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MOTIVIC DECOMPOSITIONS OF FAMILIES WITH TATE FIBERS: SMOOTH AND SINGULAR CASES

M. CAVICCHI, F. DÉGLISE, AND J. NAGEL

ABSTRACT. We apply Wildeshaus's theory of motivic intermediate extensions to the motivic decomposition conjecture, formulated by Deninger-Murre and Corti-Hanamura. We first obtain a general motivic decomposition for the Chow motive of an arbitrary smooth projective family $f : X \rightarrow S$ whose geometric fibers are Tate. Using Voevodsky's motives with rational coefficients, the formula is valid for an arbitrary regular base S , without assuming the existence of a base field or even of a prime integer ℓ invertible on S . This result, and some of Bondarko's ideas, lead us to a generalized formulation of Corti-Hanamura's conjecture. Secondly we establish the existence of the motivic decomposition when $f : X \rightarrow S$ is a projective quadric bundle over a characteristic 0 base, which is either sufficiently general or whose discriminant locus is a normal crossing divisor. This provides a motivic lift of the Bernstein-Beilinson-Deligne decomposition in this setting.

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1. INTRODUCTION

An important conjecture in the theory of motives is the *nilpotency conjecture*. It is related to the functor

$$T : \text{CHM}(k) \rightarrow \text{HM}(k)$$

from the category of Chow motives to the category of homological motives (motives modulo homological equivalence, for a choice of Weil cohomology theory e.g. Betti cohomology if $k = \mathbb{C}$.) This functor is full but not faithful. The nilpotency conjecture states that if M is a Chow motive and M_{hom} the corresponding motive modulo homological equivalence (sometimes called the Grothendieck motive), the kernel of the ring homomorphism

$$\text{End}(M) \rightarrow \text{End}(M_{\text{hom}})$$

is a nilpotent ideal. If we take $M = h(X)$, the Chow motive of a smooth projective variety X over k , the nilpotency conjecture $N(X)$ states that the kernel of the homomorphism

$$\text{Corr}^0(X, X) \rightarrow \text{End}(H^*(X, \mathbb{Q}))$$

from the ring of correspondences of degree zero to the endomorphism ring of $H^*(X)$ is a nilpotent ideal. (Note that if X has pure dimension d , the left hand side is the Chow group $\text{CH}^d(X \times X)_{\mathbb{Q}}$, the right hand side is isomorphic to $H^{2d}(X \times X, \mathbb{Q})$ by Poincaré duality and the above map is just the cycle class map.) The nilpotency conjecture has important consequences. First of all, it implies that the functor T is conservative (i.e., it detects isomorphisms), hence essentially injective (nonexistence of *phantom motives*). The nilpotency conjecture is also related to the standard conjecture $C(X)$, the Künneth conjecture, which states that the Künneth components $p_i \in H^*(X \times X, \mathbb{Q})$ of the diagonal are algebraic (i.e., they are in the image of the cycle class map). Jannsen proved that conjectures $C(X)$ and $N(X)$ imply the Chow–Künneth conjecture $CK(X)$, i.e., the Künneth projectors lift to a set of mutually orthogonal projectors $\pi_i \in \text{Corr}^0(X, X)$ ($i = 0, \dots, 2d$) such that

$$\Delta_X = \sum_{i=0}^{2d} \pi_i$$

in $\text{CH}_d(X \times X) \otimes \mathbb{Q}$ and such that π_i depends only on the motive modulo homological equivalence. In this case the motive of X admits a *Chow–Künneth decomposition*

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)$$

with $h^i(X) = (X, \pi_i)$.

The nilpotency conjecture is known to hold for finite-dimensional motives. This means in particular that isomorphisms between finite-dimensional Chow motives can be detected by passing to the corresponding homological motives. An application is the following result.

Theorem (Kimura, Vial). *Assume k is an algebraically closed field of characteristic 0. Suppose that X is a smooth projective variety over k such that the Chow motive*

$h(X)$ is finite dimensional and such that $H^*(X)$ is algebraic (i.e., the rational cycle class maps $\mathrm{CH}^i(X)_{\mathbb{Q}} \rightarrow H^{2i}(X, \mathbb{Q})$ are surjective for all i). Then

$$h(X) = \bigoplus_i \mathbb{Q}(-i)^{b_{2i}}$$

is Tate (i.e., it is a direct sum of Lefschetz motives).

This result generalises a theorem of Jannsen and was observed by Vial [Via13b] and Kimura [Kim09] (in the case where one replaces $h(X)$ by an arbitrary Chow motive M). It is a direct consequence of the conservativity of the homological realisation on finite-dimensional motives.¹

In this paper we consider possible generalisations of these results to the case of relative motives. In their fundamental work [CH00], Corti and Hanamura defined a category $\mathrm{CHM}(S)$ of relative Chow motives over a quasi-projective base scheme S defined over a field k , together with realisation functors

$$(R1) \quad \begin{aligned} \rho_B : \mathrm{CHM}(S) &\rightarrow \mathrm{D}_c^b(S^{\mathrm{an}}, \mathbb{Q}), \\ \rho_\ell : \mathrm{CHM}(S) &\rightarrow \mathrm{D}_c^b(S_{\mathrm{\acute{e}t}}, \mathbb{Q}_\ell) \end{aligned}$$

where ρ_B (resp. ρ_ℓ) is the Betti (resp. ℓ -adic) realization functor associated to a given place $\sigma : k \rightarrow \mathbb{C}$ (resp. a prime ℓ invertible in k). Depending on the field k , we can use these two realisation interchangeably, so we let Q be either \mathbb{Q} in the first case, \mathbb{Q}_ℓ in the second.

In this setup, one works with proper morphisms $f : X \rightarrow S$ with X smooth and quasi-projective over k . Relative correspondences are elements of $\mathrm{CH}_*(X \times_S X)$ and composition of correspondences is defined using refined Gysin homomorphisms. The relative analogue of the nilpotency conjecture states that the kernel of the map

$$\mathrm{Corr}_S^0(X, X) \rightarrow \mathrm{End}(Rf_*(Q_X))$$

is a nilpotent ideal. (If X has dimension d the above map can be identified with the cycle class map $\mathrm{CH}_d(X \times_S X)_{\mathbb{Q}} \rightarrow H_{2d}^{\mathrm{BM}}(X \times_S X, \mathbb{Q})$ to Borel–Moore homology.) Corti and Hanamura have considered a relative analogue of the notion of Chow–Künneth decomposition.

The smooth case. Let us first consider the case where $f : X \rightarrow S$ is in addition smooth. According to Deligne’s theorem, one gets a decomposition in $D_c^b(S, Q)$:

$$(D) \quad Rf_*(Q_X) \cong \bigoplus_i R^i f_*(Q_X)[-i].$$

One usually says that $h_S(X)$ admits a *relative Chow–Künneth decomposition* (in short CK-decomposition) if there exist relative motives $M_i \in \mathrm{CHM}(S)$ such that $h_S(X) = \bigoplus_i M_i$ and $\rho(M_i) = R^i f_*(Q_X)[-i]$ for all i .

After the seminal work of Voevodsky, finalized by Ayoub and Cisinski–Déglise, we now have at our disposal a complete formalism of constructible *rational mixed motives* $\mathrm{DM}_c(S)$, where S is a very general base scheme (noetherian quasi-excellent),

¹If $H^*(X)$ is algebraic there exists a finite set of algebraic cycles $\{Z_i\}_{i \in I}$, $Z_i \in \mathrm{CH}^{p_i}(X)$, whose cycles classes generate the cohomology. These cycles define a morphism of Chow motives $\alpha : \bigoplus_i \mathbb{Z}(-p_i) \rightarrow h(X)$ such that $T(\alpha)$ is an isomorphism, hence α is an isomorphism by conservativity.

satisfying the six functors formalism and Grothendieck-Verdier duality. Moreover, we also have triangulated realization functors:

$$(R2) \quad \begin{aligned} \rho_B &: DM_c(S, \mathbb{Q}) \rightarrow D_c^b(S^{\text{an}}, \mathbb{Q}), \\ \rho_\ell &: DM_c(S, \mathbb{Q}) \rightarrow D_c^b(S_{\text{ét}}, \mathbb{Q}_\ell) \end{aligned}$$

commuting with the six operations², where:

- ρ_B is the Betti realization functor, defined when S is a finite type k -scheme with a given complex place $\sigma : k \rightarrow \mathbb{C}$ ([Ayo10]);
- ρ_ℓ is the ℓ -adic realization functor, for a $\mathbb{Z}[1/\ell]$ -scheme S ([CD16, 7.2.24]).

Moreover, a key property of rational mixed motives is the existence of the so-called *Chow weight structure* constructed by Bondarko (see Theorem 2.1.2 for recall). One then defines the (pseudo-abelian monoidal) category $\text{CHM}(S)$ of Chow motives over a scheme S as the weight 0 part of $\text{DM}(S)$. Combining this definition with the realization functors (R2), one therefore get an extension of the Corti-Hanamura functors (R1) to more general base schemes.³

Our first theorem exploits this level of generality. It is the natural generalization of the theorem of Kimura-Vial stated before to the relative case. A first result in this direction was obtained by Vial in [Via13a, Thm. 4.2]. It states that if $f : X \rightarrow S$ is a flat morphism between k -varieties such that $\text{CH}^i(X_s) \otimes \mathbb{Q} \cong \mathbb{Q}$ for all i and for all $s \in S(\Omega)$ (where Ω is a universal domain over k) then the relative motive $h_S(X)$ is a Tate motive. Our theorem extends this result in two directions. First we consider geometric fibers: this implies that we will get more general motives than Tate motives. Secondly, we work over an arbitrary regular base scheme S , with no assumption on the existence of a base field or of a prime ℓ invertible on S .⁴

Theorem A (see Theorem 2.5.8). *Let S be a regular connected scheme with étale fundamental group π . Let $f : X \rightarrow S$ be a smooth proper morphism whose geometric fibers are Tate (i.e. the associated Chow motives are Tate). Then there exists an isomorphism of motives over S :*

$$h_S(X) \simeq \bigoplus_{i \in I} \rho_i(V_i)(-n_i)[-2n_i]$$

where I is a finite set, V_i is a simple Artin \mathbb{Q} -representation of π , n_i is a non-negative integer and $\rho_i(V_i)$ is the rational motive naturally associated with V_i .

Moreover, the set $\{([V_i], n_i), i \in I\}$ where $[V_i]$ denotes the isomorphism class in the category of Artin representations $\text{Rep}_{\mathbb{Q}}^{\text{A}}(\pi)$ of π is uniquely determined by the property that for any prime integer ℓ and any $n \geq 0$:

$$R^{2n} f'_*(\mathbb{Q}_\ell) \simeq \bigoplus_{i \in I | n_i = n} V'_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-n_i)$$

where f' and V'_i are pullbacks of f and V_i along the open immersion $U[\ell^{-1}] \rightarrow U$, and $V'_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ denotes the \mathbb{Q}_ℓ -sheaf represented by V'_i .

²for quasi-excellent finite dimensional base schemes, and morphisms of finite type between them

³Indeed, when S is quasi-projective over a perfect field, it is proved in [Jim16] that the weight 0 part of $\text{DM}_c(S)$ is equivalent to the category defined by Corti and Hanamura.

⁴This is the first example of a *BCH-decomposition*: see Def. 2.2.3 and the following paragraph for further discussion of its properties.

In the above statement, the functor $\rho_!$ sends any étale sheaf over S to its image in the category $\mathrm{DM}_c(S)$. We show that when restricted to the bounded derived category of Artin π -representations, it induces an equivalence of categories with the full subcategory $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ of *smooth Artin-Tate motives* in $\mathrm{DM}_c(S)$ (Prop. 2.3.4), extending a classical result of Voevodsky-Orgogozo (the case when S is the spectrum of a field). Hence, in particular, Theorem A shows that the naive generalisation of the theorem of Kimura-Vial to the relative situation is false: if $f : X \rightarrow S$ is a smooth, projective morphism such that $h(X_s)$ is finite dimensional and such that $H^*(X_s)$ is algebraic for all $s \in S$, the relative motive $h_S(X)$ does not necessarily decompose as a direct sum of relative Lefschetz motives⁵, since it is an object of a more general kind - an object of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$. This can be seen already on the topological level. If $f : X \rightarrow S$ is a smooth, projective morphism whose fibers have algebraic cohomology then the direct image sheaves $R^i f_* \mathbb{Q}$ are local systems with finite monodromy, but they are not necessarily constant. A concrete example is given by smooth quadric fibrations of even relative dimension (Corollary 2.5.13). We have recorded in Example 2.5.14 some constructions of smooth proper quadrics over $\mathrm{Spec}(\mathbb{Z})$, and more generally over rings of integers of number fields, to which our result applies; for rings of integers whose spectrum is non-simply connected, we give explicit cases where relative Artin-Tate motives with non-trivial monodromy actually appear.

The proof of Theorem A combines a "spreading-out" technique valid for constructible motives, a weight argument for (smooth) Artin-Tate motives (see in particular Thm. 2.4.3), and an extension argument based on Wildeshaus's analysis of weights and of minimal/intermediate extensions of pure motives. This first result relies on a new notion of *fair* extension (Def. 2.5.3) based on Wildeshaus's work. Note secondly that we rely on the proper base change theorem for étale torsion sheaves to check that the local systems have the required property for a fair extension to exist (see the end of the proof of Theorem 2.5.8).

The singular case. The general case is more complicated and involves the theory of perverse sheaves. Recall that this theory exists either in the analytical setting, or in the ℓ -adic étale one. So let us denote as above by Q either the field \mathbb{Q} or \mathbb{Q}_ℓ , intending that one uses sheaves on the associated analytical site in the first case, and étale ℓ -adic sheaves in the second.

When k is an algebraically closed field of characteristic zero or the separable closure of a finite field, the decomposition theorem of Beilinson-Bernstein-Deligne says that a decomposition of the form (D) still exists provided one replaces the canonical t-structure by the perverse one. Moreover, each perverse cohomology sheaf ${}^p Rf_*(Q_X)$ is semi-simple in the perverse heart of $D_c^b(S)$. These two statements can be expressed by the existence of a *topological decomposition* in $D_c^b(S, Q)$

$$(BBD) \quad Rf_*(Q_X) = \bigoplus_{\lambda} j_{\lambda!} (L_{\lambda})$$

where there exists an integer n_{λ} such that $L_{\lambda}[n_{\lambda}]$ is a simple local system on a smooth locally closed subscheme $U_{\lambda} \subset X$ and $j_{\lambda!} (L_{\lambda})$ denotes the *intermediate extension* (or intersection complex) defined in [BBD82a].⁶

⁵This statement is even false if one assumes that the fibers are homogeneous spaces as in [Iyer]

⁶Note in particular that one deduces isomorphisms ${}^p Rf_*(Q_X)[-n] \simeq \bigoplus_{\lambda | n_{\lambda} = n} j_{\lambda!} (L_{\lambda})$ and:

$$(D') \quad {}^p Rf_*(Q_X) \simeq \bigoplus_n {}^p R^n f_*(Q_X)[-n].$$

According to [CH00, Th. 2], it is expected from the standard conjectures that the above decomposition should lift to the category of Chow motives. In other words, there should exist a *motivic decomposition* of the form

$$(1.0.0.a) \quad h_S(X) \simeq \bigoplus_{\lambda} M_{\lambda}$$

in the pseudo-abelian category $\text{CHM}(S)$, and such that $\rho_B(M_{\lambda}) = j_{\lambda!}(L_{\lambda})$. Here again, using the more general definition of Chow motives as pure motives of weight 0 in $\text{DM}(S)$, it is not unreasonable to assume that such motivic decompositions exist when we only assume that $f : X \rightarrow S$ is proper and X is regular, without assuming that S is defined over a field. Following ideas of Bondarko in [Bon15], this can in fact be deduced from the existence of a suitable motivic t-structure. We refer the interested reader to Definition 2.2.3. Let us also recall that Theorem A gives plenty of examples of such a generalized decomposition.

