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# On estimation and feedback control of spin- $\frac{1}{2}$ systems with unknown initial states <sup>\*</sup>

Weichao Liang <sup>\*</sup> Nina H. Amini <sup>\*</sup> Paolo Mason <sup>\*</sup>

*<sup>\*</sup> Laboratoire des Signaux et Systèmes, CNRS - CentraleSupélec - Univ. Paris-Sud, Université Paris-Saclay, 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France. (e-mail: first name.family name@centralesupelec.fr).*

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**Abstract:** In this paper, we consider stochastic master equations describing the evolutions of quantum systems interacting with electromagnetic fields undergoing continuous-time measurements. In particular, we study feedback control of quantum spin- $\frac{1}{2}$  systems in the case of unawareness of initial states and in presence of measurement imperfections. We prove that the fidelity between the true quantum filter and its estimation (with arbitrary initial state) converges to one under appropriate assumption on the feedback controller. This shows the asymptotic convergence of such filters. In the more general case of spin- $J$  systems, we discuss heuristically the asymptotic behavior of the true quantum filter and the associated estimation, and the possibility of exponentially stabilizing such systems towards an eigenvector of the measurement operator by an appropriate feedback.

*Keywords:* Spin systems; Quantum filtering; Stochastic master equations; Fidelity.

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## 1. INTRODUCTION

Classical filtering (see, e.g. Kallianpur (2013); Xiong (2008)) optimizes the estimation of the state of a classical system from noisy observations. The quantum analogue was developed in the 1960s by Davies (1969, 1976) and extended by Belavkin in the 1980s Belavkin (1983, 1989, 1992), relying on quantum probability theory and quantum stochastic calculus (Hudson and Parthasarathy (1984); Hudson (2003); Meyer (2006)). The modern treatment of quantum filtering has been established in Bouten et al. (2009). Roughly speaking, quantum filtering theory gives a matrix-valued stochastic differential equation called stochastic master equation, to describe the time evolution of the state of an open quantum system interacting with an electromagnetic field under homodyne detection.

In real experiments, different types of imperfections, such as detection inefficiencies and unawareness of initial states, may be present (see e.g., Sayrin et al. (2011)). In the case of unawareness of initial states and known physical parameters such as detection inefficiencies, one considers estimated quantum filters which are designed based on measurements. The main question is whether the estimated filter “forgets” its initial state and has the same asymptotic behaviour as if it was correctly initialized. This problem can be posed when a feedback depending on the estimated quantum filter is applied. Since the observation process depends on the true quantum state, we deal with coupled stochastic master equations whose asymptotic behavior is not at all a trivial problem.

For uncontrolled system, this problem has been investigated in some recent papers. In the series of papers van Handel (2006, 2009a,b), a sufficient observability condition has been established so that the convergence of the estimation towards the true state is guaranteed. However, such condition is not easy to verify even if the system is finite dimensional. In Diósi et al. (2006), the problem of convergence is addressed assuming either the true quantum filter or its estimated state is always pure. The authors prove that the fidelity is a sub-martingale for this case. Then, in Rouchon (2011), by applying Uhlmann’s technique (Nielsen and Chuang, 2000, Theorem 9.4), the author shows that the fidelity between the state of the discrete-time quantum filter and its associated estimated state is a sub-martingale via a Kraus map. However, this sub-martingale property of the fidelity cannot ensure the convergence of the estimated filter state towards the true one. In Amini et al. (2011) the authors show that the fidelity is a sub-martingale for continuous-time quantum filters with perfect measurement for arbitrary mixed states. By the quantum repeated interaction approach, such result has been extended to the continuous-time jump-diffusion stochastic master equations with general measurement imperfections in Amini et al. (2014). Then, in Benoist and Pellegrini (2014), for quantum filters described by jump-diffusion stochastic differential equations, the authors show that when the control input is turned off, under perfect Quantum Non-Demolition (QND) measurements and a non-degeneracy assumption, the convergence is ensured.

Concerning the feedback stabilization of discrete-time QND measures, in Mirrahimi et al. (2009); Amini et al. (2013), the authors show the robustness of their feedback strategies with respect to initial states. Following Bouten

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and van Handel (2008), this is referred to as quantum separation principle. For continuous-time case, stabilization results for the case of unknown initial states have not been addressed so far. On the other hand, for angular momentum systems and without initialization imperfections, in Mirrahimi and van Handel (2007), the authors show the asymptotic convergence towards a chosen eigenvector of the measurement operator  $J_z$ . In the same context, in Liang et al. (2019a), we provide general conditions on the feedback controller and a local Lyapunov type condition which ensure exponential convergence towards a chosen eigenvector of  $J_z$ .

In this paper, we first analyze the dynamics of quantum spin- $\frac{1}{2}$  systems in presence of feedback control. We suppose imperfections in measurements and unawareness of the initial state. We show that the true filter and filter estimate have the same asymptotic behavior under appropriate assumption on the feedback. For spin- $J$  systems, we discuss heuristically the asymptotic behaviors of the true quantum filter and its associated estimated filter and the possibility of exponentially stabilizing such systems towards an eigenvector of the measurement operator  $J_z$  by a candidate feedback controller. We precise that our results don't prove the robustness properties of feedback strategies considered in Liang et al. (2019a, 2018). Numerical simulations are provided in order to illustrate our results and to support the efficiency of the proposed candidate feedback.

