(p, q)$  and  $x \in Cl^-(p, q)$ . The first result we want is that these two actions are equivalent. What does this mean?

Suppose in general, a group or algebra  $G$  acts on sets  $X$  and  $Y$  via

$$x \rightarrow \rho_g^1(x), \quad y \rightarrow \rho_g^2(y) \quad \text{with } g \in G$$

(We ignore the axioms required when we talk of an action on a set as these hold in the cases we consider.) The two actions are called *equivalent* if there exists a bijection  $f : X \rightarrow Y$  such that

$$f(\rho_g^1(x)) = \rho_g^2(f(x))$$

as in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho_g^1} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\rho_g^2} & Y \end{array}$$

More generally still, suppose  $G^1$  and  $G^2$  are isomorphic with isomorphism  $U : G^1 \rightarrow G^2$ . Suppose that  $G^1$  acts on a set  $X$  and  $G^2$  acts on a set  $Y$ . Then these actions are called equivalent if there exists a bijection  $f : X \rightarrow Y$  such that

$$f(\rho_g^1(x)) = \rho_{Ug}^2(f(x)).$$

It is easy now to show that the actions of left and right multiplication of  $Cl^+(p, q)$  on  $Cl^-(p, q)$  are equivalent.

For let  $f : Cl^-(p, q) \rightarrow Cl^-(p, q)$  be the reversion operator,  $f : x \rightarrow \tilde{x}$ . If  $u \in Cl^-(p, q)$ , put  $\rho_u^l(x) = ux$  and  $\rho_u^r(x) = x\tilde{u}$ . Then

$$f(\rho_u^l(x)) = f(ux) = \tilde{u}\tilde{x} = \tilde{x}\tilde{\tilde{u}} = \rho_{\tilde{u}}^r(\tilde{x}) = \rho_{\tilde{u}}^r(f(x))$$

so that the two actions are indeed equivalent.

Finally we can state our main result.

**Theorem 7.2** *The actions of  $Cl^+(p, q)$  on  $Cl^-(p, q)$  and  $Cl^+(q, p)$  on  $Cl^-(q, p)$  are equivalent.*

*Proof:* (From the result above, it doesn't matter whether we consider left or right actions. We will choose left action.)

Let  $\pi : Cl^+(p, q) \rightarrow Cl^+(q, p)$  be the map defined in the previous section. By theorem 1,  $\pi$  is an isomorphism between  $Cl^+(p, q)$  and  $Cl^+(q, p)$ .

We therefore need to show that for all  $u \in Cl^+(p, q)$  and  $x \in Cl^-(p, q)$

$$\pi(ux) = (\pi(u))(\pi(x)) \quad (7.46)$$

or in terms of the left multiplication operator  $\rho^l$  we have to show that the following diagramme is commutative.

$$\begin{array}{ccc} Cl^-(p, q) & \xrightarrow{\rho_u^l} & Cl^-(p, q) \\ \pi \downarrow & & \downarrow \pi \\ Cl^-(q, p) & \xrightarrow{\rho_{\pi u}^l} & Cl^-(q, p) \end{array}$$

It suffices to consider the case  $u = e_A$  where  $|A|$  is even and  $x = e_B$  where  $|B|$  is odd. Suppose also  $e_A = e_J e_H$ ,  $e_B = e_H e_K$  where  $H = A \cap B$ . Then

$$\begin{aligned} \pi(ux) &= \pi(e_A e_B) \\ &= (e_H)^2 \pi(e_J e_K) \\ &= (-1)^{|H|} (f_H)^2 \pi(e_J e_K) \quad (\text{from the lemma above}) \\ &= (-1)^{|H|} (f_H)^2 \sigma(|JK|) f_H f_K \quad \text{from (7.41)}. \end{aligned}$$

Now consider  $(\pi(u))(\pi(x))$ . We have

$$\pi(x) = \pi(e_B) = \sigma(|B|) f_B, \quad \text{by (7.41)}$$

and

$$\pi(u) = \pi(e_A) = (-1)^{\frac{|A|}{2}} f_A \quad \text{by (7.42)}$$

which gives

$$\pi(u)\pi(x) = (-1)^{\frac{|A|}{2}} \sigma(|B|) f_A f_B \quad (7.47)$$

$$= (-1)^{\frac{|A|}{2}} f_A \sigma(|B|) f_B \quad (7.48)$$

$$= (-1)^{\frac{|A|}{2}} \sigma(|B|) (f_H)^2 f_J f_K. \quad (7.49)$$



Comparing these results for  $\pi(ux)$  and  $\pi(u)\pi(x)$ , the proof reduces to showing that

$$\sigma(|JK|) = (-1)^{\frac{|A|}{2}} (-1)^{|H|} \sigma(B). \quad (7.50)$$

Put

$$\begin{aligned} |J \cup H| &= 2r, & \text{since } A \text{ is even and} \\ |H \cup K| &= 2s + 1, & \text{since } B \text{ is odd.} \end{aligned}$$

so that if  $|K| = t$ , then  $|H| = 2s - t + 1$  and  $|J| = 2r - 2s + t - 1$ .

We deduce that firstly

$$\begin{aligned} \sigma(|JK|) &= \sigma(2r - 2s + 2t - 1) \\ &= (-1)^{\frac{(2r-2s+2t-1)(2r-2s+2t-2)}{2}} \quad \text{by (7.37)} \\ &= (-1)^{(r-s+t-1)} \end{aligned}$$

while

$$\begin{aligned} (-1)^{\frac{|A|}{2}} (-1)^{|H|} \sigma(B) &= (-1)^{\frac{|A|}{2}} (-1)^{|H|} \sigma(B) \\ &= (-1)^r (-1)^{(2s-t+1)} (-1)^{(2s+1)s}, \quad \text{since } |B| = 2s + 1 \\ &= (-1)^{(r+2s-t+1+2s^2+s)} \\ &= (-1)^{(r+s-t+1)} \\ &= \sigma(|JK|). \end{aligned}$$

as required.

To summarise this section, if we restrict our view of Clifford algebras to the actions of  $Cl^+(p, q)$  on  $Cl^-(p, q)$ , then signature is not important.

## 7.3 The geometry of $SU(3)$ and $SU(4)$

### 7.3.1 Introduction

Lie groups come in different families and these are characterized by an integer, called the ‘dimension’ or ‘number of real parameters’. For example, the family of special unitary groups,  $SU(n)$ ,  $n = 1, 2, \dots$  is characterized by its dimension  $n^2 - 1$ . Similarly, spin groups  $\text{Spin}(m)$ ,  $m = 2, 3, \dots$  have dimension  $\frac{m(m-1)}{2}$ . It may happen that for two groups in different families, their dimensions coincide, in which case the natural question arises: ‘are the two groups isomorphic?’

In many cases the answer will be ‘no’, but sometimes it is ‘yes’ in which case we often say that the isomorphism is ‘an accident of small dimension’. When this happens, then since the underlying geometries of the two groups are quite different, we should not expect that the isomorphism to be a ‘natural’ one.

The best known example is the case of  $\text{Spin}(3)$  and  $SU(2)$  where the groups both have dimension 3. (These groups were discussed at length in chapter 3.) Here the relationship between  $SU(2)$  and  $R^3$  (on which  $\text{Spin}(3)$  operates) is not a natural one. Vectors in  $R^3$  have to be identified as  $2 \times 2$  trace-free, Hermitian matrices (on which matrices in  $SU(2)$  can now act) via

$$(x, y, z) \leftrightarrow \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \quad (7.51)$$

Although this is a very nice algebraic trick, it is still geometrically unnatural because (for example), it singles out the  $z$  component for special favour.<sup>6</sup>

This lack of a *geometrical* basis for the isomorphism should act as a warning that the isomorphism is not a natural one and may be expected to cause problems in interpretation later. One immediate problem which arises, is the awkward way rotation (Euler) angles are represented in  $SU(2)$ .

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<sup>6</sup>Some authors argue that this is not an issue since any component could be chosen. However it remains the case that *one* component has to be singled out.

If instead of meekly submitting to this piece of algebraic trickery we *demand* a geometrically natural isomorphism, we are irresistibly led to the conclusion that  $H^1$ , the group of unit quaternions, should be used instead of  $SU(2)$ . This is an excellent example of two groups being isomorphic (algebraically the same) but *geometrically* quite different.

The two groups  $SU(4)$  and  $\text{Spin}(6)$  (the two-fold covering group of  $SO(6)$ ), both have dimension 15, and also turn out to be isomorphic. But because this is an ‘accidental isomorphism of small dimensions’, the connection is not very obvious and constructions of this isomorphism are often quite technical and geometrically obscure. Since  $SU(4)$  occurs naturally in particle physics while  $SO(6)$  appears when combining space and momentum vectors (e.g. in describing symmetries in phase space), both areas where geometry can play a large part, it might be hoped that a more geometric connection (i.e. construction) might be found. This is the motivation behind this section.

Most authors who have considered ways to construct this isomorphism begin with the action of  $SU(4)$  on  $C^4$  and extend it to the 6-dimensional complex space  $\Lambda^2(C^4)$  - the space of all anti-symmetric, bilinear functionals on  $C^4$ . This space contains a real 6-dimensional subspace of ‘self-dual’ elements and the action of  $SU(4)$  on this subspace is easily shown to be the same as the action of  $\text{Spin}(6)$  on  $R^6$ . This leads to the required isomorphism and it is the approach we take here.

### 7.3.2 The exterior algebra <sup>7</sup>

Let  $V$  be a (complex) vector space (in our case  $V = C^4$ ). An *antisymmetric*  $k$ -linear map (usually called a  $k$ -form) is a function  $\omega: V \times \cdots \times V \rightarrow C$ , which is linear in each factor and such that

$$\omega(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$$

for all vectors  $v_1, \dots, v_k$  and all  $i$ . It follows that  $\omega(v_1, \dots, v, v, \dots, v_k) = 0$ .

We denote the vector space of  $k$ -linear, antisymmetric maps by  $\Lambda^k(V)$ . If  $k = 1$  then  $\Lambda^1(V)$  is just the dual space of linear functionals on  $V$ .

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<sup>7</sup>We closely follow Göckeler and Schücker [6], where a fuller treatment of these ideas can be found.

Suppose that  $V = C^n$  and let  $e_1, \dots, e_n$  be the standard basis for  $C^n$ . Let  $\omega$  be a 2-form. Then by linearity,  $\omega$  is determined by its values on the vectors  $(e_i, e_j)$  where  $i < j$ . It follows that  $\Lambda^2(V)$  has dimension  $\binom{n}{2}$ .

**Exercise 7.4** Show that generally,  $\Lambda^k(V)$  has dimension  $\binom{n}{k}$  when  $k \leq n$  (and 0 when  $k > n$ ).

Let  $\alpha^1$  and  $\alpha^2$  be linear maps. We define the 2-form  $\alpha^1 \wedge \alpha^2$ , called the wedge product of  $\alpha^1$  and  $\alpha^2$ , by

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \quad (7.52)$$

More generally if  $\omega \in \Lambda^p(V)$  and  $\rho \in \Lambda^q(V)$  we define  $\omega \wedge \rho \in \Lambda^{p+q}(V)$  by

$$(\omega \wedge \rho)(v_1, \dots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \text{sgn}(\pi) \omega(v_{\pi(1)}, \dots, v_{\pi(p)}) \rho(v_{\pi(p+1)}, \dots, v_{\pi(p+q)}) \quad (7.53)$$

(where  $S_n$  is the symmetric group and  $\text{sgn}(\pi)$  is the signature of  $\pi$ , i.e.  $\text{sgn}(\pi) = \pm 1$  according as  $\pi$  is even or odd.)

**Exercise 7.5** 2-forms can be identified with  $n \times n$ , anti-symmetric matrices such that if  $\omega \leftrightarrow A$  then  $\omega(x, y) = x^T A y$ .

Show that if  $\alpha^1, \alpha^2$  are linear maps defined through (column) vectors  $a^1, a^2$ , then  $\alpha^1 \wedge \alpha^2 \leftrightarrow A = a^1{}^T a^2 - a^2{}^T a^1$ .

Assume that  $\dim(V) = n$ . The direct sum

$$\Lambda(V) = \bigoplus_{k=1}^n \Lambda^k(V) \quad (7.54)$$

together with the wedge product, is an associative algebra called the exterior algebra of  $V$ . An element  $\omega \in \Lambda^k(V)$  is also said to be of degree  $k$ .

We also call  $\Lambda(C^n)$ , with multiplication defined by the wedge product, the Grassmann algebra  $G_n$ . It has a basis  $(e_i)_{i=1}^n$  satisfying

$$e_i \wedge e_j = -e_j \wedge e_i \quad \text{for all } i, j. \quad (7.55)$$

As noted above this implies that

$$e_i \wedge e_i = 0 \quad \text{for all } i. \quad (7.56)$$

(So there is no notion of signature in Grassmann algebras.)

As with Clifford algebras, the dimension of  $G_n$  is  $2^n$  with basis elements again written  $e_i, e_{ij}, \dots$  and in general as  $e_A$ .  $G_n$  also splits into a direct sum of two subspaces  $G_n^+$  and  $G_n^-$  of even and odd forms.

We write  $e_1 \wedge e_2 \cdots \wedge e_n = e$  and a basic result is that for any set  $(x_k)_{k=1}^n$  of 1-vectors,

$$x_1 \wedge x_2 \cdots \wedge x_n = \det(X)e \quad (7.57)$$

where we have (as usual) written the  $x_k$ s as column vectors and

$$X = [x_1 | x_2 | \cdots | x_n]. \quad (7.58)$$

If  $A$  is an  $n \times n$  matrix, then  $A$  lifts to an operator on  $G_n$  defined through

$$Ae_{ij} = Ae_i \wedge Ae_j \quad \text{etc.} \quad (7.59)$$

(The use of the same symbol  $A$  should not be confusing. Otherwise we can write it as  $\Lambda^k A$  when  $A$  acts on a product of  $k$ , 1-vectors.)

**Exercise 7.6** Show that  $(\Lambda^2 A)(\Lambda^2 B) = \Lambda^2 AB$ .

*Derive the following result in linear algebra*

$$\det(Ax_1 | Ax_2 | \cdots | Ax_n) = \det(A) \det(x_1 | x_2 | \cdots | x_n). \quad (7.60)$$

### 7.3.3 Two important functionals on $\Lambda^2(C^4)$

1. The (Hermitian) inner product  $\langle, \rangle$  on  $C^4$  defines an inner product on  $\Lambda^2(C^4)$  given by

$$\langle u_1 \wedge u_2, v_1 \wedge v_2 \rangle = \left| \begin{array}{cc} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle \end{array} \right|. \quad (7.61)$$

Note that this is not (complex) bilinear; it is *conjugate* linear because

$$\langle k u_1 \wedge u_2, v_1 \wedge v_2 \rangle = \bar{k} \langle u_1 \wedge u_2, v_1 \wedge v_2 \rangle. \quad (7.62)$$

If  $U$  is unitary on  $C^n$  then  $U$  lifts to a unitary matrix in  $\Lambda^2(C^n)$  (this is simple to verify). In particular this means that the group  $SU(n)$  lifts to give a representation on  $\Lambda^2(C^n)$ . In the case of  $SU(4)$  this gives a 6-dimensional representation of  $SU(4)$  on the *complex* space  $\Lambda^2(C^4)$ .