The existence of motivic decompositions, in the presence of singularities of f , has only been obtained in a few cases. First by Gordon, Hanamura, Murre in [GHM03, Th. I in §2] and then by Nagel and Saito in [NS09], for conic bundles over a surface (see also [Bou16]). Nonetheless, the conjectural existence of such decompositions, in the form (1.0.0.a), suggests a notable theoretical fact, which was of course already intended by Beilinson's conjectures: there should exist a lift of the (algebraic) intersection complexes of [BBD82b], and more generally of the intermediate extension operator $j_{!*}$. This is especially important in the motivic part of the Langlands program, where the first construction of a Chow motive realizing to intersection cohomology was provided by Scholl in [Sch90], in the case of modular curves. In recent years, Wildeshaus has started a systematic study of *motivic intermediate extensions* based on the formalism of $\text{DM}(S)$ and Bondarko's weight structures. He and his students successfully applied these methods to many families of higher dimensional Shimura varieties, the most recent one being the case of *genus two Hilbert-Siegel varieties* treated by the first author in [Cav19]. The most general construction applies to some relative motives of "abelian type" (see [Wil17]).

Wildeshaus's techniques have never been applied to the motivic decomposition problem. This is the original motivation of our paper. In our second main theorem, we adapt the construction of *loc. cit.* to extend the result of Nagel and Saito to two types of projective quadric bundles, eventually with singular fibers, and over arbitrary characteristic 0 bases.

Theorem B (see Cor. 4.1.7, Cor. 4.2.5). *Let S be a scheme over a field of characteristic 0, and $f : X \rightarrow S$ be a projective quadric bundle such that one of the following conditions holds:*

- (1) *the discriminant of X/S is a normal crossing divisor in X ;*
- (2) *X/S is regular (which means "generic" in a precise sense, see Def. 4.1.4).*

We consider the stratification $\{S_{\phi}\}$ of S given respectively by

- (1) *the intersections of the irreducible components of the discriminant (see Def. 4.2.2) in case (1);*
- (2) *corank (see after Def. 4.1.4) in case (2).*

Then $h_S(X)$ admits a motivic decomposition

$$h_S(X) \simeq \bigoplus_{i,\phi} j_{\phi!}(\rho_!(V_{i,\phi})(-n_{i,\phi}))[-2n_{i,\phi}]$$

with $V_{i,\phi}$ a simple Artin representation of $\pi_1(S_\phi)$.

Let us describe the strategy of proof and the difficulties that arise when trying to adapt the construction of [Wil17] to our situation. First recall that Wildeshaus's construction, given a stratification $\Phi = \{S_\phi\}$ of the base S , consists in constructing a glued subcategory of Chow motives starting from constant local systems upstairs. Based on a criterion of *semi-primality* (see Def. 3.1.1), this glueing procedure guarantees the existence of the functors j_{i*} when j is the immersion between (union of) strata. We have two difficulties to circumvent in order to apply this construction.

The first one is that we need to consider more general local systems, as it is known from the work of Beauville that in the case of an algebraically closed base field, *Prym motives* should appear in the motives of quadric bundles. As these correspond to an étale cover of degree 2,⁷ we are led to consider smooth Artin-Tate motives as natural *motivic* local systems. But then, in order to adapt Wildeshaus's construction, one needs to control the degeneracy of smooth Artin-Tate motives, as was already done in the smooth case, in the proof of Theorem A. Fortunately, by *purity of the branch locus* we need only to control the degeneracy in codimension 1. Then it is sufficient to assume that the smooth Artin-Tate motives are tamely ramified to ensure sufficient regularity of the degenerations; see the proof of Theorem 3.2.4.

The second difficulty is that to perform the glueing construction, Wildeshaus has to assume that the closures of strata of $\{S_\phi\}$ are smooth. This assumption is satisfied in the case (1), but not in the case (2) of regular quadric bundles. In order to deal with these singularities, we have to construct a suitable resolution of the strata, $\mathfrak{S} \rightarrow \Phi$, in such a way that we can control the pushforward of the tamely ramified smooth Artin-Tate motives appearing on the (abstract) blow-ups. This is achieved by considering the geometry of quadrics, more precisely by using the flag variety associated with the quadratic vector bundle of the regular quadric bundle. This is where the assumption *regular* appears; see Proposition 4.1.3 and Lemma 4.1.5.

Then Wildeshaus's glueing construction goes through and one obtains a suitable category of tamely ramified (\mathfrak{S}/Φ) -constructible Artin-Tate motives $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ (see Definition 3.2.7). Moreover, this category satisfies the following good properties:

- Bondarko's weight structure restricts to this subcategory. The corresponding weight 0 part, denoted by $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$, is a pseudo-abelian subcategory of $\mathrm{CHM}(S)$ which is semi-primary (Theorem 3.2.10).
- The functor j_{i*} exists, for j an inclusion of strata, and commutes with the realization functors (Theorem 3.3.1).
- The realization functors restricted to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ are conservative (Theorem 3.3.2).

Finally, we prove that, under the hypothesis of Theorem B, $h_S(X)$ belongs to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$, and get our main theorem.

⁷given by the spectrum of the associated Clifford algebra see 2.5.12, or by the Stein factorization of the Fano scheme see the proof of Theorem 4.1.6

Possible developments. We believe that our method for proving Theorem B, based in particular on Theorem A, should apply to other situations. Firstly one could investigate projective families of a different type, for example degenerations of Grassmannians or Severi-Brauer varieties. The geometry of hyper-kähler varieties or K3 surfaces should also offer a suitable playground. Lastly, the Example 2.5.14 suggests that one could look at the arithmetic case, where we work with (flat regular) schemes over rings of integers. In particular, the theory of *regular* quadric fibrations in this case still seems manageable according to [APS15].

PLAN OF THE PAPER

In the first section, we recall the theory of gluing and weights for rational mixed motives, and state several versions of *motivic decompositions*, in order of generality. We then give some basic definitions and computation of relative *smooth* Artin motives, analog of Artin motives over a field, and show that smooth Artin-Tate motives have weights, extending work of Levine and Wildeshaus. Finally, we deduce our first motivic decomposition theorem, based on the previous preparations.

The second section deals with an extension of Wildeshaus’s theory of motivic intermediate extensions in order to be able to deal with (smooth) Artin-Tate motives and their extensions along immersions. Along the lines of Wildeshaus’s theory, we first recall the abstract notion of semi-primary category and its application to intermediate extensions. We then specialize the construction to treat our own version of gluing adapted to some Artin-Tate motives. The key idea here is that the well-known condition of tame ramification allows us to control the degeneracy of this particular type of motives (see Definition 3.2.2). Then, we are able to construct a suitable category of glued well-ramified Artin-Tate motives, which satisfies several good properties, as explained in the introduction.

The last section deals with our last theorem, the motivic decomposition of certain families of projective quadric bundles. We first explain our notion of regularity for quadric bundles (Definition 4.1.4), and then deduce from the construction of Section 2 our first motivic decomposition (Corollary 3.2.13). The last part of Section 3 deals with the motivic decomposition in the case where the discriminant locus is a normal crossing divisor.

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NOTATIONS AND CONVENTIONS

Given an arbitrary group G , and a field K of characteristic 0, we will denote by $\text{Rep}_K^{\mathbb{A}}(G)$ the abelian monoidal category of finite dimensional K -representations V of G which factor through a finite quotient of G . When G is a pro-finite group, this amounts to the continuity of the action. When $G = \pi$ is the étale fundamental group of a geometrically pointed scheme X , or the usual fundamental group of a complex variety, representations in $\text{Rep}_K^{\mathbb{A}}(\pi)$ are classically called *Artin representations*. In any case, thanks to Maschke’s lemma, the category $\text{Rep}_K^{\mathbb{A}}(G)$ is semi-simple.

All our schemes are implicitly assumed to be excellent noetherian finite and dimensional.

Given (such) a scheme S , we denote by $\mathrm{DM}_c(S, \mathbb{Q})$ the triangulated category of constructible rational motives over S .⁸ We let $\mathbb{1}_S$ be the constant motive over S (which is also the unit of the monoidal structure). We will use *cohomological motives* over S : given any morphism $f : X \rightarrow S$, we put:

$$h_S(X) = f_*(\mathbb{1}_X).$$

We will also heavily rely on the Bondarko's theory of weight structure, and especially the canonical (Chow) weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$. We recall these notions in Section 2.1.

Another important property of rational mixed motives from [CD19] that we will use is the so-called *continuity property* (see Def. 4.3.2, Prop. 14.3.1 of *op. cit.*). We will use it in the following form (see Prop. 4.3.4 of *op. cit.*):

Proposition 1.0.1. *Let $(S_i)_{i \in I}$ be a projective system of schemes with a projective limit S in our category of schemes. Then the canonical functor:*

$$2 - \varinjlim_{i \in I} \mathrm{DM}_c(S_i, \mathbb{Q}) \rightarrow \mathrm{DM}_c(S, \mathbb{Q})$$

is an equivalence of categories.

We will use two kinds of triangulated realization functors:

- Let ℓ be a prime invertible on S . We have the ℓ -adic realization:

$$\rho_\ell : \mathrm{DM}_c(S, \mathbb{Q}) \rightarrow \mathrm{D}_c^b(S_{\text{ét}}, \mathbb{Q}_\ell)$$

where the right-hand side is the constructible derived category of Ekedahl's ℓ -adic étale sheaves with rational coefficients (see [CD16, 7.2.24]).

- Let E be a characteristic 0 field given with a complex embedding $\sigma : E \rightarrow \mathbb{C}$. Assume S is a finite type E -scheme. We have the Betti realization:

$$\rho_B : \mathrm{DM}_c(S, \mathbb{Q}) \rightarrow \mathrm{D}_c(S^{\text{an}}, \mathbb{Q})$$

where the right-hand side is the constructible derived category of rational sheaves over the analytical site of $S^{\text{an}} = S^\sigma(\mathbb{C})$. This realization is obtained from that of [Ayo10] as the following composite:

$$\mathrm{DM}_c(S, \mathbb{Q}) \simeq \mathrm{SH}_c(S)_{\mathbb{Q}^+} \subset \mathrm{SH}_c(S)_{\mathbb{Q}} \xrightarrow{\text{Betti}'_{S \otimes \mathbb{Q}}} \mathrm{D}_c(S^{\text{an}}, \mathbb{Q})$$

where:

- $\mathrm{SH}_c(S)$ is the constructible stable \mathbb{A}^1 -homotopy category (made of compact spectra over S).
- $\mathrm{SH}_c(S)_{\mathbb{Q}^+}$ is plus-part of the rationalization of $\mathrm{SH}_c(S)$ (see eg. [CD19, 16.2.1])
- the first equivalence is given by [CD19, Th. 16.2.13].
- the functor Betti'_S is (the obvious restriction of that) defined in [Ayo10, Def. 2.1].

⁸In [CD19], several models of this category are given: Beilinson motives (Def. 15.1.1), Voevodsky's h-motives (Th. 16.1.2), Voevodsky's motivic complexes (S geometrically unibranch scheme, Th. 16.1.4), the plus-part of the rational stable homotopy category (Th. 16.2.13 and 5.3.35), the \mathbb{P}^1 -stable \mathbb{A}^1 -derived étale category (Th. 16.2.22). All models being equivalent, the reader is free to choose his preferred one.

These two realization functors admit a right adjoint and commute with the six operations.⁹

2. SMOOTH ARTIN-TATE MOTIVES AND MOTIVIC DECOMPOSITIONS

2.1. Gluing and weights on Beilinson motives.

2.1.1. In the theory of rational mixed motives, a crucial property is given by the gluing formalism [BBD82a, Sec. 1.4.3], which is a consequence of Morel-Voevodsky's localization theorem [MV99, Th. 2.21, p. 114]. Given a closed immersion $i : Z \hookrightarrow S$ with complementary open immersion $j : U \hookrightarrow S$, one has six functors

$$(2.1.1.a) \quad \mathrm{DM}_c(U, \mathbb{Q}) \begin{array}{c} \xrightarrow{j^!} \\ \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{array} \mathrm{DM}_c(S, \mathbb{Q}) \begin{array}{c} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow{i^!} \end{array} \mathrm{DM}_c(Z, \mathbb{Q}),$$

satisfying the formalism of [BBD82a, Sec. 1.4.3].¹⁰

This property will be at the heart of our results (see in particular Section 2.1). They are the starting point of Hébert's and Bondarko's extension of the weight structure on rational motives, from the case of perfect fields to that of arbitrary bases. Let us recall this theory, from [Héb11, Thm. 3.3, thm. 3.8 (i)-(ii)], for future references.

Theorem 2.1.2. *For each scheme S , there is a canonical weight structure w on the triangulated category $\mathrm{DM}_c(S, \mathbb{Q})$ called the motivic weight structure. The family of these weight structures indexed by schemes S is uniquely characterized by the following properties.*

- (1) *The objects $\mathbb{1}_S(p)[2p]$ belong to the heart $\mathrm{DM}_c(S, \mathbb{Q})_{w=0}$ for all integers p , whenever S is regular.*
- (2) *For any morphism $f : T \rightarrow S$, the functor f^* (resp. f_*) preserves negative weights (resp. positive weights). When f is separated of finite type, the functor $f^!$ (resp. $f_!$) preserves positive weights (resp. negative weights).*

We will only use the motivic weight structure in this paper. So w will always mean weights for the motivic weight structure. Over a base scheme S , the heart of this weight structure will be denoted by $\mathrm{CHM}(S)$ and is called the category of *Chow motives over S* .

Remark 2.1.3. Gluing of (motivic) weights. Note in particular from point (2) that when f is finite (resp. étale separated of finite type), f_* (resp. f^*) preserves weights. Moreover, under the assumption of the paragraph preceding the theorem, it follows from the gluing diagram (2.1.1.a) and point (2) that the motivic weight structure on S "satisfies gluing": a motive M on S is w -positive (resp. w -negative) if and only if $j^*(M)$ and $i^!(M)$ (resp. $j^*(M)$ and $i^*(M)$) are w -positive (resp. w -negative).

Remark 2.1.4. Assumptions on base schemes. Note that in *loc. cit.*, it is assumed that schemes are of finite type over an excellent base scheme B of dimension less than 3. First, this assumption is not used in the proof of the above statement. Secondly, the only reason this assumption appeared in motivic homotopy theory is for the proof of the Grothendieck-Verdier duality (also called "local duality"),

⁹This is proved in the references indicated above.