## 2. MODEL DESCRIPTION

Here, we consider quantum spin- $\frac{1}{2}$  systems. The stochastic master equations describing the evolution of the true system state and the corresponding estimated state are given as follows,

$$d\rho_t = F_{\hat{u}_t}(\rho_t)dt + L(\rho_t)dt + G(\rho_t)(dY_t - 2\sqrt{\eta M}\text{Tr}(\sigma_z \rho_t)dt),$$

$$d\hat{\rho}_t = F_{\hat{u}_t}(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + G(\hat{\rho}_t)(dY_t - 2\sqrt{\eta M}\text{Tr}(\sigma_z \hat{\rho}_t)dt),$$

where

- the quantum state of the spin- $\frac{1}{2}$  system is denoted as  $\rho$ , and belongs to the space  $\mathcal{S}_2 := \{\rho \in \mathbb{C}^{2 \times 2} | \rho = \rho^*, \text{Tr}(\rho) = 1, \rho \geq 0\}$ . The estimated state is denoted as  $\hat{\rho} \in \mathcal{S}_2$ ,
- the matrices  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  correspond to the Pauli matrices.
- $F_u(\rho) := -i[\omega\sigma_z + u\sigma_y, \rho]$ ,  $L(\rho) := M(\sigma_z\rho\sigma_z - \rho)$  and  $G(\rho) := \sqrt{\eta M}(\sigma_z\rho + \rho\sigma_z - 2\text{Tr}(\sigma_z\rho)\rho)$ .
- $Y_t$  denotes the observation process of the quantum spin- $\frac{1}{2}$  system, which is a continuous semi-martingale whose quadratic variation is given by  $[Y, Y]_t = t$ . Its dynamics satisfies  $dY_t = dW_t + 2\sqrt{\eta M}\text{Tr}(\sigma_z \rho_t)dt$ , where  $W_t$  is a one-dimensional standard Wiener process,
- $\hat{u}_t := u(\hat{\rho}_t)$  denotes the feedback controller as a function of the estimated state  $\hat{\rho}_t$ ,
- $\omega$  is the difference between the energies of the excited state and the ground state,  $\eta \in [0, 1]$  describes the efficiency of the detector, and  $M > 0$  is the strength of the interaction between the system and the probe.

By replacing  $dY_t = dW_t + 2\sqrt{\eta M}\text{Tr}(\sigma_z \rho_t)dt$  in the equation above, we obtain the following matrix-valued stochastic differential equations describing the time evolution of

the pair  $(\rho_t, \hat{\rho}_t) \in \mathcal{S}_2 \times \mathcal{S}_2$ ,

$$d\rho_t = F_{\hat{u}_t}(\rho_t)dt + L(\rho_t)dt + G(\rho_t)dW_t, \quad (1)$$

$$d\hat{\rho}_t = F_{\hat{u}_t}(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + 2\sqrt{\eta M}G(\hat{\rho}_t)\text{Tr}(\sigma_z(\rho_t - \hat{\rho}_t))dt + G(\hat{\rho}_t)dW_t. \quad (2)$$

If  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$ , the existence and uniqueness of the solution of (1)–(2) can be shown by similar arguments as in (Mirrahimi and van Handel, 2007, Proposition 3.5). Recall that a density operator can be uniquely characterized by the Bloch sphere coordinates  $(x, y, z)$  as

$$\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2} = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}.$$

The vector  $(x, y, z)$  belongs to the ball

$$\mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}.$$

The stochastic differential equation (1) expressed in the Bloch sphere coordinates takes the following form

$$dx_t = \left(-\omega_{eg}y_t - \frac{M}{2}x_t + \hat{u}_tz_t\right)dt - \sqrt{\eta M}x_tz_t dW_t, \quad (3a)$$

$$dy_t = \left(\omega_{eg}x_t - \frac{M}{2}y_t\right)dt - \sqrt{\eta M}y_tz_t dW_t, \quad (3b)$$

$$dz_t = -\hat{u}_tx_tdt + \sqrt{\eta M}(1 - z_t^2)dW_t. \quad (3c)$$

The stochastic differential equation (2) in the Bloch sphere coordinates is given by,

$$d\hat{x}_t = \left(-\omega_{eg}\hat{y}_t - \frac{M}{2}\hat{x}_t + \hat{u}_t\hat{z}_t + \eta M\hat{x}_t\hat{z}_t(\hat{z}_t - z_t)\right)dt - \sqrt{\eta M}\hat{x}_t\hat{z}_tdW_t, \quad (4a)$$

$$d\hat{y}_t = \left(\omega_{eg}\hat{x}_t - \frac{M}{2}\hat{y}_t + \eta M\hat{y}_t\hat{z}_t(\hat{z}_t - z_t)\right)dt - \sqrt{\eta M}\hat{y}_t\hat{z}_tdW_t, \quad (4b)$$

$$d\hat{z}_t = \left(-\hat{u}_t\hat{x}_t - \eta M(1 - \hat{z}_t^2)(\hat{z}_t - z_t)\right)dt + \sqrt{\eta M}(1 - \hat{z}_t^2)dW_t. \quad (4c)$$

## 3. CONVERGENCE PROPERTY OF QUANTUM SPIN- $\frac{1}{2}$ SYSTEMS

We focus on the fidelity  $\mathcal{F}(\rho, \hat{\rho})$  which defines a “distance” between the state  $\rho$  and its estimation  $\hat{\rho}$ . In the two-level case, the fidelity can be written in the following form

$$\mathcal{F}(\hat{\rho}, \rho) = \text{Tr}(\hat{\rho}\rho) + 2\sqrt{\det(\hat{\rho})\det(\rho)}.$$

The fidelity in the Bloch sphere coordinates is given by

$$\mathcal{F}(\hat{\rho}, \rho) = \mathcal{F}(\hat{\mathbf{v}}, \mathbf{v}) = \frac{1}{2} \left(1 + \mathbf{v}^\top \hat{\mathbf{v}} + \sqrt{(1 - \|\mathbf{v}\|^2)(1 - \|\hat{\mathbf{v}}\|^2)}\right),$$

where  $\mathbf{v} := (x, y, z)$  denotes the true state and  $\hat{\mathbf{v}} := (\hat{x}, \hat{y}, \hat{z})$  denotes the estimated state in Bloch sphere coordinates. We have two following special cases,

- (1) if  $\mathcal{F}(\hat{\rho}, \rho) = 1$ , then  $\mathbf{v} = \hat{\mathbf{v}}$ ;
- (2) if  $\mathcal{F}(\hat{\rho}, \rho) = 0$ , then  $\mathbf{v} + \hat{\mathbf{v}} = 0$  and  $\|\mathbf{v}\|^2 = \|\hat{\mathbf{v}}\|^2 = 1$ .