2. The second important functional defined on pairs of elements in  $\Lambda^2(C^4)$  is obtained via the determinant function and is given by

$$\phi(u_1 \wedge u_2, v_1 \wedge v_2) = \det(u_1 | u_2 | u_3 | u_4). \quad (7.63)$$

By (7.57) we could write

$$u_1 \wedge u_2 \wedge u_3 \wedge u_4 = \phi(u_1 \wedge u_2, v_1 \wedge v_2) e. \quad (7.64)$$

Unlike the inner product,  $\phi$  is genuinely bilinear.

By (7.60)

$$\phi(A\alpha, A\beta) = \det(A)\phi(\alpha, \beta) \quad (7.65)$$

so that the bilinear functional  $\phi$  is invariant under the action of the special linear group,  $SL(4)$  and therefore also invariant under  $SU(4)$ .

## 7.4 The real 6–dimensional representations of $SU(4)$

Any bilinear, or conjugate linear, map on a space  $V$  gives a map between  $V$  and its dual space  $V^*$ . It follows that we can use *two* maps to determine a map  $V \rightarrow V$ . If we combine a bilinear and a conjugate linear map, the result is a conjugate linear map from  $V \rightarrow V$ .

In the case  $V = \Lambda^2(C^4)$  this defines the *conjugate* linear map  $J^8$  by

$$\langle J\alpha, \beta \rangle = \det(\alpha, \beta) \quad (7.66)$$

*A small change of notation.* We will write  $e_0, e_1, e_2, e_3$  for the standard basis for  $C^4$ . We choose to write the basis this way rather than  $e_1, e_2, e_3, e_4$  since we want to choose  $SU(3)$  to be the subgroup of  $SU(4)$  which fixes  $e_0$ . Then  $SU(3)$  will act on  $C^3$  which will have the basis  $e_1, e_2, e_3$ .

### 7.4.1 Properties of $J$

1.  $Je_{01} = e_{23}$ ,  $Je_{23} = e_{01}$  and cyclically for the other basis forms in  $\Lambda(C^4)$ . This follows easily, e.g. from (7.63).
2. Since  $J$  is conjugate linear, we also have  $J(ie_{01}) = -ie_{23}$ ,  $J(ie_{23}) = -ie_{01}$  and cyclically.
3.  $J^2 = 1$ . (Follows from 1. and 2. above.)
4.  $J$  commutes with every unitary operator  $U$  acting on  $\Lambda^2(C^4)$ .

*Proof:* for any 2–forms,  $\alpha$  and  $\beta$ ,

$$\langle UJ\alpha, \beta \rangle = \langle J\alpha, U^*\beta \rangle \quad (7.67)$$

$$= \det(\alpha, U^*\beta) \quad (7.68)$$

$$= \det(U\alpha, \beta) \quad (\text{as } U \text{ has determinant } 1.) \quad (7.69)$$

$$= \langle JU\alpha, \beta \rangle \quad (7.70)$$

and the result follows.

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<sup>8</sup> $J$  is known as the *Hodge* operator; see, e.g. [6], chapter 3.3.

5.  $J$  has eigenvalues  $\pm 1$ . From 1. and 2. above it follows that the eigenspace  $M^+$  corresponding to the eigenvalue  $+1$  is spanned by

$$e_{01} + e_{23}, e_{02} + e_{31}, e_{03} + e_{12}, i(e_{01} - e_{23}), i(e_{02} - e_{31}), i(e_{03} - e_{12}). \quad (7.71)$$

This is in fact an orthogonal basis for  $M^+$  but not orthonormal as every vector has norm  $\sqrt{2}$ .

Similarly  $M^-$ , the eigensubspace for  $-1$ , has a basis

$$e_{01} - e_{23}, e_{02} - e_{31}, e_{03} - e_{12}, i(e_{01} + e_{23}), i(e_{02} + e_{31}), i(e_{03} + e_{12}). \quad (7.72)$$

From this it is also clear that  $M^+ \perp M^-$ .

6.  $M^+$  and  $M^-$  are both 6– dimensional *real* spaces and from 4. above, for any unitary matrix  $U$ ,  $U : M^+ \rightarrow M^+$  and  $U : M^- \rightarrow M^-$ . So the 6–dimensional representation of  $SU(4)$  on  $\Lambda^2(C^4)$  splits into a direct sum of two 6– dimensional *real* representations on  $M^\pm$ . This gives the isomorphism between  $SU(4)$  and  $\text{Spin}(6)$ . In fact there are two such isomorphisms, one each for  $M^+$  and  $M^-$ .

With little loss of generality, we can concentrate on the representation of  $SU(4)$  on  $M^+$ . We denote this map by  $\pi_U$  so that for any  $U \in SU(4)$ ,

$$\pi_U : M^+ \rightarrow M^+. \quad (7.73)$$

Note that  $\pi$  is 2 : 1 since  $\pi_U = \pi_{-U}$ .

## 7.4.2 An illustrative example

Every unitary  $U$  is expressible as a product of elementary  $SU(2)$  matrices defined in one of the coordinate planes. Consider the general  $SU(2)$  matrix acting in the  $e_2 - e_3$  plane.

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & \bar{b} & \bar{a} \end{bmatrix}$$



with  $a, b$  complex and  $|a|^2 + |b|^2 = 1$ , and let us find the representation  $\pi_U$  on  $M^+$ .

Begin with the action of  $U$  on  $\Lambda^2(C^4)$ . It is easy to check that

$$Ue_{01} = e_{01}, Ue_{02} = ae_{02} + \bar{b}e_{03}, Ue_{03} = -be_{02} + \bar{a}e_{03}, \quad (7.74)$$

$$Ue_{23} = e_{23}, Ue_{31} = be_{12} + \bar{a}e_{31}, Ue_{12} = ae_{12} - \bar{b}e_{31}. \quad (7.75)$$

Now define the orthonormal basis  $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$  for  $M^+$  by

$$\alpha_1 = \frac{e_{01} + e_{23}}{\sqrt{2}} \text{ and cyclically,} \quad (7.76)$$

$$\beta_1 = \frac{i(e_{01} - e_{23})}{\sqrt{2}} \text{ and cyclically.} \quad (7.77)$$

**Exercise 7.7** *It is sometimes useful to know the inverse of this basis change, i.e.*

$$e_{01} = \frac{\alpha_1 - i\beta_1}{\sqrt{2}} \text{ and cyclically,} \quad (7.78)$$

$$e_{23} = \frac{\alpha_1 + i\beta_1}{\sqrt{2}} \text{ and cyclically.} \quad (7.79)$$

A little algebra gives

$$\pi_U \alpha_1 = \alpha_1,$$

$$\pi_U \alpha_2 = \operatorname{Rl}(a)\alpha_2 + \operatorname{Rl}(b)\alpha_3 + \operatorname{Im}(a)\beta_2 - \operatorname{Im}(b)\beta_3,$$

$$\pi_U \alpha_3 = -\operatorname{Rl}(b)\alpha_2 + \operatorname{Rl}(a)\alpha_3 - \operatorname{Im}(b)\beta_2 - \operatorname{Im}(a)\beta_3,$$

$$\pi_U \beta_1 = \beta_1,$$

$$\pi_U \beta_2 = -\operatorname{Im}(a)\alpha_2 + \operatorname{Im}(b)\alpha_3 + \operatorname{Rl}(a)\beta_2 + \operatorname{Rl}(b)\beta_3,$$

$$\pi_U \beta_3 = \operatorname{Im}(b)\alpha_2 + \operatorname{Im}(a)\alpha_3 - \operatorname{Rl}(b)\beta_2 + \operatorname{Rl}(a)\beta_3.$$

This can be written in a more compact form. Choose the order of the basis elements for  $M^+$  as  $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ . Then

$$\pi_U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad (7.80)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \operatorname{Rl}(a) & -\operatorname{Rl}(b) \\ 0 & \operatorname{Rl}(b) & \operatorname{Rl}(a) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \operatorname{Im}(a) & -\operatorname{Im}(b) \\ 0 & -\operatorname{Im}(b) & -\operatorname{Im}(a) \end{bmatrix}. \quad (7.81)$$

The way  $\pi_U$  breaks up naturally into this block form is a strong indication that the underlying 6– dimensional real space,  $R^6$ , should really be regarded more like  $R^3 \oplus R^3$ . Geometry is starting to tell us something (and as usual, we should be listening).

**Exercise 7.8** Consider the general  $SU(2)$  matrix acting in the  $e_3 - e_1$  plane.

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & \bar{b} \\ 0 & 0 & 1 & 0 \\ 0 & -b & 0 & a \end{bmatrix}.$$

Show that this time the representation (called  $\pi_U$ ) of  $U$  on  $M^+$  is also of the form

$$\pi_U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad (7.82)$$

where now

$$A = \begin{bmatrix} Rl(a) & 0 & Rl(b) \\ 0 & 1 & 0 \\ -Rl(b) & 0 & Rl(a) \end{bmatrix}, \quad B = \begin{bmatrix} -Im(a) & 0 & -Im(b) \\ 0 & 0 & 0 \\ -Im(b) & 0 & Im(a) \end{bmatrix}. \quad (7.83)$$

**Exercise 7.9** Finally for completeness, consider the general  $SU(2)$  matrix acting in the  $e_1 - e_2$  plane.

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & -b & 0 \\ 0 & \bar{b} & \bar{a} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Show that the corresponding matrix  $\pi_U$  is

$$\pi_U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad (7.84)$$

where

$$A = \begin{bmatrix} Rl(a) & -Rl(b) & 0 \\ Rl(b) & Rl(a) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} Im(a) & -Im(b) & 0 \\ -Im(b) & -Im(a) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (7.85)$$

### 7.4.3 A group $G$ of block matrices in $SO(6)$

(For simplicity and with only a slight abuse of notation, let us write  $U$  for  $\pi_U$ .) The three examples above all arise from  $SU(4)$  matrices which fix  $e_0$  and are complex rotations in one of the three remaining coordinate planes. As such they generate  $SU(3)$ . They are all represented as  $SO(6)$  matrices of the form

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}. \quad (7.86)$$

Since  $U \in SO(6)$ , it follows that

$$A^T A + B^T B = AA^T + BB^T = I, \quad (7.87)$$

$$A^T B \text{ and } BA^T \text{ are symmetric.} \quad (7.88)$$

Conversely this block description (7.86) - (7.88) can be used to describe  $SU(3)$  inside  $SO(6)$  as follows.

**Theorem 7.3** *The set  $G$  of matrices satisfying (7.86) - (7.88) forms a subgroup of  $SO(6)$  isomorphic to  $U(1) \times SU(3)$ .*

*Proof:* 1. Firstly check the group properties.

For  $U \in G$ ,  $U^{-1} = U^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}$  which is in  $G$ . Further the product of two elements of  $G$  is

$$\begin{bmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -B_2 \\ B_2 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 A_2 - B_1 B_2 & -(B_1 A_2 + A_1 B_2) \\ B_1 A_2 + A_1 B_2 & A_1 A_2 - B_1 B_2 \end{bmatrix} \quad (7.89)$$

which is again in  $G$ . So  $G$  is a group.

2. One way to identify  $G$  is to consider the number of degrees of freedom (parameters) needed to describe elements of  $G$ . Since  $A$  and  $B$  are real  $3 \times 3$  matrices, we start with a total of 18 degrees of freedom. Equation (7.87) imposes 6 conditions and (7.88) imposes another 3. So we are left with 9 degrees of freedom.

Recall that  $SU(3)$  can be regarded as a subset of  $G$  (since the three examples above were all of matrices in  $e_1, e_2, e_3$  space and therefore generate

$SU(3)$  and  $SU(3)$  is an 8-parameter group.

To identify the 9th parameter note that  $Z = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$  is in  $G$  but  $Z$  does not correspond to an  $SU(3)$  matrix because of the way  $SU(3)$  is chosen in  $SU(4)$ . In fact it is easy to show that the  $SU(4)$  matrix which defines  $Z$  is

$$\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{bmatrix}$$

where  $\omega = e^{-\frac{\pi}{4}}$  (so that  $\omega^2 = -i$ ) which is in  $SU(4)$  but not in  $SU(3)$ .

The matrices

$$Z(\theta) = \begin{bmatrix} \cos\theta I_3 & -\sin\theta I_3 \\ \sin\theta I_3 & \cos\theta I_3 \end{bmatrix}$$

which define  $U(1)$ , form the centre of  $G$  and  $G$  is then isomorphic to  $U(1) \times SU(3)$ .

**Exercise 7.10** (An alternate definition for  $G$ ).  $G$  is set of all matrices in  $SO(6)$  which commute with  $Z$ .

**Decomposition of  $G$  as  $U(1) \times SU(3)$ .**

Let us consider more closely, the problem of finding the decomposition of an element in  $G$  as a product of elements in  $Z(\theta) \times SU(3)$ .

Let

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in G. \quad (7.90)$$

Clearly we can also regard  $U$  as a unitary matrix, i.e. as an element of  $SU(6)$ . Let

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \quad (I = I_3). \quad (7.91)$$

Then

$$\text{Then } U' = P^{-1}UP = \dots = \begin{bmatrix} A + iB & 0 \\ 0 & A - iB \end{bmatrix}. \quad (7.92)$$

$U'$  is in fact the matrix of the transformation  $U$  when written in terms of the orthonormal basis  $\{e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\}$  in  $C^6$ .

Now  $A \pm iB \in U(3)$ . Let  $\det(A + iB) = e^{i\theta}$  so that (taking conjugates and remembering that  $A$  and  $B$  are real matrices),  $\det(A - iB) = e^{-i\theta}$ . Put

$$A + iB = e^{\frac{i\theta}{3}} X \quad \text{and} \quad (7.93)$$

$$A - iB = e^{-\frac{i\theta}{3}} \bar{X} \quad (7.94)$$

with now  $X$  and its conjugate  $\bar{X}$  in  $SU(3)$ . So we have

$$U' = \begin{bmatrix} e^{\frac{i\theta}{3}} X & 0 \\ 0 & e^{-\frac{i\theta}{3}} \bar{X} \end{bmatrix}. \quad (7.95)$$

Writing

$$X = e^{-i\frac{\theta}{3}}(A + iB) = A' + iB' \quad (A', B' \text{ real}) \quad (7.96)$$

it follows that

$$U = PU'P^{-1} \quad (7.97)$$

$$= \frac{1}{2} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \begin{bmatrix} e^{\frac{i\theta}{3}} X & 0 \\ 0 & e^{-i\frac{\theta}{3}} \bar{X} \end{bmatrix} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \quad (7.98)$$

$$= \frac{1}{2} \begin{bmatrix} e^{\frac{i\theta}{3}} X & e^{-i\frac{\theta}{3}} \bar{X} \\ -ie^{\frac{i\theta}{3}} X & ie^{-i\frac{\theta}{3}} \bar{X} \end{bmatrix} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \quad (7.99)$$

$$= \dots \quad (7.100)$$

$$= \begin{bmatrix} \cos \frac{\theta}{3} A' - \sin \frac{\theta}{3} B' & -(\sin \frac{\theta}{3} A' + \cos \frac{\theta}{3} B') \\ \sin \frac{\theta}{3} A' + \cos \frac{\theta}{3} B' & \cos \frac{\theta}{3} A' - \sin \frac{\theta}{3} B' \end{bmatrix} \quad (7.101)$$

$$= Z\left(\frac{\theta}{3}\right) \begin{bmatrix} A' & -B' \\ B' & A' \end{bmatrix}. \quad (7.102)$$

*To summarize:* Let  $U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in G$  and let  $\det(A + iB) = e^{i\theta}$ . Then

$$U = Z\left(\frac{\theta}{3}\right) \begin{bmatrix} A' & -B' \\ B' & A' \end{bmatrix} \quad (7.103)$$

where  $A' + iB'$  is the Cartesian decomposition of  $e^{-i\frac{\theta}{3}}(A + iB)$ .