¹⁰This is the so-called localization property of motives, extensively studied in [CD19, §2.3].

[CD19, Th. 4.4.21].¹¹ Since then, this result has been generalized to the schemes considered in our paper by Cisinski in [Cis19, §2.3]. It is used in [Héb11] only in the statement of Corollary 3.9.

2.2. Motivic decompositions. In this section, we formulate several forms of *motivic decompositions*, which are extensions of the classical Chow-Künneth decomposition, as well as its relative form recalled in the introduction (see Introduction, "The smooth case", page 3). Recall that the first example of a relative Chow-Künneth decomposition was obtained for abelian schemes by Deninger and Murre in [DM91].

The first generalized form of motivic decompositions is due to Corti and Hanamura [CH00]. Let us recall the definition.

Definition 2.2.1. Let k be an algebraically closed field of characteristic 0, and consider ρ either the Betti realization ρ_B associated with a fixed complex embedding of k , or the ℓ -adic realization ρ_ℓ (see our Notations and conventions, page 8). Let $f : X \rightarrow S$ be a projective morphism of quasi-projective k -schemes such that X is smooth over k , and the associated BBD-decomposition (BBD).

A *Corti-Hanamura decomposition* (in short: CH-decomposition) of the Chow motive $h_S(X)$ is a finite decomposition

$$h_S(X) \simeq \bigoplus_{\lambda} M_{\lambda}$$

in the pseudo-abelian category $\text{CHM}(S)$ such that $\rho(M_{\lambda}) = j_{\lambda!}(L_{\lambda})$.

When f is smooth, the decomposition (BBD) is a refinement of the decomposition (D) discussed in the introduction. So in the smooth case, a CH-decomposition is similarly a refinement of a CK-decomposition.

Example 2.2.2. Such decompositions have been constructed in a few cases, particularly in complex algebraic geometry: cf. [GHM03, dCM10, MSS12].

The specificity of the above definition is to incorporate the singular case as well. The statement of our first motivic decomposition theorem, 2.5.8, together with Bondarko's work [Bon15] lead us to the following more general kind of motivic decompositions of pure weight 0 motives.

Definition 2.2.3. Let $f : X \rightarrow S$ be a proper morphism such that X is regular. A *Bondarko-Corti-Hanamura decomposition* (in short: BCH-decomposition) of the Chow motive $h_S(X)$ is a finite decomposition

$$h_S(X) \simeq \bigoplus_{\lambda} j_{\lambda!}(M_{\lambda})$$

in $\text{CHM}(S)$ where $j_{\lambda} : V_{\lambda} \rightarrow S$ is a locally closed immersion, V_{λ} is regular, M_{λ} is a weight 0 motive over V_{λ} whose ℓ -adic realisation is a local system up to a shift, and $\rho_{\ell} j_{\lambda!}(M_{\lambda}) = j_{\lambda!}(\rho_{\ell}(M_{\lambda}))$.

The interest of this definition is that it does not require a base field, and moreover, it makes sense without having to choose a prime invertible on S ; in particular over \mathbb{Z} . Note also that our formulation implies the existence of a decomposition

¹¹More precisely, it is used to obtain some resolution of singularity statement; see *loc. cit.*

of the form (D') (see footnote 6, page 5), for all primes ℓ , and contains an "independence of ℓ " result: under the existence of a BCH-decomposition, the rank of ${}^p R^n f_*(\mathbb{Q}_\ell)$ will be independent of ℓ (see also Cor. 2.5.11).

2.2.4. Link with Bondarko's version of the motivic conjectures. As said before, the definition is partly motivated by analysing the work of Bondarko [Bon15]. In particular, we can conjecturally justify the existence of BCH-decompositions. Indeed, the existence of such decompositions would be a consequence of the existence of the motivic t -structure together with the assumption that it is *transversal* (see [Bon15, Def. 1.4.1]) to the Chow weight structure.¹² More precisely, to get the BCH-decomposition under the previous assumption, one can apply *op. cit.*:

- Prop. 1.4.2, point (3) to the weight 0 motive $h_S(X)$ to get the analogue of the relative Chow Künneth decomposition: $h_S(X) \simeq \bigoplus_i H^i(h_S(X))[-i]$, H^i computed for the motivic t -structure.
- Prop. 4.2.3, to get the decomposition for each $H^i(h_S(X))$ given it is pure of weight i .
- Rem. 4.2.4, point 2 to get the statement about the ℓ -adic realisation of the BCH-decomposition.

Note moreover that Bondarko obtains in his Proposition 4.2.3 the uniqueness of the set of isomorphism classes of the factors of the BCH-decomposition. We have not stated this uniqueness property in the above definition. Indeed, to give a proper statement we either need the existence of the heart of the motivic t -structure, or conservativity of the ℓ -adic realisations.¹³ However, we think that establishing BCH-decompositions is a good way to approximate the conjectural motivic t -structure, as shown by Deligne's formula (D').

Example 2.2.5. There is at least one easy example where a BCH-decomposition exists: this is the case of relative lci cellular morphisms, first addressed by Karpenko (see [ADN20, Def. 5.3.1, Cor. 5.3.7] for the more general case).¹⁴ Theorem 2.5.8 gives many more examples in the smooth case (see also 2.5.10), and has indeed served as a motivation for the previous definition.

2.3. Smooth Artin motives. It is possible to extend the known results on Artin motives over a field to the relative case. Let us start with the definition.

Definition 2.3.1. We define the category of (constructible) smooth Artin motives over S as the thick triangulated subcategory of $DM_c(S, \mathbb{Q})$ generated by motives of the form $h_S(X)$ (resp. $h_S(X)(n)$) for X/S finite and étale. We denote this category by $DM_{\text{sm}}^A(S, \mathbb{Q})$.

¹²Note also that Bondarko shows that this would be a consequence of Beilinson's conjectures (see *op. cit.*, Prop. 4.1.1). He also gives an argument to get it as a consequence of the standard conjecture D together with Murre's conjectures. Note that to get the existence of BCH-decompositions in all cases, we need to apply these conjectures for all residue fields of S , and all ℓ -adic realisations! Of course, this kind of exercise in juggling between motivic conjectures has its limit; we only hope to convince the reader that our definition of BCH-decompositions is reasonable.

¹³Note however that in our Theorems A and B below, the involved motives belong to subcategories on which the realisations can be proven to be conservative (Thm. 3.3.1). So, in our cases we *do* get uniqueness of isomorphism classes in the BCH-decompositions.

¹⁴This is not surprising as one can interpret the BCH-decomposition as an attempt to find an algebraic analogue of cellular decompositions of differential varieties.

Note that for X/S finite étale, $h_S(X)$ coincides with the Voevodsky motive $M_S(X)$.

Example 2.3.2. Let k be a perfect field. Then it follows from [VSF00, Chap. 5, Rem. 2. after 3.4.1] (see also [Org04] for details) that there exists a canonical equivalence of triangulated monoidal categories:

$$\mathrm{DM}_{\mathrm{sm}}^{\mathbb{A}}(k, \mathbb{Q}) \simeq \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(G_k))$$

where G_k is the absolute Galois group of k , and $\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(G_k)$ denotes the continuous representations of G_k with rational coefficients.

2.3.3. Let us recall that one model of $\mathrm{DM}(S, \mathbb{Q})$ is obtained as the \mathbb{P}^1 -stable \mathbb{A}^1 -derived category of the category $\mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q})$ of rational étale sheaves on Sm_S (see [CD19, 16.2.18]).

The inclusion $\rho : S_{\acute{e}t} \rightarrow Sm_{S, \acute{e}t}$ of étale sites induces a fully faithful and exact functor:

$$\rho_! : \mathrm{Sh}(S_{\acute{e}t}, \mathbb{Q}) \rightarrow \mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q}).$$

Note $\rho_!$ is moreover (symmetric) monoidal. One deduces a canonical composite functor:

$$\mathrm{D}(S_{\acute{e}t}, \mathbb{Q}) \xrightarrow{\rho_!} \mathrm{D}(\mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q})) \rightarrow \mathrm{DM}(S, \mathbb{Q})$$

the last functor being obtained by projection to the \mathbb{A}^1 -localization and then taking infinite suspensions. We will still denote the latter composite by $\rho_!$.

Then $\rho_!$ is triangulated and monoidal. By definition, it sends the sheaf represented by a finite étale scheme X/S on $S_{\acute{e}t}$ to the same object on $Sm_{S, \acute{e}t}$, seen as a motivic complex. This is just $M_S(X) = h_S(X)$.

Proposition 2.3.4. *Assume S is a regular connected scheme. Let $\pi = \pi_1(S_{\acute{e}t})$ be the étale fundamental group of S associated with some geometric base point. Then the functor $\rho_!$ is fully faithful when restricted to the full subcategory $\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(\pi))$ and induces an equivalence of triangulated monoidal categories:*

$$\rho_! : \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(\pi)) \rightarrow \mathrm{DM}_{\mathrm{sm}}^{\mathbb{A}}(S, \mathbb{Q}).$$

Moreover, the ℓ -adic realisation functor restricted to $\mathrm{DM}_{\mathrm{sm}}^{\mathbb{A}}(S, \mathbb{Q})$ lands into the bounded derived category of Artin ℓ -adic Galois representations and the composite functor:

$$\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(\pi)) \xrightarrow{\rho_!} \mathrm{DM}_{\mathrm{sm}}^{\mathbb{A}}(S, \mathbb{Q}) \xrightarrow{\rho_{\ell}} \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}_{\ell}}^{\mathbb{A}}(\pi))$$

is just the extension of scalar functor associated with $\mathbb{Q}_{\ell}/\mathbb{Q}$.

Proof. We prove the first assertion: fully faithful nature of the restriction of $\rho_!$. Note that the functor $\rho_!$ admits a right adjoint ρ^* . The functor ρ^* commutes with direct sums: this follows formally as $\mathrm{D}(S_{\acute{e}t}, \mathbb{Q})$ is compactly generated.¹⁵ We have to prove that for all complexes K, L in $\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(\pi))$, and say any $n \in \mathbb{Z}$ for the next reduction, the map:

$$\mathrm{Hom}(K, L[n]) \rightarrow \mathrm{Hom}(\rho_! K, \rho_! L[n]) = \mathrm{Hom}(K, \rho^* \rho_! L[n])$$

is an isomorphism. Now we use the fact $\mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathbb{A}}(\pi))$ is generated by shifts of sheaves representable by some finite étale cover X/S . So we are reduced to the

¹⁵In fact, it is equivalent to $\mathrm{D}(S_{\mathrm{Nis}}, \mathbb{Q})$ as π is pro-finite.

case $K = \mathbb{Q}(X)$, $L = \mathbb{Q}(Y)$ for X and Y étale cover of S . Explicitly, we have to prove that the following map is an isomorphism:

$$\mathrm{Hom}(\mathbb{Q}(X), \mathbb{Q}(Y)[n]) \rightarrow \mathrm{Hom}(h_S(X), h_S(Y)[n]).$$

In the monoidal category $\mathrm{Sh}(S_{\text{ét}}, \mathbb{Q})$, and therefore in $\mathrm{D}(S_{\text{ét}}, \mathbb{Q})$, the sheaf $\mathbb{Q}(Y)$ is auto-dual. The same result holds for $h_S(X)$ in $\mathrm{DM}(S, \mathbb{Q})$ (as for example $\rho_!$ is monoidal). Applying the formulas $\mathbb{Q}(X) \otimes \mathbb{Q}(Y) = \mathbb{Q}(X \times_S Y)$ and $h_S(X) \otimes h_S(Y) = h_S(X \times_S Y)$, we are reduced to the case $Y = S$. In other words we have to prove that the canonical map:

$$(2.3.4.a) \quad H^n(X_{\text{ét}}, \mathbb{Q}) \rightarrow \mathrm{Hom}(h_S(X), \mathbb{Q}[n]) = H_M^{n,0}(X, \mathbb{Q})$$

is an isomorphism. Note that X is regular, as it is étale over S . Thus it is geometrically unibranch and we get from [AGV73, IX, 2.14.1]:

$$H^n(X_{\text{ét}}, \mathbb{Q}) = \mathbb{Q}^{\pi_0(X)} \text{ for } n = 0, 0 \text{ otherwise.}$$

Also, according to [CD19, 14.2.14], we get:

$$H_M^{n,0}(X, \mathbb{Q}) = Gr_\gamma^0 K_{-n}(X)_\mathbb{Q} = \mathbb{Q}^{\pi_0(X)} \text{ for } n = 0, 0 \text{ otherwise,}$$

and so the map (2.3.4.a) is necessarily an isomorphism.

The other assertions are clear, by definition of the category of smooth Artin motives. \square

In the complex case, we get a simpler formulation.

Proposition 2.3.5. *Let E be a field of characteristic zero, S a smooth connected E -scheme and $\pi = \pi_1(S^{\text{an}})$ for any choice of base point of S^{an} .*

Then the Betti realisation functor:

$$\rho_B : \mathrm{DM}_c(S, \mathbb{Q}) \rightarrow \mathrm{D}(S^{\text{an}}, \mathbb{Q})$$

is fully faithful when restricted to $\mathrm{DM}_{\text{sm}}^A(S, \mathbb{Q})$ and induces an equivalence of triangulated monoidal categories:

$$\rho_B : \mathrm{DM}_{\text{sm}}^A(S, \mathbb{Q}) \rightarrow \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^A(\pi)).$$

The proof uses the same argument as in the case of the previous proposition, given that $h_S(X)$ is realized to the complex $Rf_*^{\text{an}}(\mathbb{Q}_X)$ which is concentrated in degree 0 and equal to the continuous representation of π represented by the Galois cover X^{an} over S^{an} .

Remark 2.3.6. The two previous propositions are obviously compatible: in the assumptions of the second one, we get according to a theorem of Grothendieck:

$$\pi_1(S_{\text{ét}}) \simeq \widehat{\pi_1(S^{\text{an}})}$$

where the right hand-side denotes profinite completion. In particular, we get equivalence of (abelian semi-simple monoidal) categories:

$$\mathrm{Rep}_{\mathbb{Q}}^A(\pi_1(S_{\text{ét}})) \simeq \mathrm{Rep}_{\mathbb{Q}}^A(\widehat{\pi_1(S^{\text{an}})}) \simeq \mathrm{Rep}_{\mathbb{Q}}^A(\pi_1(S^{\text{an}})).$$

2.4. Weights on smooth Artin-Tate motives. The purpose of this section is to extend results of Wildeshaus, [Wil16] on Artin-Tate motives over a field to the case of a regular base scheme S . Moreover, for the purpose of our main theorem, we will need to restrict the type of allowed Artin motives, as in *loc. cit.* So we introduce the next definition.

Definition 2.4.1. Let S be a regular connected scheme with étale fundamental group $\pi = \pi_1(S_{\text{ét}})$. Let \mathcal{A} be a full \mathbb{Q} -linear sub-category of $\text{Rep}_{\mathbb{Q}}^{\text{A}}(\pi)$ which is stable under retracts.¹⁶

We define the triangulated category of smooth Artin-Tate motives of type \mathcal{A} over S as the thick triangulated subcategory of $\text{DM}_c(S, \mathbb{Q})$ generated by motives of the form $\rho_!(A)(n)$ for an object A of \mathcal{A} and an integer $n \in \mathbb{Z}$ (see Proposition 2.3.4 for the definition of $\rho_!$). We denote it by $\text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})$.