In order to apply the Itô formula on the fidelity  $\mathcal{F}(\rho, \hat{\rho})$ , we need to show the unattainability of the boundary for  $\rho$  and  $\hat{\rho}$ . By straightforward calculations, we can show that

$$\{\rho \in \mathcal{S}_2 | \det(\rho) = 0\} = \{\rho \in \mathcal{S}_2 | \text{Tr}(\rho^2) = 1\}, \quad (5)$$

which means that the boundary  $\partial\mathcal{S}_2$  is equal to the set of all pure states  $\mathcal{P}$ . The following lemma states some invariance properties for Equations (1)–(2).

*Lemma 3.1.* If  $\rho_0 > 0$ , then  $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$ . Moreover, if  $\eta = 1$ ,  $\partial\mathcal{S}_2 \times \mathcal{S}_2$  is a.s. invariant for Equations (1)–(2). The same results hold true for  $\mathcal{S}_2 \times \partial\mathcal{S}_2$ .

**Proof.** The dynamics of the purification function  $S(\rho_t) := 1 - \text{Tr}(\rho_t^2)$  is given by

$$dS(\rho_t) = M \left( \frac{(1-\eta)(1-z_t^2)}{2} - (1-\eta z_t^2)S(\rho_t) \right) dt - 2\sqrt{\eta}Mz_t S(\rho_t) dW_t.$$

Then, if  $\eta = 1$ , it is obvious that the set of all pure states  $\mathcal{P}$  for Equation (1) is a.s. invariant.

Next, let us prove the first part of the lemma. Given  $\varepsilon > 0$ , consider any  $\mathcal{C}^2$  function on  $\mathcal{S}$  such that

$$V(\rho) = \frac{1}{S(\rho)}, \quad \text{if } S(\rho) > \varepsilon.$$

We find

$$\begin{aligned} \mathcal{L}V(\rho) &= M \left( 1 + 3\eta z^2 - (1-\eta) \frac{1-z^2}{2S(\rho)} \right) V(\rho), \\ &\leq 4MV(\rho) \quad \text{if } S(\rho) > \varepsilon, \end{aligned}$$

where  $\mathcal{L}V$  denotes the infinitesimal generator of  $V$  (see e.g., Mao (2007)). To conclude the proof, one applies standard arguments (see e.g., (Liang et al., 2019a, Lemma 4.1)). Roughly speaking, by setting  $f(\rho, t) = e^{-4Mt}V(\rho)$ , one has  $\mathcal{L}f \leq 0$  whenever  $S(\rho) > \varepsilon$ . From this fact one proves that the probability of  $S(\rho)$  becoming zero in a finite fixed time  $T$  is proportional to  $\varepsilon$  and, being the latter arbitrary, it must be 0. Due to the equality (5),  $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$  when  $\rho_0 > 0$ . The last part of the lemma can be proved in the same manner.

Next, we analyze the behavior of  $\rho_t$  (resp.  $\hat{\rho}_t$ ) whenever the corresponding initial datum  $\rho_0$  (resp.  $\hat{\rho}_0$ ) lies at the boundary of  $\mathcal{S}_2$ . Denote  $\rho_g := \text{diag}(1, 0)$  and  $\rho_e := \text{diag}(0, 1)$ , which are the eigenvectors of  $\sigma_z$  and pure states.

*Lemma 3.2.* Assume  $\eta \in (0, 1)$  and  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$ . Suppose that  $\hat{\rho}_0$  lies in  $\partial\mathcal{S}_2 \setminus \{\rho_g, \rho_e\}$ , then  $\hat{\rho}_t > 0$  for all  $t > 0$  almost surely. Moreover, if  $\hat{\rho}_0 \in \{\rho_g, \rho_e\}$  and  $u(\hat{\rho}_0) \neq 0$  then,  $\hat{\rho}_t > 0$  for all  $t > 0$  almost surely. In addition, under the assumption  $\rho_0 \in \partial\mathcal{S}_2 \setminus \{\rho_g, \rho_e\}$ , then  $\rho_t > 0$  for all  $t > 0$  almost surely. Also, if  $u(\rho_g)u(\rho_e) \neq 0$  then, for all  $\rho_0 \in \partial\mathcal{S}_2$ ,  $\rho_t$  exits the boundary in finite time and stays in the interior of  $\mathcal{S}_2$  almost surely.

**Proof.** First, consider the purification function  $S(\hat{\rho}) := 1 - \text{Tr}(\hat{\rho}^2)$  for Equation (2), whose dynamics is given by

$$dS(\hat{\rho}_t) = M \left( \frac{(1-\eta)(1-\hat{z}_t^2)}{2} - (1-\eta\hat{z}_t^2)S(\hat{\rho}_t) - 4\eta(z_t - \hat{z}_t)\hat{z}_t S(\hat{\rho}_t) \right) dt - 2\sqrt{\eta}M\hat{z}_t S(\hat{\rho}_t) dW_t.$$

Now, assume  $\hat{\rho}_0 \in \partial\mathcal{S}_2 \setminus \{B_\varepsilon(\rho_g) \cup B_\varepsilon(\rho_e)\}$ . By compactness, there exists a  $\zeta > 0$  such that  $\frac{1}{2}M(1-\eta)(1-\hat{z}^2) \geq \zeta$ . Define  $\tau := \inf\{t > 0 \mid \hat{\rho}_t \notin \partial\mathcal{S}_2 \setminus \{B_\varepsilon(\rho_g) \cup B_\varepsilon(\rho_e)\}\}$ , for all  $\hat{\rho}_0 \in \partial\mathcal{S}_2 \setminus \{B_\varepsilon(\rho_g) \cup B_\varepsilon(\rho_e)\}$  and  $t > 0$ , by Itô formula, we have

$$\mathbb{E}(S(\hat{\rho}_{t \wedge \tau})) = \mathbb{E} \left( \int_0^{t \wedge \tau} \frac{1}{2}M(1-\eta)(1-\hat{z}_s^2) ds \right) \geq \zeta \mathbb{E}(t \wedge \tau).$$

By continuity and the definition of  $\tau$ ,  $S(\hat{\rho}_{t \wedge \tau}) = 0$  almost surely. This implies that  $\mathbb{E}(t \wedge \tau) = 0$ . Since we have  $\mathbb{E}(t \wedge$

$\tau) \geq t\mathbb{P}(\tau \geq t)$  we deduce that  $\mathbb{P}(\tau \geq t) = 0$  for all  $t > 0$ . Due to the arbitrariness of  $\varepsilon$ , if  $\hat{\rho}_0 \in \partial\mathcal{S}_2 \setminus \{\rho_g, \rho_e\}$  then  $\hat{\rho}_t$  exits the boundary immediately. Combining with the strong Markov property and Lemma 3.1, we have  $\hat{\rho}_t > 0$  for all  $t > 0$ , almost surely. Moreover, if  $\hat{\rho}_0 \in \{\rho_g, \rho_e\}$  then by the condition  $u(\hat{\rho}_0) \neq 0$  we deduce the same result.

