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MOTIVIC DECOMPOSITIONS OF FAMILIES WITH TATE FIBERS: SMOOTH AND SINGULAR CASES

MATTIA CAVICCHI, FRÉDÉRIC DÉGLISE, JAN NAGEL

ABSTRACT. Motivated by the work of Corti-Hanamura and Bondarko, we introduce, for a proper morphism of schemes $f : X \rightarrow S$ with regular source, the notion of *BCH-decomposition* of the relative Chow motive $h_S(X)$ in the category $DM_c(S, \mathbb{Q})$ of constructible motives over S . When S is quasi-projective over an algebraically closed field of characteristic zero or over the separable closure of a finite field, this gives a motivic lift of the decomposition provided by the theorem of Beilinson-Bernstein-Deligne-Gabber. Using Bondarko’s theory of weight structures and Wildeshaus’ theory of motivic intermediate extensions, we show that BCH decompositions exist for any S when f is smooth with geometric fibers whose motive is Tate, and for S quasi-projective over an algebraically closed field of characteristic zero, when $f : X \rightarrow S$ is a *regular* (i.e. “sufficiently generic”) quadric bundle, with possibly singular fibers.

CONTENTS

1. Introduction	2
Notations and conventions	9
2. Smooth Artin-Tate motives	10
2.1. Gluing and weights on Beilinson motives	10
2.2. Smooth Artin motives	11
2.3. Weights on smooth Artin-Tate motives	13
2.4. First decomposition theorem	15
3. Recall on Wildeshaus’ motivic intermediate extension	18
3.1. Semi-primary categories and motivic intermediate extensions.	18
3.2. Semi-primary categories of Chow motives.	19
3.3. Compatibility with realizations and conservativity	24
4. Corti-Hanamura decomposition of general quadrics	26
4.1. Corti-Hanamura decomposition of general quadrics	26
References	29

1. INTRODUCTION

An important conjecture in the theory of motives is the *nilpotency conjecture*. It is related to the functor

$$T : \text{CHM}(k) \rightarrow \text{HM}(k)$$

from the category of Chow motives to the category of homological motives (motives modulo homological equivalence, for a choice of Weil cohomology theory e.g. Betti cohomology if $k = \mathbb{C}$.) This functor is full but not faithful. The nilpotency conjecture states that if M is a Chow motive and M_{hom} the corresponding motive modulo homological equivalence (sometimes called the Grothendieck motive), the kernel of the ring homomorphism

$$\text{End}(M) \rightarrow \text{End}(M_{\text{hom}})$$

is a nilpotent ideal. If we take $M = h(X)$, the Chow motive of a smooth projective variety X over k , the nilpotency conjecture $N(X)$ states that the kernel of the homomorphism

$$\text{Corr}^0(X, X) \rightarrow \text{End}(H^*(X, \mathbb{Q}))$$

from the ring of correspondences of degree zero to the endomorphism ring of $H^*(X)$ is a nilpotent ideal. (Note that if X has pure dimension d , the left hand side is the Chow group $\text{CH}^d(X \times X)_{\mathbb{Q}}$, the right hand side is isomorphic to $H^{2d}(X \times X, \mathbb{Q})$ by Poincaré duality and the above map is just the cycle class map.) The nilpotency conjecture has important consequences. First of all, it implies that the functor T is conservative (i.e., it detects isomorphisms), hence essentially injective (nonexistence of *phantom motives*). The nilpotency conjecture is also related to the standard conjecture $C(X)$, the Künneth conjecture, which states that the Künneth components $p_i \in H^*(X \times X, \mathbb{Q})$ of the diagonal are algebraic (i.e., they are in the image of the cycle class map). Jannsen proved that conjectures $C(X)$ and $N(X)$ imply the Chow–Künneth conjecture $CK(X)$, i.e., the Künneth projectors lift to a set of mutually orthogonal projectors $\pi_i \in \text{Corr}^0(X, X)$ ($i = 0, \dots, 2d$) such that

$$\Delta_X = \sum_{i=0}^{2d} \pi_i$$

in $\text{CH}_d(X \times X) \otimes \mathbb{Q}$ and such that π_i depends only on the motive modulo homological equivalence. In this case the motive of X admits a *Chow–Künneth decomposition*

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)$$

with $h^i(X) = (X, \pi_i)$.

The nilpotency conjecture is known to hold for finite-dimensional motives. This means in particular that isomorphisms between finite-dimensional Chow motives can be detected by passing to the corresponding homological motives. An application is the following result.

Theorem. (*) *Suppose that X is a smooth projective variety over k such that the Chow motive $h(X)$ is finite dimensional and such that $H^*(X)$ is algebraic (i.e., the rational cycle class maps $\text{CH}^i(X)_{\mathbb{Q}} \rightarrow H^{2i}(X, \mathbb{Q})$ are surjective for all i). Then*

$$h(X) = \bigoplus_i \mathbb{Q}(-i)^{b_{2i}}$$

is Tate (i.e., it is a direct sum of Lefschetz motives).

This result generalises a theorem of Jannsen and was observed by Vial and Kimura (in the case where one replaces $h(X)$ by an arbitrary Chow motive M). It is a direct consequence of the conservativity of the homological realisation on finite-dimensional motives. ¹

In this paper we consider possible generalisations of these results to the case of relative motives. In their fundamental work [CH00], Corti and Hanamura defined a category $\text{CHM}(S)$ of relative Chow motives over a quasi-projective base scheme S defined over a field k , together with a realisation functor

$$\rho : \text{CHM}(S) \rightarrow D_c^b(S, Q)$$

where:

- $D_c^b(S, Q)$ is the derived category of constructible complexes of \mathbb{Q} -sheaves on the analytical space $S(\mathbb{C})$, with constructible cohomology, if $k = \mathbb{C}$;
- $D_c^b(S, Q)$ is the derived category of constructible complexes of ℓ -adic étale sheaves on S in general, after the choice of a prime ℓ invertible on k .

In this setup, one works with proper morphisms $f : X \rightarrow S$ with X smooth and quasi-projective over k . Relative correspondences are elements of $\text{CH}_*(X \times_S X)$ and composition of correspondences is defined using refined Gysin homomorphisms. The relative analogue of the nilpotency conjecture states that the kernel of the map

$$\text{Corr}_S^0(X, X) \rightarrow \text{End}(Rf_*(Q_X))$$

is a nilpotent ideal. (If X has dimension d the above map can be identified with the cycle class map $\text{CH}_d(X \times_S X)_{\mathbb{Q}} \rightarrow H_{2d}^{\text{BM}}(X \times_S X, \mathbb{Q})$ to Borel–Moore homology.) Corti and Hanamura have considered a relative analogue of the notion of Chow–Künneth decomposition. Let us first consider the case where $f : X \rightarrow S$ is in addition smooth. According to Deligne’s theorem, one gets a decomposition in $D_c^b(S, Q)$:

$$(D) \quad Rf_*(Q_X) \cong \bigoplus_i R^i f_*(Q_X)[-i].$$

One usually says that $h_S(X)$ admits a *relative Chow–Künneth decomposition* (in short CK-decomposition) if there exist relative motives $M_i \in \text{CHM}(S)$ such that $h_S(X) = \bigoplus_i M_i$ and $\rho(M_i) = R^i f_*(Q_X)[-i]$ for all i .

The general case is more complicated and involves the theory of perverse sheaves. When k is an algebraically closed field of characteristic zero or the separable closure of a finite field, the decomposition theorem of Beilinson–Bernstein–Deligne says that a decomposition of the form (D) still exists provided one replaces the canonical t-structure by the perverse one. Moreover, each perverse cohomology sheaf ${}^p Rf_*(Q_X)$ is semi-simple in the perverse heart of $D_c^b(S)$. These two statements can be expressed by the existence of a decomposition in $D_c^b(S, Q)$

$$(BBD) \quad Rf_*(Q_X) = \bigoplus_{\lambda} j_{\lambda!} (L_{\lambda})$$

¹If $H^*(X)$ is algebraic there exists a finite set of algebraic cycles $\{Z_i\}_{i \in I}$, $Z_i \in \text{CH}^{p_i}(X)$, whose cycles classes generate the cohomology. These cycles define a morphism of Chow motives $\alpha : \bigoplus_i \mathbb{Z}(-p_i) \rightarrow h(X)$ such that $T(\alpha)$ is an isomorphism, hence α is an isomorphism by conservativity.

where there exists an integer n_λ such that $L_\lambda[n_\lambda]$ is a simple local system on a smooth locally closed subscheme $j_\lambda : U_\lambda \subset X$ and $j_{\lambda!}(L_\lambda)$ denotes the *intermediate extension* (or intersection complex) defined in [BBD82].² We believe the following definition is useful to express the ideas of [CH00].

Definition 1. *Let $f : X \rightarrow S$ be a projective morphism of quasi-projective k -schemes such that X is smooth over k . Consider the above notations (and in particular choose a prime ℓ invertible in k in the ℓ -adic case).*

A Corti-Hanamura decomposition (in short: CH-decomposition) of the Chow motive $h_S(X)$ is a finite decomposition

$$h_S(X) \simeq \bigoplus_{\lambda} M_{\lambda}$$

in the pseudo-abelian category $\text{CHM}(S)$ such that $\rho(M_{\lambda}) = j_{\lambda!}(L_{\lambda})$.

Such decompositions have been constructed in several cases, particularly in complex algebraic geometry [MSS12, dCM10, GHM03]. The results of the present work are precisely in line with these works: our original goal was to obtain a Corti-Hanamura decomposition in the case of singular quadric fibrations (Theorem B below).

Nevertheless, our proofs have led us to a generalisation of Corti-Hanamura's considerations that is interesting on its own, based on Beilinson's conjectures on motivic complexes. Nowadays, following the seminal work of Voevodsky, finalized by Ayoub and Cisinski-Dégliise, we have at our disposal the (triangulated) category of rational mixed motives $\text{DM}(S)$, equipped with Grothendieck's six functors formalism over any noetherian quasi-excellent base scheme S . Moreover, for any prime ℓ invertible on S , there is the ℓ -adic realisation functor

$$\rho_{\ell} : \text{DM}_c(S) \rightarrow D_c^b(S, \mathbb{Q}_{\ell}),$$

which is compatible with the six operations (see [CD19], [CD16]). While we do not have the motivic t -structure on the left hand-side, which would mirror the perverse t -structure on the right hand-side, a key structure of mixed motives is the existence of the so-called *Chow weight structure* constructed by Bondarko.³ One defines the (pseudo-abelian monoidal) category $\text{CHM}(S)$ of Chow motives over S as the weight 0 part of $\text{DM}(S)$.⁴ Bondarko's speculative work [Bon15] leads us to the following extension of Definition 1.

Definition 2. *Let $f : X \rightarrow S$ be a proper morphism such that X is regular. A BCH-decomposition⁵ of the Chow motive $h_S(X)$ is a finite decomposition*

$$h_S(X) \simeq \bigoplus_{\lambda} j_{\lambda!}(M_{\lambda})$$

²Note in particular that one deduces isomorphisms ${}^p Rf_*(Q_X)[-n] \simeq \bigoplus_{\lambda | n_{\lambda} = n} j_{\lambda!}(L_{\lambda})$ and: (D') ${}^p Rf_*(Q_X) \simeq \bigoplus_n {}^p R^n f_*(Q_X)[-n]$.

³The existence of this weight structure was in fact predicted by Beilinson's conjectures (see [Bon10]). However, one does not expect that an analogous structure exists in general on $D_c^b(S, \mathbb{Q}_{\ell})$.

⁴When S is quasi-projective over a perfect field, it is proved in [Jin16] that the latter category is equivalent to the category defined by Corti and Hanamura.

⁵extensively: Bondarko-Corti-Hanamura decomposition!

in $\mathrm{CHM}(S)$ where $j_\lambda : V_\lambda \rightarrow S$ is a locally closed immersion, V_λ is regular, M_λ is a weight 0 motive over V_λ whose ℓ -adic realisation is a local system up to a decalage, and $\rho_\ell j_{\lambda!}(M_\lambda) = j_{\lambda!}(\rho_\ell(M_\lambda))$.

The novelty of this definition is that it does not require a base field, and moreover, it makes sense without having to choose a prime invertible on S ; in particular over \mathbb{Z} . Note also that our formulation implies the existence of a decomposition of the form (D'), for all primes ℓ , and contains an "independence of ℓ " result: under the existence of a BCH-decomposition, the rank of ${}^p R^n f_*(\mathbb{Q}_\ell)$ will be independent of ℓ (see also Cor. 2.4.11).