This is the decomposition of  $G$  as  $G = U(1) \times SU(3)$ .

#### 7.4.4 Identification of $G$ and $SU(3)$ inside $\text{Spin}(6)$

Here we want to understand more precisely, how the matrices in  $G = U(1) \times SU(3)$  can be described inside  $\text{Spin}(6)$ . To do this it is useful to remind ourselves of an important result for orthogonal matrices, known as the ‘Cosine-Sine’ or C-S decomposition (see e.g. [7], chapter 2).

The Cosine-Sine (C-S) decomposition of orthogonal matrices can be extremely useful. For our purposes this decomposition will make it easy to pick out which matrices in  $SO(6)$  arise from unitary matrices in  $SU(3)$ .

The C-S decomposition of a  $2N \times 2N$  orthogonal matrix  $U$  is as follows

$$U = \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix} D \begin{bmatrix} R_0^T & 0 \\ 0 & R_1^T \end{bmatrix} \quad (7.104)$$

where the matrices  $L_0, L_1, R_0, R_1$  are  $N \times N$  orthogonal and

$$D = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \quad (7.105)$$

with  $C = \text{diag}(\cos\theta_1, \cos\theta_2, \dots, \cos\theta_N)$  and  $S = \text{diag}(\sin\theta_1, \sin\theta_2, \dots, \sin\theta_N)$  for some angles  $\theta_1, \theta_2, \dots, \theta_N$ .

(In our case,  $N = 3$  so that all the matrices appearing on the right inside the block will be  $3 \times 3$ .)

Now suppose that

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in G.$$

Consider first, the sub-matrix  $A$ . Its singular-value decomposition (SVD) means that we can write

$$A = LCR^T \quad (7.106)$$

where  $L, R$  are  $3 \times 3$  orthogonal matrices and  $C$  is a diagonal matrix whose elements lie in the interval  $[-1, 1]$ . (This last follows from (7.87) since  $0 \leq A^T A \leq I$ .)

To find the SVD for  $B$ , start with (7.87) which gives

$$BB^T = I - AA^T = L(I - C^2)L^T = LS^2L^T$$

for some diagonal matrix  $S$  whose values satisfy  $S^2 = I - C^2$ .

Using (7.88), it is straightforward to show (and left as an exercise), that the SVD decomposition of  $B$  is

$$B = LSR^T$$

and finally because  $C^2 + S^2 = I$ , we can write

$$C = \begin{bmatrix} \cos\theta_1 & 0 & 0 \\ 0 & \cos\theta_2 & 0 \\ 0 & 0 & \cos\theta_3 \end{bmatrix}, \quad S = \begin{bmatrix} \sin\theta_1 & 0 & 0 \\ 0 & \sin\theta_2 & 0 \\ 0 & 0 & \sin\theta_3 \end{bmatrix} \quad (7.107)$$

for angles  $\theta_1, \theta_2, \theta_3$ .

Putting all these decompositions together means that we have

$$U = \begin{bmatrix} LCR^T & -LSR^T \\ LSR^T & LCR^T \end{bmatrix} \quad (7.108)$$

$$= \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} \quad (7.109)$$

and we have the C-S decomposition for  $U$ .

So matrices  $U \in G$  are singled out as those matrices in  $SO(6)$  whose C-S decomposition (7.104) has  $L_0 = L_1$  and  $R_0 = R_1$ .

### An example:

Consider the example given in section 7.4.2; the  $SU(3)$  matrix  $U$  which defines a ‘rotation’ in the  $e_2 - e_3$  plane.

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & \bar{b} & \bar{a} \end{bmatrix}.$$

As we saw above  $U$  defines the matrix  $\pi_U$  (or more simply, still written as  $U$ ) in  $SO(6)$

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad (7.110)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Rl(a) & -Rl(b) \\ 0 & Rl(b) & Rl(a) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Im(a) & -Im(b) \\ 0 & -Im(b) & -Im(a) \end{bmatrix}. \quad (7.111)$$

A straightforward calculation gives

$$U = \begin{bmatrix} \mathbb{L} & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix} \quad (7.112)$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} \quad (7.113)$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & -\sin \theta \end{bmatrix}. \quad (7.114)$$

(Note that the three angles in  $C$  and  $S$  are therefore  $0, \theta, -\theta$ .)

The connection with these angles and the entries in  $U$  can be written as

$$\begin{aligned} Rl(a) &= \cos \theta \cos(\alpha - \beta) \\ Im(a) &= \sin \theta \cos(\alpha + \beta) \\ Rl(b) &= \cos \theta \sin(\alpha - \beta) \\ Im(b) &= -\sin \theta \sin(\alpha + \beta) \end{aligned}$$

From these we can find the various rotation angles. To find the angles defining  $L$  and  $R$  we see that

$$\tan(\alpha - \beta) = \frac{Rl(b)}{Rl(a)}, \quad \text{and} \quad \tan(\alpha + \beta) = -\frac{Im(b)}{Im(a)}. \quad (7.115)$$

which give  $\alpha$  and  $\beta$ .

$\theta$  can be found from the equations

$$\begin{aligned} \cos \theta &= Rl(a) \cos(\alpha - \beta) + Rl(b) \sin(\alpha - \beta) \\ \sin \theta &= Im(a) \cos(\alpha + \beta) - Im(b) \sin(\alpha + \beta). \end{aligned}$$



*To Summarize:* Elements of  $SU(3)$  (more generally, elements of  $U(1) \times SU(3)$ ) can be realised as orthogonal matrices acting on  $R^3 \oplus R^3$ . If we take bases  $\alpha_1, \alpha_2, \alpha_3$  and  $\beta_1, \beta_2, \beta_3$  for each of the component spaces  $R^3$ , then the matrices are products of rotations of two types.

1. Rotations (denoted above by  $L$  or  $R$ ) which act independently on either of the two subspaces  $R^3$ .

2. Rotations (written above as  $\begin{bmatrix} C & -S \\ S & C \end{bmatrix}$ ) in the three coordinate planes spanned by  $\{\alpha_i, \beta_i\}$ ,  $i = 1, 2, 3$ .

#### 7.4.5 Representations of the Gell-Mann matrices in $SO(6)$

In 1961, Gell-Mann and Nishijima independently proposed that  $SU(3)$  had a central role to play in the classification of elementary particles; in particular it gave rise to the suspicion of a substructure of these so-called fundamental particles. (See e.g. [8], chapter 2.)

The lie algebra  $su(3)$  is an 8-dimensional real algebra and perhaps the best known and most useful basis, consists of the Gell-Mann matrices  $\lambda_1, \dots, \lambda_8$ , defined as

$$\begin{aligned} \lambda_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_2 &= \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \lambda_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, & \lambda_5 &= \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, & \lambda_6 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ \lambda_7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

These matrices are Hermitian and generate eight, 1-parameter families of groups in  $SU(3)$  according to the formula

$$\Lambda_j(t) = \exp it\lambda_j, \quad j = 1, \dots, 8. \quad (7.116)$$

In the following we will be interested in showing how the  $\Lambda_j$ s can be represented in  $SO(6)$ , following on from the general representation of  $SU(3)$  as a subgroup of  $SO(6)$ .

We will look at the case of  $\Lambda_1$  in some detail; the other cases are reasonably similar and we will content ourselves with just stating the results.

So begin with

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we obtain

$$\Lambda_1(t) = e^{it\lambda_1} = \begin{bmatrix} \cos t & i \sin t & 0 \\ i \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The next step is to describe the action of  $\Lambda_1$  in the way given in section 7.4.2. Recall that this means firstly regarding  $\Lambda_1$  as acting on  $C^4$ , fixing a basis element  $e_0$  in the basis  $\{e_0, e_1, e_2, e_3\}$  and then finding the action of  $\Lambda_1$  on  $\Lambda^2(C^4)$ . A straightforward calculation leads to

$$\Lambda_1 e_{01} = (\Lambda_1 e_0) \wedge (\Lambda_1 e_1) = e_0(\cos t e_1 + i \sin t e_2) = \cos t e_{01} + i \sin t e_{02}$$

and similarly

$$\Lambda_1 e_{02} = i \sin t e_{01} + \cos t e_{02}$$

$$\Lambda_1 e_{03} = e_{03}$$

$$\Lambda_1 e_{23} = \cos t e_{23} - i \sin t e_{31}$$

$$\Lambda_1 e_{31} = -i \sin t e_{23} + \cos t e_{31}$$

$$\Lambda_1 e_{12} = e_{12}.$$

In terms of the basis  $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$  (see definitions (7.76), (7.77)), this

can be written as

$$\begin{aligned}
\Lambda_1 \alpha_1 &= \cos t \alpha_1 + \sin t \beta_2 \\
\Lambda_1 \alpha_2 &= \cos t \alpha_2 + \sin t \beta_1 \\
\Lambda_1 \alpha_3 &= \alpha_3 \\
\Lambda_1 \beta_1 &= -\sin t \alpha_2 + \cos t \beta_1 \\
\Lambda_1 \beta_2 &= -\sin t \alpha_1 + \cos t \beta_2 \\
\Lambda_1 \beta_3 &= \beta_3.
\end{aligned}$$

So we have the representation of  $\Lambda_1$  as an  $SO(6)$  matrix

$$\Lambda_1 = \begin{bmatrix} \cos t & 0 & 0 & 0 & -\sin t & 0 \\ 0 & \cos t & 0 & -\sin t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sin t & 0 & \cos t & 0 & 0 \\ \sin t & 0 & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad (7.117)$$

where

$$A = \begin{bmatrix} \cos t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \sin t & 0 \\ \sin t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally we find the C-S decomposition for  $\Lambda_1$  in the form given by (7.109). That is, we look for  $3 \times 3$  *rotation* matrices  $L$  and  $R$ , and diagonal matrices  $C$  and  $S$  such that

$$\Lambda_1 = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} R^T & 0 \\ 0 & R^T \end{bmatrix}$$

or, more usefully, such that

$$A = LCR^T, \quad B = LSR^T. \quad (7.118)$$

These matrices are easy to find since they describe rotations in a plane in  $R^3$ .

**Exercise 7.11** Use (7.118) to show that

$$L = R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} \cos t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} \sin t & 0 & 0 \\ 0 & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, regarded as a matrix in  $SO(6)$ ,  $\Lambda_1$  decomposes as a product of three rotations.  $L$  and  $R$  are rotations in the  $\{\alpha_1, \alpha_2\}$  and  $\{\beta_1, \beta_2\}$  planes while

$$\begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

describes rotations through an angle  $t$  in the  $\{\alpha_1, \beta_1\}$  and  $\{\alpha_2, \beta_2\}$  planes and the null rotation in the  $\{\alpha_3, \beta_3\}$  plane.

**Exercise 7.12** Use symmetry to obtain a similar decomposition in the cases of  $\lambda_4$  and  $\lambda_6$ .

The case of  $\lambda_2$  is a little different. Using the same technique as with  $\lambda_1$ , we obtain, after a few calculations, the  $SO(6)$  representation

$$\Lambda_2(t) = e^{it\lambda_2} = \begin{bmatrix} \cos t & \sin t & 0 & 0 & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t & \sin t & 0 \\ 0 & 0 & 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this case,  $\Lambda_2$  is already in block diagonal form and we do not need to find its C-S decomposition<sup>9</sup> It is clear that  $\Lambda_2$  is a direct sum of two rotations, through an angle  $t$ , in each of the component subspaces  $\{\alpha_1, \alpha_2, \alpha_3\}$  and  $\{\beta_1, \beta_2, \beta_3\}$ .

**Exercise 7.13** As in the previous exercise, use symmetry arguments to find the descriptions for  $\Lambda_5$  and  $\Lambda_7$ .

---

<sup>9</sup>This is because the centre matrix  $\begin{bmatrix} C & -S \\ S & C \end{bmatrix}$  is the identity.

There now remain two cases  $\lambda_3$  and  $\lambda_8$ . Since both are in diagonal form, the analysis is a little easier.

In the first case we have

$$\Lambda_3(t) = e^{it\lambda_3} = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which leads to the  $SO(6)$  representation

$$\Lambda_3(t) = \begin{bmatrix} \cos t & 0 & 0 & -\sin t & 0 & 0 \\ 0 & \cos t & 0 & 0 & \sin t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin t & 0 & 0 & \cos t & 0 & 0 \\ 0 & -\sin t & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is the direct sum of three rotations; the first through an angle  $t$  in the  $\{\alpha_1, \beta_1\}$  plane, the second through an angle  $-t$  in the  $\{\alpha_2, \beta_2\}$  plane, and the third is the null rotation in the  $\{\alpha_3, \beta_3\}$  plane.

Finally, consider  $\lambda_8$ , a little different to the others as may be expected. Here we firstly have

$$\Lambda_8(t) = e^{\frac{it\lambda_8}{\sqrt{3}}} = \begin{bmatrix} e^{\frac{it}{\sqrt{3}}} & 0 & 0 \\ 0 & e^{\frac{it}{\sqrt{3}}} & 0 \\ 0 & 0 & e^{-\frac{2it}{\sqrt{3}}} \end{bmatrix}$$

which then gives for the  $SO(6)$  matrix

$$\Lambda_8(t) = \begin{bmatrix} \cos \frac{t}{\sqrt{3}} & 0 & 0 & -\sin \frac{t}{\sqrt{3}} & 0 & 0 \\ 0 & \cos \frac{t}{\sqrt{3}} & 0 & 0 & -\sin \frac{t}{\sqrt{3}} & 0 \\ 0 & 0 & \cos \frac{2t}{\sqrt{3}} & 0 & 0 & \sin \frac{2t}{\sqrt{3}} \\ \sin \frac{t}{\sqrt{3}} & 0 & 0 & \cos \frac{t}{\sqrt{3}} & 0 & 0 \\ 0 & \sin \frac{t}{\sqrt{3}} & 0 & 0 & \cos \frac{t}{\sqrt{3}} & 0 \\ 0 & 0 & -\sin \frac{2t}{\sqrt{3}} & 0 & 0 & \cos \frac{2t}{\sqrt{3}} \end{bmatrix}.$$

This is a decomposition of  $\Lambda_8$  into a product of three rotations: one is in the  $\{\alpha_1, \beta_1\}$  plane, through an angle  $\frac{t}{\sqrt{3}}$ , the second is similar but in the  $\{\alpha_2, \beta_2\}$  plane, while the third is through an angle  $-\frac{2t}{\sqrt{3}}$ , in the  $\{\alpha_3, \beta_3\}$  plane.

**Exercise 7.14** Find anyway, the C-S decompositions for  $\Lambda_3$  and  $\lambda_8$  and verify that they are in fact trivial.

The different descriptions of these  $\Lambda_i$  matrices is perhaps unfortunate but is wholly due to the original asymmetric choice of the Gell-Mann matrices.

## 7.4.6 Conservation of commutation relations

As an illustrative example of the usefulness of this representation of  $SU(3)$  (more generally of  $G$ ) by matrices in  $SO(6)$ , let us see how  $SU(3)$  ‘naturally’ preserves the Poisson product in 3-space.