For the case  $\rho_0 \in \partial\mathcal{S}_2 \setminus \{\rho_g, \rho_e\}$ , the above arguments can be repeated so that  $\rho_t > 0$  for all  $t > 0$  almost surely. Moreover, if  $u(\rho_g)u(\rho_e) \neq 0$  and  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$ , then there exists a neighborhood of  $\rho_e$  denoted by  $B_{r_e}(\rho_e)$  and a neighborhood of  $\rho_g$  denoted by  $B_{r_g}(\rho_g)$  such that,  $u(\hat{\rho}) \neq 0$  for all  $\rho \in B_{r_e}(\rho_e) \cup B_{r_g}(\rho_g)$ . By applying the same arguments as in (Liang et al., 2019a, Lemma 6.1), we can show that if  $\rho_0 \in \partial\mathcal{S}_2$ ,  $\hat{\rho}_t$  enters in  $B_{r_e}(\rho_e) \cup B_{r_g}(\rho_g)$  in finite time almost surely, which means that  $u(\hat{\rho}_t)$  becomes non-zero in finite time almost surely. As a consequence,  $\rho_t$  exits the boundary and stays in the interior of  $\mathcal{S}$  almost surely. The proof is then complete.

*Proposition 3.3.* Assume  $\eta \in (0, 1]$  and let  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$ . Then for all  $(\rho_0, \hat{\rho}_0) \in \mathcal{S}_2 \times \mathcal{S}_2$ , either  $\hat{\rho}_t$  converges to  $\{\rho_g, \rho_e\}$  or  $\mathcal{F}(\rho_t, \hat{\rho}_t)$  converges to one, almost surely. In particular, if  $u(\rho_e)u(\rho_g) \neq 0$  then  $\mathcal{F}(\rho_t, \hat{\rho}_t)$  converges to one almost surely.

**Proof.** In order to study the asymptotic behavior of  $\mathcal{F}(\rho, \hat{\rho})$ , in the following we will apply Itô formula. For this purpose, we first need to show the  $\mathcal{C}^2$  regularity of  $\mathcal{F}$  on appropriate invariant sets. If  $\eta = 1$ , then by Lemma 3.1,  $\partial\mathcal{S}_2 \times \partial\mathcal{S}_2$ ,  $\partial\mathcal{S}_2 \times \text{int}(\mathcal{S}_2)$ ,  $\text{int}(\mathcal{S}_2) \times \partial\mathcal{S}_2$  and  $\text{int}(\mathcal{S}_2) \times \text{int}(\mathcal{S}_2)$  are invariant for the coupled system (1)–(2) almost surely. Moreover, if either  $\rho$  or  $\hat{\rho}$  belongs to the boundary of  $\mathcal{S}_2$ , then the fidelity takes the form  $\mathcal{F}(\rho, \hat{\rho}) = \text{Tr}(\rho\hat{\rho})$ , which is a  $\mathcal{C}^2$  function. For the case  $\eta \in (0, 1)$ , under the assumptions that  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$  and  $u(\rho_e)u(\rho_g) \neq 0$ , by Lemma 3.2,  $(\rho_t, \hat{\rho}_t)$  exits the boundary in finite time and stays in  $\text{int}(\mathcal{S}_2) \times \text{int}(\mathcal{S}_2)$  afterwards almost surely. Note that the fidelity function is  $\mathcal{C}^2$  in  $\text{int}(\mathcal{S}_2) \times \text{int}(\mathcal{S}_2)$ .

Consider the Lyapunov function  $\mathcal{V}(\rho, \hat{\rho}) := 1 - \mathcal{F}(\rho, \hat{\rho})$ . Denote  $\Xi := \sqrt{(1 - \|\mathbf{v}\|^2)(1 - \|\hat{\mathbf{v}}\|^2)}$ . For any  $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$ , the infinitesimal generator of  $\mathcal{F}(\rho, \hat{\rho})$  is given by

$$\begin{aligned} \mathcal{L}\mathcal{F}(\rho, \hat{\rho}) &= \frac{M(1-\eta)}{4\Xi} \left( (1-\hat{z}^2)(1-\|\mathbf{v}\|^2) \right. \\ &\quad \left. + (1-z^2)(1-\|\hat{\mathbf{v}}\|^2) + 2\hat{z}^2(1-\mathbf{v}^\top \hat{\mathbf{v}} - \Xi)\Xi - 2(1-z\hat{z})\Xi \right) \\ &\quad + \frac{M}{2} (1-\hat{z}^2)(1-\mathbf{v}^\top \hat{\mathbf{v}} - \Xi). \end{aligned} \quad (6)$$

In particular, for  $\eta = 0$ , we have

$$\begin{aligned} \mathcal{L}\mathcal{F}(\rho, \hat{\rho}) &= \frac{M}{2} \left( \frac{(1-\hat{z}^2)(1-\|\mathbf{v}\|^2) + (1-z^2)(1-\|\hat{\mathbf{v}}\|^2)}{2\Xi} \right. \\ &\quad \left. + z\hat{z} - \mathbf{v}^\top \hat{\mathbf{v}} - \Xi \right) \\ &\geq \frac{M}{2} \left( \frac{(1-\hat{z}^2)(1-\|\mathbf{v}\|^2) + (1-z^2)(1-\|\hat{\mathbf{v}}\|^2)}{2\Xi} \right. \\ &\quad \left. - \sqrt{(\|\mathbf{v}\|^2 - z^2)(\|\hat{\mathbf{v}}\|^2 - \hat{z}^2)} - \Xi \right) \\ &= \frac{M}{4\Xi} \left( \sqrt{(\|\hat{\mathbf{v}}\|^2 - \hat{z}^2)(1-\|\mathbf{v}\|^2)} \right. \\ &\quad \left. - \sqrt{(\|\mathbf{v}\|^2 - z^2)(1-\|\hat{\mathbf{v}}\|^2)} \right)^2. \end{aligned}$$