We arrived at this definition by analysing the work of Bondarko [Bon15]. In particular, the existence of BCH-decompositions would be a consequence of the existence of the motivic t -structure with the assumption that it is *transversal* (see [Bon15, Def. 1.4.1]) to the Chow weight structure.⁶ More precisely, to get the BCH-decomposition under the previous assumption, one can apply *op. cit.*:

- Prop. 1.4.2, point (3) to the weight 0 motive $h_S(X)$ to get the analogue of the relative Chow Künneth decomposition: $h_S(X) \simeq \bigoplus_i H^i(h_S(X))[-i]$, H^i computed for the motivic t -structure.
- Prop. 4.2.3, to get the decomposition for each $H^i(h_S(X))$ given it is pure of weight i .
- Rem. 4.2.4, point 2 to get the statement about the ℓ -adic realisation of the BCH-decomposition.

Note moreover that Bondarko obtains in his Proposition 4.2.3 the uniqueness of the set of isomorphism classes of the factors of the BCH-decomposition. We have not formulated this uniqueness in the above definition. Indeed, to give a proper statement we either need the existence of the heart of the motivic t -structure, or conservativity of the ℓ -adic realisations.⁷ However we think that establishing BCH-decompositions might be easier than getting the motivic t -structure.

In fact, the main problem in getting BCH-decompositions without having the motivic t -structure at our disposal is to be able to construct the required *motivic intermediate extension functor* $j_{\lambda!}$. Fortunately, this problem has been extensively studied in a series of works by Wildeshaus: [Wil16], [Wil09], [Wil12], [Wil17]. His approach will be fundamental in the proof of our results, as we are going to explain.

There is at least one easy example where a BCH-decomposition exists: this is the case of relative lci cellular morphisms (see [ADN20, Def. 5.3.1, Cor. 5.3.7] for the more general case).⁸ Our first theorem gives many more examples (see also 2.4.10):

⁶Note also that Bondarko shows that this would be consequence of Beilinson's conjectures (see *op. cit.*, Prop. 4.1.1). He also gives an argument to get it as a consequence of the standard conjecture D together with Murre's conjectures. Note that to get the existence of BCH-decompositions in all cases, we need to apply these conjectures for all residue fields of S , and all ℓ -adic realisations! Of course, this kind of exercise in juggling between motivic conjectures has its limit; we only hope to convince the reader that our definition of BCH-decompositions is reasonable.

⁷Note however that in our Theorems A and B below, the involved motives belong to subcategories on which the realisations can be proven to be conservative (Thm. 3.3.1). So, in our cases we *do* get uniqueness of isomorphism classes in the BCH-decompositions.

⁸This is not surprising as one can interpret the BCH-decomposition as an attempt to find an algebraic analogue of the existence cellular decompositions of differential varieties.

Theorem A (see Theorem 2.4.8). *Let S be a regular connected scheme with étale fundamental group π . Let $f : X \rightarrow S$ be a smooth proper morphism whose geometric fibers are Tate. Then there exists an isomorphism of motives over S :*

$$h_S(X) \simeq \bigoplus_{i \in I} \rho_!(V_i)(-n_i)[-2n_i]$$

where I is a finite set, V_i is a simple Artin \mathbb{Q} -representation of π and n_i is a non-negative integer.

Moreover, this decomposition is a BCH-decomposition of the Chow motive $h_S(X)$.

In the above statement, the functor $\rho_!$ sends any étale sheaf over S to its image in the category $\mathrm{DM}_c(S)$. We show that when restricted to the bounded derived category of Artin π -representations, it induces an equivalence of categories with the full subcategory $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ of *smooth Artin-Tate motives* in $\mathrm{DM}_c(S)$ (Prop. 2.2.4), generalizing a result of Orgogozo when S is the spectrum of a field. Hence, in particular, Theorem A shows that the naive generalisation of Theorem (*) of the present introduction to the relative situation is false: if $f : X \rightarrow S$ is a smooth, projective morphism such that $h(X_s)$ is finite dimensional and such that $H^*(X_s)$ is algebraic for all $s \in S$, the relative motive $h_S(X)$ does not necessarily decompose as a direct sum of relative Lefschetz motives⁹, since it is an object of a more general kind - an object of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$. This can be seen already on the topological level. If $f : X \rightarrow S$ is a smooth, projective morphism whose fibers have algebraic cohomology then the direct image sheaves $R^i f_* \mathbb{Q}$ are local systems with finite monodromy, but they are not necessarily constant. A concrete example is given by smooth quadric fibrations of even relative dimension; see Corollary 2.4.13.

Let us outline the proof of Theorem A, since it uses some intermediate results which are interesting in their own. Namely, it rests on three crucial ingredients. First, the structural properties of the category $\mathrm{DM}_c(S)$ allow us to: (a) “spread out” the description of the motive of the fiber of f over a generic point of S (as a direct sum of Tate twists) to an analogous description of $h_V(X_V)$, where V is some étale cover of a dense open subscheme U of S ; (b) deduce that $h_U(X_U)$ is a smooth Artin-Tate motive over U . Second, we employ a generalization of the work of Wildeshaus (on Artin-Tate motives over a field) to the case of smooth Artin-Tate motives over a base. In fact, our Thm. 2.3.3 identifies the heart of the induced weight structure on $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ and shows that $h_U(X_U)$, as any weight zero smooth Artin-Tate motive, admits a decomposition analogous to the one in the right hand side of the isomorphism claimed by Theorem A. Third, we rely upon other ideas of Wildeshaus in order to show that there exists a *minimal* extension of $h_U(X_U)$ to a weight zero, smooth Artin-Tate motive h over the whole of S . We have baptized it a *fair* extension (Def. 2.4.3) and its minimality property consists in the fact that $h_S(X)$, being another such extension, decomposes as a direct sum of $h_U(X_U)$ and an object supported on the boundary $Z := S \setminus U$. The rest of the proof boils down to showing that this latter object is zero, so that the decomposition of h as a weight zero smooth Artin-Tate motive gives the desired decomposition of $h_S(X)$.

When S is a quasi-projective scheme over an algebraically closed field k of characteristic zero, we are able to treat cases where the proper morphism f , with Tate

⁹This statement is even false if one assumes that the fibers are homogeneous spaces as in [Iyer]

geometric fibers, is not anymore supposed to be smooth. In particular, we succeed in constructing the required motivic intermediate extension functors $j_{\lambda!*$, of which the fair extensions used in the proof of Theorem A represent a first occurrence. To perform our construction, we restrict to dealing with *regular* (i.e. “sufficiently generic”) *quadric bundles* over S (see Def. 4.1.2), thus generalizing the work of Nagel-Saito [NS09] on conic bundles over a surface (see also [Bou16]): in this case, the perverse sheaves appearing in the decomposition of $Rf_*\mathbb{Q}$ are either constant or of the form $i_*(L)$ with L a local system on a smooth closed subvariety $Z \hookrightarrow S$, so no intermediate extensions are needed. Our results provide, to our knowledge, the first new examples of motivic intermediate extensions in the non-smooth case, after the ones for Chow motives of *abelian type* defined by Wildeshaus and applied to the context of Shimura varieties by him and his students (see [Wil17] and the references cited therein; earlier instances of intermediate extensions in the latter setting were constructed by hand by Scholl [Sch90] and Gordon-Hanamura-Murre [GHM03]). Concretely, our second main theorem reads as follows:

Theorem B (see Cor. 4.1.5). *Let $f : X \rightarrow S$ be a regular quadric bundle. Let $\{S_\phi\}$ be the stratification of S given by corank (defined after Def. 4.1.2). Then $h_S(X)$ admits a Corti-Hanamura decomposition*

$$h_S(X) \simeq \bigoplus_{i,\phi} j_{\phi!}(\rho_l(V_{i,\phi})(-n_{i,\phi}))[-2n_{i,\phi}]$$

with $V_{i,\phi}$ a simple Artin representation of $\pi_1(S_\phi)$.

In order to achieve our goal, we once again follow the path laid out by Wildeshaus. Recall that over a base S with a smooth, dense open subscheme $j : U \hookrightarrow S$ and closed complement $i : Z \hookrightarrow S$, and considering subcategories $\mathcal{C}(S)$ of $\text{CHM}(S)$, obtained by *gluing* appropriate subcategories $\mathcal{C}(U)$ of $\text{CHM}(U)$ and $\mathcal{C}(Z)$ of $\text{CHM}(Z)$, Wildeshaus highlighted conditions allowing to define an intermediate extension $j_{i!}M \in \text{CHM}(S)$ of any object M in $\text{CHM}(U)$, which can be in fact uniquely characterized. More precisely, he has shown that it is sufficient to ask $\mathcal{C}(Z)$ to be *semi-primary*, i.e. “semi-simple up to a nilpotent radical” (see Def. 3.1.1). Note that by André-Kahn [AK02], any subcategory of Chow motives, whose objects are finite-dimensional, is semi-primary. Moreover, over a field k , the semi-primality of the whole of $\text{CHM}(k)$ would imply the nilpotence conjecture, thus tying up these notions with the ideas discussed at the beginning of this introduction.

If $\mathcal{C}(Z)$ is semi-primary, then every object M of $\mathcal{C}(S)$ is (non-uniquely) isomorphic to $j_{i!}j^*M \oplus i_*N$, with N in $\mathcal{C}(Z)$ (Prop. 3.1.4). This is the key point that lends itself to iteration, ultimately allowing us to obtain a CH-decomposition as desired in Theorem B. In a first approximation, our task can then be divided into the following steps:

- (1) when S is equipped with a stratification \mathfrak{S} into locally closed, regular subschemes, show that the categories of smooth Artin-Tate motives over each stratum can be *glued* to give a category $\mathcal{C}(S)$ of \mathfrak{S} -*constructible* smooth Artin-Tate motives over S ;
- (2) prove that the categories of smooth Artin-Tate motives over each stratum are semi-primary, and conclude by Wildeshaus’ formalism that the glued one is semi-primary as well;
- (3) show that the motive $h_S(X)$ of a regular quadric bundle belongs to $\mathcal{C}(S)$.

Points (1) and (2) are taken care of in Section 3, by suitably adapting Wildeshaus' methods, but with some technical complications. First of all, in order to glue Artin-Tate motives, we are forced to impose a condition of being *well-ramified* along the closures of strata (Def. 3.2.2), and to assume that these closures are regular themselves (Assumption 3.2.1). The gluing is then worked out in Thm. 3.2.3. Moreover, since the latter assumption will not be verified in the context of quadric bundles, we have to use a suitable base change $\pi : S' \rightarrow S$ such that the desired conditions are met on S' , then glue everything there, and finally glue those objects on the strata of S which arise as push-forward via π of objects on the strata of S' . On the other hand, to show that a suitable form of point (2) holds, we have to check that under some hypotheses on π (see Ass. 3.2.4), push-forward preserves weight-zero smooth Artin-Tate motives, hence finite dimensionality, hence semi-primality. This is the content of (the proof of) Thm. 3.2.9. Wildeshaus' techniques also can be adapted to show two other important facts: the realization functors commute with the intermediate extension functors (Thm. 3.3.1) and they are *conservative* on the categories that we obtain from the gluing process (Thm. 3.3.2).

Proving point (3) is the principal task of Section 4 (Thm. 4.1.4). We have enough control on the geometry of regular quadric bundles over an algebraically closed field of characteristic zero to understand completely the possible degenerations of smooth fibers to singular ones, as well as the associated stratification on S by corank. In particular, we are able to construct a suitable resolution of singularities $\pi : S' \rightarrow S$ (Lemma 4.1.3). Combining this with the results about *smooth* Artin-Tate motives established in Section 2, this turns out to be enough to show that the pullback of $h_S(X)$ to each stratum belongs to the correct subcategory of Chow motives and to complete the proof of Theorem B.

NOTATIONS AND CONVENTIONS

Given an arbitrary group G , and a field K of characteristic 0, we will denote by $\text{Rep}_K^{\mathbb{A}}(G)$ the abelian monoidal category of finite dimensional K -representations V of G which factor through a finite quotient of G . When G is a pro-finite group, this amounts to the continuity of the action. When $G = \pi$ is the étale fundamental group of a geometrically pointed scheme X , or the usual fundamental group of a complex variety, representations in $\text{Rep}_K^{\mathbb{A}}(\pi)$ are classically called *Artin representations*. In any case, thanks to Maschke's lemma, the category $\text{Rep}_K^{\mathbb{A}}(G)$ is semi-simple.

All our schemes are implicitly assumed to be excellent noetherian finite and dimensional.

Given (such) a scheme S , we denote by $\text{DM}_c(S, \mathbb{Q})$ the triangulated category of constructible rational motives over S .¹⁰ We let $\mathbb{1}_S$ be the constant motive over S (which is also the unit of the monoidal structure). We will use *cohomological motives* over S : given any morphism $f : X \rightarrow S$, we put:

$$h_S(X) = f_*(\mathbb{1}_X).$$

We will also heavily rely on the Bondarko's theory of weight structure, and especially the canonical (Chow) weight structure on $\text{DM}_c(S, \mathbb{Q})$. We recall these notions in Section 2.1.