If  $f$  and  $g$  are functions defined on  $n$ -dimensional phase space, i.e. are functions of  $(x_i, p_i)$ , the Poisson bracket is defined as

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right). \quad (7.119)$$

It follows that

$$\{x_i, x_j\} = \{p_i, p_j\} = 0 \quad \text{and} \quad (7.120)$$

$$\{x_i, p_j\} = \delta_{ij}. \quad (7.121)$$

The idea in this section is to show that these Poisson brackets on  $(x_i, p_i)$  are preserved by  $G$ .

As above let  $U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$  and

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}. \quad (7.122)$$

Then

$$x'_i = (Ax - Bp)_i \quad (7.123)$$

$$= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 - (b_{i1}p_1 + b_{i2}p_2 + b_{i3}p_3) \quad (7.124)$$

and similarly

$$p'_j = (Bx + Ap)_j \quad (7.125)$$

$$= b_{j1}x_1 + b_{j2}x_2 + b_{j3}x_3 + a_{j1}p_1 + a_{j2}p_2 + a_{j3}p_3 \quad (7.126)$$

so that using (7.120) and (7.121)

$$\{x'_i, p'_j\} = a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} + b_{i1}b_{j1} + b_{i2}b_{j2} + b_{i3}b_{j3} \quad (7.127)$$

$$= (AA^T + BB^T)_{ij} = \delta_{ij} \quad \text{by (7.87)}. \quad (7.128)$$

Likewise

$$\{x'_i, x'_j\} = (AB^T)_{ij} \quad \text{and} \quad (7.129)$$

$$\{p'_i, x'_j\} = (BA^T)_{ij} \quad (7.130)$$

and both expressions are 0 as the Poisson bracket is anti-symmetric while  $AB^T$  and  $BA^T$  are symmetric after (7.88).

So  $G$  preserves the Poisson bracket.

An important, if simple, example of a transformation which keeps the Poisson bracket invariant is

$$\alpha_1 \rightarrow \beta_1, \beta_1 \rightarrow -\alpha_1 \quad (7.131)$$

and the remaining  $\alpha_i, \beta_i$  unchanged. Its matrix representation shows that this is an element of  $G$ . The matrix is

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and it is easy to show that  $A$  and  $B$  satisfy conditions (7.87) and (7.88) so that  $U \in G$ .

**Exercise 7.15** Show that the corresponding matrix in  $SU(4)$  is

$$\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{bmatrix} \quad \text{where } \omega = e^{i\frac{\pi}{4}}. \quad (7.132)$$

## 7.5 Notes and references

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# Chapter 8

## The hydrogen atom

### 8.1 Introduction

By now, we have developed enough of Clifford algebra theory to give a good foundation for applications. The examples we have chosen have been selected to give an idea of how Clifford algebras may be used to illustrate (especially) any underlying geometry. We have restricted ourselves to areas in classical and quantum physics but many more examples can be found. For example, chapter 5 by J. M. Selig in [1] gives a number of examples in engineering and computer science which are not only highly interesting in their own right, but also indicative of the breadth of the whole subject area.

We have to be careful too (mathematicians especially), that we should not be seen to be satisfied with talking to the converted. The author's contention is that Clifford algebras should play a major role in the description of many results in physics. Against that, is the undeniable fact that a large majority of physicists would claim that they can get along quite well without this approach. Their claim is based on the view that (e.g.) quantum mechanics can be described perfectly well using the modern, and well understood, apparatus of complex Hilbert space and that the physical phenomena can all be described quite adequately with this approach. So why should they bother using a new and unfamiliar language when the old one serves perfectly well?

The only viable answer to that, is that we enthusiasts of Clifford algebra, must convince those recalcitrant souls that there is a *real benefit* in using Clifford algebra, because it provides a deeper insight into their subject area. This deeper insight *must* not only shed more light but must also hold the promise of a better way of obtaining results. It seems self-evident that without this approach, the existing schism (real, geometric versus complex) will remain.<sup>1</sup>

So in this chapter we consider the fundamental problem; that of solving the Dirac equation for the hydrogen atom (to the extent of finding the values of the hydrogen spectrum). These are of course well-known and we use this example to illustrate the use of a Clifford algebra approach to an important problem.

Because of the spherical symmetry in the geometry of the hydrogen atom, it makes a great deal of sense to write the Dirac equation as well as a number of operators associated with it, in spherical polar coordinate form. So we will start by describing this coordinate system, as it may be expressed in Clifford algebras.

## 8.2 Spherical polar coordinates

We begin by reviewing the basic definitions and ideas and write them in Clifford algebra terms. Then we can consider the various angular momentum and spin operators and see how they can be defined from this perspective.

As we are dealing with 3-space only, the appropriate algebra will be taken to be  $Cl(0,3)$ , though  $Cl(3,0)$  works identically. Start with the usual definition

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta. \quad (8.1)$$

---

<sup>1</sup>Mathematicians are sometimes put off by the apparently cavalier approach that they see in physicists, because it smacks of a non-rigorous attitude. As an example, consider the way tensors are introduced in physics. Typically you come across statements along the line “tensors are objects which transform in such-and-such a way.” This seems nonsense as a definition but in fact physicists almost invariably prefer a *description* to a *definition*. As Gian-Carlo Rota pointed out “descriptions are more fundamental than definitions, even if they are less rigorous, because descriptions motivate and ultimately determine definitions, not the other way around.”

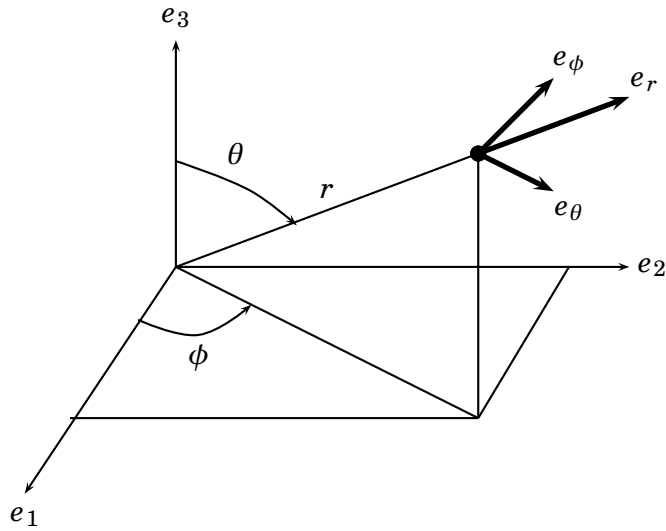


Figure 8.1

The change of basis can be written as <sup>2</sup>

$$\begin{aligned} e_r &= \sin\theta \cos\phi e_1 + \sin\theta \sin\phi e_2 + \cos\theta e_3, \\ e_\theta &= \cos\theta \cos\phi e_1 + \cos\theta \sin\phi e_2 - \sin\theta e_3, \\ e_\phi &= -\sin\phi e_1 + \cos\phi e_2. \end{aligned}$$

or

$$\begin{bmatrix} e_r \\ e_\theta \\ e_\phi \end{bmatrix} = R_{\theta\phi} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad (8.2)$$

where

$$R_{\theta\phi} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}. \quad (8.3)$$

(Note that  $R_{\theta\phi}$  is a rotation matrix. It could therefore be put in the form

$$R_{\theta\phi} x = qx\tilde{q}$$

for some unit quaternion  $q$ . We will look at this shortly.)

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<sup>2</sup>Beware that some authors interchange the roles of  $\phi$  and  $\theta$ .

$R_{\theta\phi}$  also provides the link between the components of a vector in the two representations. If we write

$$x = x^1 e_1 + x^2 e_2 + x^3 e_3 = x^r e_r + x^\theta e_\theta + x^\phi e_\phi$$

then a comparison of coefficients gives

$$\begin{bmatrix} x^r \\ x^\theta \\ x^\phi \end{bmatrix} = R_{\theta\phi} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}. \quad (8.4)$$

Note that

$$R_{\theta\phi}^{-1} = R_{\theta\phi}^T = \begin{bmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}. \quad (8.5)$$

**Exercise 8.1** Show that

$$e_r e_\theta e_\phi (= e_{r\theta\phi}) = e_{123}$$

(as expected!) Deduce that the space  $M$  of Pauli spinors,  $M = \langle e_1, e_2, e_3, e_{123} \rangle = \langle e_r, e_\theta, e_\phi, e_{r\theta\phi} \rangle$  (also as expected.)

**Exercise 8.2** (Writing  $R_{\theta\phi}$  in quaternion form.)

Show that

$$R_{\theta\phi} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Deduce that  $R_{\theta\phi}$  is a product of three rotations and each one can be written in quaternion form  $x : qx\tilde{q}$ . So  $R_{\theta\phi}$  is itself of this form with the quaternion being

$$\begin{aligned} q &= \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} e_{12} \right) \left( \cos \frac{\phi}{2} - \sin \frac{\phi}{2} e_{23} \right) \left( \frac{1}{4} (1 + e_{23} + e_{31} + e_{12}) \right) \\ &= \dots \\ &= \frac{1}{2} \left[ \left( \cos \frac{\theta+\phi}{2} + \sin \frac{\theta+\phi}{2} \right) + \left( \cos \frac{\theta-\phi}{2} + \sin \frac{\theta-\phi}{2} e_{23} \right) \right. \\ &\quad \left. + \left( \cos \frac{\theta-\phi}{2} - \sin \frac{\theta-\phi}{2} e_{31} \right) + \left( \cos \frac{\theta+\phi}{2} - \sin \frac{\theta+\phi}{2} \right) e_{12} \right]. \end{aligned}$$

### 8.2.1 Some differential formulae

The change to spherical polar coordinates leads to the differential formula

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{bmatrix} \quad (8.6)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \sin \theta \end{bmatrix} R_{\theta\phi} \begin{bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{bmatrix} \quad (8.7)$$

so that,

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{bmatrix} &= R_{\theta\phi}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \sin \theta \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix} \\ &= \dots \\ &= \begin{bmatrix} \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\ \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\ \cos \theta & -\frac{\sin \theta}{r} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix}. \end{aligned}$$

**Exercise 8.3** Deduce the polar form of the gradient operator

$$\nabla = e_1 \frac{\partial}{\partial x^1} + e_2 \frac{\partial}{\partial x^2} + e_3 \frac{\partial}{\partial x^3} = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}. \quad (8.8)$$

Another useful result comes from writing<sup>3</sup>

$$x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3$$

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<sup>3</sup>We often use the abbreviations  $\partial_i$  for  $\frac{\partial}{\partial x^i}$  and similarly  $\partial_r$  for  $\frac{\partial}{\partial r}$  etc.

in spherical polar terms.

$$x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3 = \begin{bmatrix} x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \quad (8.9)$$

$$= \begin{bmatrix} r \sin \theta \cos \phi & r \sin \theta \sin \phi & r \cos \theta \end{bmatrix} \quad (8.10)$$

$$\times \begin{bmatrix} \sin \theta \cos \phi \partial_r + \frac{\cos \theta \cos \phi}{r} \partial_\theta - \frac{\sin \phi}{r \sin \theta} \partial_\phi \\ \sin \theta \sin \phi \partial_r + \frac{\cos \theta \sin \phi}{r} \partial_\theta + \frac{\cos \phi}{r \sin \theta} \partial_\phi \\ \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \end{bmatrix} \quad (8.11)$$

$$= \dots \quad (8.12)$$

$$= r \partial_r. \quad (8.13)$$

**Exercise 8.4** Show that

$$\nabla^2 = \frac{1}{r^2} \left( \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right). \quad (8.14)$$

### 8.3 The angular momentum operators

Let us begin with a brief summary of the way angular momentum operators are usually introduced. With a small abuse of convention we define the operators  $L_i$  by <sup>4</sup>

$$L_1 = -\hbar(x^2 \partial_3 - x^3 \partial_2); \quad L_2 = -\hbar(x^3 \partial_1 - x^1 \partial_3); \quad L_3 = -\hbar(x^1 \partial_2 - x^2 \partial_1).$$

---

<sup>4</sup>The more standard notation is to define

$$L_1 = -i\hbar(x^2 \partial_3 - x^3 \partial_2) \quad \text{etc.}$$

but we want to avoid using complex numbers as much as possible. In any case, it is not always clear what  $i$  would correspond to in a Clifford algebra setting.

Writing these in polar coordinates gives for  $L_1$

$$\begin{aligned}
L_1 &= -\hbar(x^2\partial_3 - x^3\partial_2) \\
&= -\hbar(r\sin\theta\sin\phi)\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) \\
&\quad + \hbar r\cos\theta\left(\sin\theta\sin\phi\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\cos\phi}{r\sin\theta}\partial_\phi\right) \\
&= \dots \\
&= \hbar\left(\sin\phi\partial_\theta + \frac{\cos\theta\cos\phi}{\sin\theta}\partial_\phi\right).
\end{aligned}$$

Similarly for  $L_2$  and  $L_3$ . In summary,

$$L_1 = \hbar\left(\sin\phi\partial_\theta + \frac{\cos\theta\cos\phi}{\sin\theta}\partial_\phi\right), \quad (8.15)$$

$$L_2 = \hbar\left(-\cos\phi\partial_\theta + \frac{\cos\theta\sin\phi}{\sin\theta}\partial_\phi\right), \quad (8.16)$$

$$L_3 = -\hbar\partial_\phi. \quad (8.17)$$

Note that the  $L_i$ s do not involve  $\frac{\partial}{\partial r}$ , a useful property as we will see.

It follows also that<sup>5</sup>

$$L_1^2 + L_2^2 + L_3^2 = \frac{\hbar^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\hbar^2}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \quad (8.18)$$

The operators  $L_i$  can be written in more physical terms if we replace the partial derivatives  $\partial_i$  by momentum operators  $p_i$ . In the conventional treatment we would define

$$p_i = -i\hbar\partial_i$$

which leads to the momentum (row) vector

$$\underline{p} = -i\hbar\nabla.$$

---

<sup>5</sup>We don't write  $L_1^2 + L_2^2 + L_3^2$  as  $L^2$  - which is the traditional notation - since in Clifford algebra notation, this is ambiguous (see the discussion below). Note too that traditionally, a minus sign occurs on the right hand side, due to the extra factor of  $i$  in the definitions of the  $L_i$ s.

The situation is a little more delicate in the Clifford algebra setting. The obvious step is to define something like

$$p_i = -u \hbar \partial_i, \quad (u^2 = -1)$$

but the problem with that definition is that  $p_i$  is a bivector (since  $u$  is) and it is not at all clear how we could obtain a 1-vector  $p$  from this definition. Nor is it clear why we should choose any particular  $u$  since this contains a preferred “rotation axis”.

As the imaginary  $i$  is only introduced into the definition of  $p_i$  to obtain a Hermitian operator (from a skew-Hermitian one), the easiest solution is not to demand this and to simply define

$$p_i = -\hbar \partial_i.$$

Then we define the momentum vector (operator) to be

$$p = e_1 p_1 + e_2 p_2 + e_3 p_3 \quad (8.19)$$

and the angular momentum operators become

$$L_1 = x^2 p_3 - x^3 p_2, \quad L_2 = x^3 p_1 - x^1 p_3, \quad L_3 = x^1 p_2 - x^2 p_1.$$

**Exercise 8.5** *Verify that*

$$[L_1, L_2] = \hbar L_3, \quad \text{and cyclically.} \quad (8.20)$$

(Note that conventionally the right hand side would have an extra  $i$  term which is missing here because of our definition of  $L_i$ .)