When $\mathcal{A} = \text{Rep}_{\mathbb{Q}}^{\text{A}}(\pi)$, the above category is simply the category of smooth Artin-Tate motives over S , denoted by $\text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})$.

Remark 2.4.2. Note that compared to Definition 1.6 of [Wil16], we do not assume that \mathcal{A} is closed under tensor product. As a consequence $\text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})$ is not monoidal in general.

Theorem 2.4.3. *Consider the above notations and assumptions of the above definition.*

Then the weight structure on $\text{DM}_c(S, \mathbb{Q})$ restricts to a weight structure on the triangulated sub-category $\text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})$. Moreover, any motive M in $\text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})$ of weight 0 admits a decomposition:

$$(2.4.3.a) \quad M \simeq \bigoplus_{i \in I} \rho_!(V_i)(n_i)[2n_i]$$

where I is a finite set, V_i is a simple Artin representation of π in \mathcal{A} and n_i is an integer. (See Paragraph 2.3.3 for $\rho_!$).

Remark 2.4.4. (1) Another way of stating the second assertion is that the canonical functor

$$\begin{aligned} \text{D}^b(\mathcal{A}) &\rightarrow \text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q})_{w=0} = \text{DM}_{\text{sm}}^{\text{AT}}(S, \mathbb{Q}) \cap \text{CHM}(S, \mathbb{Q}) \\ (V_n)_{n \in \mathbb{Z}} &\mapsto \bigoplus_{n \in \mathbb{Z}} \rho_!(V_n)(n)[2n] \end{aligned}$$

is essentially surjective. Contrary to what happens in [Wil16], over a perfect field, this functor is not an equivalence of triangulated categories. The problem comes from the non-triviality of $\text{CH}^n(V)_{\mathbb{Q}}$ for $n > 0$ and V/S finite étale. So in particular, the preceding functor is an equivalence when S is (regular) semi-local.

(2) The decomposition (2.4.3.a) is a particular case of the Corti-Hanamura decomposition: If we apply the ℓ -adic realization functor ρ_{ℓ} to an Artin representation V_n we get (according to the last assertion of Prop. 2.3.4)

$$\rho_{\ell}(M) \simeq \bigoplus_{i \in I} (V_i \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})(n_i)[2n_i].$$

¹⁶And therefore stable under kernel and cokernel as $\text{Rep}_{\mathbb{Q}}^{\text{A}}(\pi)$ is abelian semi-simple. In particular, \mathcal{A} is abelian semi-simple.

In particular, if the decomposition (2.4.3.a) of M is not unique. But the pairs (V_i, n_i) for $i \in I$ are uniquely determined by M (or its realization).

Proof. Let us show the first assertion. Let \mathcal{H} be the full \mathbb{Q} -linear sub-category of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ whose objects are of the form (2.4.3.a), where V_i is a simple object of \mathcal{A} . As S is regular, all motives in \mathcal{H} are of weight 0. In particular, given such motives M, N , one has $\mathrm{Hom}(M, N[i]) = 0$ for $i > 0$. This condition implies that there is a unique weight structure on $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ (see eg. [Wil09, 1.5]), whose heart is the pseudo-abelianization \mathcal{K} of \mathcal{H} . It follows from the axioms of weight structures that this weight structure is just the restriction of the Chow weight structure, thus proving the first assertion.

To prove the second assertion, it is sufficient to prove that $\mathcal{K} = \mathcal{H}$, i.e. that \mathcal{H} is pseudo-abelian. Let $e : M \rightarrow M$ be an idempotent of \mathcal{H} . By assumption, we can write $M = \sum_{i=1}^r \rho_!(V_i)(n_i)[2n_i]$ where V_i is an object of \mathcal{A} (semi-simple, but not necessary simple) and $n_1 < n_2 < \dots < n_r$. We prove by induction on the number of factors r of M that e admits a kernel in \mathcal{H} .

We treat the case $r = 1$. According to Proposition 2.3.4, and the fact that twists are invertible, the category of motives of the form $\rho_!(V_1)(n_1)[2n_1]$ for an object V_1 of \mathcal{A} and an integer n_1 is equivalent to the abelian (semi-simple) category \mathcal{A} . This implies that any idempotent of a motive of this form admits a kernel, giving the case $r = 1$.

Assume the result is known for any integer less than $r > 1$, and prove the case where M has exactly r factors as above. Put $P = \rho_!(V_1)(n_1)[2n_1]$ and $Q = \sum_{i=2}^r \rho_!(V_i)(n_i)[2n_i]$ so that $M = P \oplus Q$. Given that decomposition, we can write the idempotent e as a 2 by 2 matrix:

$$e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then c belongs to

$$\bigoplus_{i=2}^r \mathrm{Hom}(\rho_!(V_i)(n_i)[2n_i], \rho_!(V_1)(n_1)[2n_1]) = \mathrm{CH}^{n_1 - n_i}(V_i \times_S V_1)$$

using the auto-duality of $\rho_!(V)$ (see the proof of Prop. 2.3.4) — in the right hand-side, we identify the sheaf V_i with the finite étale S -scheme which represent it. As by assumption, $n_1 < n_i$ for $i > 1$, we get that $c = 0$. Note that a and d are idempotent, respectively of P and Q . According to the induction case, these idempotents admit a kernel in \mathcal{H} . One easily deduces from the matrix form of e that $\mathrm{Ker}(a) \oplus \mathrm{Ker}(d)$ is a kernel of e in \mathcal{H} . This concludes the induction step, and the proof. \square

2.5. First decomposition theorem. For the needs of the proof of our first decomposition theorem, we will extend some definitions of [Wil12]. Let us first recall the following theorem, Th. 1.7 of *loc. cit.*

Theorem 2.5.1 ((Wildeshaus)). *Let $j : U \rightarrow S$ be an open immersion. Then the weight-exact functor j^* induces an additive exact functor*

$$j^* : \mathrm{CHM}(S) \rightarrow \mathrm{CHM}(U)$$

which is essentially surjective and full.

2.5.2. Consider an open immersion $j : U \rightarrow S$ and a Chow motive M over U . A *Chow extension* of M along j will be a pair (\bar{M}, α) where \bar{M} is a Chow motive

over S and α is an isomorphism $j^*(\bar{M}) \xrightarrow{\sim} M$. Morphisms of such extensions are defined in the obvious way.

According to the previous theorem, extensions of M along j always exist. The goal of the work of Wildeshaus is to find (and to define) the *intermediate extension* of M along j : see [Wil17], Summary 2.12. For the needs of our first decomposition theorem, we will use a special type of Chow extensions that we now introduce.

Definition 2.5.3. Consider the above notations, assuming that j is dense. Then a Chow extension (\bar{M}, α) of M along j will be called *fair* if the induced map

$$\mathrm{End}(\bar{M}) \rightarrow \mathrm{End}(j^*(M)) \xrightarrow{\alpha_*} \mathrm{End}(M)$$

is an isomorphism — *i.e.* a monomorphism according to the previous theorem.

The proof of the following result is identical to that of [Wil12, Th. 3.1(a)].

Theorem 2.5.4. *Consider the notations of the previous definition. Assume a fair Chow extension (\bar{M}, α) of M along j exists.*

Then for any extension (P, β) of M along j , there exists a decomposition of P of the form:

$$\psi : P \xrightarrow{\sim} \bar{M} \oplus i_*(L_Z)$$

where L_Z is a Chow motive over Z , satisfying the relation: $j^*(\psi) = \alpha^{-1} \circ \beta$.

If moreover (P, β) is fair, then $L_Z = 0$ and the isomorphism ψ is uniquely determined by the preceding relation.

Example 2.5.5. A fair Chow extension as above is a particular instance of Wildeshaus's theory of intermediate extension $j_{!*}(M)$, as shown for example by the characterizing property (4a) of Summary 2.12 of [Wil17]. The preceding theorem can also be interpreted as a minimality property of the extension (\bar{M}, α) .

One cannot always expect that such minimal extensions exist (the correct hope is formulated in *loc. cit.*, Conjecture 3.4; cfr. Rmk. 3.1.5). However, here are some interesting examples.

- (1) Assume that U is regular, and the normalization of X is regular. Then $\mathbb{1}_U$ admits a fair Chow extension: this is [Wil12, Th. 3.11(a)].
- (2) Assume that both U and S are regular. Let V/U be an étale cover. Using a classical terminology, we will say that V is *non-ramified along $(S - U)$* if there exists an étale cover \bar{V}/S whose restriction to U is V . In that case, for any integer $n \in \mathbb{Z}$, the Artin-Tate motive $\bar{M} = h_S(\bar{V})(n)[2n]$ is a fair Chow extension of $M = h_U(V)(n)[2n]$ along j .

Indeed, \bar{M} is obviously a Chow extension of M along j (using the six functors formalism). Moreover, as in the proof of Prop. 2.3.4, one gets:

$$\mathrm{End}(\bar{M}) \simeq \mathbb{Q}^{\pi_0(\bar{V} \times_S \bar{V})}, \quad \mathrm{End}(M) \simeq \mathbb{Q}^{\pi_0(V \times_U V)}.$$

As $V \times_U V$ is a dense open of $\bar{V} \times_S \bar{V}$, we get that \bar{M} is fair as claimed.

Remark 2.5.6. The reader can check that one can replace, in [Wil12, Th. 3.11] points (a) and (b), the constant motive $\mathbb{1}_U$ by a smooth Artin-Tate motive $h_U(V)$ such that V is unramified along $(S - U)$.

2.5.7. Recall also that a Chow motive over a field k is said to be *Tate* if it is isomorphic to a finite sum of motives of the form $\mathbb{1}(i)[2i]$ for an integer $i \in \mathbb{Z}$. By extension, a smooth proper k -scheme is said to be Tate if its associated Chow motive is Tate.

Theorem 2.5.8. *Let S be a regular connected scheme with étale fundamental group π . Let $f : X \rightarrow S$ be a smooth proper morphism whose geometric fibers are Tate.*

Then there exists an isomorphisms of motives over S :

$$h_S(X) \simeq \bigoplus_{i \in I} \rho_!(V_i)(-n_i)[-2n_i]$$

where I is a finite set, V_i is a simple Artin representation of π and n_i is a non-negative integer. In other words, $h_S(X)$ is a smooth Artin-Tate motive over S of weight 0 — see Theorem 2.4.3.

Moreover, this decomposition is a BCH-decomposition of the Chow motive $h_S(X)$ (see Def. 2.2.3), and the set $\{([V_i], n_i), i \in I\}$ where $[V_i]$ denotes the isomorphism class in $\text{Rep}_{\mathbb{Q}}^{\text{A}}(\pi)$ (or what amount to the same in the corresponding category of simple étale S -covers) is uniquely determined by the property that for any prime integer ℓ and any $n \geq 0$:

$$(2.5.8.a) \quad \mathbb{R}^{2n} f'_*(\mathbb{Q}_\ell) \simeq \bigoplus_{i \in I | n_i = n} V'_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-n_i)$$

where f' and V'_i are pullback of f and V_i along the open immersion $U[\ell^{-1}] \rightarrow U$.

Proof. Let $\bar{\eta}$ be a geometric generic point of S . By assumption, one gets:

$$h_{\bar{\eta}}(X_{\bar{\eta}}) \simeq \sum_{i \in I} \mathbb{1}_{\bar{\eta}}(-n_i)[-2n_i].$$

According to the continuity property of DM_c (see Proposition 1.0.1), there exists a dense open $j : U \rightarrow S$ and an étale cover $p : V \rightarrow U$ such that the above isomorphism lifts to:

$$h_V(X_V) \simeq \bigoplus_{j \in J} \mathbb{1}_V(-m_j)[-2m_j].$$

In particular, $h_U(X_U) = p_*(h_V(X_V)) \simeq \sum_{j \in J} p_*(\mathbb{1}_V)(-m_j)[-2m_j]$ is a smooth Artin-Tate motive over U . As p is finite étale, the natural map

$$p^* : h_U(X_U) \rightarrow h_U(X_V)$$

is a split monomorphism, with splitting $\frac{1}{d} p_*$ where p_* is the Gysin morphism associated with p (see e.g. [CD19, 13.7.4 and 13.7.6]). In particular, $h_U(X_U)$ is a smooth Artin-Tate motive, and it follows from Theorem 2.4.3 that there exists a decomposition:

$$h_U(X_U) \simeq \bigoplus_{i \in I} \rho_!(W_i)(-n_i)[-2n_i]$$

where W_i is a simple Artin representation of $\pi_1(U)$. Given a prime ℓ , we let $U' = U[\ell^{-1}]$, $f_{U'}$ the pullback of f over U' , W'_i the Artin representation of $\pi_1(U')$ induced by W_i . According to Remark 2.4.4(2), and because $\rho_\ell(h_{U'}(X_{U'})) \simeq \mathbb{R} f_{U'*}(\mathbb{Q}_\ell)$, one gets:

$$(2.5.8.b) \quad \mathbb{R}^{2n} f_{U'*}(\mathbb{Q}_\ell) \simeq \bigoplus_{i \in I | n_i = n} W'_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-n_i).$$

This is the decomposition of the locally constant sheaf $\mathbb{R}^{2n} f_{U'*}(\mathbb{Q}_\ell)$ into semi-simple components (beware the W'_i might not be simple). As this sheaf admits an extension to all S , namely $\mathbb{R}^{2n} f'_*(\mathbb{Q}_\ell)$, $f' = f[\ell^{-1}]$, it follows that each representation W'_i is unramified along $(S - U)$. As this is true for any prime ℓ , one deduces

that W_i is unramified along $(S - U)$; *i.e.* it admits an extension V_i to S . According to Example 2.5.5(2), one deduces that

$$\bigoplus_{i \in I} \rho_l(V_i)(-n_i)[-2n_i]$$

is a fair Chow extension of $h_U(X_U)$. As $h_S(X)$ is obviously a Chow extension of $h_U(X_U)$, one deduces from Theorem 2.5.4 that there exists a decomposition:

$$(2.5.8.c) \quad h_S(X) \simeq \bigoplus_{i \in I} \rho_l(V_i)(-n_i)[-2n_i] \oplus i_*(M_Z)$$

for some Chow motive M_Z over Z . Note that Relation (2.5.8.b) implies Relation (2.5.8.a).¹⁷

It remains to prove that $M_Z = 0$. As the morphism $p : S' = \sum_{\ell \text{ prime}} S[\ell^{-1}] \rightarrow S$ is a pro-open cover, the pullback functor $p^* : \text{DM}(S) \rightarrow \text{DM}(S')$ is conservative (use the continuity property of DM [CD19, Th. 14.3.1] and the Zariski separation property as in the proof of [CD19, Prop. 4.3.9]). In particular, we can fix a prime ℓ and work over $S[\ell^{-1}]$. To simplify notation, let us assume $S = S[\ell^{-1}]$. Consider a point $x \in Z$, and let $i_x : \{x\} \rightarrow Z$ be the canonical immersion. Let \bar{x} be a geometric point over x . By assumption, $h_{\bar{x}}(X_{\bar{x}})$ is a Tate motive. In other words, the motive $i_x^* h_S(X) = h_x(X_x)$ is an Artin-Tate motive over the residue field $\kappa(x)$. Decomposition (2.5.8.c) gives:

$$h_x(X_x) \simeq \bigoplus_{i \in I} \rho_l(V_{i,x})(-n_i)[-2n_i] \oplus i_x^*(M_Z)$$

where $V_{i,x}$ denotes the pullback of V_i to x (seen as an étale sheaf). As recalled in Remark 2.4.4(2), applied over $\kappa(x)$, the ℓ -adic realization of $h_x(X_x)$ determines the factors of the decomposition of $h_x(X_x)$ into twists of Artin motives (up to isomorphisms and permutations). Thus relation (2.5.8.a) (which we have already established), specialized at x using the smooth base change theorem in ℓ -adic étale cohomology, implies that the set of isomorphism classes of the $V_{i,x}$ describes all the possible factors of the weight 0 Artin-Tate motive $h_x(X_x)$ and this implies $i_x^*(M_Z) = 0$. One concludes as $(i_x^*)_{x \in Z}$ is a conservative family of functors on $\text{DM}_c(Z, \mathbb{Q})$ (see [CD19, 4.3.17]). \square

Remark 2.5.9. Note also that we deduce from Example 2.5.5 that for any dense open immersion $j : U \rightarrow S$, $h_S(X)$ is a fair Chow extension of $h_U(X_U)$. In particular, $h_S(X) = j_*(h_U(X_U))$.