For  $\eta = 1$ , we have

$$\mathcal{L}\mathcal{F}(\rho, \hat{\rho}) = \frac{M}{2}(1 - \hat{z}^2)(1 - \mathbf{v}^\top \hat{\mathbf{v}} - \Xi) = M(1 - \hat{z}^2)(1 - \mathcal{F}(\rho, \hat{\rho})).$$

Therefore, for all  $\eta \in [0, 1]$  and  $(\rho, \hat{\rho}) \in \text{int}(\mathcal{S}_2) \times \text{int}(\mathcal{S}_2)$ , we have  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho}) \geq 0$  which implies that  $\mathcal{L}\mathcal{V}(\rho, \hat{\rho}) \leq 0$ . By the stochastic LaSalle-type theorem in Mao (1999), we deduce that  $\lim_{t \rightarrow \infty} \mathcal{L}\mathcal{F}(\rho_t, \hat{\rho}_t) = 0$  almost surely. Since  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$  for any  $\eta \in (0, 1]$  can be written as a convex combination of the expressions of  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$  for  $\eta = 0$  and  $\eta = 1$  given above, we have that either  $|\hat{z}|$  converges to one or  $\mathcal{F}(\rho, \hat{\rho})$  converges to one almost surely. This concludes the proof of the first part of the proposition. The additional assumption  $u(\rho_e)u(\rho_g) \neq 0$  rules out the first possibility, completing the proof of the proposition.

*Remark 3.4.* The fact that the fidelity is a sub-martingale in  $\text{int}(\mathcal{S}_2) \times \text{int}(\mathcal{S}_2)$ , as shown in the above proof, suggests that the convergence of  $F(\rho_t, \hat{\rho}_t)$  to one should not require the technical condition  $u(\rho_e)u(\rho_g) \neq 0$ . This will be discussed more in detail in Section 4.

### 3.1 Simulations

In this section, we illustrate Proposition 3.3 through simulations of the system (1)–(2) in the case  $u(\hat{\rho}) \equiv 1$  and with parameters  $\omega = 0.3$ ,  $\eta = 0.3$  and  $M = 1$ . We set  $\rho_e$  as the initial state of the quantum filter and  $\rho_g$  as the initial state of the quantum filter estimate. In Fig. 1, we simulate the fidelity  $\mathcal{F}(\rho_t, \hat{\rho}_t)$ . Fig. 2 represents the behavior of a sample trajectory  $\rho_t$  and its corresponding estimation  $\hat{\rho}_t$  in Bloch sphere coordinates.

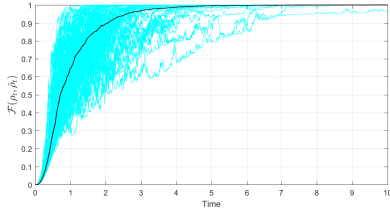


Fig. 1. Convergence of the fidelity  $\mathcal{F}(\rho_t, \hat{\rho}_t)$  towards one with the feedback law  $u(\hat{\rho}) = \hat{z}^2$  starting at  $(\rho_0, \hat{\rho}_0) = (\rho_e, \rho_g)$ , when  $\omega = 0.3$ ,  $\eta = 0.3$  and  $M = 1$ : the black curve represents the mean value of 100 arbitrary samples.

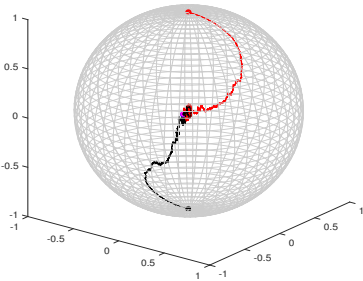


Fig. 2. Behavior of the trajectories  $\rho_t$  and  $\hat{\rho}_t$  with the feedback law  $u(\hat{\rho}) = \hat{z}^2$ , starting at  $(\rho_e, \rho_g)$ , when  $\omega = 0.3$ ,  $\eta = 0.3$  and  $M = 1$ . The black curve corresponds to a sample trajectory  $\rho_t$ , the blue point represents its end point; the red curve corresponds to the estimated trajectory  $\hat{\rho}_t$ , the magenta point represents its end point.

## 4. ASYMPTOTICS AND FEEDBACK CONTROL OF THE COUPLED SPIN- $J$ SYSTEMS

In this section, we discuss the asymptotic behavior of the true quantum filter and its estimate for spin- $J$  systems with unknown initial states. The stochastic master equations are given by

$$\begin{aligned} d\rho_t &= F_{\hat{u}_t}(\rho_t)dt + L(\rho_t)dt + G(\rho_t)(dY_t - 2\sqrt{\eta M}\text{Tr}(J_z\rho_t)dt), \\ d\hat{\rho}_t &= F_{\hat{u}_t}(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + G(\hat{\rho}_t)(dY_t - 2\sqrt{\eta M}\text{Tr}(J_z\hat{\rho}_t)dt), \end{aligned}$$