In  $Cl(0, 3)$ , the orbital angular momentum operators  $L_i$  can be defined as the bivector part of  $x p$ , i.e.

$$\begin{aligned} x p &= (x_1 e_1 + x_2 e_2 + x_3 e_3)(e_1 p_1 + e_2 p_2 + e_3 p_3) \\ &= -(x_1 p_1 + x_2 p_2 + x_3 p_3) + (L_1 e_{23} + L_2 e_{31} + L_3 e_{12}) \\ &= L_0 + L_1 e_{23} + L_2 e_{31} + L_3 e_{12} \\ &= L_0 + L \quad \text{as a scalar plus bivector.} \end{aligned}$$

So  $L = L_1 e_{23} + L_2 e_{31} + L_3 e_{12}$  is a bivector (operator) which is the analogue of the (axial) vector  $\underline{x} \times \underline{p}$  which gives the usual definition of (orbital) angular momentum.



**Exercise 8.6** Show that in polar form,

$$xp = -\hbar r e_r \left( e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + \frac{1}{r \sin \theta} e_\phi \partial_\phi \right). \quad (8.21)$$

Deduce that  $L_0 = \hbar r \partial_r$  and that  $L = -\hbar \left( e_{r\theta} \partial_\theta + \frac{1}{\sin \theta} e_{r\phi} \partial_\phi \right)$ .

Notice (yet again) how much simpler, angular momentum is described, when using spherical polar coordinates.

An interesting property of  $L_0$  is that it commutes with all the other  $L_i$ s. This is easy to verify using the cartesian expressions. <sup>6</sup>

**Exercise 8.7** Show, for example, that

$$\begin{aligned} [L_0, L_1] \psi &= -\hbar^2 (x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3) (x^2 \partial_3 \psi - x^3 \partial_2 \psi) \\ &\quad + (x^2 \partial_3 - x^3 \partial_2) (x^1 \partial_1 \psi + x^2 \partial_2 \psi + x^3 \partial_3 \psi) \\ &= \dots \\ &= 0. \end{aligned}$$

In the usual treatment, we would define  $L^2 = L_1^2 + L_2^2 + L_3^2$ . But here,  $L$  is a bivector whose terms do not commute. So we have to be a little careful. In fact we have

$$L^2 = (L_1 e_{23} + L_2 e_{31} + L_3 e_{12})^2 \quad (8.22)$$

$$= \dots \quad (8.23)$$

$$= -(L_1^2 + L_2^2 + L_3^2) + (L_1 L_2 e_{12} + L_2 L_1 e_{21} \dots) \quad (8.24)$$

$$= -(L_1^2 + L_2^2 + L_3^2) + \hbar L \quad \text{using Exercise 5.} \quad (8.25)$$

---

<sup>6</sup>It is even easier in polar coordinates as  $L_0$  is expressed solely in terms of  $r$ , while the  $L_i$ 's are expressed in terms of  $\theta$  and  $\phi$  only.

## 8.4 Commutators of the position, momentum and spin operators

The Dirac Hamiltonian which we will look at later in this chapter, contains a gradient type term

$$e_{01}\partial_1 + e_{02}\partial_2 + e_{03}\partial_3.$$

In order to find the energy eigenvectors of the Hamiltonian, it is important to know the commutators of the Hamiltonian with various spin operators. In particular, when looking at either orbital or intrinsic spin, we observe that since each  $L_i$  commutes with  $e_0$ , the problem usually simplifies to finding the commutators of the spin operators and the momentum operator  $p = e_1p_1 + e_2p_2 + e_3p_3$ .

Begin with a technical result. which we leave as an exercise to verify.

**Exercise 8.8** (*The following commutator products are useful when looking at operators which commute with the Dirac Hamiltonian.*) Show that

1.

$$[x^i, p_j] = \begin{cases} \hbar & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

2.

$$[x^2p_3 - x^3p_2, e_1p_1] = 0 \quad \text{and cyclically,}$$

3.

$$[x^2p_3 - x^3p_2, e_2p_2] = \hbar e_2p_3 \quad \text{and cyclically,}$$

4.

$$[x^2p_3 - x^3p_2, e_3p_3] = -\hbar e_3p_2 \quad \text{and cyclically,}$$

5.

$$\left[ e_{23}, \sum_i e_i p_i \right] = 2(e_3p_2 - e_2p_3), \quad \text{and cyclically.}$$

It follows that,

$$\begin{aligned}
[L_1, p] &= \left[ x^2 p_3 - x^3 p_2, \sum_i e_i p_i \right] \\
&= \sum_i [x^2 p_3 - x^3 p_2, e_i p_i] \\
&= \hbar(e_2 p_3 - e_3 p_2) \text{ by cases 2 and 3.} \tag{8.26}
\end{aligned}$$

Similar results hold for  $L_2$  and  $L_3$ .

So the components of angular momentum do not commute with  $p$  (and therefore do not commute with the Dirac Hamiltonian) which means that we cannot hope to obtain common eigenvectors. But if we look at total angular momentum, the picture is much better. This is because (8.26) can be combined with Exercise 8.8 to give

$$\left[ L_1 + \frac{\hbar}{2} e_{23}, p \right] = 0 \tag{8.27}$$

again with similar results for  $L_2$  and  $L_3$ .

Results like (8.27) have an immediate physical interpretation. In quantum mechanics, the total angular momentum of a particle is the sum of its orbital angular momentum and its intrinsic angular momentum (its spin). We have seen that the orbital angular momentum operators are the components of the bivector operator  $L$ . Traditionally, the spin vector denoted by  $\underline{S}$  is given as

$$\underline{S} = \frac{\hbar}{2}(S_x, S_y, S_z)$$

with  $S_x, S_y, S_z$ , the Pauli spin matrices. In the Clifford algebra context, (8.27) tells us that the spin vector should be taken to be

$$\underline{S} = \frac{\hbar}{2}(e_{23}, e_{31}, e_{12})$$

and the total angular momentum operator is then defined as

$$\underline{J} = \underline{L} + \underline{S} \tag{8.28}$$

$$= \left( L_1 + \frac{\hbar}{2} e_{23}, L_2 + \frac{\hbar}{2} e_{31}, L_3 + \frac{\hbar}{2} e_{12} \right). \tag{8.29}$$

We then have

$$[J_i, p] = 0, \quad i = 1, 2, 3. \tag{8.30}$$

**Exercise 8.9** Show that  $J$  and  $L$  satisfy the same commutator relations, i.e. show that

$$[J_1, J_2] = \hbar J_3 \quad \text{and cyclically.}$$

### 8.4.1 The spin-orbit operator $K$

The two angular momentum operators discussed above ( $\underline{L}$  and  $\underline{S}$ ), act on Pauli spinors and so also on Dirac spinors when they are regarded as pairs of Pauli spinors. The spin-orbit operator  $K$  on the other hand, is defined directly on Dirac spinors with no Pauli counterpart. It is particularly useful when discussing the hydrogen atom because when looking at common eigenvectors of the Dirac Hamiltonian and  $K$ , it turns out that the angular part of the Hamiltonian factors out and we are left with the radial part only. This simplifies the problem of finding the energy values considerably as we will see.

In the Clifford algebra setting, we define the spin-orbit operator  $K$  by

$$K = e_0(L - \hbar) \tag{8.31}$$

where as above,  $L = L_1 e_{23} + L_2 e_{31} + L_3 e_{12}$ .

We want to show that  $K$  commutes with  $e_0 p$  and with each  $J_i$ .

Firstly consider the case  $e_0 p$ . To this end we first note that

$$\sum_i L_i p_i = (x^2 p_3 - x^3 p_2) p_1 + (x^3 p_1 - x^1 p_3) p_2 + (x^1 p_2 - x^2 p_1) p_3 = \dots = 0.$$

Similarly  $\sum_i p_i L_i = 0$ .

Then

$$\begin{aligned} Lp &= (L_1 e_{23} + L_2 e_{31} + L_3 e_{12})(e_1 p_1 + e_2 p_2 + e_3 p_3) \\ &= e_1 (L_2 p_3 - L_3 p_2) + \dots \\ &= e_1 ((x^3 p_1 - x^1 p_3) p_3 - (x^1 p_2 - x^2 p_1) p_2) + \dots \end{aligned}$$

while (using the result  $[x^i, p_j] = \hbar \delta_j^i$ ),

$$\begin{aligned}
pL &= (e_1 p_1 + e_2 p_2 + e_3 p_3)(L_1 e_{23} + L_2 e_{31} + L_3 e_{12}) \\
&= e_1(p_2 L_3 - p_3 L_2) + \cdots \\
&= e_1(p_2(x^1 p_2 - x^2 p_1) - p_3(x^3 p_1 - x^1 p_3)) + \cdots \\
&= e_1(x^1 p_2 p_2 - (x^2 p_2 - \hbar)p_1 - (x^3 p_3 - \hbar)p_1 + x^1 p_3 p_3) + \cdots \\
&= -e_1((x^3 p_1 - x^1 p_3)p_3 - (x^1 p_2 - x^2 p_1)p_2 - 2\hbar p_1) + \cdots.
\end{aligned}$$

So we have the result

$$pL = -Lp + 2\hbar p. \quad (8.32)$$

We can now calculate the commutator  $[K, e_0 p]$ .

$$\begin{aligned}
[K, e_0 p] &= K e_0 p - e_0 p K = e_0(L - \hbar)e_0 p - e_0 p e_0(L - \hbar) \\
&= Lp - \hbar p + pL - \hbar p = 0 \quad \text{by (8.32)}
\end{aligned}$$

(where here we used the results  $e_0 p = -p e_0$  and  $e_0 L = L e_0$ .)

So  $K$  commutes with  $e_0 p$ .

Next consider the commutator of  $K$  with (for example)  $J_3$ .

$$\begin{aligned}
[K, J_3] &= e_0(L_1 e_{23} + L_2 e_{31} + L_3 e_{12} - \hbar) \left( L_3 + \frac{\hbar}{2} e_{12} \right) \\
&\quad - \left( L_3 + \frac{\hbar}{2} e_{12} \right) e_0(L_1 e_{23} + L_2 e_{31} + L_3 e_{12} - \hbar) \\
&= e_0(L_1 L_3 e_{23} + L_2 L_3 e_{31} + L_3^2 e_{12} - \hbar L_3) \\
&\quad + \frac{\hbar}{2} e_0(-L_1 e_{31} + L_2 e_{23} - L_3 - \hbar e_{12}) \\
&\quad - e_0(L_3 L_1 e_{23} + L_3 L_2 e_{31} + L_3^2 e_{12} - \hbar L_3) \\
&\quad - \frac{\hbar}{2} e_0(L_1 e_{31} - L_2 e_{23} - L_3 - \hbar e_{12}) \\
&= e_0([L_1, L_3]e_{23} + [L_2, L_3]e_{31}) + \hbar e_0(-L_1 e_{31} + L_2 e_{23}) \\
&= e_0(-\hbar L_2 e_{23} + \hbar L_1 e_{31} - \hbar L_1 e_{31} + \hbar L_2 e_{23}) \\
&= 0
\end{aligned}$$

and  $K$  commutes with  $J_3$ . Similarly with  $J_1$  and  $J_2$ .

**Exercise 8.10** *Show that*

$$\sum_i L_i x^i = \sum_i x^i L_i = 0.$$

### 8.4.2 Remarks on some raising and lowering operators

Traditionally, solving the Dirac equation for the hydrogen atom involves finding common eigenvectors for  $L_z$  and  $L^2$ . This is possible as the operators commute.

We have seen (above) that  $L_0$  commutes with each  $L_i$ ,  $i = 1, 2, 3$  and so will also commute with  $\sum_i L_i^2$ . A plausible alternative procedure would be to find common eigenvalues with  $L_z$  replaced by  $L_0$ . This has the advantage of not requiring a fixed choice of direction for the angular momentum operator.

Following this idea, the problem then becomes one of finding the eigenvalues of these two operators. In the  $L^2$  and  $L_z$  case, this is achieved through the use of raising and lowering operators for  $L_z$ . So the question arises “are there corresponding operators for  $L_0$ ?” Whether or not these ideas bear fruit, it is instructive to look at this problem in its Clifford algebra setting where we can essentially replace  $L_0$  by the scalar part of  $xp$ . The following exercise illustrates the idea.

**Exercise 8.11** *Show that*

$$1. \quad \left[ L_0, \sum_i x^i e_i \right] = \hbar \sum_i x^i e_i, \quad (8.33)$$

$$2. \quad \left[ L_0, \sum_i e_i p_i \right] = -\hbar \sum_i e_i p_i. \quad (8.34)$$

*So we have raising and lowering operators for  $L_0$  which traditionally would allow us to move up and down the range of eigenvalues adding or subtracting  $\hbar$  at each step. However we must proceed with a certain degree of caution since in the Clifford algebra setting, the operators are not Hermitian.*

## 8.5 Eigenvalues of the angular momentum operators

We will restrict ourselves to looking at the eigenvalues of the operators  $L$  and  $K$ . Eigenvalues for other operators which arise, such as  $J^2$  may be derived from these.

In the conventional treatment, these operators are defined in such a way as to be Hermitian. This involves the occasional judicious introduction of an  $i$  factor, whose role is basically to change a skew-Hermitian operator to a Hermitian one. This approach is usually not available in the Clifford algebra setting; an example being the momentum operator  $p$ , used to define  $L$  (see the discussion above).

We are particularly interested in the operator  $K$  whose eigenvectors can be used to find solutions of the radial equation for the hydrogen atom.

The action of  $K$  on a Dirac spinor  $\Psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  is (see (8.31)),

$$K\Psi = K \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} (L - \hbar)\xi \\ -(L - \hbar)\eta \end{bmatrix} \quad (8.35)$$

i.e. the action decomposes into the action of  $L$  on the components of  $\Psi$ .

(Actually it pays to verify first that  $L$  is indeed a Hermitian operator and therefore will have real eigenvalues. To do this, note that the operator  $L_1 = x_2\partial_3 - x_3\partial_2$  is skew-Hermitian and therefore  $L_1e_{23} = (x_2\partial_3 - x_3\partial_2)e_{23}$  is Hermitian - as  $e_{23}$  plays the role of the imaginary  $i$ . Similarly for  $L_2e_{31}$  and  $L_3e_{12}$ . So their sum  $L$  is Hermitian.)

### 8.5.1 Eigenvalues and eigenvectors of $L$

The study of the eigenvalues and eigenvectors of the various angular momentum operators is usually restricted to finding the common eigenvectors of the operators  $L_3$  and  $L^2 = L_1^2 + L_2^2 + L_3^2$ . These operators are scalar-valued (meaning that they are maps on spaces of scalar-valued functions) and this

approach is often sufficient to solve problems such as classifying the stationary states of the hydrogen atom in the non-relativistic form given by Schrodinger's equation.

However in the relativistic case where the hydrogen atom is described by the Dirac equation, it becomes important to look at this problem more deeply and in particular to find an eigenvalue decomposition of  $L$  itself.

In the conventional treatment,  $L$  is described by a vector  $\underline{L}$  and this means solving a problem of the form

$$\underline{\sigma} \cdot \underline{L} = \begin{bmatrix} L_3 & L_1 + iL_2 \\ L_1 - iL_2 & -L_3 \end{bmatrix} \psi = \lambda \hbar \psi \quad (8.36)$$

where  $\psi$  is some suitable vector. We will start by looking at how this equation is normally solved, as it introduces new ideas which carry over very naturally to the Clifford algebra setting.