Another important property of rational mixed motives from [CD19] that we will use is the so-called *continuity property* (see Def. 4.3.2, Prop. 14.3.1 of *op. cit.*). We will use it in the following form (see Prop. 4.3.4 of *op. cit.*):

Proposition 1.0.1. *Let $(S_i)_{i \in I}$ be a projective system of schemes with a projective limit S in our category of schemes. Then the canonical functor:*

$$2 - \varinjlim_{i \in I} \text{DM}_c(S_i, \mathbb{Q}) \rightarrow \text{DM}_c(S, \mathbb{Q})$$

is an equivalence of categories.

We will use two kinds of triangulated realization functors:

- Let ℓ be a prime invertible on S . We have the ℓ -adic realization:

$$\rho_{\ell} : \text{DM}_c(S, \mathbb{Q}) \rightarrow \text{D}_c^b(S_{\text{ét}}, \mathbb{Q}_{\ell})$$

where the right-hand side is the constructible derived category of Ekedahl's ℓ -adic étale sheaves with rational coefficients (see [CD16, 7.2.24]).

- Let E be a characteristic 0 field given with a complex embedding $\sigma : E \rightarrow \mathbb{C}$. Assume S is a finite type E -scheme. We have the Betti realization:

$$\rho_B : \text{DM}_c(S, \mathbb{Q}) \rightarrow \text{D}_c(S^{\text{an}}, \mathbb{Q})$$

where the right-hand side is the constructible derived category of rational sheaves over the analytical site of $S^{\text{an}} = S^{\sigma}(\mathbb{C})$. This realization is obtained from that of [Ayo10] as the following composite:

$$\text{DM}_c(S, \mathbb{Q}) \simeq \text{SH}_c(S)_{\mathbb{Q}+} \subset \text{SH}_c(S)_{\mathbb{Q}} \xrightarrow{\text{Betti}'_S \otimes \mathbb{Q}} \text{D}_c(S^{\text{an}}, \mathbb{Q})$$

¹⁰In [CD19], several models of this category are given: Beilinson motives (Def. 15.1.1), Voevodsky's h-motives (Th. 16.1.2), Voevodsky's motivic complexes (S geometrically unibranch scheme, Th. 16.1.4), the plus-part of the rational stable homotopy category (Th. 16.2.13 and 5.3.35), the \mathbb{P}^1 -stable \mathbb{A}^1 -derived étale category (Th. 16.2.22). All models being equivalent, the reader is free to choose his preferred one.

where:

- $\mathrm{SH}_c(S)$ is the constructible stable \mathbb{A}^1 -homotopy category (made of compact spectra over S).
- $\mathrm{SH}_c(S)_{\mathbb{Q}^+}$ is plus-part of the rationalization of $\mathrm{SH}_c(S)$ (see eg. [CD19, 16.2.1])
- the first equivalence is given by [CD19, Th. 16.2.13].
- the functor Betti_S is (the obvious restriction of that) defined in [Ayo10, Def. 2.1].

These two realization functors admit a right adjoint and commute with the six operations.¹¹

2. SMOOTH ARTIN-TATE MOTIVES

2.1. Gluing and weights on Beilinson motives.

2.1.1. In the theory of rational mixed motives, a crucial property is given by the gluing formalism [BBD82, Sec. 1.4.3], which is a consequence of Morel-Voevodsky's localization theorem [MV99, Th. 2.21, p. 114]. Given a closed immersion $i : Z \hookrightarrow S$ with complementary open immersion $j : U \hookrightarrow S$, one has six functors

$$(2.1.1.a) \quad \mathrm{DM}_c(U, \mathbb{Q}) \begin{array}{c} \xrightarrow{j_!} \\ \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{array} \mathrm{DM}_c(S, \mathbb{Q}) \begin{array}{c} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow{i^!} \end{array} \mathrm{DM}_c(Z, \mathbb{Q}),$$

satisfying the formalism of [BBD82, Sec. 1.4.3].¹²

This property will be at the heart of our results (see in particular Section ??). They are the starting point of Hébert's and Bondarko's extension of the weight structure on rational motives, from the case of perfect fields to that of arbitrary bases. Let us recall this theory, from [Héb11, Thm. 3.3, thm. 3.8 (i)-(ii)], for future references.

Theorem 2.1.2. *For each scheme S , there is a canonical weight structure w on the triangulated category $\mathrm{DM}_c(S, \mathbb{Q})$ called the motivic weight structure. The family of these weight structures indexed by schemes S is uniquely characterized by the following properties.*

(1) *The objects $\mathbf{1}_S(p)[2p]$ belong to the heart $\mathrm{DM}_c(S, \mathbb{Q})_{w=0}$ for all integers p , whenever S is regular.*

(2) *For any morphism $f : T \rightarrow S$, the functor f^* (resp. f_*) preserves negative weights (resp. positive weights). When f is separated of finite type, the functor $f^!$ (resp. $f_!$) preserves positive weights (resp. negative weights).*

We will only use the motivic weight structure in this paper. So w will always mean weights for the motivic weight structure. Over a base scheme S , the heart of this weight structure will be denoted by $\mathrm{CHM}(S)$ and is called the category of *Chow motives over S* .

Remark 2.1.3. Gluing of (motivic) weights. Note in particular from point (2) that when f is finite (resp. étale separated of finite type), f_* (resp. f^*) preserves weights. Moreover, under the assumption of the paragraph preceding the theorem, it follows from the gluing diagram (2.1.1.a) and point (2) that the motivic weight

¹¹This is proved in the references indicated above.

¹²This is the so-called localization property of motives, extensively studied in [CD19, §2.3].

structure on S "satisfies gluing": a motive M on S is w -positive (resp. w -negative) if and only if $j^*(M)$ and $i^!(M)$ (resp. $j^*(M)$ and $i^*(M)$) are w -positive (resp. w -negative).

Remark 2.1.4. Assumptions on base schemes. Note that in *loc. cit.*, it is assumed that schemes are of finite type over an excellent base scheme B of dimension less than 3. First, this assumption is not used in the proof of the above statement. Secondly, the only reason this assumption appeared in motivic homotopy theory is for the proof of the Grothendieck-Verdier duality (also called "local duality"), [CD19, Th. 4.4.21].¹³ Since then, this result has been generalized to the schemes considered in our paper by Cisinski in [Cis19, §2.3]. It is used in [Héb11] only in the statement of Corollary 3.9.

2.2. Smooth Artin motives. It is possible to extend the known results on Artin motives over a field to the relative case. Let us start with the definition.

Definition 2.2.1. We define the category of (constructible) smooth Artin motives over S as the thick triangulated subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by motives of the form $h_S(X)$ (resp. $h_S(X)(n)$) for X/S finite and étale. We denote this category by $\mathrm{DM}_{\mathrm{sm}}^A(S, \mathbb{Q})$.

Note that for X/S finite étale, $h_S(X)$ coincides with the Voevodsky motive $M_S(X)$.

Example 2.2.2. Let k be a perfect field. Then it follows from [VSF00, Chap. 5, Rem. 2. after 3.4.1] (see also [Org04] for details) that there exists a canonical equivalence of triangulated monoidal categories:

$$\mathrm{DM}_{\mathrm{sm}}^A(k, \mathbb{Q}) \simeq D^b(\mathrm{Rep}_{\mathbb{Q}}^A(G_k))$$

where G_k is the absolute Galois group of k , and $\mathrm{Rep}_{\mathbb{Q}}^A(G_k)$ denotes the continuous representations of G_k with rational coefficients.

2.2.3. Let us recall that one model of $\mathrm{DM}(S, \mathbb{Q})$ is obtained as the \mathbb{P}^1 -stable \mathbb{A}^1 -derived category of the category $\mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q})$ of rational étale sheaves on Sm_S (see [CD19, 16.2.18]).

The inclusion $\rho : S_{\acute{e}t} \rightarrow Sm_{S, \acute{e}t}$ of étale site induces a fully faithful and exact functor:

$$\rho_! : \mathrm{Sh}(S_{\acute{e}t}, \mathbb{Q}) \rightarrow \mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q}).$$

Note $\rho_!$ is moreover (symmetric) monoidal. One deduces a canonical composite functor:

$$D(S_{\acute{e}t}, \mathbb{Q}) \xrightarrow{\rho_!} D(\mathrm{Sh}(Sm_{S, \acute{e}t}, \mathbb{Q})) \rightarrow \mathrm{DM}(S, \mathbb{Q})$$

the last functor being obtained by projection to the \mathbb{A}^1 -localization and then taking infinite suspensions. We will still denote the later composite by $\rho_!$.

Then $\rho_!$ is triangulated and monoidal. By definition, it sends the sheaf represented by a finite étale scheme X/S on $S_{\acute{e}t}$ to the same object on $Sm_{S, \acute{e}t}$, seen as a motivic complex. This is just $M_S(X) = h_S(X)$.

Proposition 2.2.4. *Assume S is a regular connected scheme. Let $\pi = \pi_1(S_{\acute{e}t})$ be the étale fundamental group of S associated with some geometric base point. Then*

¹³More precisely, it is used to obtain some resolution of singularity statement; see *loc. cit.*

the functor $\rho_!$ is fully faithful when restricted to the full subcategory $D^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi))$ and induces an equivalence of triangulated monoidal categories:

$$\rho_! : D^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)) \rightarrow \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}}(S, \mathbb{Q}).$$

Moreover, the ℓ -adic realisation functor restricted to $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}}(S, \mathbb{Q})$ lands into the bounded derived category of Artin ℓ -adic Galois representations and the composite functor:

$$D^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)) \xrightarrow{\rho_!} \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}}(S, \mathbb{Q}) \xrightarrow{\rho_{\ell}} D^b(\mathrm{Rep}_{\mathbb{Q}_{\ell}}^{\mathrm{A}}(\pi))$$

is just the extension of scalar functor associated with $\mathbb{Q}_{\ell}/\mathbb{Q}$.

Proof. We prove the first assertion: fully faithful nature of the restriction of $\rho_!$. Note that the functor $\rho_!$ admits a right adjoint ρ^* . The functor ρ^* commutes with direct sums: this follows formally as $D(S_{\mathrm{ét}}, \mathbb{Q})$ is compactly generated.¹⁴ We have to prove that for all complexes K, L in $D^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi))$, and say any $n \in \mathbb{Z}$ for the next reduction, the map:

$$\mathrm{Hom}(K, L[n]) \rightarrow \mathrm{Hom}(\rho_! K, \rho_! L[n]) = \mathrm{Hom}(K, \rho^* \rho_! L[n])$$

is an isomorphism. Now we use the fact $D^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi))$ is generated by shifts of sheaves representable by some finite étale cover X/S . So we are reduced to the case $K = \mathbb{Q}(X)$, $L = \mathbb{Q}(Y)$ for X and Y étale cover of S . Explicitly, we have to prove that the following map is an isomorphism:

$$\mathrm{Hom}(\mathbb{Q}(X), \mathbb{Q}(Y)[n]) \rightarrow \mathrm{Hom}(h_S(X), h_S(Y)[n]).$$

In the monoidal category $\mathrm{Sh}(S_{\mathrm{ét}}, \mathbb{Q})$, and therefore in $D(S_{\mathrm{ét}}, \mathbb{Q})$, the sheaf $\mathbb{Q}(Y)$ is auto-dual. The same result holds for $h_S(X)$ in $\mathrm{DM}(S, \mathbb{Q})$ (as for example $\rho_!$ is monoidal). Applying the formulas $\mathbb{Q}(X) \otimes \mathbb{Q}(Y) = \mathbb{Q}(X \times_S Y)$ and $h_S(X) \otimes h_S(Y) = h_S(X \times_S Y)$, we are reduced to the case $Y = S$. In other words we have to prove that the canonical map:

$$(2.2.4.a) \quad H^n(X_{\mathrm{ét}}, \mathbb{Q}) \rightarrow \mathrm{Hom}(h_S(X), \mathbb{Q}[n]) = H_M^{n,0}(X, \mathbb{Q})$$

is an isomorphism. Note that X is regular, as it is étale over S . Thus it is geometrically unibranch and we get from [AGV73, IX, 2.14.1]:

$$H^n(X_{\mathrm{ét}}, \mathbb{Q}) = \mathbb{Q}^{\pi_0(X)} \text{ for } n = 0, 0 \text{ otherwise.}$$

Also, according to [CD19, 14.2.14], we get:

$$H_M^{n,0}(X, \mathbb{Q}) = Gr_{\gamma}^0 K_{-n}(X)_{\mathbb{Q}} = \mathbb{Q}^{\pi_0(X)} \text{ for } n = 0, 0 \text{ otherwise,}$$

and so the map (2.2.4.a) is necessarily an isomorphism.