where

- the quantum state of the spin system is denoted as  $\rho$ , and belongs to the space  $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N \times N} \mid \rho = \rho^*, \text{Tr}(\rho) = 1, \rho \geq 0\}$ . The estimated state is denoted as  $\hat{\rho} \in \mathcal{S}_N$ ,
- $F_u(\rho) := -i[\omega J_z + u J_y, \rho]$ ,  $L(\rho) := \frac{M}{2}(2J_z\rho J_z - J_z^2\rho - \rho J_z^2)$  and  $G(\rho) := \sqrt{\eta M}(J_z\rho + \rho J_z - 2\text{Tr}(J_z\rho)\rho)$ ,
- $Y_t$  denotes the observation process of the quantum spin system, which is a continuous semi-martingale whose quadratic variation is given by  $[Y, Y]_t = t$ . Its dynamics satisfies  $dY_t = dW_t + 2\sqrt{\eta M}\text{Tr}(J_z\rho_t)dt$ , where  $W_t$  is a one-dimensional standard Wiener process,
- $\hat{u}_t := u(\hat{\rho}_t)$  denotes the feedback controller as a function of the estimated state  $\hat{\rho}_t$ ,
- $J_z$  is the (self-adjoint) angular momentum along the  $z$  axis, and it is defined by  $J_z e_n = (J - n)e_n$ ,  $n \in \{0, \dots, 2J\}$ , where  $J = \frac{N-1}{2}$  represents the fixed angular momentum and  $\{e_0, \dots, e_{2J}\}$  corresponds to an orthonormal basis of  $\mathbb{C}^N$ . With respect to this basis, the matrix form of  $J_z$  is given by

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix},$$

We define the pure states  $\rho_n := e_n e_n^*$  for  $n \in \{0, \dots, 2J\}$  corresponding to the eigenvectors of  $J_z$ .

- $J_y$  is the (self-adjoint) angular momentum along the  $y$  axis, and it is defined by

$$J_y e_n = -ic_n e_{n-1} + ic_{n+1} e_{n+1}, \quad n \in \{0, \dots, 2J\},$$

where  $c_m = \frac{1}{2}\sqrt{(2J+1-m)m}$ . The matrix form of  $J_y$  is given by

$$J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2J-1} & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix},$$

- $\eta \in (0, 1]$  measures the efficiency of the detectors,  $M > 0$  is the strength of the interaction between the system and the probe, and  $\omega \geq 0$  is a parameter characterizing the free Hamiltonian.

By replacing  $dY_t = dW_t + 2\sqrt{\eta M}\text{Tr}(J_z\rho_t)dt$  in the above equation, we obtain the following matrix-valued stochastic differential equations describing the time evolution of the pair  $(\rho_t, \hat{\rho}_t) \in \mathcal{S}_N \times \mathcal{S}_N$ ,

$$d\rho_t = F_{\hat{u}_t}(\rho_t)dt + L(\rho_t)dt + G(\rho_t)dW_t, \quad (7)$$

$$d\hat{\rho}_t = F_{\hat{u}_t}(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + 2\sqrt{\eta M}G(\hat{\rho}_t)\text{Tr}(J_z(\rho_t - \hat{\rho}_t))dt + G(\hat{\rho}_t)dW_t. \quad (8)$$

If  $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$ , the existence and uniqueness of solutions of (7)–(8) can be shown by similar arguments as in (Mirrahimi and van Handel, 2007, Proposition 3.5).

Note that, if we turn off the feedback controller, there are  $N^2$  equilibria  $(\rho_n, \rho_m)$  with  $n, m \in \{0, \dots, 2J\}$  for the coupled system. However, since the system (7) satisfies the non-demolition condition and the measurement operator  $J_z$  satisfies the non-degeneracy condition (Benoist and Pellegrini, 2014, Definition 2 and Assumption (ND)), we may state the following result.

*Theorem 4.1.* (Benoist and Pellegrini (2014)). If  $u \equiv 0$  and  $\eta = 1$ ,  $(\rho_t, \hat{\rho}_t)$  converges exponentially towards the set  $\{(\rho_0, \rho_0), \dots, (\rho_{2J}, \rho_{2J})\}$ .

Furthermore, the following holds true.

*Lemma 4.2.* Consider the coupled system (7)–(8) with  $\eta \in (0, 1]$  and  $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$ . Then the fidelity  $\mathcal{F}(\rho, \hat{\rho})$  is a sub-martingale.

Indeed, note that the value  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$  only depends on the instantaneous value of the Hamiltonian  $\omega J_z + u(\hat{\rho})J_y$ . Then by (Amini et al., 2014, Theorem 5), one has  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho}) \geq 0$  for the constant Hamiltonian instantaneously equal to  $\omega J_z + u(\hat{\rho})J_y$ , which proves the result.

*Remark 4.3.* In Proposition 3.3, by explicit computations, we have shown that for spin- $\frac{1}{2}$  systems  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho}) \geq 0$ , which is consistent with Lemma 4.2. This implied that  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$  should converge to zero almost surely. Moreover, we were able to analyze the zeros of  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$  and deduce some information about the asymptotic behavior of  $\mathcal{F}(\rho, \hat{\rho})$ . Similar results for spin- $J$  systems cannot be obtained easily because of the complicated form of  $\mathcal{L}\mathcal{F}(\rho, \hat{\rho})$ .

The above results give some intuitions about the asymptotic behaviors of  $\rho$  and  $\hat{\rho}$  at least for the case  $\eta = 1$  or for spin- $\frac{1}{2}$  systems. In these cases, without loss of generality, one may assume that the fidelity  $\mathcal{F}(\rho_0, \hat{\rho}_0)$  is close to one. If  $\eta = 1$ , this can be obtained for example by turning off the feedback controller for a large enough time in view of Theorem 4.1. For spin- $\frac{1}{2}$  systems, one can exploit Proposition 3.3 and apply a feedback controller satisfying  $u(\rho_g)u(\rho_e) \neq 0$  for a large enough time. Now, by Lemma 4.2, we have  $\mathbb{E}(\mathcal{F}(\rho_t, \hat{\rho}_t)) \geq \mathcal{F}(\rho_0, \hat{\rho}_0)$  which leads to the conclusion that  $\mathbb{E}(\mathcal{F}(\rho_t, \hat{\rho}_t))$  is close to one, independently of the chosen feedback law.