The eigenvector  $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$  of (8.36), is a Pauli spinor and has components which are functions of the angular variables  $\theta$  and  $\phi$ . In the usual treatment they are of the form

$$\psi_i = P_l^m(\cos\theta) e^{im\phi}$$

where  $P_l^m(x)$  is an associated Legendre polynomial and  $l, m$  are integers such that  $m$  is in the range  $-l, \dots, l$ .  $\psi_i$  may also include a constant dependent on the normalisation of  $P_l^m$ .

Turn now to the Clifford algebra setting. The first point to note is that the operator

$$L = L_1 e_{23} + L_2 e_{31} + L_3 e_{12}$$

should also act on a Pauli spinor which we recall is an element of the space  $M = Cl^-(0, 3)$  spanned by the elements  $e_1, e_2, e_3, e_{123}$ . We also need an element which corresponds to the complex  $i$ .

This last point is easily resolved. The Dirac equation includes a pure, unit quaternion  $u$  whose choice forces a particular direction (or rather, since  $u$  is a bivector, a particular axis). Since we are choosing to work in spherical polar coordinates, where the  $z$  axis is singled out, this implies that we should



choose not just  $u = e_{12}$  but also use  $u$  instead of the conventional imaginary  $i$ .

We will make this assumption for the rest of this chapter.

The eigenvalue problem for  $L$  now becomes one of solving the equation

$$L\psi = \lambda\hbar\psi \quad (8.37)$$

where the eigenvector<sup>7</sup>  $\psi$  is of the form

$$\psi = e_A P_l^m(\cos\theta) e^{um\phi} \quad (8.38)$$

with  $e_A$  an odd vector and  $P_l^m(x)$  is the *associated Legendre Polynomial*. Since there is some variation in the way these polynomials are defined (depending on the choice of any normalizing factor), we choose the following

for  $l = 0, 1, \dots$  and  $m = -l, \dots, l$ , define

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$

In the case  $m = 0$  we recover the ordinary Legendre polynomials

$$P_l(x) = P_l^0(x)$$

which may also be defined in terms of a generating function,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l.$$

Because  $L$  is a differential operator, we will need two recurrence relations among the various  $P_l^m(\theta)$  polynomials<sup>8</sup>, one for raising  $m$  and one for lowering. The results we need are these

$$\frac{d}{d\theta} P_l^m = P_l^{m+1} + m \cot\theta P_l^m, \quad (8.39)$$

$$\frac{d}{d\theta} P_l^m = -(l+m)(l-m+1)P_l^{m-1} - m \cot\theta P_l^m. \quad (8.40)$$

<sup>7</sup>Such eigenvectors  $\psi$  are called *spinor spherical harmonics*.

<sup>8</sup>For simplicity we write  $P_l^m(\theta)$  as shorthand for  $P_l^m(\cos\theta)$ .

*Proof:* These are standard results. A proof may be found, e.g. in Belousov [1].

$P_l^m$  can be defined for negative  $l$  by

$$P_{-l}^m = P_{l-1}^m.$$

It follows that for  $l \leq -1$ ,  $P_l^m$  is only defined for  $|m| \leq -(l+1)$ .

In the extreme case of  $l = 1$ ,  $P_{-1}^m = P_0^m$  which requires too that  $m = 0$ , i.e.  $P_{-1}^0 = P_0^0 (= 1)$ . We'll come back to this point later.

Using this definition, the recurrence relations (8.39) and (8.40) are also valid for negative  $l$ .

**Exercise 8.12** *Show this!*

**Exercise 8.13** *(A parity property)*

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x)$$

so that each  $P_l^m$  is either even or odd.

To clarify the action of  $L$  on spinor spherical harmonics, it is useful to break  $L$  into two parts, one comprised of  $L_3 e_{12}$  and the other  $L_1 e_{23} + L_2 e_{31}$ . The later can then be written in exponential form.

Dividing by  $\hbar$  for convenience, and recalling the polar coordinate results for  $L_i$ , (see (8.15) - (8.17))

$$\begin{aligned} \frac{1}{\hbar}(L_1 e_{23} + L_2 e_{31}) &= e_{23} (\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi) + e_{31} (-\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi) \\ &= -e_{31} (\cos \phi - e_{12} \sin \phi) \partial_\theta + e_{23} \cot \theta (\cos \phi - e_{12} \sin \phi) \partial_\phi \\ &= -e_{31} e^{-u\phi} \partial_\theta + e_{23} \cot \theta e^{-u\phi} \partial_\phi \end{aligned} \quad (8.41)$$

(recalling that we write  $u$  for  $e_{12}$ ).

(So  $L_1 e_{23} + L_2 e_{31}$  behaves much like the operators  $L^+$  or  $L^-$ , for example as in Sakurai [2], chapter 3.6.)

Now fix  $l$  and  $m$ . Write

$$Y_l^m(\theta, \phi) = P_l^m(\theta) e^{um\phi}$$

We then have the following result which we leave as an exercise,

**Exercise 8.14**

$$(L_1 e_{23} + L_2 e_{31})(e_3 Y_l^m) = \hbar e_1(l+m)l - m + 1) Y_l^{m-1} \quad (8.42)$$

$$(L_1 e_{23} + L_2 e_{31})(e_{123} Y_l^m) = -\hbar e_2(l+m)l - m + 1) Y_l^{m-1} \quad (8.43)$$

$$(L_1 e_{23} + L_2 e_{31})(e_1 Y_l^{m-1}) = \hbar e_3 Y_l^m \quad (8.44)$$

$$(L_1 e_{23} + L_2 e_{31})(e_2 Y_l^{m-1}) = -\hbar e_{123} Y_l^m \quad (8.45)$$

$$(L_3 e_{12})(e_3 Y_l^m) = \hbar m e_3 Y_l^m \quad (8.46)$$

$$(L_3 e_{12})(e_{123} Y_l^m) = \hbar m e_{123} Y_l^m \quad (8.47)$$

$$(L_3 e_{12})(e_1 Y_l^{m-1}) = -\hbar(m-1)e_1 Y_l^{m-1} \quad (8.48)$$

$$(L_3 e_{12})(e_2 Y_l^{m-1}) = -\hbar(m-1)e_2 Y_l^{m-1} \quad (8.49)$$

*These results are not independent (for example, 8.45 can be derived from 8.44 by multiplying on the left by  $e_{12}$  and rearranging terms.) They are listed here because they are all necessary for a complete description of  $L$ . Also because  $L_3$  is a simple differential operator, the last four equations are trivial.*

*Note that we have to bear in mind, commutation properties such as  $e_3$  commutes with  $u = e_{12}$  and  $e_1 u = -u e_1$ .*

The results of exercise 8.14 can be expressed more concisely and usefully. Recalling that  $M$  is the space of Pauli spinors, we can extend the definition of this space by defining (for fixed  $l, m$ ),  $M_l^m$  as the space of spherical harmonic spinors spanned by the basis  $\{e_{123} Y_l^m, e_3 Y_l^m, e_1 Y_l^{m-1}, e_2 Y_l^{m-1}\}$ . Then

$$L : M_l^m \rightarrow M_l^m$$

and (relative to this basis) can be written as the matrix

$$L = \hbar \begin{bmatrix} m & 0 & 0 & -1 \\ 0 & m & 1 & 0 \\ 0 & (l+m)(l-m+1) & -(m-1) & 0 \\ -(l+m)(l-m+1) & 0 & 0 & -(m-1) \end{bmatrix}. \quad (8.50)$$

This gives a decomposition of the angular momentum operator  $L$  into  $4 \times 4$  matrices each acting on a space  $M_l^m$ .

**Exercise 8.15** Show that if  $l \neq 0$ ,

1. the eigenvalues  $\lambda$  of  $L$  are  $-\hbar l$  and  $\hbar(l+1)$  each with multiplicity 2, and that
2. a basis for the respective eigenvectors can be taken as

$$(case \lambda = -\hbar l) \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ l+m \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -(l+m) \\ 0 \end{bmatrix}, \quad (8.51)$$

$$(case \lambda = \hbar(l+1)) \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -(l-m+1) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ l-m+1 \\ 0 \end{bmatrix}. \quad (8.52)$$

(The case  $l = 0$  requires some careful handling because now  $Y_0^m$  only exists when  $m = 0$ . So the space  $V_l^m$  is now only the two-dimensional, spanned by  $\{e_{123}, e_3\}$  and consequently  $L = 0$ .)

The (basis) eigenvectors of  $L$  can be expressed in terms of the spinor spherical harmonic functions by

$$(\lambda = -\hbar l) \quad e_{123}Y_l^m + (l+m)e_2Y_l^{m-1}, e_3Y_l^m - (l+m)e_1Y_l^{m-1}, \quad (8.53)$$

$$(\lambda = \hbar(l+1)) \quad e_{123}Y_l^m - (l-m+1)e_2Y_l^{m-1}, e_3Y_l^m + (l-m+1)e_1Y_l^{m-1}. \quad (8.54)$$

This can be simplified. First note that

$$e_{123}Y_l^m + (l+m)e_2Y_l^{m-1} = (e_3Y_l^m - (l+m)e_1Y_l^{m-1}) e_{12} \quad \text{and} \quad (8.55)$$

$$e_{123}Y_l^m - (l-m+1)e_2Y_l^{m-1} = (e_3Y_l^m + (l-m+1)e_1Y_l^{m-1}) e_{12}. \quad (8.56)$$

Next suppose that in general we write

$$\lambda = \hbar l \quad (l = 0, \pm 1, \pm 2, \dots)$$

then in the case where  $\lambda > 0$  (i.e.  $l > 0$ ), we have from (8.56) the basis eigenvector

$$e_3 Y_{l-1}^m + (l-m)e_1 Y_{l-1}^{m-1}.$$

(We call this *the* basis eigenvector as the other one can be obtained by right multiplication by  $e_{12}$  as we have just seen.)

On the other hand, if  $\lambda < 0$  (i.e.  $l < 0$ ), we have (replacing  $l$  by  $-l$  in (8.54) the basis eigenvector

$$e_3 Y_{-l}^m + (l-m)e_1 Y_{-l}^{m-1}.$$

Using the result  $P_{-l}^m = P_{l-1}^m$ , we again have  $e_3 Y_{l-1}^m + (l-m)e_1 Y_{l-1}^{m-1}$ .

We can then summarise the following result for the eigenvalues and eigenvectors of  $L$ .

**Theorem 8.1** 1. *The eigenvalues  $\lambda$  of  $L$  are of the form  $\lambda = \hbar l$  ( $l = 0, \pm 1, \pm 2, \dots$ )*

2. *The eigenspace for  $\lambda = \hbar l$  is spanned by  $\{e_3 Y_{l-1}^m + (l-m)e_1 Y_{l-1}^{m-1}, (e_3 Y_{l-1}^m - (l-m)e_1 Y_{l-1}^{m-1})e_{12}\}$  with  $m$  ranging between the allowed limits.*

**Exercise 8.16** *Recall the definition*

$$\mathcal{L}^2 = -(L_1^2 + L_2^2 + L_3^2)$$

(so that  $\mathcal{L}^2 = L^2$  with the conventional definition of  $L^2$ ).

*Use the result (8.27) that*

$$L^2 = \mathcal{L}^2 + \hbar L$$

*to derive the well-known result that the eigenvalues of  $\mathcal{L}^2$  are of the form  $\hbar^2 l(l+1)$  for  $l = 0, 1, 2, \dots$*

*What happens with the negative eigenvalues of  $L$ ?*

## 8.5.2 Eigenvalues and eigenvectors of $K$

It is now easy to find the eigenvalues and eigenvectors of the spin orbital operator  $K$ . Knowing these greatly simplifies the calculation of the energy spectrum of the hydrogen atom as we will see.

Recall that  $K = e_0 L$  so that in matrix form

$$K = \begin{bmatrix} L - \hbar \\ -(L - \hbar) \end{bmatrix}.$$

Consider the eigenvalue equation  $K\Psi = \hbar k\Psi$ , i.e.

$$\begin{bmatrix} L - \hbar \\ -(L - \hbar) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \hbar k \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \quad (8.57)$$

Then

$$\begin{aligned} L\xi &= \hbar(1+k)\xi, \\ L\eta &= \hbar(1-k)\eta \end{aligned}$$

where  $k$  is an integer.

Define now

$$\xi_k^m = e_3 Y_k^m + (k+1-m)e_1 Y_k^{m-1}. \quad (8.58)$$

Then from theorem 38.1,

1.  $\xi$  is in the subspace spanned by  $\{\xi_k^m, \xi_k^m e_{12}\}$  while,
2.  $\eta$  is in the subspace spanned by  $\{\xi_{-k}^m, \xi_{-k}^m e_{12}\}$ .

In other words, the eigenvectors  $\Psi$  corresponding to the eigenvalue  $k$  are of the form

$$\Psi = \Psi_k^m = \begin{bmatrix} \xi_k^m (a + b e_{12}) \\ \xi_{-k}^m (c + d e_{12}) \end{bmatrix} \quad (8.59)$$

where  $a, b, c, d$  are real and independent of  $\theta$  and  $\phi$ . (They may depend on  $r$  though, an important point which we will use in the next theorem.)