The other assertions are clear, by definition of the category of smooth Artin motives. \square

In the complex case, we get a simpler formulation.

Proposition 2.2.5. *Let E be a field of characteristic zero, S a smooth connected E -scheme and $\pi = \pi_1(S^{\mathrm{an}})$ for any choice of base point of S^{an} .*

Then the Betti realisation functor:

$$\rho_B : \mathrm{DM}_c(S, \mathbb{Q}) \rightarrow D(S^{\mathrm{an}}, \mathbb{Q})$$

¹⁴In fact, it is equivalent to $D(S_{\mathrm{Nis}}, \mathbb{Q})$ as π is pro-finite.

is fully faithful when restricted to $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}}(S, \mathbb{Q})$ and induces an equivalence of triangulated monoidal categories:

$$\rho_B : \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}}(S, \mathbb{Q}) \rightarrow \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)).$$

The proof uses the same argument as in the case of the previous proposition, given that $h_S(X)$ is realized to the complex $\mathrm{R}f_*^{\mathrm{an}}(\mathbb{Q}_X)$ which is concentrated in degree 0 and equal to the continuous representation of π represented by the Galois cover X^{an} over S^{an} .

Remark 2.2.6. The two previous propositions are obviously compatible: in the assumptions of the second one, we get according to a theorem of Grothendieck:

$$\pi_1(S_{\mathrm{ét}}) \simeq \widehat{\pi_1(S^{\mathrm{an}})}$$

where the right hand-side denotes profinite completion. In particular, we get equivalence of (abelian semi-simple monoidal) categories:

$$\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi_1(S_{\mathrm{ét}})) \simeq \mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\widehat{\pi_1(S^{\mathrm{an}})}) \simeq \mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi_1(S^{\mathrm{an}})).$$

2.3. Weights on smooth Artin-Tate motives. The purpose of this section is to extend results of Wildeshaus, [Wil16] on Artin-Tate motives over a field to the case of a regular base scheme S . Moreover, for the purpose of our main theorem, we will need to restrict the type of allowed Artin motives, as in *loc. cit.* So we introduce the next definition.

Definition 2.3.1. Let S be a regular connected scheme with étale fundamental group $\pi = \pi_1(S_{\mathrm{ét}})$. Let \mathcal{A} be a full \mathbb{Q} -linear sub-category of $\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)$ which is stable under retracts.¹⁵

We define the triangulated category of smooth Artin-Tate motives of type \mathcal{A} over S as the thick triangulated subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by motives of the form $\rho_!(A)(n)$ for an object A of \mathcal{A} and an integer $n \in \mathbb{Z}$ (see Proposition 2.2.4 for the definition of $\rho_!$). We denote it by $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$.

When $\mathcal{A} = \mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)$, the above category is simply the category of smooth Artin-Tate motives over S , denoted by $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$.

Remark 2.3.2. Note that compared to Definition 1.6 of [Wil16], we do not assume that \mathcal{A} is closed under tensor product. As a consequence $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ is not monoidal in general.

Theorem 2.3.3. *Consider the above notations and assumptions of the above definition.*

Then the weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$ restricts to a weight structure on the triangulated sub-category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$. Moreover, any motive M in $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ of weight 0 admits a decomposition:

$$(2.3.3.a) \quad M \simeq \bigoplus_{i \in I} \rho_!(V_i)(n_i)[2n_i]$$

where I is a finite set, V_i is a simple Artin representation of π in \mathcal{A} and n_i is an integer. (See Paragraph 2.2.3 for $\rho_!$).

¹⁵And therefore stable under kernel and cokernel as $\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)$ is abelian semi-simple. In particular, \mathcal{A} is abelian semi-simple.

Remark 2.3.4. (1) Another way of stating the second assertion is that the canonical functor

$$\begin{aligned} D^b(\mathcal{A}) &\rightarrow \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})_{w=0} = \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q}) \cap \mathrm{CHM}(S, \mathbb{Q}) \\ (V_n)_{n \in \mathbb{Z}} &\mapsto \bigoplus_{n \in \mathbb{Z}} \rho_!(V_n)(n)[2n] \end{aligned}$$

is essentially surjective. Contrary to what happens in [Wil16], over a perfect field, this functor is not an equivalence of triangulated categories. The problem comes from the non-triviality of $CH^n(V)_{\mathbb{Q}}$ for $n > 0$ and V/S finite étale. So in particular, the preceding functor is an equivalence when S is (regular) semi-local.

(2) The decomposition (2.3.3.a) is a particular case of the Corti-Hanamura decomposition: If we apply the ℓ -adic realization functor ρ_{ℓ} to an Artin representation V_n we get (according to the last assertion of Prop. 2.2.4)

$$\rho_{\ell}(M) \simeq \bigoplus_{i \in I} (V_i \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})(n_i)[2n_i].$$

In particular, if the decomposition (2.3.3.a) of M is not unique. But the pairs (V_i, n_i) for $i \in I$ are uniquely determined by M (or its realization).

Proof. Let us show the first assertion. Let \mathcal{H} be the full \mathbb{Q} -linear sub-category of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ whose objects are of the form (2.3.3.a), where V_i is a simple object of \mathcal{A} . As S is regular, all motives in \mathcal{H} are of weight 0. In particular, given such motives M, N , one has $\mathrm{Hom}(M, N[i]) = 0$ for $i > 0$. This condition implies that there is a unique weight structure on $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ (see eg. [Wil09, 1.5]), whose heart is the pseudo-abelianization \mathcal{K} of \mathcal{H} . It follows from the axioms of weight structures that this weight structure is just the restriction of the Chow weight structure, thus proving the first assertion.

To prove the second assertion, it is sufficient to prove that $\mathcal{K} = \mathcal{H}$, i.e. that \mathcal{H} is pseudo-abelian. Let $e : M \rightarrow M$ be an idempotent of \mathcal{H} . By assumption, we can write $M = \sum_{i=1}^r \rho_!(V_i)(n_i)[2n_i]$ where V_i is an object of \mathcal{A} (semi-simple, but not necessary simple) and $n_1 < n_2 < \dots < n_r$. We prove by induction on the number of factors r of M that e admits a kernel in \mathcal{H} .

We treat the case $r = 1$. According to Proposition 2.2.4, and the fact that twists are invertible, the category of motives of the form $\rho_!(V_1)(n_1)[2n_1]$ for an object V_1 of \mathcal{A} and an integer n_1 is equivalent to the abelian (semi-simple) category \mathcal{A} . This implies that any idempotent of a motive of this form admits a kernel, giving the case $r = 1$.

Assume the result is known for any integer less than $r > 1$, and prove the case where M has exactly r factors as above. Put $P = \rho_!(V_1)(n_1)[2n_1]$ and $Q = \sum_{i=2}^r \rho_!(V_i)(n_i)[2n_i]$ so that $M = P \oplus Q$. Given that decomposition, we can write the idempotent e as a 2 by 2 matrix:

$$e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then c belongs to

$$\bigoplus_{i=2}^r \mathrm{Hom}(\rho_!(V_i)(n_i)[2n_i], \rho_!(V_1)(n_1)[2n_1]) = \mathrm{CH}^{n_1-n_i}(V_i \times_S V_1)$$

using the auto-duality of $\rho_!(V)$ (see the proof of Prop. 2.2.4) — in the right hand-side, we identify the sheaf V_i with the finite étale S -scheme which represent it.

As by assumption, $n_1 < n_i$ for $i > 1$, we get that $c = 0$. Note that a and d are idempotent, respectively of P and Q . According to the induction case, these idempotents admit a kernel in \mathcal{H} . One easily deduces from the matrix form of e that $\text{Ker}(a) \oplus \text{Ker}(d)$ is a kernel of e in \mathcal{H} . This concludes the induction step, and the proof. \square

2.4. First decomposition theorem. For the needs of the proof of our first decomposition theorem, we will extend some definitions of [Wil12]. Let us first recall the following theorem, Th. 1.7 of *loc. cit.*

Theorem 2.4.1 (Wildeshaus). *Let $j : U \rightarrow S$ be an open immersion. Then the weight-exact functor j^* induces an additive exact functor*

$$j^* : \text{CHM}(S) \rightarrow \text{CHM}(U)$$

which is essentially surjective and full.

2.4.2. Consider an open immersion $j : U \rightarrow S$ and a Chow motive M over U . A *Chow extension* of M along j will be a pair (\bar{M}, α) where \bar{M} is a Chow motive over S and α is an isomorphism $j^*(\bar{M}) \xrightarrow{\sim} M$. Morphisms of such extensions are defined in the obvious way.

According to the previous theorem, extensions of M along j always exist. The goal of the work of Wildeshaus is to find (and to define) the *intermediate extension* of M along j : see [Wil17], Summary 2.12. For the needs of our first decomposition theorem, we will use a special type of Chow extensions that we now introduce.

Definition 2.4.3. Consider the above notations, assuming that j is dense. Then a Chow extension (\bar{M}, α) of M along j will be called *fair* if the induced map

$$\text{End}(\bar{M}) \rightarrow \text{End}(j^*(M)) \xrightarrow{\alpha_*} \text{End}(M)$$

is an isomorphism — *i.e.* a monomorphism according to the previous theorem.

The proof of the following result is identical to that of [Wil12, Th. 3.1(a)].

Theorem 2.4.4. *Consider the notations of the previous definition. Assume a fair Chow extension (\bar{M}, α) of M along j exists.*

Then for any extension (P, β) of M along j , there exists a decomposition of P of the form:

$$\psi : P \xrightarrow{\sim} \bar{M} \oplus i_*(L_Z)$$

where L_Z is a Chow motive over Z , satisfying the relation: $j^(\psi) = \alpha^{-1} \circ \beta$.*

If moreover (P, β) is fair, then $L_Z = 0$ and the isomorphism ψ is uniquely determined by the preceding relation.

Example 2.4.5. A fair Chow extension as above is a particular instance of Wildeshaus's theory of intermediate extension $j_{i*}(M)$, as shown for example by the characterizing property (4a) of Summary 2.12 of [Wil17]. The preceding theorem can also be interpreted as a minimality property of the extension (\bar{M}, α) .

One cannot always expect that such minimal extensions exist (the correct hope is formulated in *loc. cit.*, Conjecture 3.4; cfr. Rmk. 3.1.5). However, here are some interesting examples.

- (1) Assume that U is regular, and the normalization of X is regular. Then $\mathbb{1}_U$ admits a fair Chow extension: this is [Wil12, Th. 3.11(a)].

- (2) Assume that both U and S are regular. Let V/U be an étale cover. Using a classical terminology, we will say that V is *non-ramified along* $(S - U)$ if there exists an étale cover \bar{V}/S whose restriction to U is V . In that case, for any integer $n \in \mathbb{Z}$, the Artin-Tate motive $\bar{M} = h_S(\bar{V})(n)[2n]$ is a fair Chow extension of $M = h_U(V)(n)[2n]$ along j .

Indeed, \bar{M} is obviously a Chow extension of M along j (using the six functors formalism). Moreover, as in the proof of Prop. 2.2.4, one gets:

$$\mathrm{End}(\bar{M}) \simeq \mathbb{Q}^{\pi_0(\bar{V} \times_S \bar{V})}, \quad \mathrm{End}(M) \simeq \mathbb{Q}^{\pi_0(V \times_U V)}.$$

As $V \times_U V$ is a dense open of $\bar{V} \times_S \bar{V}$, we get that \bar{M} is fair as claimed.

Remark 2.4.6. The reader can check that one can replace, in [Wil12, Th. 3.11] points (a) and (b), the constant motive $\mathbb{1}_U$ by a smooth Artin-Tate motive $h_U(V)$ such that V is unramified along $(S - U)$.

2.4.7. Recall also that a Chow motive over a field k is said to be *Tate* if it is isomorphic to a finite sum of motives of the form $\mathbb{1}(i)[2i]$ for an integer $i \in \mathbb{Z}$. By extension, a smooth proper k -scheme is said to be Tate if its associated Chow motive is Tate.

Theorem 2.4.8. *Let S be a regular connected scheme with étale fundamental group π . Let $f : X \rightarrow S$ be a smooth proper morphism whose geometric fibers are Tate.*

Then there exists an isomorphisms of motives over S :

$$h_S(X) \simeq \bigoplus_{i \in I} \rho_i(V_i)(-n_i)[-2n_i]$$

where I is a finite set, V_i is a simple Artin representation of π and n_i is a non-negative integer. In other words, $h_S(X)$ is a smooth Artin-Tate motive over S of weight 0 — see Theorem 2.3.3.