Example 2.5.10. Relative Severi-Brauer schemes, étale locally cellular schemes, relative homogeneous varieties, smooth quadrics.

Corollary 2.5.11. *Under the assumption of the previous theorem, for any integer $n \geq 0$, the integer:*

$$\dim_{\mathbb{Q}_\ell} (\mathbb{R}^n f_*(\mathbb{Q}_\ell))$$

is independent of the prime ℓ invertible on S .

¹⁷In particular, we know that $\rho_\ell(M_Z)$ vanishes but this is not sufficient to conclude (as we do not know yet that the ℓ -realization is conservative).

2.5.12. An important corollary for us is the case of smooth quadrics: following [DK73, XII, Def. 2.4], a smooth quadric over a scheme S is a smooth proper morphism $f : X \rightarrow S$ whose geometric fibers are smooth quadric hypersurfaces in the classical sense.

Let us introduce some notations in order to state the next corollary. Assume f has constant relative dimension n . According to *op. cit.*, 2.6, there exists a Severi Brauer S -scheme $P(X)$ which contains X as an effective Cartier divisor of degree 2. If $n = 2m$, the center of the Clifford \mathcal{O}_S -algebra associated with the closed pair $(P(X), X)$ is an étale cover $Z(X)$ over S of degree 2; *op. cit.*, 2.7.

Then, applying Theorem 2.5.8 and [DK73, Th. 3.3] one gets:

Corollary 2.5.13. *Consider the above notations. Assume that S is regular and that there exists a prime ℓ invertible on S .*

Then there exists a decomposition of Chow motives over S as follows:

$$h_S(X) \simeq \begin{cases} \bigoplus_{i=0}^n \mathbb{1}_S(i)[2i] & \text{if } n = 2m + 1, \\ h_S(Z(X)) \oplus \bigoplus_{i=0, i \neq m}^n \mathbb{1}_S(i)[2i] & \text{if } n = 2m. \end{cases}$$

Example 2.5.14. Let us give some concrete examples illustrating the above Corollary when S is an arithmetic scheme.

(1) It is known that smooth quadrics over $\text{Spec}(\mathbb{Z})$ exist: one way to see this has been explained by Buzzard in [hb]. One considers a quadric hypersurface X in $\mathbb{P}_{\mathbb{Q}}^N$ whose Hessian matrix H is a symmetric, integer valued $(N+1) \times (N+1)$ matrix with $\det H = \pm 1$ and even entries down the diagonal; e.g., the Gram matrix of any even unimodular lattice (these exist in any positive dimension divisible by 8) provides such an H . As $\det H$ is non-zero mod p for any prime p , the quadric X admits a proper and smooth model over \mathbb{Z} .

(2) Since $\text{Spec}(\mathbb{Z})$ is simply connected (a consequence of Minkowski's theorem), the previous construction yields quadrics whose motive decomposes, according to Cor. 2.5.13, as a sum of Tate motives (the Artin factors are necessarily trivial). In order to find cases where genuine Artin-Tate motives appear, we have to place ourselves over a non-simply connected base. Let then F be a number field with ring of integers \mathcal{O}_F , such that $S := \text{Spec}(\mathcal{O}_F)$ has non-trivial étale fundamental group. In this setting, examples abound, as we are going to argue following a line of thought suggested to us by G. Ancona.

Let N be a positive integer. Analogously to Example (1) above, we look to a \mathcal{O}_F -valued $2N \times 2N$ matrix H , such that any entry on the diagonal is divisible by 2, and $\det H$ is invertible in \mathcal{O}_F . We may take for example $N = 2$, $\mathcal{O}_F = \mathbb{Z}[\sqrt{-5}]$ (we have then $\pi_{1, \text{ét}}(S) \simeq \mathbb{Z}/2\mathbb{Z}$) and

$$(2.5.14.a) \quad H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A = \begin{pmatrix} -4\sqrt{-5} & 9 \\ 9 & 4\sqrt{-5} \end{pmatrix}, B = \begin{pmatrix} 2 & \sqrt{-5} \\ \sqrt{-5} & -2 \end{pmatrix}$$

The matrix H is then the Hessian matrix of a smooth quadric X in \mathbb{P}_S^{2N-1} . We want to exhibit examples where the motive $h_S(Z(X))$ of Cor. 2.5.13 is non-Tate. By proper base change and Thm. 2.3.4, it is enough to make sure that there exists a prime \mathfrak{p} of \mathcal{O}_F such that the Galois representation on $H_{\text{ét}}^{N-1}(X_{\mathfrak{p}}, \mathbb{Q}_{\ell})$, for $\ell \neq p := \text{car } \mathcal{O}_F/\mathfrak{p}$, is non-trivial. The Lefschetz fixed point formula shows that the latter property is equivalent to the condition

$$\text{Card}(X_{\mathfrak{p}}(\mathbb{F}_q)) \neq 1 + q + \dots + 2q^{N-1} + \dots q^{2N-2},$$

where \mathbb{F}_q denotes the field $\mathcal{O}_F/\mathfrak{p}$ and $q = p^r$ for some r . This can be checked in specific cases, using the following elementary fact: a quadratic form in at least three variables over a finite field always admits an isotropic vector, hence always contains an hyperbolic plane.

We make explicit this principle when $N = 2$. Let m be the quadratic form in four variables over \mathbb{F}_q obtained by reducing the coefficients of H modulo \mathfrak{p} . By the fact just recalled, m can be written in the form $ut = rx^2 + sy^2$, with $r, s \in \mathbb{F}_q^\times$. Now a computation shows that if $rx^2 + sy^2$ has no isotropic vectors, then $\text{Card}(X_{\mathfrak{p}}(\mathbb{F}_q)) = 1 + q^2$, thus giving us the desired quadric. For example, if we choose A and B as in (2.5.14.a) and suppose that $p \neq 2$, we have $r = s = 1$, and the fact that the bilinear form $x^2 + y^2$ has no isotropic vectors is equivalent to ask that $1 = \text{disc}(x^2 + y^2) \neq -1 \pmod{(\mathbb{F}_q^\times)^2}$, which is equivalent to ask -1 not being a square in \mathbb{F}_q . Now choose as \mathfrak{p} one of the two prime ideals above the completely split prime $p = 3$. Then, $\mathbb{F}_q = \mathbb{Z}/3\mathbb{Z}$, and -1 is not a square mod 3.

3. RECALL AND COMPLEMENT ON WILDESHAUS'S MOTIVIC INTERMEDIATE EXTENSIONS

3.1. Semi-primary categories and motivic intermediate extensions. Recall the following definition.

Definition 3.1.1. A \mathbb{Q} -linear category \mathfrak{C} is *semi-primary* if

(1) for all objects B of \mathfrak{C} , the *radical*

$$\text{rad}_{\mathfrak{C}}(B, B) := \{f \in \text{Hom}_{\mathfrak{C}}(B, B) \mid \forall g \in \text{Hom}_{\mathfrak{C}}(B, B), \text{id}_B - gf \text{ is invertible}\}$$

is nilpotent;

(2) the quotient category $\mathfrak{C}/\text{rad}_{\mathfrak{C}}$ is semisimple.

Adopt the notations of subsection 2.1. We are now going to explain how the notion of semi-primality leads to a definition of an intermediate extension functor, following Wildeshaus.

Let \cdot denote any of the schemes U , S or Z , and fix $\mathcal{C}(\cdot)$ full pseudo-abelian subcategories of the categories $\text{DM}_{\mathcal{C}}(\cdot, \mathbb{Q})$, related by gluing. Assume that they inherit a weight structure (automatically compatible with the gluing) from the restrictions of the motivic weight structure. The subscript $w = 0$ will mean that we are taking the *heart* of such weight structures.

Moreover, denote by $\mathcal{C}(S)_{w=0}^u$ the quotient of the category $\mathcal{C}(S)_{w=0}$ by the two-sided ideal¹⁸ \mathfrak{g} generated by $\text{Hom}_{\mathcal{C}(S)}(A, i_*B)$ and $\text{Hom}_{\mathcal{C}(S)}(i_*B, A)$, with (A, B) varying on the collection of objects of $\mathcal{C}(S)_{w=0} \times \mathcal{C}(Z)_{w=0}$ such that A admits no non-zero direct factor belonging to $\mathcal{C}(Z)_{w=0}$. Finally, denote by $\mathcal{C}(Z)_{w=0}^u$ the quotient of the category $\mathcal{C}(Z)_{w=0}$ by the restriction of \mathfrak{g} to $\mathcal{C}(Z)_{w=0}$ (with respect to the fully faithful inclusion $i_* : \mathcal{C}(Z)_{w=0} \hookrightarrow \mathcal{C}(S)_{w=0}$).

Theorem 3.1.2. [Wil17, Theorem 2.9]

(1) If $\mathcal{C}(Z)_{w=0}$ is semi-primary, then the functors j^* and i_* induce a canonical equivalence of categories

$$(3.1.2.a) \quad \mathcal{C}(S)_{w=0}^u \simeq \mathcal{C}(U)_{w=0} \times \mathcal{C}(Z)_{w=0}^u$$

(2) If both $\mathcal{C}(Z)_{w=0}$ and $\mathcal{C}(U)_{w=0}$ are semi-primary, then so is $\mathcal{C}(S)_{w=0}$.

¹⁸Note that in our setting, this ideal will always be contained in $\text{rad}_{\mathcal{C}(S)_{w=0}}$ ([Wil17, Cor. 1.5 (a)]).

Definition 3.1.3. Suppose that $\mathcal{C}(Z)_{w=0}$ is semi-primary. The *intermediate extension* is the fully faithful functor

$$j_{!*} : \mathcal{C}(U)_{w=0} \hookrightarrow \mathcal{C}(S)_{w=0}^u$$

corresponding to the functor $(\mathrm{id}_{\mathcal{C}(U)_{w=0}}, 0)$ under the equivalence of categories (3.1.2.a).

It follows from its very definition that the motivic intermediate extension functor $j_{!*}$ enjoys the following property ([Wil17, Summary 2.12 (b)]):

Proposition 3.1.4. *Consider the notation and the assumptions of the previous definition. Then, any object M of $\mathcal{C}(S)_{w=0}$ is isomorphic to a direct sum $j_{!*}M_U \oplus i_*N$, for an object M_U of $\mathcal{C}(U)_{w=0}$ and an object N of $\mathcal{C}(Z)_{w=0}$. The object M_U is such that $j^*M \simeq M_U$ (hence unique up to unique isomorphism) and N is unique up to an isomorphism, which becomes unique in $\mathcal{C}(Z)_{w=0}^u$.*

Remark 3.1.5. It is believed that for any S , the heart $\mathrm{CHM}(S)$ of the motivic weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$ is semi-primary (cfr. [Wil17, Conj. 3.4]). This conjecture is at the moment completely out of reach, but it would permit, by choosing as $\mathcal{C}(\cdot)$ the whole of the categories $\mathrm{DM}_c(\cdot, \mathbb{Q})$, to define (up to non-unique isomorphism) the intermediate extension to S of any Chow motive on U .

3.2. Semi-primary categories of Chow motives. The aim of this section is to single out some subcategories of Chow motives which can actually be shown to be semi-primary and which will be suited for our geometric applications. We will adapt Wildeshaus' methods from [Wil17], in order to show semiprimality of subcategories which are different¹⁹ from the ones considered in *op. cit.*.

Fix a scheme S admitting a *good stratification* \mathfrak{S} , i.e. such that S may be written as a finite (set-theoretic) disjoint union $\bigsqcup_{\sigma \in \mathfrak{S}} S_\sigma$ of locally closed subschemes such that the closure \overline{S}_σ of each stratum S_σ is a union of strata.

We make the following assumption on our good stratification:

Assumption 3.2.1. For all $\sigma \in \mathfrak{S}$, the strata S_σ are nilregular²⁰, with nilregular closure.

Fix a stratum S_σ and consider the categories $\mathrm{DM}_{\mathrm{sm}}^{\mathcal{A}\mathrm{T}}(S_\sigma, \mathbb{Q})$ introduced in Definition 2.4.1.

Definition 3.2.2. The category of *tamely ramified* smooth Artin-Tate motives over S_σ is the category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{tr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ obtained by choosing $\mathcal{A} = \mathrm{A}_{\mathrm{tr}}$, where A_{tr} is the full subcategory of direct factors of objects corresponding to finite étale morphisms $q : X_{S_\sigma} \rightarrow S_\sigma$ which are *tamely ramified in codimension 1* ([dJ, 57.31, tag 0BSE]).

Remark 3.2.3. In the previous definition, suppose that S_σ is a scheme of characteristic zero. Then any finite étale morphism with target S_σ is tamely ramified in codimension 1, and the category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{tr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ equals $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S_\sigma, \mathbb{Q})$.

Theorem 3.2.4. *Let S be a scheme with a good stratification \mathfrak{S} satisfying Assumption 3.2.1. Then:*

¹⁹See Rmk. 3.2.11 for a comment on these differences.

²⁰I.e., the underlying reduced scheme is regular.

(1) the categories $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(S_\sigma, \mathbb{Q})$ of tamely ramified smooth Artin-Tate motives over S_σ , $\sigma \in \mathfrak{S}$, can be glued to give a full, triangulated sub-category $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$ of $\mathrm{DM}_c(S, \mathbb{Q})$, called the category of \mathfrak{S} -constructible tamely ramified Artin-Tate motives over S . This subcategory is dense.

(2) Let $M \in \mathrm{DM}_c(S, \mathbb{Q})$. Then the following conditions are equivalent.

(a) $M \in \mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$.