We will need the following result which enables us to interchange the two components of a Dirac spinor. The result in turn depends on the following two recurrence identities for associated Legendre polynomials (for details, see e.g. [1]).

$$P_{k-1}^m = \cos\theta P_k^m + (k+1-m)\sin\theta P_k^{m-1}, \quad (8.60)$$

$$\sin\theta P_k^m = (k+1-m)\cos\theta P_k^{m-1} - (k+m-1)P_{k-1}^{m-1}. \quad (8.61)$$

**Theorem 8.2** *If  $e_r$  is expressed in  $2 \times 2$  matrix form and*

$$\Psi_k^m = \begin{bmatrix} \xi_k^m f(r) \\ \xi_{-k}^m g(r) \end{bmatrix}$$

*as above, with  $f$  and  $g$  taking values in  $R \oplus Re_{12}$  then*

$$e_r \Psi_k^m = \begin{bmatrix} \xi_k^m g(r) \\ \xi_{-k}^m f(r) \end{bmatrix} e_{12}. \quad (8.62)$$

*Proof:* Since

$$e_r = \frac{x}{r} = \sin\theta \cos\phi e_1 + \sin\theta \sin\phi e_2 + \cos\theta e_3, \quad (\text{see section 8.2})$$

then in matrix form

$$e_r = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix} \quad (8.63)$$

where  $p$  is the pure quaternion

$$p = \sin\theta \cos\phi e_{23} + \sin\theta \sin\phi e_{31} + \cos\theta e_{12}$$

and so

$$e_r \Psi_k^m = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix} \begin{bmatrix} \xi_k^m f(r) \\ \xi_{-k}^m g(r) \end{bmatrix} = \begin{bmatrix} p \xi_{-k}^m g(r) \\ p \xi_k^m f(r) \end{bmatrix}. \quad (8.64)$$

So it suffices to show that  $p \xi_k^m = \xi_{-k}^m e_{12}$ . We can write

$$p = \cos\theta e_{12} + \sin\theta (\cos\phi + e_{12} \sin\phi) e_{23} = \cos\theta e_{12} + \sin\theta (e^{e_{12}\phi}) e_{23}.$$

Then

$$\begin{aligned}
p \xi_k^m &= (\cos \theta e_{12} + \sin \theta (e^{e_{12}\phi}) e_{23}) (e_3 Y_k^m + (k+1-m)e_1 Y_k^{m-1}) \\
&= e_{123} (\cos \theta Y_k^m + (k+1-m) \sin \theta e^{e_{12}\phi} Y_k^{m-1}) \\
&\quad + ((k+1-m) \cos \theta e_2 Y_k^{m-1} - \sin \theta e^{e_{12}\phi} e_2 Y_k^m) \\
&= e_{123} Y_{k-1}^m + e_2 ((k+1-m) \cos \theta Y_k^{m-1} - \sin \theta e^{-e_{12}\phi} Y_k^m) \\
&\quad (\text{using (8.61) and } e^{e_{12}\phi} e_2 = e_2 e^{-e_{12}\phi}) \\
&= e_{123} Y_{k-1}^m + e_2 (k+m-1) Y_{k-1}^{m-1} \quad \text{by (8.62)} \\
&= (e_3 Y_{k-1}^m - (k+m-1) e_1 Y_{k-1}^{m-1}) e_{12} \\
&= (e_3 Y_{-k}^m - (k+m-1) e_1 Y_{-k}^{m-1}) e_{12} \quad \text{from (8.???) } \\
&= \xi_{-k}^m e_{12} \quad \text{as required.}
\end{aligned}$$

## 8.6 The Dirac equation for the hydrogen atom

### 8.6.1 Note on central potentials

A useful result about the  $L_i$ s ( $i = 1, 2, 3$ ) is that they commute with all central potentials. To see this, if  $V = V(r)$ , then

$$\partial_i V = \frac{\partial V}{\partial x^i} = V'(r) \frac{x^i}{r}$$

so that for all  $i, j$  and functions  $\psi$  (which could be the component of a Dirac spinor)

$$\begin{aligned}
[x^i \partial_j, V] \psi &= x^i \partial_j (V \psi) - V x^i \partial_j \psi \\
&= x^i (\partial_j V) \psi + x^i V \partial_j \psi - V x^i \partial_j \psi \\
&= x^i (\partial_j V) \psi \\
&= \frac{x^i x^j}{r} V'(r) \psi.
\end{aligned}$$

It follows that for all  $i$ ,

$$\begin{aligned}
[L_i, V] \psi &= -\hbar [x^j \partial_k - x^k \partial_j, V] \psi \quad (\text{for suitable } j, k) \\
&= -\hbar \left( \frac{x^j x^k}{r} V'(r) - \frac{x^k x^j}{r} V'(r) \right) \psi = 0
\end{aligned}$$



so that  $L_i$  commutes with  $V$ .

*Corollary:* It is an easy exercise to show that

$$[K, V]\Psi = 0 \quad (8.65)$$

for any Dirac spinor  $\Psi$ , i.e.  $K$  commutes with  $V$ .

**Exercise 8.17** *Derive these results again, this time using the polar coordinate expression for  $L_i$ . See how easier the problem becomes!*

In the case of  $L_0$  however, this is not the case.

**Exercise 8.18** *Show that*

$$[L_0, V] = \hbar r V'(r)$$

and in the case of the Coulomb potential,

$$[L_0, V] = -\hbar V. \quad (8.66)$$

(So we could say that the Coulomb potential  $V$  acts as a lowering operator for  $L_0$ .)

## 8.6.2 The conventional treatment

The usual way to treat this case is to start with the (free) Dirac equation in the form

$$i \hbar \gamma^\mu \partial_\mu \psi - mc\psi = 0. \quad (8.67)$$

For a particle of charge  $q$  in an electromagnetic field  $A = (A, \underline{A})$ , we replace the partial derivatives by the covariant derivatives, i.e. make the transformation<sup>9</sup>

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + i \frac{q}{\hbar c} A_\mu$$

and the Dirac equation then becomes<sup>10</sup>

$$i \hbar \gamma^\mu \left( \partial_\mu + i \frac{q}{\hbar c} A_\mu \right) \psi - mc\psi = 0. \quad (8.68)$$

<sup>9</sup>Griffiths 10.38 (p 360)

<sup>10</sup>See e.g. Cottingham and Greenwood Chapter 7.2

In the case of the hydrogen atom, we choose (i.e. with the appropriate choice of units),  $A_0 = \frac{-Zq}{r}$  with  $Z$  a constant and  $\underline{A} = \underline{0}$ . We then have

$$i\hbar\gamma^0\left(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{Zq^2}{\hbar cr}\right)\psi + i\hbar\gamma^i\partial_i\psi - mc\psi = 0. \quad (8.69)$$

so that we have (multiplying by  $\gamma^0$ )

$$i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{Zq^2}{\hbar cr}\right)\psi + i\hbar\gamma^0\gamma^i\partial_i\psi - mc\gamma^0\psi = 0 \quad (8.70)$$

or

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{Zq^2}{r}\right)\psi + i\hbar c\gamma^0\gamma^i\partial_i\psi - mc^2\gamma^0\psi = 0. \quad (8.71)$$

We want stationary states, i.e. states  $\psi$  such that  $\psi(t, \underline{x}) = \phi(t)\Psi(\underline{x})$ . (Here  $\phi$  is a complex-valued function of  $t$  only and  $\Psi$  is a Dirac spinor which depends only on  $\underline{x}$ .) Then from (8.70),

$$i\hbar\frac{d\phi}{dt}\Psi + \frac{Zq^2}{r}\phi\Psi + i\hbar c\phi\gamma^0\gamma^i\partial_i\Psi - mc^2\phi\gamma^0\Psi = 0 \quad (8.72)$$

i.e.<sup>11</sup>

$$i\hbar\frac{1}{\phi}\frac{d\phi}{dt} + \frac{Zq^2}{r} + i\hbar c\gamma^0\gamma^i(\partial_i\Psi)\Psi^{-1} - mc^2(\gamma^0\Psi)\Psi^{-1} = 0. \quad (8.73)$$

From this we find that

$$i\hbar\frac{d\phi}{dt} = E\phi \quad (8.74)$$

where  $E$  is a constant, i.e.

$$\phi(t) = e^{-i\frac{Et}{\hbar}} \quad (8.75)$$

and  $\Psi$  satisfies the equation

$$E + \frac{Zq^2}{r} + i\hbar c\gamma^0\gamma^i(\partial_i\Psi)\Psi^{-1} - mc^2(\gamma^0\Psi)\Psi^{-1} = 0 \quad (8.76)$$

or

$$E\Psi = mc^2\gamma^0\Psi - i\hbar c\gamma^0\gamma^i\partial_i\Psi - \frac{Zq^2}{r}\Psi. \quad (8.77)$$

---

<sup>11</sup>We won't worry too much here just what  $\Psi^{-1}$  means when  $\Psi$  is a Dirac spinor.

In the spherically symmetric case, we would then write (8.76) in spherical coordinates, i.e. write  $\gamma^i \partial_i \Psi$  in spherical coordinate form. Then separation of variables again, leads to (8.76) splitting into a radial and an angular part, each of which is then solved. We omit the details since they occur again in the Clifford algebra setting which we now consider.

## 8.7 Clifford algebra version

We begin with the Dirac equation in the form

$$d\psi + \frac{mc}{\hbar} \psi u = 0 \quad (8.78)$$

where again we take  $u = e_{12}$  and we write the Dirac spinor  $\psi$  in the usual form

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

with  $\xi, \eta$  Pauli spinors, i.e. elements of  $Cl^-(0,3)$ .

Consider the coupling of a Dirac particle with an electromagnetic field given by  $A = A_0 e_0 + \underline{A}$ , which is a 1-vector in  $Cl(1,3)$  (written, like  $d$  as a  $2 \times 2$  matrix.)

The coupling means that we should make the transformation (see chapter 6)

$$d\psi \rightarrow d\psi + qA\psi u.$$

In the case of the hydrogen atom, we are interested in the case where  $A_0 = V(r)$  and  $\underline{A} = \underline{0}$ . Then with this transformation, the Dirac equation becomes

$$e_0 \partial_0 \psi + qV e_0 \psi u - (e_1 \partial_1 \psi + \dots) + \frac{mc}{\hbar} \psi u = 0. \quad (8.79)$$

To bring this into a more usual form, write  $\partial_0 = \frac{1}{c} \partial_t$  so that

$$e_0 \frac{\partial \psi}{\partial t} u = c (e_1 \partial_1 \psi + \dots) u + \frac{mc^2}{\hbar} \psi + q c V e_0 \psi. \quad (8.80)$$

To find the energy levels, (which means separating out the time component), we firstly write this as

$$\hbar \frac{\partial \psi}{\partial t} u = \hbar c (e_{01} \partial_1 \psi + \dots) u + mc^2 e_0 \psi + q \hbar c V \psi \quad (8.81)$$

and now separate out the time variable, i.e. put

$$\psi(t) = \Psi(\underline{x})\phi(t)$$

with  $\phi$  taking values in the 2-space defined by 1,  $u$  and  $\Psi$  a Dirac spinor. Then we have from (8.80) <sup>12</sup>

$$\hbar\Psi\frac{\partial\phi}{\partial t}u = \hbar c(e_{01}\partial_1\Psi + \dots)\phi u + mc^2e_0\Psi\phi + q\hbar cV\Psi\phi \quad (8.82)$$

i.e.

$$\hbar\Psi\frac{1}{\phi}\frac{\partial\phi}{\partial t}u = \hbar c(e_{01}\partial_1\Psi + \dots)u + mc^2e_0\Psi + q\hbar cV\Psi. \quad (8.83)$$

This gives the pair of equations

$$\hbar\frac{1}{\phi}\frac{\partial\phi}{\partial t}u = E \quad (8.84)$$

$$\hbar c(e_{01}\partial_1\Psi + \dots)u + mc^2e_0\Psi + q\hbar cV\Psi = E\Psi \quad (8.85)$$

for some real constant  $E$ .

The first equation gives as expected

$$\phi(t) = \exp\frac{-uEt}{\hbar} \quad (8.86)$$

while the second equation (8.84) is the time-independent Dirac equation and defines the Dirac Hamiltonian  $H_D$  by

$$H_D\Psi = \hbar c(e_{01}\partial_1\Psi + \dots)u + mc^2e_0\Psi + q\hbar cV\Psi, \quad (8.87)$$

or in terms of the momentum vector  $p$ ,

$$H_D\Psi = -ce_0p\Psi u + mc^2e_0\Psi + q\hbar cV\Psi. \quad (8.88)$$

So the problem is now an eigenvalue problem, i.e. to find the eigenvalues  $E$  of  $H_D$ . This problem is considerably simplified if we know of a suitable operator which commutes with  $H_D$ , for then we can use the eigenvectors of this operator.

---

<sup>12</sup>We must be careful in the order of terms since  $\phi$  and  $\Psi$  need not commute.

### 8.7.1 Calculating the energy levels

There is a number of operators which commute with  $H_D$  and with each other. Their eigenvalues and eigenvectors are relatively easy to find and this means that solving the time-dependent Dirac equation becomes easier too as we need only look for solutions  $\Psi$  corresponding to those eigenvectors.

Among these operators the ones usually considered, are the angular momentum operator  $L$  together with the spin-orbit operator  $K$ . Other operators such as the total angular momentum operator  $J$  can be derived in terms of these. We will see that in fact, only the operator  $K$  need be involved.

Firstly let us note that  $K$  commutes with  $H_D$  and therefore will have common eigenvectors. To see this, recall from section 8.4.1 that  $K$  commutes with  $e_0 p$ , while from the corollary in section 8.6.1,  $K$  commutes with  $V$ . Putting these two together gives the required result.

To see the importance of this result, note that the (stationary) Dirac equation

$$H_D \Psi = E \Psi$$

can be written as a system of three equations, one for each of the polar coordinates. So far as the energy spectrum is concerned, the most important is the radial equation in  $r$ . The three equations are naturally intertwined and to separate out the radial equation, we will rewrite the momentum operator  $p$  in  $H_D$  in terms of  $K$ . Then (since  $K$  commutes with  $H_D$ ) we can choose  $\Psi$  to be an eigenvector of  $K$ . The result is that the angular partial derivatives  $(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$  disappear and we obtain the radial equation. Let us look at this in more detail.

Firstly recall that

$$xp = L_0 + L = L_0 + L_1 e_{23} + L_2 e_{31} + L_3 e_{12}$$

so that (see exercise 8.6)

$$\begin{aligned} -r^2 p &= x^2 p = x(xp) \\ &= x(L_0 + L) \\ &= x(\hbar r \frac{\partial}{\partial r} + L) \end{aligned}$$

or

$$p = -\frac{x}{r^2} \left( \hbar r \frac{\partial}{\partial r} + L \right). \quad (8.89)$$

Since  $r \frac{\partial \Psi}{\partial r} = \frac{\partial(r\Psi)}{\partial r} - \Psi$  and because  $L - \hbar = e_0 K$  we have

$$p\Psi = -\frac{x}{r^2} \left( \hbar \frac{\partial(r\Psi)}{\partial r} + (L - \hbar)\Psi \right) \quad (8.90)$$

$$= -\frac{1}{r} e_r \left( \hbar \frac{\partial(r\Psi)}{\partial r} + e_0 K \Psi \right) \quad (\text{since } x = r e_r). \quad (8.91)$$

The Dirac Hamiltonian is then (see (8.87))

$$H_D \Psi = -c e_0 p \Psi u + m c^2 e_0 \Psi + q \hbar c V \Psi \quad (8.92)$$

$$= \frac{c}{r} e_0 e_r \left( \hbar \frac{\partial(r\Psi)}{\partial r} + e_0 K \Psi \right) u + m c^2 e_0 \Psi + q \hbar c V \Psi. \quad (8.93)$$

We see that in this last equation, the angular partial derivatives are both contained in the operator  $K$  term. It follows that if we take  $\Psi$  to be an eigenvector of  $K$  (as we may since  $K$  commutes with  $H_D$ ) we are left with precisely the radial equation.

Suppose that  $\Psi (= \Psi_k^m)$  is an eigenvector of  $K$ , i.e.

$$K \Psi = \hbar k \Psi \quad (8.94)$$

so that the stationary Dirac equation becomes

$$E \Psi = H_D \Psi = \frac{c}{r} e_0 e_r \left( \hbar \frac{\partial(r\Psi)}{\partial r} + \hbar k e_0 \Psi \right) u + m c^2 e_0 \Psi + q \hbar c V \Psi. \quad (8.95)$$

Let us put this in vector form to make it clearer just how the two components of  $\Psi$  behave. (If you prefer, you can always leave it in the usual Clifford algebra notation, noting that the vector notation  $\Psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$  is only another way of writing  $\Psi = \xi + \eta e_{0123}$ .)

Since  $\Psi$  is an eigenvector of  $K$  corresponding to the eigenvalue  $\hbar k$  and by (8.58) we can write

$$\Psi = \begin{bmatrix} \xi_k^m \frac{F(r)}{r} \\ \xi_{-k}^m \frac{G(r)}{r} \end{bmatrix} \quad (8.96)$$

where  $F$  and  $G$  take values in the subspace spanned by  $\{1, u = e_{12}\}$ . (The factor  $\frac{1}{r}$  is included to simplify later calculations.)