Moreover, this decomposition is a BCH-decomposition of the Chow motive $h_S(X)$ (see Def. 2), and the set $\{([V_i], n_i), i \in I\}$ where $[V_i]$ denotes the isomorphism class in $\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{A}}(\pi)$ (or what amount to the same in the corresponding category of simple étale S -covers) is uniquely determined by the property that for any prime integer ℓ and any $n \geq 0$:

$$(2.4.8.a) \quad \mathbb{R}^{2n} f'_*(\mathbb{Q}_{\ell}) \simeq \bigoplus_{i \in I | n_i = n} V'_i \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}(-n_i)$$

where f' and V'_i are pullback of f and V_i along the open immersion $U[\ell^{-1}] \rightarrow U$.

Proof. Let $\bar{\eta}$ be a geometric generic point of S . By assumption, one gets:

$$h_{\bar{\eta}}(X_{\bar{\eta}}) \simeq \sum_{i \in I} \mathbb{1}_{\bar{\eta}}(-n_i)[-2n_i].$$

According to the continuity property of DM_c (see Proposition 1.0.1), there exists a dense open $j : U \rightarrow S$ and an étale cover $p : V \rightarrow U$ such that the above isomorphism lifts to:

$$h_V(X_V) \simeq \bigoplus_{j \in J} \mathbb{1}_V(-m_j)[-2m_j].$$

In particular, $h_U(X_U) = p_*(h_V(X_V)) \simeq \sum_{j \in J} p_*(\mathbb{1}_V)(-m_j)[-2m_j]$ is a smooth Artin-Tate motive over U . As p is finite étale, the natural map

$$p^* : h_U(X_U) \rightarrow h_U(X_V)$$

is a split monomorphism, with splitting $\frac{1}{d} \cdot p_*$ where p_* is the Gysin morphism associated with p (see e.g. [CD19, 13.7.4 and 13.7.6]). In particular, $h_U(X_U)$ is a smooth Artin-Tate motive, and it follows from Theorem 2.3.3 that there exists a decomposition:

$$h_U(X_U) \simeq \bigoplus_{i \in I} \rho!(W_i)(-n_i)[-2n_i]$$

where W_i is a simple Artin representation of $\pi_1(U)$. Given a prime ℓ , we let $U' = U[\ell^{-1}]$, $f_{U'}$ the pullback of f over U' , W'_i the Artin representation of $\pi_1(U')$ induced by W_i . According to Remark 2.3.4(2), and because $\rho_\ell(h_{U'}(X_{U'})) \simeq \mathbf{R}f_{U'*}(\mathbb{Q}_\ell)$, one gets:

$$(2.4.8.b) \quad \mathbf{R}^{2n} f_{U'*}(\mathbb{Q}_\ell) \simeq \bigoplus_{i \in I | n_i = n} W'_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-n_i).$$

This is the decomposition of the locally constant sheaf $\mathbf{R}^{2n} f_{U'*}(\mathbb{Q}_\ell)$ into semi-simple components (beware the W'_i might not be simple). As this sheaf admits an extension to all S , namely $\mathbf{R}^{2n} f'_*(\mathbb{Q}_\ell)$, $f' = f[\ell^{-1}]$, it follows that each representation W'_i is unramified along $(S - U)$. As this is true for any prime ℓ , one deduces that W_i is unramified along $(S - U)$; *i.e.* it admits an extension V_i to S . According to Example 2.4.5(2), one deduces that

$$\bigoplus_{i \in I} \rho!(V_i)(-n_i)[-2n_i]$$

is a fair Chow extension of $h_U(X_U)$. As $h_S(X)$ is obviously a Chow extension of $h_U(X_U)$, one deduces from Theorem 2.4.4 that there exists a decomposition:

$$(2.4.8.c) \quad h_S(X) \simeq \bigoplus_{i \in I} \rho!(V_i)(-n_i)[-2n_i] \oplus i_*(M_Z)$$

for some Chow motive M_Z over Z . Note that Relation (2.4.8.b) implies Relation (2.4.8.a).¹⁶

It remains to prove that $M_Z = 0$. As the morphism $p : S' = \sum_{\ell \text{ prime}} S[\ell^{-1}] \rightarrow S$ is a pro-open cover, the pullback functor $p^* : \text{DM}(S) \rightarrow \text{DM}(S')$ is conservative (use the continuity property of DM [CD19, Th. 14.3.1] and the Zariski separation property as in the proof of [CD19, Prop. 4.3.9]). In particular, we can fix a prime ℓ and work over $S[\ell^{-1}]$. To simplify notation, let us assume $S = S[\ell^{-1}]$. Consider a point $x \in Z$, and let $i_x : \{x\} \rightarrow Z$ be the canonical immersion. Let \bar{x} be a geometric point over x . By assumption, $h_{\bar{x}}(X_{\bar{x}})$ is a Tate motive. In other words, the motive $i_x^* h_S(X) = h_x(X_x)$ is an Artin-Tate motive over the residue field $\kappa(x)$. Decomposition (2.4.8.c) gives:

$$h_x(X_x) \simeq \bigoplus_{i \in I} \rho!(V_{i,x})(-n_i)[-2n_i] \oplus i_x^*(M_Z)$$

where $V_{i,x}$ denotes the pullback of V_i to x (seen as an étale sheaf). As recalled in Remark 2.3.4(2), applied over $\kappa(x)$, the ℓ -adic realization of $h_x(X_x)$ determines the factors of the decomposition of $h_x(X_x)$ into twists of Artin motives (up to isomorphisms and permutations). Thus relation (2.4.8.a) (which we have already established), specialized at x using the smooth base change theorem in ℓ -adic étale cohomology, implies that the set of isomorphism classes of the $V_{i,x}$ describes all

¹⁶In particular, we know that $\rho_\ell(M_Z)$ vanishes but this is not sufficient to conclude (as we do not know yet that the ℓ -realization is conservative).

the possible factors of the weight 0 Artin-Tate motive $h_x(X_x)$ and this implies $i_x^*(M_Z) = 0$. One concludes as $(i_x^*)_{x \in Z}$ is a conservative family of functors on $\mathrm{DM}_c(Z, \mathbb{Q})$ (see [CD19, 4.3.17]). \square

Remark 2.4.9. Note also that we deduce from Example 2.4.5 that for any dense open immersion $j : U \rightarrow S$, $h_S(X)$ is a fair Chow extension of $h_U(X_U)$. In particular, $h_S(X) = j_{!*}(h_U(X_U))$.

Example 2.4.10. Relative Severi-Brauer schemes, étale locally cellular schemes, relative homogeneous varieties, smooth quadrics.

Corollary 2.4.11. *Under the assumption of the previous theorem, for any integer $n \geq 0$, the integer:*

$$\dim_{\mathbb{Q}_\ell} (\mathrm{R}^n f_*(\mathbb{Q}_\ell))$$

is independent of the prime ℓ invertible on S .

2.4.12. An important corollary for us is the case of smooth quadrics: following [DK73, XII, Def. 2.4], a smooth quadric over a scheme S is a smooth proper morphism $f : X \rightarrow S$ whose geometric fibers are smooth quadric hypersurfaces in the classical sense.

Let us introduce some notations in order to state the next corollary. Assume f has constant relative dimension n . According to .op. cit., 2.6, there exists a Severi Brauer S -scheme $P(X)$ which contains X as an effective Cartier divisor of degree 2. If $n = 2m$, the center of the Clifford \mathcal{O}_S -algebra associated with the closed pair $(P(X), X)$ is an étale cover $Z(X)$ over S of degree 2; *op. cit.*, 2.7.

Then, applying Theorem 2.4.8 and [DK73, Th. 3.3] one gets:

Corollary 2.4.13. *Consider the above notations. Assume that S is regular and that there exists a prime ℓ invertible on S .*

Then there exists a decomposition of Chow motives over S as follows:

$$h_S(X) \simeq \begin{cases} \bigoplus_{i=0}^n \mathbb{1}_S(i)[2i] & \text{if } n = 2m + 1, \\ h_S(Z(X)) \oplus \bigoplus_{i=0, i \neq m}^n \mathbb{1}_S(i)[2i] & \text{if } n = 2m. \end{cases}$$

3. RECALL ON WILDESCHAUS' MOTIVIC INTERMEDIATE EXTENSION

3.1. Semi-primary categories and motivic intermediate extensions. Recall the following definition.

Definition 3.1.1. A \mathbb{Q} -linear category \mathfrak{C} is *semi-primary* if

(1) for all objects B of \mathfrak{C} , the *radical*

$$\mathrm{rad}_{\mathfrak{C}}(B, B) := \{f \in \mathrm{Hom}_{\mathfrak{C}}(B, B) \mid \forall g \in \mathrm{Hom}_{\mathfrak{C}}(B, B), \mathrm{id}_B - gf \text{ is invertible}\}$$

is nilpotent;

(2) the quotient category $\mathfrak{C}/\mathrm{rad}_{\mathfrak{C}}$ is semisimple.

Adopt the notations of subsection 2.1. We are now going to explain how the notion of semi-primality leads to a definition of an intermediate extension functor, following Wildeshaus.

Let \cdot denote any of the schemes U , S or Z , and fix $\mathcal{C}(\cdot)$ full pseudo-abelian subcategories of the categories $\mathrm{DM}_c(\cdot, \mathbb{Q})$, related by gluing. Assume that they inherit a weight structure (automatically compatible with the gluing) from the

restrictions of the motivic weight structure. The subscript $w = 0$ will mean that we are taking the *heart* of such weight structures.

Moreover, denote by $\mathcal{C}(S)_{w=0}^u$ the quotient of the category $\mathcal{C}(S)_{w=0}$ by the two-sided ideal¹⁷ \mathfrak{g} generated by $\mathrm{Hom}_{\mathcal{C}(S)}(A, i_*B)$ and $\mathrm{Hom}_{\mathcal{C}(S)}(i_*B, A)$, with (A, B) varying on the collection of objects of $\mathcal{C}(S)_{w=0} \times \mathcal{C}(Z)_{w=0}$ such that A admits no non-zero direct factor belonging to $\mathcal{C}(Z)_{w=0}$. Finally, denote by $\mathcal{C}(Z)_{w=0}^u$ the quotient of the category $\mathcal{C}(Z)_{w=0}$ by the restriction of \mathfrak{g} to $\mathcal{C}(Z)_{w=0}$ (with respect to the fully faithful inclusion $i_* : \mathcal{C}(Z)_{w=0} \hookrightarrow \mathcal{C}(S)_{w=0}$).

Theorem 3.1.2. [Wil17, Theorem 2.9]

(1) If $\mathcal{C}(Z)_{w=0}$ is semi-primary, then the functors j^* and i_* induce a canonical equivalence of categories

$$(3.1.2.a) \quad \mathcal{C}(S)_{w=0}^u \simeq \mathcal{C}(U)_{w=0} \times \mathcal{C}(Z)_{w=0}^u$$

(2) If both $\mathcal{C}(Z)_{w=0}$ and $\mathcal{C}(U)_{w=0}$ are semi-primary, then so is $\mathcal{C}(S)_{w=0}$.

Definition 3.1.3. Suppose that $\mathcal{C}(Z)_{w=0}$ is semi-primary. The *intermediate extension* is the fully faithful functor

$$j_{!*} : \mathcal{C}(U)_{w=0} \hookrightarrow \mathcal{C}(S)_{w=0}^u$$

corresponding to the functor $(\mathrm{id}_{\mathcal{C}(U)_{w=0}}, 0)$ under the equivalence of categories (3.1.2.a).

It follows from its very definition that the motivic intermediate extension functor $j_{!*}$ enjoys the following property ([Wil17, Summary 2.12 (b)]):

Proposition 3.1.4. Consider the notation and the assumptions of the previous definition. Then, any object M of $\mathcal{C}(S)_{w=0}$ is isomorphic to a direct sum $j_{!*}M_U \oplus i_*N$, for an object M_U of $\mathcal{C}(U)_{w=0}$ and an object N of $\mathcal{C}(Z)_{w=0}$. The object M_U is such that $j^*M \simeq M_U$ (hence unique up to unique isomorphism) and N is unique up to an isomorphism, which becomes unique in $\mathcal{C}(Z)_{w=0}^u$.

Remark 3.1.5. It is believed that for any S , the heart $\mathrm{CHM}(S)$ of the motivic weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$ is semi-primary (cfr. [Wil17, Conj. 3.4]). This conjecture is at the moment completely out of reach, but it would permit, by choosing as $\mathcal{C}(\cdot)$ the whole of the categories $\mathrm{DM}_c(\cdot, \mathbb{Q})$, to define (up to non-unique isomorphism) the intermediate extension to S of any Chow motive on U .

3.2. Semi-primary categories of Chow motives. The aim of this section is to single out some subcategories of Chow motives which can actually be shown to be semi-primary and which will be suited for our geometric applications. We will adapt Wildeshaus' methods from [Wil17], in order to show semiprimality of subcategories which are different¹⁸ from the ones considered in *op. cit.*.