(b) $j^*M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(S_\sigma, \mathbb{Q})$ for all $\sigma \in \mathfrak{S}$, where j denotes the immersion $S_\sigma \hookrightarrow S$.

(c) $j^!M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(S_\sigma, \mathbb{Q})$ for all $\sigma \in \mathfrak{S}$.

In particular, the triangulated category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ of smooth Artin-Tate motives over S is contained in $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$.

(3) The category $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$ is pseudo-abelian.

Proof. One proceeds by induction on the number of strata. If there is only one stratum, all claims are trivial, except for the last claim of part (1) and for part (3). Of course, it suffices to prove part (3). But in the case we are treating, the category in question is just $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$, and it has been shown in the proof of Thm. 2.4.3 that this category has a (bounded) weight structure whose heart is pseudo-abelian. Hence (by [Bon10, Lemma 5.2.1]), the category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ itself is pseudo-abelian.

As for the induction step, we have that the theorem is true for the complement Z of any open stratum U , with its stratification \mathfrak{S}_Z , by the induction hypothesis. Write j_U , resp. i_Z , for the open, resp. locally closed immersion of U , resp. Z , in S .

In order to prove (1), we will prove that the criterion given by [Wil17, Prop. 4.1. (a)] is satisfied; it says that $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(U, \mathbb{Q})$ and $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$ can be glued if and only if for all objects $M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(U, \mathbb{Q})$, $i_Z^*j_{U,*}M$ belongs to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. For this, we can suppose that the closure of U in S is the whole of S , that S is connected, and that S and Z are regular (and not just nilregular, cfr. Assumption 3.2.1). Then, we take a direct factor M_U of some motive $h_U(X_U/U)$, with $q : X_U \rightarrow U$ finite étale and tamely ramified in codimension 1. Suppose now that Z is of codimension 1 in S . Then by [dJ, Lem. 57.31.5, tag 0EYH], the normalization $q' : X \rightarrow S$ of X in X_U is a finite morphism such that the restriction of X to U is isomorphic to X_U and such that $Z' := ((q')^{-1}(Z))_{\mathrm{red}}$ is an effective Cartier divisor which is a regular scheme. This shows that X is regular, because otherwise, any singular point of X would lie in Z' , and by a standard argument, it would be a singular point of Z' . We can now apply absolute purity to the closed immersion $Z' \hookrightarrow X$, and by proper base change and the fact that q' induces an isomorphism $Z' \simeq Z$, we get that for any $p \in Z$, both $i_Z^!h_S(X/S)(p)$ and $i_Z^*h_S(X/S)(p)$ belong to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. Now, again using proper base change, we see that $i_Z^*j_{U,*}h_U(X_U/U)(p)$ is a cone of the canonical morphism $i_Z^!h_S(X/S)(p) \rightarrow i_Z^*h_S(X/S)(p)$. Hence, it belongs to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. But the latter category is dense by the induction hypothesis, so $i_Z^*j_{U,*}M_U(p)$ belongs to it as well. This means that the functor $i_Z^*j_{U,*}$ maps the generators of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(U, \mathbb{Q})$ to $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. As a consequence, the whole of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AtrT}}(U, \mathbb{Q})$ is mapped to the latter category under $i_Z^*j_{U,*}$, and the criterion is fulfilled.

If Z is of codimension ≥ 2 in S , then q extends to a finite étale morphism $q' : X \rightarrow S$ by purity of the branch locus (see [dJ, Lem. 53.20.4, tag 0BMB]) and we conclude by reasoning as above.

The proof of the remaining points carries over word by word from the proof of the analogous points in [Wil17, Thm. 4.5]. \square

Let us now set up a slightly different notation, in order to treat the more general geometric situations which we are interested in. Suppose that S is equipped with a good stratification, denoted from now on by Φ . The subcategories we are interested in will depend on the choice of a proper morphism $\pi : S' \rightarrow S$ from a scheme S' which admits a good stratification \mathfrak{S} , such that the preimage via π of any stratum of Φ is a union of strata (i.e. π is a *morphism of good stratifications*). Whenever T is a subscheme of S , we will denote by \mathfrak{S}_T the stratification induced by \mathfrak{S} on S'_T .

It will be necessary to make the following assumptions on the nature of the stratifications \mathfrak{S} and Φ and of the morphism π :

Assumption 3.2.5. (cfr. [Wil17, Ass. 5.6])

- (1) the good stratification \mathfrak{S} on S' satisfies Assumption 3.2.1;
- (2) for all $\phi \in \Phi$, the strata S_ϕ are nilregular;
- (3) the morphism π is surjective, and for all $\phi \in \Phi$ and $\sigma \in \mathfrak{S}$ such that S'_σ is a stratum of $\pi^{-1}(S_\phi)$, the morphism $\pi_\sigma : S'_\sigma \rightarrow S_\phi$ is proper with geometrically connected fibres, smooth, and such that the motive $h(S'_\sigma/S_\phi)$ belongs to the category of Tate motives over S_ϕ .

Remark 3.2.6. Since we ask π_σ to be proper and smooth, the previous assumption actually implies that $h(S'_\sigma/S_\phi)$ belongs to the category of *weight zero, smooth* Tate motives over S_ϕ .

Fix a morphism $\pi : S' \rightarrow S$ of good stratifications Φ and \mathfrak{S} , satisfying Assumption 3.2.5.(1).

Definition 3.2.7. The category $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is the pseudo-abelian completion of the strict, full, \mathbb{Q} -linear triangulated subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by the images under π_* of the objects of $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$.

Suppose moreover that Assumption 3.2.5.(2) is satisfied. Then, reasoning in the same way as in the proof [Wil17, Cor. 4.11], we see:

Lemma 3.2.8. (1) *The restriction of the motivic weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$ induces a bounded weight structure on $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$. For $\pi = \mathrm{id}$, this gives a bounded weight structure on $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$.*

(2) *The heart of the above weight structure on $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is the pseudo-Abelian completion of the strict, full, \mathbb{Q} -linear additive subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by the images under π_* of the objects of the heart of the weight structure on $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$.*

Definition 3.2.9. The heart of the weight structure on $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ given by the preceding Lemma is denoted by $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$.

Now we can finally state the result that we want to employ.

Theorem 3.2.10. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} , Φ satisfying Assumption 3.2.5. Then, the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$ is semi-primary.*

Proof. By Thm. 3.2.4 and proper base change (cfr. the analogous [Wil17, Cor. 4.10 (b)]), the category $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is obtained by gluing the categories $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S}_{S_\phi}/S_\phi)$ for all $\phi \in \Phi$. Hence, to prove our claim, it is enough to prove semi-primality of $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ for each stratum S_ϕ and then to apply Thm. 3.1.2.(2) iteratively.

We first observe that by Lemma 3.2.8.(2), the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ is the pseudo-Abelian completion of the strict, full, \mathbb{Q} -linear triangulated subcategory of

$\mathrm{DM}_c(S_\phi)$ of objects, which are isomorphic to images under π_σ of tamely ramified Artin-Tate motives over S'_σ , for $\sigma \in \mathfrak{S}$ such that S'_σ is a stratum of $\pi^{-1}(S_\phi)$. This implies that, by reasoning as in [Wil17, Thm. 5.4], the claim will follow as soon as we prove that the objects of the latter form are *finite dimensional* in the sense of Kimura.

Choose a couple of strata $\pi_\sigma : S'_\sigma \rightarrow S_\phi$ as above and take a tamely ramified finite étale morphism $q : D \rightarrow S'_\sigma$. The Stein factorization of the morphism $\pi_\sigma \circ q$ gives rise to a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{q} & S'_\sigma \\ p \downarrow & & \downarrow \pi_\sigma \\ \tilde{S}_\phi & \xrightarrow{r} & S_\phi \end{array}$$

where p is proper with connected fibres and r is finite étale. Moreover, the fiber of r over each point s of S_ϕ is in set-theoretic bijection with the set of connected components of the fiber of $\pi_\sigma \circ q$ over s . Thus, because of our assumption 3.2.5.(3) on the properties of π_σ , the degree of r is the same as the degree of q , say equal to d . We get a diagram

$$\begin{array}{ccccc} D & & & & \\ & \searrow \iota & & \searrow q & \\ & & \tilde{S}_\phi \times_{S_\phi} S'_\sigma & \xrightarrow{q'} & S'_\sigma \\ & \searrow p & \downarrow p' & & \downarrow \pi_\sigma \\ & & \tilde{S}_\phi & \xrightarrow{r} & S_\phi \end{array}$$

where q' is finite étale of degree d , so that ι has to be finite étale of degree 1, i.e. it embeds D as a connected component of $\tilde{S}_\phi \times_{S_\phi} S'_\sigma$. Call the latter scheme D' . Then, using proper base change and the fact that q' and r are finite étale, we get

$$h(D'/\tilde{S}_\phi) = p'_* \mathbb{1}_{D'} \simeq p'_* q'^* \mathbb{1}_{S'_\sigma} \simeq r^* \pi_{\sigma,*} \mathbb{1}_{S'_\sigma} = r^* h(S'_\sigma/S_\phi)$$

Since $h(S'_\sigma/S_\phi)$ belongs to the category of weight zero, smooth Tate motives over S_ϕ (again by Assumption 3.2.5.(3) and Rem. 3.2.6), we obtain that $h(D'/\tilde{S}_\phi)$ belongs to the category of weight zero, smooth Tate motives over \tilde{S}_ϕ . The motive $h(D/\tilde{S}_\phi)$, being a direct factor of $h(D'/\tilde{S}_\phi)$, belongs to the same category as well. Now $h(D/S_\phi)$ is isomorphic to the direct image of $h(D/\tilde{S}_\phi)$ under the finite étale morphism r , and as a consequence, it is actually a weight zero, smooth Artin-Tate motive over S_ϕ . As such, it is indeed finite dimensional (apply for example [Wil17, Prop. 5.8 (c)]). As the objects we were interested in are direct factors of objects of the form $h(D/S_\phi)$, we conclude. \square

Remark 3.2.11. (1) If we relax point (3) of Assumption 3.2.5 by asking that $h(S'_\sigma/S_\phi)$ be simply *finite dimensional*, the proof of the above theorem carries through, with the following adjustments. First, one exploits the commutative diagram coming from the Stein factorization and invokes proper base change and [Wil17, Prop. 5.8

(a)] in order to show that $h(D/\tilde{S}_\phi)$ is also finite dimensional. Then, one shows that the same holds for $h(D/S_\phi)$, by applying [Wil17, Prop. 5.8 (c)].

(2) The above theorem differs in the following way from the analogous Thm. 5.4 in [Wil17]. On the one hand, in order to deal with the gluing, we are forced to be more restrictive on the choice of possible morphisms $S' \rightarrow S$. In fact, we ask for regularity of the closure of the strata of S' , whereas in *loc. cit.*, it is only asked the weaker condition that for every immersion i_σ of a stratum S'_σ in the closure of a stratum, the functor $i_\sigma^!$ send the unit object to a Tate motive. Moreover, the morphisms π_σ in *loc. cit.* can belong to a more general class than the one considered here. On the other hand, our stronger restrictions are necessary because, for a fixed S' which fulfils our requirements, the categories that we glue along the strata of S' are more general than the ones of *loc. cit.*

The proof of the above theorem shows in particular that for each $\phi \in \Phi$, denoting by Z_ϕ the complement of a stratum S_ϕ in its closure $\overline{S_\phi}$, the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{Z_\phi}/Z_\phi)$ is semiprimary. So we get:

Corollary 3.2.12. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.5. For each $\phi \in \Phi$, denote by $j_\phi : S_\phi \hookrightarrow \overline{S_\phi}$ the open immersion of a stratum in its closure. Then for each $\phi \in \Phi$, the intermediate extension functor*

$$j_{\phi,!*} : \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi) \hookrightarrow \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{\overline{S_\phi}}/\overline{S_\phi})^u$$

is defined (as in Def. 3.2.12).

Corollary 3.2.13. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.5. Let M be an object of the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$. Consider the notations of the previous Corollary, and denote moreover by $i_\phi : \overline{S_\phi} \hookrightarrow S$ the closed immersion of the closure of a stratum. Then, there exist a subset $\Phi' \subset \Phi$, objects N_ϕ in $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$, $\phi \in \Phi'$, and a non-canonical isomorphism*

$$M \simeq \bigoplus_{\phi \in \Phi'} i_{\phi,*} j_{\phi,!*} N_\phi$$

Proof. We may always suppose, for simplicity, that there is only one open stratum $U := S_{\phi_1}$ of Φ , and denote j_{ϕ_1} by $j : U \hookrightarrow S$. By using Cor. 3.2.12 and applying Prop. 3.1.4, we know that we have an isomorphism

$$M \simeq j_{!*} M_U \oplus i_* N$$

with N an object of $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_Z/Z)$. By proper base change and part (2) of Thm. 3.2.4, and the fact that pullback along open immersions sends weight-zero objects to weight-zero objects, we know that the pullback of N to any stratum S_ϕ which is open in Z belongs to $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$. Thus, we can apply to it the functor $j_{\phi,!*}$ (defined using again Cor. 3.2.12). The statement then follows by an iterated application of Prop. 3.1.4. \square

Notations 3.2.14. To ease notation, in future applications we will often write j_ϕ for the immersion $i_\phi \circ j_\phi$, and write $j_{\phi,!*}$ for the functors $i_{\phi,*} j_{\phi,!*}$ appearing in the decomposition of the previous Corollary.

3.3. Compatibility with realizations and conservativity. . In this paragraph, we will fix a generic point $\mathrm{Spec}(k) \rightarrow S$ of our base S , and we will make use of the two realization functors with target the categories $D_c^b(S_{k,\acute{e}t}, \mathbb{Q}_\ell)$ and $D_c^b(S_k^{\mathrm{an}}, \mathbb{Q})$, obtained by composition with base change through $\mathrm{Spec}(k) \rightarrow S$ from the functors ρ_ℓ, ρ_B introduced in the “Notations and conventions” section. Whenever we employ one of these two functors, we will implicitly assume that the hypotheses on S and ℓ are satisfied. These functors will still be denoted by the same symbols.

Let us denote any of the two families of categories $D_c^b(S_{k,\acute{e}t}, \mathbb{Q}_\ell)$ and $D_c^b(S_k^{\mathrm{an}}, \mathbb{Q})$ by the same symbol $D_c^b(S_k)$. Both families of categories are equipped with a perverse t -structure, whose heart (the corresponding category of *perverse sheaves*) will be denoted $\mathrm{Perv}_c(S_k)$ in both cases. We will then denote by

$$H^m : D_c^b(S_k) \rightarrow \mathrm{Perv}_c(S_k)$$

the perverse cohomology functors, and if $j : U \hookrightarrow S$ is an open immersion, by

$$j_{!*} : \mathrm{Perv}_c(U_k) \rightarrow \mathrm{Perv}_c(S_k)$$

the *intermediate extension* of perverse sheaves ([BBD82a, Déf. 1.4.22]). The composition of the collection of the perverse cohomology functors with one of the realization functors will be called the corresponding *perverse cohomological realization* functor.