Then (by theorem 8.2 and recalling that  $e_0 e_r = -e_r e_0$ ) (8.95) becomes

$$E \begin{bmatrix} \xi_k^m \frac{F(r)}{r} \\ \xi_{-k}^m \frac{G(r)}{r} \end{bmatrix} = \frac{\hbar c}{r} e_0 e_r \begin{bmatrix} \xi_k^m \frac{\partial F}{\partial r} \\ \xi_{-k}^m \frac{\partial G}{\partial r} \end{bmatrix} e_{12} - \frac{\hbar k c}{r} e_r \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix} e_{12} \quad (8.97)$$

$$+ m c^2 e_0 \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix} + q \hbar c V \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix} \quad (8.98)$$

$$= -\frac{\hbar c}{r} e_0 \begin{bmatrix} \xi_k^m \frac{\partial G}{\partial r} \\ \xi_{-k}^m \frac{\partial F}{\partial r} \end{bmatrix} + \frac{\hbar c k}{r} \begin{bmatrix} \xi_k^m \frac{G}{r} \\ \xi_{-k}^m \frac{F}{r} \end{bmatrix} \quad (8.99)$$

$$+ m c^2 e_0 \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix} + q \hbar c V \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix} \quad (8.100)$$

$$= -\frac{\hbar c}{r} \begin{bmatrix} \xi_k^m \frac{\partial G}{\partial r} \\ -\xi_{-k}^m \frac{\partial F}{\partial r} \end{bmatrix} + \frac{\hbar c k}{r} \begin{bmatrix} \xi_k^m \frac{G}{r} \\ \xi_{-k}^m \frac{F}{r} \end{bmatrix} \quad (8.101)$$

$$+ m c^2 \begin{bmatrix} \xi_k^m \frac{F}{r} \\ -\xi_{-k}^m \frac{G}{r} \end{bmatrix} + q \hbar c V \begin{bmatrix} \xi_k^m \frac{F}{r} \\ \xi_{-k}^m \frac{G}{r} \end{bmatrix}. \quad (8.102)$$

From this we have the system of equations

$$E F = -\hbar c \frac{\partial G}{\partial r} + \frac{\hbar k c}{r} G + m c^2 F + q \hbar c V F$$

$$E G = \hbar c \frac{\partial F}{\partial r} + \frac{\hbar k c}{r} F - m c^2 G + q \hbar c V G$$

or in more recognisable form

$$(E - m c^2 - q \hbar c V) F = -\hbar c \left( \frac{\partial G}{\partial r} - \frac{k G}{r} \right) \quad (8.103)$$

$$(E + m c^2 - q \hbar c V) G = \hbar c \left( \frac{\partial F}{\partial r} + \frac{k F}{r} \right). \quad (8.104)$$

To simplify these equations somewhat, put

$$k_1 = \sqrt{\frac{m c^2 + E}{\hbar c}}, \quad k_2 = \sqrt{\frac{m c^2 - E}{\hbar c}}$$

and write the Coulomb potential  $V$  in the form

$$qV = -\frac{\alpha}{r}$$

to obtain (inverting the order of the equations)

$$\frac{\partial F}{\partial r} + \frac{kF}{r} = \left(k_1^2 + \frac{\alpha}{r}\right) G \quad (8.105)$$

$$\frac{\partial G}{\partial r} - \frac{kG}{r} = \left(k_2^2 - \frac{\alpha}{r}\right) F. \quad (8.106)$$

In the limiting case where  $r \rightarrow \infty$ , these equations simplify to

$$\frac{\partial F}{\partial r} = k_1^2 G \quad (8.107)$$

$$\frac{\partial G}{\partial r} = k_2^2 F \quad (8.108)$$

so that

$$\frac{d^2 F}{dr^2} = k_1^2 k_2^2 F.$$

So bearing in mind that  $F$  should be normalisable,  $F$  behaves like  $e^{-\lambda r}$  where  $\lambda = k_1 k_2$ . The same result holds for  $G$ .

Next define  $U_1, U_2$  by

$$F = k_1 e^{-\lambda r} (U_1 + U_2) \quad (8.109)$$

$$G = k_2 e^{-\lambda r} (U_1 - U_2). \quad (8.110)$$

Then

$$U_1 = \frac{e^{\lambda r}}{2} \left( \frac{F}{k_1} + \frac{G}{k_2} \right)$$

$$U_2 = \frac{e^{\lambda r}}{2} \left( \frac{F}{k_1} - \frac{G}{k_2} \right).$$



Using (8.106) and (8.107) and recalling the definitions of  $k_1, k_2$ ,

$$\begin{aligned}
\frac{dU_1}{dr} &= \frac{\lambda e^{\lambda r}}{2} \left( \frac{F}{k_1} + \frac{G}{k_2} \right) + \frac{e^{\lambda r}}{2} \left( \frac{1}{k_1} \frac{dF}{dr} + \frac{1}{k_2} \frac{dG}{dr} \right) \\
&= \lambda U_1 + \frac{e^{\lambda r}}{2k_1} \left( \left( k_1^2 + \frac{\alpha}{r} \right) G - \frac{kF}{r} \right) + \frac{e^{\lambda r}}{2k_2} \left( \left( k_2^2 - \frac{\alpha}{r} \right) F + \frac{kG}{r} \right) \\
&= \lambda U_1 + \frac{e^{\lambda r}}{2} (k_1 G + k_2 F) + \frac{\alpha e^{\lambda r}}{2r} \left( \frac{G}{k_1} - \frac{F}{k_2} \right) - \frac{k e^{\lambda r}}{2r} \left( \frac{F}{k_1} - \frac{G}{k_2} \right) \\
&= \lambda U_1 + k_1 k_2 U_1 + \frac{\alpha}{2r} \left( \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) U_1 - \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) U_2 \right) - \frac{k U_2}{r} \\
&= \left( 2\lambda + \frac{\alpha}{2r} \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) \right) U_1 - \left( \frac{\alpha}{2r} \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) + \frac{k}{r} \right) U_2 \\
&= \left( 2\lambda - \frac{\alpha E}{\hbar c \lambda r} \right) U_1 - \left( \frac{m c^2 \alpha}{\hbar c \lambda r} + \frac{k}{r} \right) U_2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{dU_2}{dr} &= \frac{\lambda e^{\lambda r}}{2} \left( \frac{F}{k_1} - \frac{G}{k_2} \right) + \frac{e^{\lambda r}}{2} \left( \frac{1}{k_1} \frac{dF}{dr} - \frac{1}{k_2} \frac{dG}{dr} \right) \\
&= \lambda U_2 + \frac{e^{\lambda r}}{2k_1} \left( \left( k_1^2 + \frac{\alpha}{r} \right) G - \frac{kF}{r} \right) - \frac{e^{\lambda r}}{2k_2} \left( \left( k_2^2 - \frac{\alpha}{r} \right) F + \frac{kG}{r} \right) \\
&= \lambda U_2 + \frac{e^{\lambda r}}{2} (k_1 G - k_2 F) + \frac{\alpha e^{\lambda r}}{2r} \left( \frac{G}{k_1} + \frac{F}{k_2} \right) - \frac{k e^{\lambda r}}{2r} \left( \frac{F}{k_1} + \frac{G}{k_2} \right) \\
&= \lambda U_2 - k_1 k_2 U_2 + \frac{\alpha}{2r} \left( \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) U_1 - \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) U_2 \right) - \frac{k U_1}{r} \\
&= \left( \frac{\alpha}{2r} \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) - \frac{k}{r} \right) U_1 - \frac{\alpha}{2r} \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) U_2 \\
&= \left( \frac{m c^2 \alpha}{\hbar c \lambda r} - \frac{k}{r} \right) U_1 + \frac{\alpha E}{\hbar c \lambda r} U_2.
\end{aligned}$$

Simplifying further, let  $\rho = 2\lambda r$  and obtain the system

$$\frac{dU_1}{d\rho} = \left( 1 - \frac{\alpha E}{\hbar c \lambda \rho} \right) U_1 - \left( \frac{m c^2 \alpha}{\hbar c \lambda \rho} + \frac{k}{\rho} \right) U_2 \quad (8.111)$$

$$\frac{dU_2}{d\rho} = \left( \frac{m c^2 \alpha}{\hbar c \lambda \rho} - \frac{k}{\rho} \right) U_1 + \frac{\alpha E}{\hbar c \lambda \rho} U_2 \quad (8.112)$$

We can now use the first order system (8.111), (8.112) to obtain a second order equation for  $U_2$ . From (8.112),

$$U_1 = \frac{\hbar c \lambda}{m c^2 \alpha - \hbar c \lambda k} \left( \rho \frac{dU_2}{d\rho} - \frac{\alpha E}{\hbar c \lambda} U_2 \right)$$

and substituting this for  $U$  in (8.111),

$$\begin{aligned} & \frac{\hbar c \lambda}{m c^2 \alpha - \hbar c \lambda k} \left( \rho \frac{d^2 U_2}{d\rho^2} + \frac{dU_2}{d\rho} - \frac{\alpha E}{\hbar c \lambda} \frac{dU_2}{d\rho} \right) \\ &= \left( 1 - \frac{\alpha E}{\hbar c \lambda \rho} \right) \left( \frac{\hbar c \lambda}{m c^2 \alpha - \hbar c \lambda k} \right) \left( \rho \frac{dU_2}{d\rho} - \frac{\alpha E}{\hbar c \lambda} U_2 \right) - \left( \frac{m c^2 \alpha}{\hbar c \lambda \rho} + \frac{k}{\rho} \right) U_2. \end{aligned}$$

A little algebra now gives this in a more digestible form as

$$\rho \frac{d^2 U_2}{d\rho^2} + (1 - \rho) \frac{dU_2}{d\rho} + \left( \frac{\alpha E}{\hbar c \lambda} + \frac{(k_1 k_2 \alpha)^2 - (k \lambda)^2}{\lambda^2 \rho} \right) U_2 = 0. \quad (8.113)$$

Consider finally the asymptotic case when  $\rho$  is small. Returning for the moment to the equations for  $F$  and  $G$ , (8.105) and (8.106) become

$$\frac{dF}{d\rho} + \frac{kF}{\rho} = \frac{\alpha G}{\rho} \quad (8.114)$$

$$\frac{dG}{d\rho} - \frac{kG}{\rho} = -\frac{\alpha F}{\rho}. \quad (8.115)$$

These equations can be separated as two second order equations. To find the equation for  $F$ , start with (8.114)

$$\begin{aligned} & \rho \frac{dF}{d\rho} = \alpha G - kF \quad \text{so that} \\ & \rho \frac{d^2 F}{d\rho^2} + \frac{dF}{d\rho} = \alpha \frac{dG}{d\rho} - k \frac{dF}{d\rho} \\ & \quad = \alpha \left( \frac{kG}{\rho} - \frac{\alpha F}{\rho} \right) - k \frac{dF}{d\rho} \quad \text{by (8.115)} \\ \text{i.e. } & \rho^2 \frac{d^2 F}{d\rho^2} + \rho \frac{dF}{d\rho} = \alpha k G - \alpha^2 F - k \rho \frac{dF}{d\rho} \\ & \quad = k \left( \rho \frac{dF}{d\rho} + kF \right) - \alpha^2 F - k \rho \frac{dF}{d\rho} \quad \text{by (8.114)} \end{aligned}$$

which simplifies to

$$\rho^2 \frac{d^2 F}{d\rho^2} + \rho \frac{dF}{d\rho} + (\alpha^2 - k^2) F = 0. \quad (8.116)$$

This is a homogeneous Euler-Cauchy differential equation which has the two basic solutions  $\rho^{\gamma_1}, \rho^{\gamma_2}$  where  $\gamma, \gamma_2$  are the roots of the characteristic equation

$$\gamma(\gamma - 1) + \gamma + (\alpha^2 - k^2) = 0$$

i.e.  $\gamma_i = \pm \sqrt{k^2 - \alpha^2}$ . Since normalisability implies that  $F$  is finite at  $\rho = 0$ , we choose the basic (asymptotic) solution to be  $F = \rho^\gamma$  where  $\gamma = \sqrt{k^2 - \alpha^2}$ .

**Exercise 8.19** Show that  $G$  also satisfies the differential equation (8.116). Deduce that  $U$  and  $V$  are also asymptotically of this form.

So now write

$$U_2(\rho) = \rho^\gamma W(\rho). \quad (8.117)$$

Equation (8.113) becomes

$$\begin{aligned} 0 = & \rho^{\gamma+1} \frac{d^2 W}{d\rho^2} + 2\gamma\rho^\gamma \frac{dW}{d\rho} + \gamma(\gamma - 1)\rho^{\gamma-1} \\ & + (1 - \rho) \left( \gamma\rho^{\gamma-1} W + \rho^\gamma \frac{dW}{d\rho} \right) \\ & + \left( \frac{\alpha E}{\hbar c \lambda} + \frac{(k_1 k_2 \alpha)^2 - (k \lambda)^2}{\lambda^2 \rho} \right) \rho^\gamma W. \end{aligned}$$

Dividing by  $\rho^\gamma$  and reorganising terms gives

$$\rho^\gamma \frac{d^2 W}{d\rho^2} + (2\gamma + 1 - \rho) W + \left( \frac{\alpha E}{\hbar c \lambda} - \gamma + \frac{(\gamma \lambda)^2 + (k_1 k_2 \alpha)^2 - (k \lambda)^2}{\lambda^2 \rho} \right) W = 0$$

which simplifies to

$$\rho \frac{d^2 W}{d\rho^2} + (2\gamma + 1 - \rho) W + \left( \frac{\alpha E}{\hbar c \lambda} - \gamma \right) W = 0. \quad (8.118)$$

This equation will give us the allowed values of  $E$ . (8.118) is Kummer's (or Laguerre's associated) differential equation (see e.g. [3]) usually written in the form

$$x \frac{d^2 y}{dx^2} + (v + 1 - x) \frac{dy}{dx} + \lambda y = 0$$

which has a basic solution (a confluent hypergeometric function)

$$y = 1 - \frac{\lambda}{\nu+1} \frac{x}{1!} - \frac{\lambda(\lambda-1)}{(\nu+1)(\nu+2)} \frac{x^2}{2!} - \frac{\lambda(\lambda-1)(\lambda-2)}{(\nu+1)(\nu+2)(\nu+3)} \frac{x^3}{3!} - \dots$$

The solution grows exponentially unless  $\lambda$  is a non-negative integer, in which case the solution is a finite series which can be written in the form

$$y = \frac{\lambda! L_\lambda^\nu(x)}{(\nu+1)_\lambda}$$

where we have used Pochhammer's symbol  $(a)_n = a(a+1)\cdots(a+n-1)$ .

By (8.118), all this implies that if  $E$  is an allowable energy level, then

$$\frac{\alpha E}{\hbar c \lambda} - \gamma = n = 0, 1, 2, 3, \dots \quad (8.119)$$

Since  $\gamma = \sqrt{k^2 - \alpha^2}$  and  $\lambda = k_1 k_2 = \frac{\sqrt{m^2 c^4 - E^2}}{\hbar c}$ ,

$$\frac{\alpha E}{\sqrt{m^2 c^4 - E^2}} - \sqrt{k^2 - \alpha^2} = n$$

and rearranging terms gives

$$E = E_{nk} = \frac{m c^2 \sqrt{n + \sqrt{k^2 - \alpha^2}}}{\sqrt{\alpha^2 + n + \sqrt{k^2 - \alpha^2}}}, \quad n, k = 0, 1, 2, 3, \dots \quad (8.120)$$

for the allowed energy levels. This is the result we were after.

## 8.8 Notes and references

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There are a number of excellent articles which are all aimed at finding the energy levels of the hydrogen atom. Among them, the following have been particularly useful to the author.

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