Fix a scheme S admitting a *good stratification* \mathfrak{S} , i.e. such that S may be written as a finite (set-theoretic) disjoint union $\bigsqcup_{\sigma \in \mathfrak{S}} S_\sigma$ of locally closed subschemes such that the closure $\overline{S_\sigma}$ of each stratum S_σ is a union of strata.

We make the following assumption on our good stratification:

Assumption 3.2.1. For all $\sigma \in \mathfrak{S}$, the strata S_σ are regular, with regular closure.

¹⁷Note that in our setting, this ideal will always be contained in $\mathrm{rad}_{\mathcal{C}(S)_{w=0}}$ ([Wil17, Cor. 1.5 (a)]).

¹⁸See Rmk. 3.2.10 for a comment on these differences.

Fix a stratum S_σ and consider the categories $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S_\sigma, \mathbb{Q})$ introduced in Definition 2.3.1. In order to define the subcategories \mathcal{A} that we need, observe that by applying Zariski's main theorem, any finite étale morphism $q : X_{S_\sigma} \rightarrow S_\sigma$ extends to a finite morphism $q' : X \rightarrow \overline{S_\sigma}$.

Definition 3.2.2. The category of *well ramified* smooth Artin-Tate motives over S_σ is the category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ obtained by choosing $\mathcal{A} = \mathrm{A}_{\mathrm{wr}}$, where A_{wr} is the full subcategory of direct factors of objects corresponding to finite étale morphisms $q : X_{S_\sigma} \rightarrow S_\sigma$ verifying the following condition (cfr. the terminology of 2.4.5 (2)):

- either X_{S_σ} is non-ramified along $Z_\sigma := \overline{S_\sigma} \setminus S_\sigma$,
- or there exists a q' as above, inducing $X_{Z_\sigma} \simeq Z_\sigma$.

Theorem 3.2.3. (cfr. [Wil17, Thm. 4.5]) *Let S be a scheme with a good stratification \mathfrak{S} satisfying Assumption 3.2.1. Then:*

(1) *the categories $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ of well ramified smooth Artin-Tate motives over S_σ , $\sigma \in \mathfrak{S}$, can be glued to give a full, triangulated sub-category $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$ of $\mathrm{DM}_c(S, \mathbb{Q})$, called the category of \mathfrak{S} -constructible well ramified Artin-Tate motives over S . This subcategory is dense.*

(2) *Let $M \in \mathrm{DM}_c(S, \mathbb{Q})$. Then the following conditions are equivalent.*

- (a) $M \in \mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$.
- (b) $j^*M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ for all $\sigma \in \mathfrak{S}$, where j denotes the immersion $S_\sigma \hookrightarrow S$.
- (c) $j^!M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(S_\sigma, \mathbb{Q})$ for all $\sigma \in \mathfrak{S}$.

In particular, the triangulated category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ of smooth Artin-Tate motives over S is contained in $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$.

(3) *The category $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$ is pseudo-abelian.*

Proof. The proof imitates the one of [Wil17, Thm. 4.5], and proceeds by induction on the number of strata. If there is only one stratum, all claims are trivial, except for the last claim of part (1) and for part (3). Of course, it suffices to prove part (3). But in the case we are treating, the category in question is just $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$, and it has been shown in the proof of Thm. 2.3.3 that this category has a (bounded) weight structure whose heart is pseudo-abelian. Hence (by [Bon10, Lemma 5.2.1]), the category $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S, \mathbb{Q})$ itself is pseudo-abelian.

As for the induction step, we have that the theorem is true for the complement Z of any open stratum U , with its stratification \mathfrak{S}_Z , by the induction hypothesis. Write j_U , resp. i_Z , for the open, resp. locally closed immersion of U , resp. Z , in S .

In order to prove (1), we will prove that the criterion given by [Wil17, Prop. 4.1. (a)] is satisfied; it says that $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(U, \mathbb{Q})$ and $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$ can be glued if and only if for all objects $M \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(U, \mathbb{Q})$, $i_Z^*j_{U,*}M$ belongs to $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. For this, we can suppose that the closure of U in S is the whole of S , and we take a direct factor M_U of some motive $h_U(X_U/U)$, with $q : X_U \rightarrow U$ finite étale satisfying the conditions in Def. 3.2.2. We will treat only the case in which q extends to a finite morphism $q' : X \rightarrow S$ inducing $X_Z \simeq Z$ (the other one being easier). Since Z is regular, by our Assumption 3.2.1, we have that X is regular, too. By proper base change and absolute purity, this implies that for any $p \in \mathbb{Z}$, both $i_Z^!h_S(X/S)(p)$ and $i_Z^*h_S(X/S)(p)$ belong to $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. Now, again using proper base change, we see that $i_Z^*j_{U,*}h_U(X_U/U)(p)$ is a cone of the canonical morphism $i_Z^!h_S(X/S)(p) \rightarrow i_Z^*h_S(X/S)(p)$. Hence, it belongs to $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. But

the latter category is dense by the induction hypothesis, so $i_Z^* j_{U,*} M_U(p)$ belongs to it as well. This means that the functor $i_Z^* j_{U,*}$ maps the generators of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(U, \mathbb{Q})$ to $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z)$. As a consequence, the whole of $\mathrm{DM}_{\mathrm{sm}}^{\mathrm{A}_{\mathrm{wr}}\mathrm{T}}(U, \mathbb{Q})$ is mapped to the latter category under $i_Z^* j_{U,*}$, and the criterion is fulfilled.

The proof of the remaining points carries over word by word from the proof of the analogous points in [Wil17, Thm. 4.5]. \square

Let us now set up a slightly different notation, in order to treat the more general geometric situations which we are interested in. Come back to the setting $U \hookrightarrow S \leftarrow Z$ of subsection 2.1 and suppose that S is equipped with a good stratification Φ . The subcategories we are interested in will depend on the choice of a proper morphism $\pi : S' \rightarrow S$ from a scheme S' which admits a good stratification, such that the preimage via π of any stratum of Φ is a union of strata (i.e. π is a *morphism of good stratifications*). We may always suppose that U is the only open stratum of Φ (with an induced, trivial stratification denoted by the same symbol U). The closed complement Z inherits a good stratification Φ_Z ; moreover, we have that S'_U , resp. S'_Z , inherit good stratifications \mathfrak{S}_U , resp. \mathfrak{S}_Z , and that π induces morphisms of good stratifications $S'_U \rightarrow U$, $S'_Z \rightarrow Z$.

It will be necessary to make the following assumptions on the nature of the stratifications \mathfrak{S} and Φ and of the morphism π :

Assumption 3.2.4. (cfr. [Wil17, Ass. 5.6])

- (1) the good stratification \mathfrak{S} on S' satisfies Assumption 3.2.1;
- (2) for all $\phi \in \Phi$, the strata S_ϕ are regular;
- (3) the morphism π is surjective, and for all $\phi \in \Phi$ and $\sigma \in \mathfrak{S}$ such that S'_σ is a stratum of $\pi^{-1}(S_\phi)$, the morphism $\pi_\sigma : S'_\sigma \rightarrow S_\phi$ is proper with geometrically connected fibres, smooth, and such that the motive $h(S'_\sigma/S_\phi)$ belongs to the category of Tate motives over S_ϕ .

Remark 3.2.5. Since we ask π_σ to be proper and smooth, the previous assumption actually implies that $h(S'_\sigma/S_\phi)$ belongs to the category of *weight zero, smooth* Tate motives over S_ϕ .

Fix a morphism $\pi : S' \rightarrow S$ of good stratifications Φ and \mathfrak{S} , satisfying Assumption 3.2.4.(1). For any $\phi \in \Phi$, we will denote by \mathfrak{S}_{S_ϕ} the stratification induced by \mathfrak{S} on the pullback of S_ϕ via π .

Definition 3.2.6. (cfr. [Wil17, Def. 4.9]) The category $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is the pseudo-abelian completion of the strict, full, \mathbb{Q} -linear triangulated subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by the images under π_* of the objects of $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$.

Suppose moreover that Assumption 3.2.4.(2) is satisfied. Then, reasoning in the same way as in the proof [Wil17, Cor. 4.11], we see:

Lemma 3.2.7. (1) *The restriction of the motivic weight structure on $\mathrm{DM}_c(S, \mathbb{Q})$ induces a bounded weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$. For $\pi = \mathrm{id}$, this gives a bounded weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$.*

(2) *The heart of the above weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is the pseudo-Abelian completion of the strict, full, \mathbb{Q} -linear additive subcategory of $\mathrm{DM}_c(S, \mathbb{Q})$ generated by the images under π_* of the objects of the heart of the weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S})$.*

Definition 3.2.8. The heart of the weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ given by the preceding Lemma is denoted by $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$.

Now we can finally state the result that we want to employ.

Theorem 3.2.9. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} , Φ satisfying Assumption 3.2.4. Then, the categories $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_U/U)$, $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ and $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z/\Phi_Z)$ are related by gluing, and the category $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_Z/\Phi_Z)$ is semi-primary.*

Proof. The first claim follows from Thm. 3.2.3 and proper base change (cfr. the analogous [Wil17, Cor. 4.10 (b)]). So, the category $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}_Z/\Phi_Z)$ itself is obtained by gluing. Hence, in order to prove the second claim, it is enough to prove semi-primality of $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ for each stratum S_ϕ of Z and then to apply Thm. 3.1.2.(2).

We first observe that by Lemma 3.2.7.(2), the category $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ is the pseudo-Abelian completion of the strict, full, \mathbb{Q} -linear triangulated subcategory of $\mathrm{DM}_c(S_\phi)$ of objects, which are isomorphic to images under π_σ of well-ramified Artin-Tate motives over S'_σ , for $\sigma \in \mathfrak{S}$ such that S'_σ is a stratum of $\pi^{-1}(S_\phi)$. This implies that, by reasoning as in [Wil17, Thm. 5.4], the claim will follow as soon as we prove that the objects of the latter form are *finite dimensional* in the sense of Kimura.

Choose a couple of strata $\pi_\sigma : S'_\sigma \rightarrow S_\phi$ as above and take a well-ramified finite étale morphism $q : D \rightarrow S'_\sigma$. The Stein factorization of the morphism $\pi_\sigma \circ q$ gives rise to a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{q} & S'_\sigma \\ p \downarrow & & \downarrow \pi_\sigma \\ \tilde{S}_\phi & \xrightarrow{r} & S_\phi \end{array}$$

where p is proper with connected fibres and r is finite étale. Moreover, the fiber of r over each point s of S_ϕ is in set-theoretic bijection with the set of connected components of the fiber of $\pi_\sigma \circ q$ over s . Thus, because of our assumption 3.2.4.(3) on the properties of π_σ , the degree of r is the same as the degree of q , say equal to d . We get a diagram

$$\begin{array}{ccccc} D & & & & \\ & \searrow \iota & & \searrow q & \\ & & \tilde{S}_\phi \times_{S_\phi} S'_\sigma & \xrightarrow{q'} & S'_\sigma \\ & \searrow p & \downarrow p' & & \downarrow \pi_\sigma \\ & & \tilde{S}_\phi & \xrightarrow{r} & S_\phi \end{array}$$

where q' is finite étale of degree d , so that ι has to be finite étale of degree 1, i.e. it embeds D as a connected component of $\tilde{S}_\phi \times_{S_\phi} S'_\sigma$. Call the latter scheme D' . Then, using proper base change and the fact that q' and r are finite étale, we get

$$h(D'/\tilde{S}_\phi) = p'_* \mathbb{1}_{D'} \simeq p'_* q'^* \mathbb{1}_{S'_\sigma} \simeq r^* \pi_{\sigma,*} \mathbb{1}_{S'_\sigma} = r^* h(S'_\sigma/S_\phi)$$

Since $h(S'_\sigma/S_\phi)$ belongs to the category of weight zero, smooth Tate motives over S_ϕ (again by Assumption 3.2.4.(3) and Rem. 3.2.5), we obtain that $h(D'/\tilde{S}_\phi)$ belongs to the category of weight zero, smooth Tate motives over \tilde{S}_ϕ . The motive $h(D/\tilde{S}_\phi)$, being a direct factor of $h(D'/\tilde{S}_\phi)$, belongs to the same category as well. Now $h(D/S_\phi)$ is isomorphic to the direct image of $h(D/\tilde{S}_\phi)$ under the finite étale morphism r , and as a consequence, it is actually a weight zero, smooth Artin-Tate motive over S_ϕ . As such, it is indeed finite dimensional (apply for example [Wil17, Prop. 5.8 (c)]). As the objects we were interested in are direct factors of objects of the form $h(D/S_\phi)$, we conclude. \square

Remark 3.2.10. (1) The proof of the above theorem actually uses a little less than Assumption 3.2.4: denoting by π_Z the restriction of π to Z , we only need the requirements of point (3) to be met for all $\phi \in \Phi_Z$ and $\sigma \in \mathfrak{S}$ such that S'_σ is a stratum of $\pi_Z^{-1}(S_\phi)$. Admitting Assumption 3.2.4 to be verified as it is, the proof shows that in fact, *the whole* category $\text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)$ is semi-primary.