The following result gives the compatibility of the functor of Def. 3.2.12 (when available) with the realization functors:

Theorem 3.3.1. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.5. Denote by ρ any of the two realization functors ρ_ℓ or ρ_B . For each $\phi \in \Phi$, denote by $j_\phi : S_\phi \hookrightarrow \overline{S_\phi}$ the open immersion of a stratum in its closure. Then:*

(1) *for any $\phi \in \Phi$, for any integer m , the restriction of the composition*

$$H^m \circ \rho : \mathrm{DM}_c(\overline{S_\phi}, \mathbb{Q}) \rightarrow \mathrm{Perv}_c(\overline{S_{\phi_k}})$$

to $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{\overline{S_\phi}}/\overline{S_\phi})$ factors over $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{\overline{S_\phi}}/\overline{S_\phi})^u$;

(2) *for any $\phi \in \Phi$, for any integer m , the diagram*

$$\begin{array}{ccc} \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi) & \xrightarrow{j_{\phi,*}} & \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{\overline{S_\phi}}/\overline{S_\phi})^u \\ H^m \circ \rho \downarrow & & \downarrow H^m \circ \rho \\ \mathrm{Perv}_c(S_{\phi,k}) & \xrightarrow{j_{!*}} & \mathrm{Perv}_c(\overline{S_{\phi_k}}) \end{array}$$

commutes.

Proof. The same proof of [Wil17, Thm. 7.2] applies. Indeed, the only ingredient occurring in that proof, which has to be generalized, is *op. cit.*, Cor. 7.13: we need to obtain an analogous statement in our situation, i.e. we need to show that the radical (cfr. Def. 3.1.1) of our categories $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{\overline{S_\phi}}/\overline{S_\phi})$ is mapped to zero under the perverse cohomological realization. The proof of Thm. 3.2.10 tells us that over each stratum S_ϕ of S , the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ is actually contained in the category $\mathrm{AT}(S_\phi)$ of weight zero, smooth Artin-Tate motives over S_ϕ . Then, the same strategy of proof of *op. cit.*, Cor. 7.13, based on *op. cit.*, Thm. 7.12,

shows that we can obtain the desired statement as soon as we prove the following claim: for any couple of finite étale morphisms

$$q_1 : D_1 \rightarrow X, \quad q_2 : D_2 \rightarrow X$$

the ideal

$$\text{rad}_{\text{CHM}(X)}(h_X(D_1), h_X(D_2))$$

consists of morphisms which are mapped to zero under the perverse cohomological realization. Now, by the same reasoning as in the proof of *loc. cit.*, Thm. 7.12, we may assume that X is the spectrum of a field k . But then, $h_X(D_1)$ and $h_X(D_2)$ are Artin motives over k , i.e. objects of a full *semisimple* subcategory of $\text{CHM}(X) = \text{CHM}(k)$. Hence, the radical in question is zero. \square

We end this section by discussing the conservativity of the restriction of the realization functors to the categories $\text{DM}_{\text{tr}}^{\text{AT}}(\mathfrak{S}/\Phi)$.

Theorem 3.3.2. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.5. Denote by ρ any of the two realization functors ρ_ℓ or ρ_B . Then, the restriction of ρ to $\text{DM}_{\text{tr}}^{\text{AT}}(\mathfrak{S}/\Phi)$,*

$$\rho : \text{DM}_{\text{tr}}^{\text{AT}}(\mathfrak{S}/\Phi) \rightarrow \text{D}_c^b(S_k)$$

is conservative.

Proof. We will adopt the strategy of [Wil18, Thm. 4.3], which adapts to our setting and shows that conservativity of both realizations (ℓ -adic and Betti) can be deduced if the following properties hold²¹: (1) the weight structure on $\text{DM}_{\text{tr}}^{\text{AT}}(\mathfrak{S}/\Phi)$ is bounded, (2) its heart $\text{AT}_{\text{tr}}(\mathfrak{S}/\Phi)$ is semi-primary and pseudo-Abelian, (3) the restriction of ρ to $\text{AT}_{\text{tr}}(\mathfrak{S}/\Phi)$ maps the radical to zero, and (4) zero is the only object of $\text{AT}_{\text{tr}}(\mathfrak{S}/\Phi)$ mapped to zero by ρ . The first two properties have been verified before (Lemma 3.2.8 and Thm. 3.2.10), and property (3) has been seen to hold in the course of the proof of Thm. 3.3.1. With this in our hands, we can imitate step by step the proof of *op. cit.*, Thm. 4.2 to show that property (4) is also verified and to conclude. In fact, to argue as in *loc. cit.* we only need the existence of the intermediate extension functor defined in Cor. 3.2.12, the fact that objects in $\text{AT}_{\text{tr}}(\mathfrak{S}/\Phi)$ decompose as direct sums of intermediate extensions (Cor. 3.2.13), and the finite dimensional nature of the objects of the categories $\text{AT}_{\text{tr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ (proof of Thm. 3.2.10). \square

4. MOTIVIC DECOMPOSITIONS OF PROJECTIVE QUADRIC BUNDLES

4.1. Corti-Hanamura decomposition of general quadrics.

4.1.1. In this section we work over a field k of characteristic zero. Let V be a k -vector space of dimension n . A quadratic form $Q \in S^2V^\vee$ can be viewed as a symmetric map $Q : V \rightarrow V^\vee$. Its radical is the subspace $\text{Rad}(Q) = \ker(Q)$. It is known from the theory of quadratic forms that Q descends to a symmetric bilinear form \overline{Q} on the quotient space $\overline{V} = V/\text{Rad}(Q)$ (cf. [Cas78, 2.6]). We say that Q has *corank* r if $\dim(\text{Rad}(Q)) = r$. The zero locus of Q defines a quadric $X \subset \mathbb{P}(V)$

²¹These are precisely all of the assumptions which are shown to be sufficient for conservativity in *op. cit.*, Thm. 2.10, except for one: *strictness* of any morphism in the image of the perverse cohomological realization, with respect to the *weight filtration* of the latter functor. Only the ℓ -adic realization is known to satisfy this last assumption. Building on this, Thm. 4.3 of *loc. cit.* then shows how to obtain conservativity for the Betti realization, too.

which is smooth if and only if $r = 0$. If $r > 0$ its singular locus is the linear subspace $\Lambda = \mathbb{P}(\text{Rad}(Q))$, and X is a cone with vertex Λ over the smooth quadric $\overline{X} = V(\overline{Q}) \subset \mathbb{P}(\overline{V}) \cong \mathbb{P}_k^{n-r-1}$.

Let

$$\Delta_k = \{Q \in S^2V^\vee \mid \dim(\ker Q) \geq k\}$$

be the space of quadratic forms of corank $\geq k$. The standard desingularization of Δ_k is

$$\widehat{\Delta}_k = \{(Q, F) \in S^2V^\vee \times G(k, V) \mid F \subset \ker Q\}.$$

For our purposes it is convenient to use a slightly different construction. We denote by $\text{Fl}(V) = \text{Fl}(1, \dots, n-1, V)$ the variety of complete flags

$$F_1 \subset \dots \subset F_{n-1} \subset F_n = V$$

where $\dim(F_i) = i$. Define

$$\widetilde{\Delta}_k = \{(Q, F_\bullet) \in S^2V^\vee \times \text{Fl}(V) \mid F_k \subset \ker Q\}.$$

The fiber of the projection map $p_2 : \widetilde{\Delta}_k \rightarrow \text{Fl}(V)$ over F_\bullet is the vector space $H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(F_k)}(2))$. Hence $\widetilde{\Delta}_k$ is smooth. The advantage over the previous construction is that we have inclusions

$$\widetilde{\Delta}_{k+1} \subset \widetilde{\Delta}_k \subset \text{Fl}(V)$$

for all k .

4.1.2. We now discuss the analogue of these constructions in the relative case. Let S be a quasi-projective scheme over k . Let E be a rank n vector bundle over S , and let L be a line bundle over S . A quadratic form on E with values in L is a global section $q \in H^0(S, S^2E^\vee \otimes L)$, or equivalently a symmetric homomorphism

$$q : E \rightarrow E^\vee \otimes L.$$

Let $\rho : \mathbb{P}(E) \rightarrow S$ be the associated projective bundle with tautological line bundle $\xi_E = \mathcal{O}_{\mathbb{P}(E)}(1)$. Using the isomorphism

$$H^0(S, S^2E^\vee \otimes L) \cong H^0(\mathbb{P}(E), \xi_E^2 \otimes \rho^*L)$$

we can identify q with a global section (still denoted by q) of $\xi_E^2 \otimes \rho^*L$. The associated quadric bundle is $X = V(q) \subset \mathbb{P}(E)$. The fiber of $f : X \rightarrow S$ over $s \in S$ is the zero locus of $q(s) \in S^2E_s^\vee \otimes L_s$.

We write

$$\Delta_i(q) = \{s \in S \mid \text{corank } q(s) \geq i\}, \quad U_i = \Delta_i(q) \setminus \Delta_{i+1}(q).$$

The restriction of q to U_i defines a homomorphism of vector bundles

$$q_i : E_i \rightarrow E_i^\vee \otimes L_i$$

whose kernel $F_i = \ker(q_i)$ is a subbundle of E_i of rank i . As before, the quadratic form q_i descends to a quadratic form \bar{q}_i on the quotient $\bar{E}_i = E_i/F_i$. Geometrically this means that the subscheme $X_i = f^{-1}(U_i)$ is a relative cone over $\bar{X}_i = V(\bar{q}_i) \subset \mathbb{P}(\bar{E}_i)$ with vertex $\mathbb{P}(F_i)$ i.e., for every $s \in U_i$ the quadric $X_s = f^{-1}(s)$ is a cone with vertex $\mathbb{P}(F_{i,s})$ over $\bar{X}_s \subset \mathbb{P}(\bar{E}_{i,s})$.

Let $\mathrm{Fl}(E)$ be the bundle of complete flags in E with projection map $\pi : \mathrm{Fl}(E) \rightarrow S$. The vector bundle π^*E has a flag of universal subbundles S_i of rank i , $i = 1, \dots, n-1$. The composition of

$$\pi^*q : \pi^*E \rightarrow \pi^*E^\vee \otimes \pi^*L$$

and the inclusion $\lambda_i : S_i \rightarrow \pi^*E$ defines

$$\tilde{q}_i = \pi^*(q) \circ \lambda_i : S_i \rightarrow \pi^*E^\vee \otimes \pi^*L.$$

Define

$$\tilde{\Delta}_i(q) = V(\tilde{q}_i) \subset \mathrm{Fl}(E).$$

Proposition 4.1.3. *If $E^\vee \otimes E^\vee \otimes L$ is generated by global sections and $q \in H^0(X, S^2E^\vee \otimes L)$ is general, then*

- (1) $\Delta_i(q)$ is empty or has the expected codimension $\binom{i+1}{2}$ and $\mathrm{Sing}(\Delta_i(q)) = \Delta_{i+1}(q)$.
- (2) $\tilde{\Delta}_i(q)$ is smooth.

Proof. Note that if $E^\vee \otimes E^\vee \otimes L$ is generated by global sections, then the bundles $S^2E^\vee \otimes L$, $\pi^*(E^\vee \otimes E^\vee \otimes L)$ and $S_i^\vee \otimes \pi^*E^\vee \otimes \pi^*L$ are globally generated. Part 1 is proved by adapting the argument of [B91, 4.1] to the symmetric case; cf. [Ott95, 2.17]. Part 2 follows from Bertini's theorem. \square

Definition 4.1.4. We say that $X \rightarrow S$ is a *regular quadric bundle* if it satisfies the conditions of the Proposition 4.1.3.

A quadric bundle $f : X \rightarrow S$ admits a natural stratification by corank. Write $U_i = \Delta_i(q) \setminus \Delta_{i+1}(q)$, $X_i = f^{-1}(U_i)$. As the stratification $\Phi = \{U_i\}_{i \in I}$ does not verify Assumption 1, we have to pass to a suitable base change and verify Assumption 2. Define $S' = \mathrm{Fl}(E)$ and consider the stratification \mathfrak{S} given by $U'_i = \tilde{\Delta}_i(q) \setminus \tilde{\Delta}_{i+1}(q)$. Write $X' = X \times_S S'$ and $X'_i = X_i \times_{U_i} U'_i$.

Lemma 4.1.5. *Let $X \rightarrow S$ be a regular quadric bundle. Then the stratification $\mathfrak{S} = \{U'_i\}_{i \in I}$ satisfies Assumption 3.2.5.*

Proof. Proposition 4.1.3 implies that the stratification \mathfrak{S} satisfies conditions (1) and (2) of Assumption 3.2.5. The definition of $\tilde{\Delta}_i(q)$ shows that the fiber of $\pi_i : \tilde{\Delta}_i(q) \rightarrow \Delta_i(q)$ over $s \in \Delta_i(q)$ is

$$\{(W_\bullet) \in \mathrm{Fl}(E_s) \mid W_i \subset \ker q(s)\}.$$

If $s \in U_i$ then $\ker(q(s))$ has dimension i , hence the fiber of the induced map $U'_i \rightarrow U_i$ over $s \in U_i$ is

$$\{(W_\bullet) \in \mathrm{Fl}(E_s) \mid W_i = \ker(q(s))\} \cong \mathrm{Fl}(\ker(q(s)) \times \mathrm{Fl}(i+1, \dots, n, E_s)).$$

Over U_i we have an injective homomorphism of flag bundles

$$\mathrm{Fl}(F_i) \times \mathrm{Fl}(i+1, \dots, n; E_i) \rightarrow \mathrm{Fl}(E_i)$$

whose image is U'_i . Hence $U'_i \rightarrow U_i$ is a relative homogeneous space and $h_{U_i}(U'_i)$ is a relative Tate motive. This implies that the stratification \mathfrak{S} also satisfies condition (3) of Assumption 3.2.5. \square

Theorem 4.1.6. *Let $X \rightarrow S$ be a regular quadric bundle. Then $h_S(X) \in \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$.*

Proof. We consider the stratifications Φ and \mathfrak{S} introduced above and write $X_i = f^{-1}(U_i)$, $X'_i = X_i \times_{U_i} U'_i$. As we have seen before, X_i is a relative cone over \overline{X}_i with vertex $\mathbb{P}(F_i)$. The complement $V_i = X_i \setminus \mathbb{P}(F_i)$ is a locally trivial fiber bundle over \overline{X}_i with affine fibers.²² A similar result holds after base change via $\pi_i : U'_i \rightarrow U_i$: X'_i contains the projective bundle $\mathbb{P}(F'_i)$, where $F'_i = \pi_i^*(F_i)$, and the projection of the complement $V'_i = X'_i \setminus \mathbb{P}(F'_i)$ to U'_i factors as $V'_i \xrightarrow{\rho_i} \overline{X}'_i \xrightarrow{\sigma_i} U'_i$ where ρ_i is a locally trivial fiber bundle with affine fibers and σ_i is a smooth quadric bundle. Hence $h_{U'_i}^c(V'_i) \cong h_{U'_i}(\overline{X}'_i)(-i)[-2i]$. As we have seen in Corollary 2.5.13, the motive $h_{U'_i}(\overline{X}'_i)$ is either a relative Tate motive or a relative Artin-Tate motive associated to a double étale covering $Z(\overline{X}'_i) \rightarrow U'_i$. The latter case arises if $n - i$ is even, say $n - i = 2m$. In this case the double covering $Z(\overline{X}'_i) \rightarrow U'_i$ comes from the Stein factorisation of the relative Fano scheme of m -planes $F_m(\overline{X}'_i/U'_i) \rightarrow U'_i$. Since we work in characteristic zero, the map $Z(\overline{X}'_i) \rightarrow U'_i$ is tamely ramified by Remark 3.2.3. As $h_{U'_i}(\mathbb{P}(F'_i))$ is a relative Tate motive, the localisation triangle

$$h_{U'_i}^c(V'_i) \rightarrow h_{U'_i}(X'_i) \rightarrow h_{U'_i}(\mathbb{P}(F'_i)) \rightarrow h_{U'_i}^c(V'_i)[1]$$

then shows that $h_{U'_i}(X'_i) \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{ArtT}}(U'_i, \mathbb{Q})$ for all i . As the map $h_{U'_i}(\mathbb{P}(F'_i)) \rightarrow h_{U'_i}^c(V'_i)[1]$ is zero for weight reasons, it follows that $h_{U'_i}(X'_i)$ has weight zero. Hence $h_{S'}(X') \in \mathrm{DM}_{\mathrm{tr}}^{\mathrm{ArtT}}(\mathfrak{S})$ thanks to part (2) of Thm. 3.2.4 and has weight zero by the gluing property of motivic weights.