In the following, we discuss the possibility of designing a feedback controller  $u(\hat{\rho})$  which stabilizes exponentially almost surely the coupled system (7)–(8) towards a given target state  $(\rho_{\bar{n}}, \rho_{\bar{n}})$  with  $\bar{n} \in \{0, \dots, 2J\}$ . We note that the two subsystems (7)–(8) share the same feedback controller  $u(\hat{\rho})$ , only depending on the estimated state  $\hat{\rho}$ . Hence, if we suppose that the feedback  $u$  satisfies the assumption  $u(\rho_{\bar{n}}) = 0$  and  $u(\rho_k) \neq 0$  for all  $k \neq \bar{n}$ , then the coupled system (7)–(8) possesses exactly  $N$  equilibria, given by  $(\rho_k, \rho_{\bar{n}})$  for  $k \in \{0, \dots, 2J\}$ .

Concerning feedback exponential stabilization of the coupled spin- $J$  systems (7)–(8) towards  $(\rho_{\bar{n}}, \rho_{\bar{n}})$ , we propose a conjecture inspired by (Liang et al., 2019a, Theorem 6.4

and Theorem 6.5). These results were developed for the case  $\rho_0 = \hat{\rho}_0$ .

*Conjecture 4.4.* Consider the coupled system (7)–(8) with  $(\rho_0, \hat{\rho}_0) \in \mathcal{S}_N \times \mathcal{S}_N \setminus (\rho_{\bar{n}}, \rho_{\bar{n}})$  and assume  $\eta \in (0, 1)$ . Then, the feedback controller

$$u_{\bar{n}}(\hat{\rho}) = \alpha(1 - \text{Tr}(\hat{\rho}\rho_{\bar{n}}))^\beta, \quad \alpha > 0, \quad \beta \geq 1, \quad (9)$$

exponentially stabilizes  $(\rho_t, \hat{\rho}_t)$  to  $(\rho_{\bar{n}}, \rho_{\bar{n}})$  almost surely for the special case  $\bar{n} \in \{0, 2J\}$  with sample Lyapunov exponent less or equal than  $-\eta M$ . Moreover, the feedback controller

$$u_{\bar{n}}(\hat{\rho}) = \alpha(J - \bar{n} - \text{Tr}(J_z\hat{\rho}))^\beta, \quad \alpha > 0, \quad \beta \geq 1, \quad (10)$$

exponentially stabilizes  $(\rho_t, \hat{\rho}_t)$  to  $(\rho_{\bar{n}}, \rho_{\bar{n}})$  almost surely for the general case  $\bar{n} \in \{0, \dots, 2J\}$  with sample Lyapunov exponent less or equal than  $-\eta M/2$  for  $\bar{n} \in \{1, \dots, 2J-1\}$  and  $-\eta M$  for  $\bar{n} \in \{0, 2J\}$ .

### Simulations

Here, we illustrate the above conjecture through simulations for a three-level quantum spin system. In Fig. 3, we consider the target state  $(\rho_0, \rho_0)$ , and the Lyapunov function  $V_0(\rho, \hat{\rho}) = \sqrt{1 - \text{Tr}(\rho\rho_0)\text{Tr}(\hat{\rho}\hat{\rho}_0)}$ . We define the distance for the coupled system via the Bures distance  $d_B(\cdot, \cdot)$  in  $S_3$  (Bengtsson and Życzkowski (2017)) as follows  $\mathbf{d}_B((\rho, \hat{\rho}), \rho_0) := d_B(\rho, \rho_0) + d_B(\hat{\rho}, \rho_0)$ . We have  $\frac{\sqrt{2}}{4}V_0(\rho, \hat{\rho}) \leq \mathbf{d}_B((\rho, \hat{\rho}), \rho_0) \leq \sqrt{2}V_0(\rho, \hat{\rho})$ . In Fig 4, we choose  $(\rho_1, \rho_1)$  as the target state and  $V_1(\rho, \hat{\rho}) = \sum_{k \neq 1} (\sqrt{\text{Tr}(\rho\rho_k)} + \sqrt{\text{Tr}(\hat{\rho}\hat{\rho}_k)})$  as the Lyapunov function.

We can show that  $\frac{\sqrt{2}}{2}V_1(\rho, \hat{\rho}) \leq \mathbf{d}_B((\rho, \hat{\rho}), \rho_1) \leq \sqrt{2}V_1(\rho, \hat{\rho})$ .

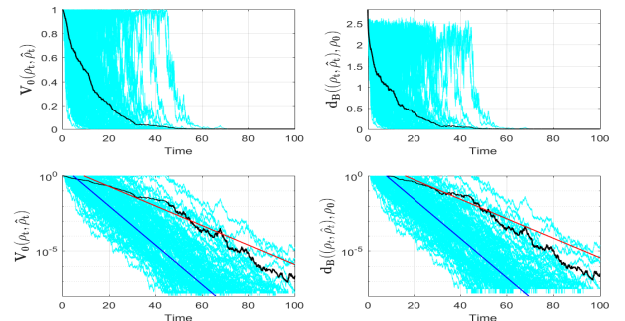


Fig. 3. Exponential stabilization of a three-level quantum spin system towards  $(\rho_0, \rho_0)$  with the feedback law (9) starting at  $(\rho_0, \hat{\rho}_0) = (\rho_2, \rho_1)$  with  $\omega = 0.3$ ,  $\eta = 0.3$ ,  $M = 1$ ,  $\alpha = 5$  and  $\beta = 2$ . The black curve represents the mean value of 100 arbitrary sample trajectories, the red and blue curves represent the exponential references with exponents  $-\eta M/2$  and  $-\eta M$  respectively. The figures at the bottom are the semi-log versions of the ones at the top.