(2) If we relax point (3) of Assumption 3.2.4 by asking that $h(S'_\sigma/S_\phi)$ be simply *finite dimensional*, the proof of the above theorem carries through, with the following adjustments. First, one exploits the commutative diagram coming from the Stein factorization and invokes proper base change and [Wil17, Prop. 5.8 (a)] in order to show that $h(D/\tilde{S}_\phi)$ is also finite dimensional. Then, one shows that the same holds for $h(D/S_\phi)$, by applying [Wil17, Prop. 5.8 (c)].

(3) The above theorem differs in the following way from the analogous Thm. 5.4 in [Wil17]. On the one hand, in order to deal with the gluing, we are forced to be more restrictive on the choice of possible morphisms $S' \rightarrow S$. In fact, we ask for regularity of the closure of the strata of S' , whereas in *loc. cit.*, it is only asked the weaker condition that for every immersion i_σ of a stratum S'_σ in the closure of a stratum, the functor $i_\sigma^!$ send the unit object to a Tate motive. Moreover, the morphisms π_σ in *loc. cit.* can belong to a more general class than the one considered here. On the other hand, our stronger restrictions are necessary because, for a fixed S' which fulfils our requirements, the categories that we glue along the strata of S' are more general than the ones of *loc. cit.*

Corollary 3.2.11. (*cfr.* [Wil17, Cor. 5.7]) *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.4. Then, the intermediate extension functor*

$$j_{!*} : \text{AT}_{\text{wr}}(\mathfrak{S}_U/U) \hookrightarrow \text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)^u$$

is defined (as in Def. 3.2.11).

Corollary 3.2.12. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.4. For each $\phi \in \Phi$, denote by $j_\phi : S_\phi \hookrightarrow \overline{S}_\phi$ the open immersion of a stratum in its closure, by $i_\phi : \overline{S}_\phi \hookrightarrow S$ the closed immersion of the closure of a stratum.*

Let M be an object of the category $\text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)$. Then, there exist a subset $\Phi' \subset \Phi$, objects N_ϕ in $\text{AT}_{\text{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$, $\phi \in \Phi'$, and a non-canonical isomorphism

$$M \simeq \bigoplus_{\phi \in \Phi'} i_{\phi,*} j_{\phi,!} N_\phi$$

Proof. By applying Prop. 3.1.4, we know that we have an isomorphism

$$M \simeq j_{!*}M_U \oplus i_*N$$

with N an object of $\text{AT}_{\text{wr}}(\mathfrak{S}_Z/Z)$. By proper base change and part (2) of Thm. 3.2.3, and the fact that pullback along open immersions sends weight-zero objects to weight-zero objects, we know that the pullback of N to any stratum S_ϕ which is open in Z belongs to $\text{AT}_{\text{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$. Thus, we can apply to it the functor $j_{\phi,!}^*$ (defined applying Cor. 3.2.11). The statement then follows by an iterated application of Prop. 3.1.4. \square

3.3. Compatibility with realizations and conservativity. . In this paragraph, we will fix a generic point $\text{Spec}(k) \rightarrow S$ of our base S , and we will make use of the two realization functors with target the categories $D_c^b(S_{k,\text{ét}}, \mathbb{Q}_\ell)$ and $D_c^b(S_k^{\text{an}}, \mathbb{Q})$, obtained by composition with base change through $\text{Spec}(k) \rightarrow S$ from the functors ρ_ℓ, ρ_B introduced in the “Notations and conventions” section. Whenever we employ one of these two functors, we will implicitly assume that the hypotheses on S and ℓ are satisfied. These functors will still be denoted by the same symbols.

Let us denote any of the two families of categories $D_c^b(S_{k,\text{ét}}, \mathbb{Q}_\ell)$ and $D_c^b(S_k^{\text{an}}, \mathbb{Q})$ by the same symbol $D_c^b(S_k)$. Both families of categories are equipped with a perverse t -structure, whose heart (the corresponding category of *perverse sheaves*) will be denoted $\text{Perv}_c(S_k)$ in both cases. We will then denote by

$$H^m : D_c^b(S_k) \rightarrow \text{Perv}_c(S_k)$$

the perverse cohomology functors, and by

$$j_{!*} : \text{Perv}_c(U_k) \rightarrow \text{Perv}_c(S_k)$$

the *intermediate extension* of perverse sheaves ([BBD82, Déf. 1.4.22]). The composition of the collection of the perverse cohomology functors with one of the realization functors will be called the corresponding *perverse cohomological realization* functor.

The following result gives the compatibility of the functor of Def. 3.2.11 (when available) with the realization functors:

Theorem 3.3.1. (cfr. [Wil17, Thm. 7.2]) *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.4. Denote by ρ any of the two realization functors ρ_ℓ or ρ_B . Then:*

(1) *for any integer m , the restriction of the composition*

$$H^m \circ \rho : \text{DM}_c(S, \mathbb{Q}) \rightarrow \text{Perv}_c(S_k)$$

to $\text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)$ factors over $\text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)^u$;

(2) *for any integer m , the diagram*

$$\begin{array}{ccc} \text{AT}_{\text{wr}}(\mathfrak{S}_U/U) & \xrightarrow{j_{!*}} & \text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)^u \\ \downarrow H^m \circ \rho & & \downarrow H^m \circ \rho \\ \text{Perv}_c(U_k) & \xrightarrow{j_{!*}} & \text{Perv}_c(S_k) \end{array}$$

commutes.

Proof. The same proof of [Wil17, Thm. 7.2] applies. Indeed, the only ingredient occurring in that proof, which has to be generalized, is *op. cit.*, Cor. 7.13: we

need to obtain an analogous statement in our situation, i.e. we need to show that the radical (cfr. Def. 3.1.1) of our category $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ is mapped to zero under the perverse cohomological realization. The proof of Thm. 3.2.9 tells us that over each stratum S_ϕ of S , the category $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ is actually contained in the category $\mathrm{AT}(S_\phi)$ of weight zero, smooth Artin-Tate motives over S_ϕ . Then, the same strategy of proof of *op. cit.*, Cor. 7.13, based on *op. cit.*, Thm. 7.12, shows that we can obtain the desired statement as soon as we prove the following claim: for any couple of finite étale morphisms

$$q_1 : D_1 \rightarrow X, \quad q_2 : D_2 \rightarrow X$$

the ideal

$$\mathrm{rad}_{\mathrm{CHM}(X)}(h_X(D_1), h_X(D_2))$$

consists of morphisms which are mapped to zero under the perverse cohomological realization. Now, by the same reasoning as in the proof of *loc. cit.*, Thm. 7.12, we may assume that X is the spectrum of a field k . But then, $h_X(D_1)$ and $h_X(D_2)$ are Artin motives over k , i.e. objects of a full *semisimple* subcategory of $\mathrm{CHM}(X) = \mathrm{CHM}(k)$. Hence, the radical in question is zero. \square

We end this section by discussing the conservativity of the restriction of the realization functors to the categories $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$.

Theorem 3.3.2. *Let $\pi : S' \rightarrow S$ be a proper morphism of good stratifications \mathfrak{S} and Φ , satisfying Assumption 3.2.4. Denote by ρ any of the two realization functors ρ_ℓ or ρ_B . Then, the restriction of ρ to $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$,*

$$\rho : \mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi) \rightarrow \mathrm{D}_c^b(S_k)$$

is conservative.

Proof. We will adopt the strategy of [Wil18, Thm. 4.3], which adapts to our setting and shows that conservativity of both realizations (ℓ -adic and Betti) can be deduced if the following properties hold¹⁹: (1) the weight structure on $\mathrm{DM}_{\mathrm{wr}}^{\mathrm{AT}}(\mathfrak{S}/\Phi)$ is bounded, (2) its heart $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ is semi-primary and pseudo-Abelian, (3) the restriction of ρ to $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ maps the radical to zero, and (4) zero is the only object of $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ mapped to zero by ρ . The first two properties have been verified before (Lemma 3.2.7 and Rmk. 3.2.10.(1)), and property (3) has been seen to hold in the course of the proof of Thm. 3.3.1. With this in our hands, we can imitate step by step the proof of *op. cit.*, Thm. 4.2 to show that property (4) is also verified and to conclude. In fact, to argue as in *loc. cit.* we only need the existence of the intermediate extension functor defined in Cor. 3.2.11, the fact that objects in $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ decompose as direct sums of intermediate extensions (Cor. 3.2.12), and the finite dimensional nature of the objects of the categories $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_{S_\phi}/S_\phi)$ (proof of Thm. 3.2.9). \square

¹⁹These are precisely all of the assumptions which are shown to be sufficient for conservativity in *op. cit.*, Thm. 2.10, except for one: *strictness* of any morphism in the image of the perverse cohomological realization, with respect to the *weight filtration* of the latter functor. Only the ℓ -adic realization is known to satisfy this last assumption. Building on this, Thm. 4.3 of *loc. cit.* then shows how to obtain conservativity for the Betti realization, too.

4. CORTI-HANAMURA DECOMPOSITION OF GENERAL QUADRICS

4.1. Corti-Hanamura decomposition of general quadrics. In this section we work over an algebraically closed field k of characteristic zero. Let V be a k -vector space of dimension n . A quadratic form $Q \in S^2V^\vee$ can be viewed as a symmetric map $Q : V \rightarrow V^\vee$. Its radical is the subspace $\text{Rad}(Q) = \ker(Q)$. It is known from the theory of quadratic forms that Q descends to a symmetric bilinear form \bar{Q} on the quotient space $\bar{V} = V/\text{Rad}(Q)$ (cf. [Cas78, 2.6]). The corank of Q is $n - r$, where $r = \dim(\text{Rad}(Q))$. The zero locus of Q defines a quadric $X \subset \mathbb{P}(V)$ which is smooth if and only if $r = 0$. If $r > 0$ its singular locus is the linear subspace $\Lambda = \mathbb{P}(\text{Rad}(Q))$, and X is a cone with vertex Λ over the smooth quadric $\bar{X} = V(\bar{Q}) \subset \mathbb{P}(\bar{V}) \cong \mathbb{P}_k^{n-r-1}$.

Let

$$\Delta_k = \{Q \in S^2V^\vee \mid \dim(\ker Q) \geq k\}$$

be the space of quadratic forms of corank $\geq k$. The standard desingularization of Δ_k is

$$\hat{\Delta}_k = \{(Q, F) \in S^2V^\vee \times G(k, V) \mid F \subset \ker Q\}.$$

For our purposes it is convenient to use a slightly different construction. We denote by $\text{Fl}(V) = \text{Fl}(1, \dots, n-1, V)$ the variety of complete flags

$$F_1 \subset \dots \subset F_{n-1} \subset V$$

where $\dim(F_i) = i$. Define

$$\tilde{\Delta}_k = \{(Q, F_\bullet) \in S^2V^\vee \times \text{Fl}(V) \mid F_k \subset \ker Q\}.$$

The fiber of the projection map $p_2 : \tilde{\Delta}_k \rightarrow \text{Fl}(V)$ over F_\bullet is the vector space $H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(F_k)}(2))$. Hence $\tilde{\Delta}_k$ is smooth. The advantage over the previous construction is that we have inclusions

$$\tilde{\Delta}_{k+1} \subset \tilde{\Delta}_k \subset \text{Fl}(V)$$

for all k .

We now discuss the analogue of these constructions in the relative case. Let S be a quasi-projective scheme over k . Let E be a rank n vector bundle over S , and let L be a line bundle over S . A quadratic form on E with values in L is a global section $q \in H^0(S, S^2E^\vee \otimes L)$, or equivalently a symmetric homomorphism

$$q : E \rightarrow E^\vee \otimes L.$$

Let $\rho : \mathbb{P}(E) \rightarrow S$ be the associated projective bundle with tautological line bundle $\xi_E = \mathcal{O}_{\mathbb{P}(E)}(1)$. Using the isomorphism

$$H^0(S, S^2E^\vee \otimes L) \cong H^0(\mathbb{P}(E), \xi_E^2 \otimes \rho^*L)$$

we can identify q with a global section (still denoted by q) of $\xi_E^2 \otimes \rho^*L$. The associated quadric bundle is $X = V(q) \subset \mathbb{P}(E)$. The fiber of $f : X \rightarrow S$ over $s \in S$ is the zero locus of $q(s) \in S^2E_s^\vee \otimes L_s$.