As $S' = \mathrm{Fl}(E) \xrightarrow{\pi} S$ is a relative homogeneous space we have

$$\pi_* \mathbb{1}_{S'} \cong \mathbb{1}_S \oplus \left(\bigoplus_{i:n_i > 0} \mathbb{1}_S(-n_i)[-2n_i] \right).$$

By the projection formula we obtain

$$\pi_* h_{S'}(X') = \pi_* \pi^* h_S(X) \cong h_S(X) \oplus \left(\bigoplus_{i:n_i > 0} h_S(X)(-n_i)[-2n_i] \right).$$

Hence $h_S(X) \in \mathrm{DM}_{\mathrm{tr}}^{\mathrm{ArtT}}(\mathfrak{S}/\Phi)$. Since proper morphisms respect weights and $h_{S'}(X')$ is of weight zero, $\pi_* h_{S'}(X')$ is of weight zero, as well as any of its direct factors. So $h_S(X) \in \mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$. \square

Corollary 4.1.7. *A regular quadric bundle $X \rightarrow S$ admits a CH-decomposition (see Def. 2.2.1).*

Proof. By Corollary 3.2.13 every object M of the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}/\Phi)$ admits a decomposition

$$M \simeq \bigoplus_{\phi \in \Phi'} i_{\phi,*} j_{\phi,!} N_{\phi}$$

with N_{ϕ} in $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S}_{S_{\phi}}/S_{\phi})$. The proof of Theorem 3.2.10 shows that $N_{\phi} \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{ArtT}}(S_{\phi}, \mathbb{Q})$. Using Theorem 2.4.3 we obtain a decomposition

$$N_{\phi} \simeq \bigoplus_{i, \phi \in I_{\phi}} \rho_i(V_{i,\phi})(n_{i,\phi})[2n_{i,\phi}]$$

²²In fact, if $\pi_i : \mathbb{P}(E_i) \rightarrow U_i$ is the projection map and if one chooses local trivialisations $F_i|_U \cong W \otimes \mathcal{O}_U$, $E_i|_U \cong V \otimes \mathcal{O}_U$ over an open subset $U \subset U_i$ then $V_i \cap \pi_i^{-1}(U)$ is the total space of the vector bundle $\mathcal{O}_U(-1) \oplus W \otimes \mathcal{O}_U$; cf. [EH16, 9.3.2]

with $V_{i,\phi}$ a simple Artin representation of $\pi_1(S_\phi)$. Taking $M = h_S(X)$ we get

$$h_S(X) \simeq \bigoplus_{i,\phi} i_{\phi,*} j_{\phi,!} (\rho!(V_{i,\phi})(n_{i,\phi}))[2n_{i,\phi}]$$

which is a decomposition of $h_S(X)$ into simple objects of weight zero. Applying the realisation functor ρ_B and using Thm. 3.3.1, we obtain a decomposition of $Rf_*\mathbb{Q}_X$ into a sum of simple objects in $D_c^b(S, \mathbb{Q})$. The decomposition theorem of Beilinson-Bernstein-Deligne gives isomorphisms

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_k {}^p R^k f_*\mathbb{Q}_X[-k] \simeq \bigoplus_{k,\lambda} i_{\lambda,*} j_{\lambda,!} (L_{k,\lambda})[-k]$$

where $L_{k,\lambda}$ is a local system on S_λ . Since the simple objects appearing in this decomposition are unique, we conclude that they are realisations of Chow motives. Hence $X \rightarrow S$ admits a Corti-Hanamura decomposition. \square

Remark 4.1.8. Our proof gives a more precise statement: $h_S(X)$ is a direct sum of motivic intermediate extensions of Artin-Tate motives of degree at most 2, hence $Rf_*\mathbb{Q}_X$ decomposes as a sum of intersection complexes of local systems whose monodromy is either trivial or $\mathbb{Z}/2\mathbb{Z}$. This has been observed in the example of regular conic bundles over a surface [NS09]. In this case

$$Rf_*\mathbb{Q}_X \simeq \mathbb{Q}_S \oplus \mathbb{Q}_S[-2] \oplus i_* L[-1]$$

where L is a local system on the smooth discriminant curve $\Delta \subset S$. The underlying motivic decomposition is

$$h_S(X) \simeq \mathbb{1}_S \oplus \mathbb{1}_S(-1)[-2] \oplus \mathrm{Prym}(\tilde{\Delta}/\Delta)(-1)[-2]$$

where $\mathrm{Prym}(\tilde{\Delta}/\Delta)$ is the Prym motive, an Artin-Tate motive of degree 2.

4.2. Corti-Hanamura decomposition of quadric bundles with normal crossing discriminant.

4.2.1. The aim of this final section is to show that given the previous results, it is also quite immediate to get a Corti-Hanamura decomposition for a class of quadric bundles $f : X \rightarrow S$ over a field k of characteristic zero, which don't satisfy anymore the regularity assumption. We will instead suppose that the discriminant locus $i : D \hookrightarrow S$ of f is a *normal crossing divisor*. To fix our conventions, this means that D is a finite union of irreducible components

$$D = \bigcup_{i \in I} D_i$$

such that for each finite subset $J \subseteq I$, the underlying reduced scheme of the scheme $D_J := \bigcap_{j \in J} D_j$ is regular.

Fix now an ordering D_{J_1}, \dots, D_{J_r} of the set $\{D_J\}_{J \subseteq I}$, in such a way that the function $\dim D_{J_i}$ is (non-strictly) increasing with respect to i , and pose the following:

Definition 4.2.2. The stratification \mathfrak{S} of S is the one obtained by choosing, for each $\sigma \in \mathfrak{S} := \{1, \dots, r\}$,

$$S_\sigma := D_{J_\sigma} \setminus \left(\bigcup_{k=1, \dots, \sigma-1} D_{J_k} \right)$$

Note that the above defined \mathfrak{S} is a well-defined stratification (i.e., the closure of each stratum is a union of strata). Then, it is clear from the definitions that we have:

Lemma 4.2.3. *The stratification \mathfrak{S} satisfies Assumption 3.2.1.*

Hence, we can define the full, triangulated subcategory $\mathrm{DM}_{\mathrm{tr}}^{\mathrm{AT}}(\mathfrak{S})$ of $\mathrm{DM}_c(S)$ as in Thm. 3.2.4.

Theorem 4.2.4. *Let $X \rightarrow S$ be a quadric bundle whose discriminant is a divisor with normal crossings. Then, the motive $h_S(X)$ belongs to the category $\mathrm{AT}_{\mathrm{tr}}(\mathfrak{S})$.*

Proof. For each $\sigma \in \mathfrak{S}$, the fibers of X_{S_σ} over S_σ have constant corank. Hence, the same computation as in the proof of Thm. 4.1.6 shows that $h_{S_\sigma}(X_{S_\sigma})$ belongs to $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S_\sigma, \mathbb{Q})$ for each $\sigma \in \mathfrak{S}$, and gives the conclusion. \square

Finally, the proof of Corollary 4.1.7 can be repeated in order to show:

Corollary 4.2.5. *A quadric bundle $X \rightarrow S$ whose discriminant is a divisor with normal crossings admits a CH-decomposition.*

We present an example where several nonconstant intersection motives appear in the decomposition. First we need a Lemma.

Lemma 4.2.6. *Let $\pi : S' \rightarrow S$ be a finite morphism. Write $U' = \pi^{-1}(U)$, $\pi' = \pi|_{U'}$ and let $j : U \rightarrow S$, $j' : U' \rightarrow S$ be the inclusions. If F is a perverse sheaf on U' then*

$$j'_*(\pi'_*F) \simeq \pi_*(j'_*(F)).$$

Proof. As π and π' are finite morphisms, the functors π_* and π'_* are t-exact for the perverse t-structure [BBD82b, Cor. 2.2.6 (i)]. The result then follows from the isomorphisms

$$\pi_* \circ j'_* \simeq j_* \circ \pi'_*, \quad \pi_* \circ j'_! \simeq j_! \circ \pi'_*$$

using [loc.cit., 1.3.17 (iv) and 1.4.16]. \square

Example 4.2.7. Let r be a nonnegative integer. Given a $(r+1) \times (2n+3)$ matrix $A = (a_{ij})$ we define quadrics

$$Q_i = \{x \in \mathbb{P}^{2n+2} \mid \sum_j a_{ij}x_j^2 = 0\} \subset \mathbb{P}^{2n+2}, \quad i = 0, \dots, r.$$

The quadric bundle associated to the linear system of quadrics spanned by Q_0, \dots, Q_r is $f : X \rightarrow \mathbb{P}^r$ with

$$X = \{(x, \lambda) \in \mathbb{P}^{2n+2} \times \mathbb{P}^r \mid \sum_{i,j} \lambda_i a_{ij} x_j^2 = 0\}.$$

The fiber Q_λ of f over $\lambda = (\lambda_0 : \dots : \lambda_r) \in \mathbb{P}^r$ is the zero locus of

$$q_\lambda = \sum_j \left(\sum_i a_{ij} \lambda_i \right) x_j^2.$$

If A is a generic matrix, the discriminant locus of f is a simple normal crossing divisor which is the union of $2n+3$ hyperplanes: $\Delta = D_0 \cup \dots \cup D_{2n+2}$ where

$$D_j = \{\lambda \in \mathbb{P}^r \mid \sum_i a_{ij} \lambda_i = 0\}, \quad j = 0, \dots, 2n+2.$$

By the previous Corollary $f : X \rightarrow S$ admits a CH-decomposition. We shall show that it contains several nonconstant intersection motives by looking at the topological realisation. Write $d_X = \dim X = 2n + r + 1$. By the decomposition theorem²³

$$Rf_*(\mathbb{Q}_X[d_X]) \simeq \bigoplus_{i=-2n-1}^{2n+1} {}^p R^i f_*(\mathbb{Q}_X[d_X])[-i]$$

and the perverse direct images ${}^p R^i f_*(\mathbb{Q}_X[d_X])$ are direct sums of intermediate extensions. By the perverse versions of the Lefschetz hyperplane theorem and the hard Lefschetz theorem we have

$${}^p R^i f_*(\mathbb{Q}_X[d_X]) = \begin{cases} \mathbb{Q}[r] & i \neq 0, i \text{ even} \\ 0 & i \neq 0, i \text{ odd} . \end{cases}$$

hence

$$Rf_*\mathbb{Q}_X[d_X] \simeq \bigoplus_{k=-n}^n \mathbb{Q}[r - 2k - 1] \oplus {}^p R^0 f_*(\mathbb{Q}_X[d_X]).$$

Write

$$\Delta_i = \cup_{j \neq i} D_i \cap D_j, \quad S_i = D_i \setminus \Delta_i.$$

If $\lambda \in S_i$ then Q_λ has an ordinary double point p_λ and Q_λ is a cone with vertex p_λ over a smooth quadric $\overline{Q_\lambda} \subset \mathbb{P}^{2n+1}$. Hence

$$R^{2n+2} f_* \mathbb{Q}|_{S_i} = \mathbb{Q} \oplus L_i$$

where L_i is a nontrivial rank one local system on S_i with monodromy $\mathbb{Z}/2\mathbb{Z}$. The decomposition theorem then shows that

$$R^{2n+2} f_* \mathbb{Q}|_{S_i} = \mathcal{H}^{-r+1}(Rf_* \mathbb{Q}_X[d_X]|_{S_i}) = \mathbb{Q}[r-1] \oplus \mathcal{H}^{-r+1}({}^p R^0 f_*(\mathbb{Q}_X[d_X]))$$

for all i . Thus, writing $j_\alpha : S_\alpha \rightarrow \mathbb{P}^r$ for the inclusion maps,

$${}^p R^0 f_*(\mathbb{Q}_X[d_X]) \simeq \bigoplus_{\alpha=0}^r (j_\alpha)_! (L_\alpha) \oplus R$$

where R is a perverse sheaf supported in codimension ≥ 2 . Hence the CH-decomposition of $X \rightarrow S$ is of the form

$$h_S(X) \simeq \bigoplus_{k=0}^n \mathbb{1}(-k)[-2k] \oplus \bigoplus_{i=0}^r M_i \oplus M$$

where $\rho(M) = R$ and $\rho(M_i) = j_{i*}(L_i)$ for all i .

Remark 4.2.8. In the case $r = 2$ there is a direct construction of the motive M_i underlying the intermediate extension of L_i . In this case $D_i \cong \mathbb{P}^1$ and $\Delta_i \subset D_i$ is a set of $2n + 2$ points. Let $D'_i \rightarrow D_i$ be the double covering ramified along Δ_i . Then D'_i is a smooth hyperelliptic curve of genus n . Applying the previous lemma to the diagram

$$\begin{array}{ccc} S'_i & \xrightarrow{j'} & D'_i \\ \downarrow \pi' & & \downarrow \pi \\ S_i & \xrightarrow{j} & D_i \end{array}$$

²³For ease of comparison with the existing literature we have adopted the usual convention of shifting by d_X .

we obtain $\pi_*(j'_{!*}\mathbb{Q}) \simeq \mathbb{Q} \oplus j_{!*}(L_i)$ since $\pi'_*\mathbb{Q} \simeq \mathbb{Q} \oplus L_i$.

If we denote τ the hyperelliptic involution on D'_i , the intermediate extension $j_{!*}(L_i)$ is the realisation of the Prym motive $M_i = (D'_i, \frac{1}{2}(\tau^* + \text{id}))$.

It is not clear whether there exists a similar description of the motives M_i if $r > 2$. In this case the variety D'_i is singular, hence M_i should be constructed as a suitable direct factor of the "intersection motive" of D'_i .

Remark 4.2.9. If we take $n = 0$ and $r = 2$ in the example above (conic bundles over \mathbb{P}^2) the result follows from [NS09] (The result in [loc.cit.] is more general and applies to conic bundle over a surface with a reducible discriminant locus that is not necessarily a simple normal crossing divisor.)

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