## 5. CONCLUSION

In this paper, for spin- $\frac{1}{2}$  systems, we have shown that under appropriate assumption on the feedback controller, we can guarantee the same asymptotics for the quantum filter and its estimate. For the general case of spin- $J$  systems, we provided an heuristic approach regarding the asymptotic behavior of  $\rho$  and  $\hat{\rho}$  and feedback exponential stabilization of the coupled system (7)–(8). Future goals will be to



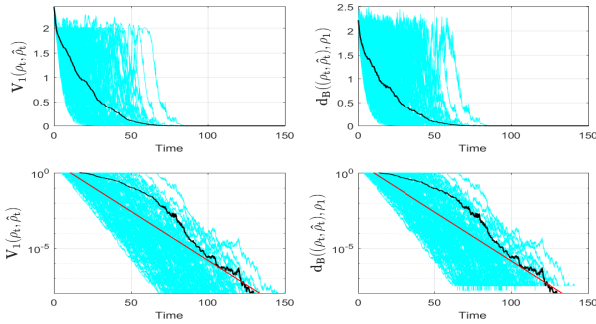


Fig. 4. Exponential stabilization of a three-level quantum spin system towards  $(\rho_1, \rho_1)$  with the feedback law (10) starting at  $(\rho_0, \hat{\rho}_0) = (\text{diag}(0.2, 0.2, 0.6), \text{diag}(0.8, 0.1, 0.1))$  with  $\omega = 0.3$ ,  $\eta = 0.3$ ,  $M = 1$ ,  $\alpha = 2$  and  $\beta = 2$ . The black curve represents the mean value of 100 arbitrary sample trajectories, the red curve represents the exponential reference with exponent  $-\eta M/2$ . The figures at the bottom are the semi-log versions of the ones at the top.

provide a rigorous proof of Conjecture 4.4 and present systematic methods ensuring exponential stabilization of the coupled spin systems. Other research lines include generalization to more general forms of filter equations which are driven by Wiener and/or Poisson processes. Also, exponential stabilization of entangled states such as GHZ states, with unknown initial states is included in our research lines (see Liang et al. (2019b) for exponential stabilization of two-qubit systems with perfect initialization).

## REFERENCES

- Amini, H., Mirrahimi, M., and Rouchon, P. (2011). On stability of continuous-time quantum filters. In *IEEE Conference on Decision and Control*, 6242–6247.
- Amini, H., Pellegrini, C., and Rouchon, P. (2014). Stability of continuous-time quantum filters with measurement imperfections. *Russian Journal of Mathematical Physics*, 21(3), 297–315.
- Amini, H., Somaraju, R.A., Dotsenko, I., et al. (2013). Feedback stabilization of discrete-time quantum systems subject to non-demolition measurements with imperfections and delays. *Automatica*, 49(9), 2683–2692.
- Belavkin, V.P. (1983). On the theory of controlling observable quantum systems. *Automation and Remote Control*, 44(2), 178–188.
- Belavkin, V.P. (1989). Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes. In *Modeling and Control of Systems*, 245–265. Springer.
- Belavkin, V.P. (1992). Quantum stochastic calculus and quantum nonlinear filtering. *Journal of Multivariate Analysis*, 42(2), 171–201.
- Bengtsson, I. and Życzkowski, K. (2017). *Geometry of quantum states: an introduction to quantum entanglement*. Cambridge University Press.
- Benoist, T. and Pellegrini, C. (2014). Large time behavior and convergence rate for quantum filters under standard non demolition conditions. *Communications in Mathematical Physics*, 331(2), 703–723.
- Bouten, L. and van Handel, R. (2008). On the separation principle in quantum control. In *Quantum stochasticity and information: statistics, filtering and control*, 206–238. World Scientific.
- Bouten, L., van Handel, R., and James, M.R. (2009). A discrete invitation to quantum filtering and feedback control. *SIAM review*, 51(2), 239–316.
- Davies, E.B. (1969). Quantum stochastic processes. *Communications in Mathematical Physics*, 15(4), 277–304.
- Davies, E.B. (1976). *Quantum theory of open systems*. Academic Press.
- Diósi, L., Konrad, T., Scherer, A., et al. (2006). Coupled Ito equations of continuous quantum state measurement and estimation. *Journal of Physics A: Mathematical and General*, 39(40), L575.
- Hudson, R.L. (2003). An introduction to quantum stochastic calculus and some of its applications. In *Quantum Probability Communications: QP-PQ (Volumes XI)*, 221–271. World Scientific.
- Hudson, R.L. and Parthasarathy, K.R. (1984). Quantum Ito’s formula and stochastic evolutions. *Communications in Mathematical Physics*, 93(3), 301–323.
- Kallianpur, G. (2013). *Stochastic filtering theory*, volume 13. Springer Science & Business Media.
- Liang, W., Amini, N.H., and Mason, P. (2018). On exponential stabilization of spin- $\frac{1}{2}$  systems. In *IEEE Conference on Decision and Control*, 6602–6607.
- Liang, W., Amini, N.H., and Mason, P. (2019a). On exponential stabilization of  $N$ -level quantum angular momentum systems. *Siam Journal on Control and Optimization*, 57(6), 3939–3960.
- Liang, W., Amini, N.H., and Mason, P. (2019b). On exponential stabilization of two-qubit systems. In *2019 IEEE Conference on Decision and Control*, 2304–2309.
- Mao, X. (1999). Stochastic versions of the LaSalle theorem. *Journal of Differential Equations*, 153(1), 175–195.
- Mao, X. (2007). *Stochastic differential equations and applications*. Woodhead Publishing.
- Meyer, P.A. (2006). *Quantum probability for probabilists*. Springer.
- Mirrahimi, M., Dotsenko, I., and Rouchon, P. (2009). Feedback generation of quantum Fock states by discrete QND measures. In *IEEE Conference on Decision and Control*, 1451–1456.
- Mirrahimi, M. and van Handel, R. (2007). Stabilizing feedback controls for quantum systems. *SIAM Journal on Control and Optimization*, 46(2), 445–467.
- Nielsen, M.A. and Chuang, I. (2000). *Quantum computation and quantum information*. Cambridge University Press.
- Rouchon, P. (2011). Fidelity is a sub-martingale for discrete-time quantum filters. *IEEE Transactions on automatic control*, 56(11), 2743–2747.
- Sayrin, C., Dotsenko, I., Zhou, X., et al. (2011). Real-time quantum feedback prepares and stabilizes photon number states. *Nature*, 477(7362), 73–77.
- van Handel, R. (2006). Filtering, stability, and robustness. *PhD thesis, California Institute of Technology*.
- van Handel, R. (2009a). Observability and nonlinear filtering. *Probability theory and related fields*, 145(1-2), 35–74.
- van Handel, R. (2009b). The stability of quantum Markov filters. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 12(01), 153–172.
- Xiong, J. (2008). *An introduction to stochastic filtering theory*, volume 18. Oxford University Press.