We write

$$\Delta_i(q) = \{s \in S \mid \text{corank } q(s) \geq i\}, \quad U_i = \Delta_i(q) \setminus \Delta_{i+1}(q).$$

The restriction of q to U_i defines a homomorphism of vector bundles

$$q_i : E_i \rightarrow E_i^\vee \otimes L_i$$

whose kernel $F_i = \ker(q_i)$ is a subbundle of E_i of rank i . As before, the quadratic form q_i descends to a quadratic form \bar{q}_i on the quotient $\bar{E}_i = E_i/F_i$. Geometrically this means that the subscheme $X_i = f^{-1}(U_i)$ is a relative cone over $\bar{X}_i = V(\bar{q}_i) \subset \mathbb{P}(\bar{E}_i)$ with vertex $\mathbb{P}(F_i)$ i.e., for every $s \in U_i$ the quadric $X_s = f^{-1}(s)$ is a cone with vertex $\mathbb{P}(F_{i,s})$ over $\bar{X}_s \subset \mathbb{P}(\bar{E}_{i,s})$.

Let $\text{Fl}(E)$ be the bundle of complete flags in E with projection map $\pi : \text{Fl}(E) \rightarrow S$. The vector bundle π^*E has a flag of universal subbundles S_i of rank i , $i = 1, \dots, n-1$. The composition of

$$\pi^*q : \pi^*E \rightarrow \pi^*E^\vee \otimes \pi^*L$$

and the inclusion $\lambda_i : S_i \rightarrow \pi^*E$ defines

$$\tilde{q}_i = \pi^*(q) \circ \lambda_i : S_i \rightarrow \pi^*E^\vee \otimes \pi^*L.$$

Define

$$\tilde{\Delta}_i(q) = V(\tilde{q}_i) \subset \text{Fl}(E).$$

Proposition 4.1.1. *If $E^\vee \otimes E^\vee \otimes L$ is generated by global sections and $q \in H^0(X, S^2E^\vee \otimes L)$ is general, then*

- (1) $\Delta_i(q)$ is empty or has the expected codimension $\binom{i+1}{2}$ and $\text{Sing}(\Delta_i(q)) = \Delta_{i+1}(q)$.
- (2) $\tilde{\Delta}_i(q)$ is smooth.

Proof. Note that if $E^\vee \otimes E^\vee \otimes L$ is generated by global sections, then the bundles $S^2E^\vee \otimes L$, $\pi^*(E^\vee \otimes E^\vee \otimes L)$ and $S_i^\vee \otimes \pi^*E^\vee \otimes \pi^*L$ are globally generated. Part 1 is proved by adapting the argument of [B91, 4.1] to the symmetric case; cf. [Ott95, 2.17]. Part 2 follows from Bertini's theorem. \square

Definition 4.1.2. We say that $X \rightarrow S$ is a *regular* quadric bundle if it satisfies the conditions of the Proposition 4.1.1.

A quadric bundle $f : X \rightarrow S$ admits a natural stratification by corank. Write $U_i = \Delta_i(q) \setminus \Delta_{i+1}(q)$, $X_i = f^{-1}(U_i)$. As the stratification $\Phi = \{U_i\}_{i \in I}$ does not verify Assumption 1, we have to pass to a suitable base change and verify Assumption 2. Define $S' = \text{Fl}(E)$ and consider the stratification \mathcal{U}' given by $U'_i = \tilde{\Delta}_i(q) \setminus \tilde{\Delta}_{i+1}(q)$. Write $X' = X \times_S S'$ and $X'_i = X_i \times_{U_i} U'_i$.

Lemma 4.1.3. *Let $X \rightarrow S$ be a regular quadric bundle. Then the stratification $\mathfrak{S} = \{U'_i\}_{i \in I}$ satisfies Assumption 3.2.4.*

Proof. Proposition 4.1.1 implies that the stratification \mathfrak{S} satisfies conditions (1) and (2) of Assumption 3.2.4. The definition of $\tilde{\Delta}_i(q)$ shows that the fiber of $\pi_i : \tilde{\Delta}_i(q) \rightarrow \Delta_i(q)$ over $s \in \Delta_i(q)$ is

$$\{(W_\bullet) \in \text{Fl}(E_s) \mid W_i \subset \ker q(s)\}.$$

If $s \in U_i$ then $\ker(q(s))$ has dimension i , hence the fiber of the induced map $U'_i \rightarrow U_i$ over $s \in U_i$ is

$$\{(W_\bullet) \in \text{Fl}(E_s) \mid W_i = \ker(q(s))\} \cong \text{Fl}(\ker(q(s)) \times \text{Fl}(i+1, \dots, n, E_s)).$$

Over U_i we have an injective homomorphism of flag bundles

$$\text{Fl}(F_i) \times \text{Fl}(i+1, \dots, n; E_i) \rightarrow \text{Fl}(E_i)$$

whose image is U'_i . Hence $U'_i \rightarrow U_i$ is a relative homogeneous space and $h_{U_i}(U'_i)$ is a relative Tate motive. This implies that the stratification \mathfrak{S} also satisfies condition (3) of Assumption 3.2.4. \square

Theorem 4.1.4. *Let $X \rightarrow S$ be a regular quadric bundle. Then $h_S(X) \in \text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)$.*

Proof. We consider the stratifications Φ and \mathfrak{S} introduced above and write $X_i = f^{-1}(U_i)$, $X'_i = X_i \times_{U_i} U'_i$. As we have seen before, X_i is a relative cone over \overline{X}_i with vertex $\mathbb{P}(F_i)$. The complement $V_i = X_i \setminus \mathbb{P}(F_i)$ is a locally trivial fiber bundle over \overline{X}_i with affine fibers.²⁰ A similar result holds after base change via $\pi_i : U'_i \rightarrow U_i$: X'_i contains the projective bundle $\mathbb{P}(F'_i)$, where $F'_i = \pi_i^*(F_i)$, and the projection of the complement $V'_i = X'_i \setminus \mathbb{P}(F'_i)$ to U'_i factors as $V'_i \xrightarrow{\rho_i} \overline{X}'_i \xrightarrow{\sigma_i} U'_i$ where ρ_i is a locally trivial fiber bundle with affine fibers and σ_i is a smooth quadric bundle. Hence $h_{U'_i}^c(V'_i) \cong h_{U'_i}(\overline{X}'_i)(-i)[-2i]$. As we have seen in Corollary 2.4.13, the motive $h_{U'_i}(\overline{X}'_i)$ is either a relative Tate motive or a relative Artin-Tate motive associated to a double étale covering $Z(\overline{X}'_i) \rightarrow U'_i$. The latter case arises if $n - i$ is even, say $n - i = 2m$. In this case the double covering $Z(\overline{X}'_i) \rightarrow U'_i$ comes from the Stein factorisation of the relative Fano scheme of m -planes $F_m(\overline{X}'_i/U'_i) \rightarrow U'_i$. This map is ramified over the boundary $\widetilde{\Delta}_{i+1}(q)$ since the quadrics over the boundary become singular, and it is known that the Fano scheme $F_m(Q)$ of a singular quadric Q of dimension $2m$ is connected [Ter88, Prop. 4.1.2]. Hence $Z(\overline{X}'_i) \rightarrow U'_i$ is well-ramified. Since $h_{U'_i}(\mathbb{P}(F'_i))$ is a relative Tate motive, the localisation triangle

$$h_{U'_i}^c(V'_i) \rightarrow h_{U'_i}(X'_i) \rightarrow h_{U'_i}(\mathbb{P}(F'_i)) \rightarrow h_{U'_i}^c(V'_i)[1]$$

then shows that $h_{U'_i}(X'_i) \in \text{DM}_{\text{sm}}^{\text{AwrT}}(U'_i, \mathbb{Q})$ for all i . As the map $h_{U'_i}(\mathbb{P}(F'_i)) \rightarrow h_{U'_i}^c(V'_i)[1]$ is zero for weight reasons, it follows that $h_{U'_i}(X'_i)$ has weight zero. Hence $h_{S'}(X') \in \text{DM}_{\text{wr}}^{\text{AT}}(\mathfrak{S})$ thanks to part (2) of Thm. 3.2.3 and has weight zero by the gluing property of motivic weights.

As $S' = \text{Fl}(E) \xrightarrow{\pi} S$ is a relative homogeneous space we have

$$\pi_* \mathbb{1}_{S'} \cong \mathbb{1}_S \oplus \left(\bigoplus_{i:n_i>0} \mathbb{1}_S(-n_i)[-2n_i] \right).$$

By the projection formula we obtain

$$\pi_* h_{S'}(X') = \pi_* \pi^* h_S(X) \cong h_S(X) \oplus \left(\bigoplus_{i:n_i>0} h_S(X)(-n_i)[-2n_i] \right).$$

Hence $h_S(X) \in \text{DM}_{\text{wr}}^{\text{AT}}(\mathfrak{S}/\Phi)$. Since proper morphisms respect weights and $h_{S'}(X')$ is of weight zero, $\pi_* h_{S'}(X')$ is of weight zero, as well as any of its direct factors. So $h_S(X) \in \text{AT}_{\text{wr}}(\mathfrak{S}/\Phi)$. \square

Corollary 4.1.5. *A regular quadric bundle $X \rightarrow S$ admits a CH-decomposition (see Def. 1).*

²⁰In fact, if $\pi_i : \mathbb{P}(E_i) \rightarrow U_i$ is the projection map and if one chooses local trivialisations $F_i|_U \cong W \otimes \mathcal{O}_U$, $E_i|_U \cong V \otimes \mathcal{O}_U$ over an open subset $U \subset U_i$ then $V_i \cap \pi_i^{-1}(U)$ is the total space of the vector bundle $\mathcal{O}_U(-1) \oplus W \otimes \mathcal{O}_U$; cf. [EH16, 9.3.2]

Proof. By Corollary 3.2.12 every object M of the category $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}/\Phi)$ admits a decomposition

$$M \simeq \bigoplus_{\phi \in \Phi'} i_{\phi,*} j_{\phi,!} N_{\phi}$$

with N_{ϕ} in $\mathrm{AT}_{\mathrm{wr}}(\mathfrak{S}_{S_{\phi}}/S_{\phi})$. The proof of Theorem 3.2.9 shows that $N_{\phi} \in \mathrm{DM}_{\mathrm{sm}}^{\mathrm{AT}}(S_{\phi}, \mathbb{Q})$. Using Theorem 2.3.3 we obtain a decomposition

$$N_{\phi} \simeq \bigoplus_{i, \phi \in I_{\phi}} \rho!(V_{i,\phi})(n_{i,\phi})[2n_{i,\phi}]$$

with $V_{i,\phi}$ a simple Artin representation of $\pi_1(S_{\phi})$. Taking $M = h_S(X)$ we get

$$h_S(X) \simeq \bigoplus_{i, \phi} i_{\phi,*} j_{\phi,!} (\rho!(V_{i,\phi})(n_{i,\phi})[2n_{i,\phi}])$$

which is a decomposition of $h_S(X)$ into simple objects of weight zero. Applying the realisation functor ρ_B and using Thm. 3.3.1, we obtain a decomposition of $Rf_*\mathbb{Q}_X$ into a sum of simple objects in $D_c^b(S, \mathbb{Q})$. The decomposition theorem of Beilinson-Bernstein-Deligne gives isomorphisms

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_k {}^p R^k f_*\mathbb{Q}_X[-k] \simeq \bigoplus_{k,\lambda} i_{\lambda,*} j_{\lambda,!} (L_{k,\lambda})[-k]$$

where $L_{k,\lambda}$ is a local system on S_{λ} . Since the simple objects appearing in this decomposition are unique, we conclude that they are realisations of Chow motives. Hence $X \rightarrow S$ admits a Corti-Hanamura decomposition. \square

Remark 4.1.6. Our proof gives a more precise statement: $h_S(X)$ is a direct sum of motivic intermediate extensions of Artin-Tate motives of degree at most 2, hence $Rf_*\mathbb{Q}_X$ decomposes as a sum of intersection complexes of local systems whose monodromy is either trivial or $\mathbb{Z}/2\mathbb{Z}$. This has been observed in the example of regular conic bundles over a surface [NS09]. In this case

$$Rf_*\mathbb{Q}_X \simeq \mathbb{Q}_S \oplus \mathbb{Q}_S[-2] \oplus i_*L[-1]$$

where L is a local system on the smooth discriminant curve $\Delta \subset S$. The underlying motivic decomposition is

$$h_S(X) \simeq \mathbb{1}_S \oplus \mathbb{1}_S(-1)[-2] \oplus \mathrm{Prym}(\tilde{\Delta}/\Delta)(-1)[-2]$$

where $\mathrm{Prym}(\tilde{\Delta}/\Delta)$ is the Prym motive, an Artin-Tate motive of degree 2.

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