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# INVARIANCE PRINCIPLES FOR CLOCKS

MARIA-EMILIA CABALLERO<sup>1</sup> AND ALAIN ROUAULT<sup>2</sup>

ABSTRACT. We show an invariance principle for rescaled clocks of positive semi-stable Markov processes, proving a conjecture presented in Remark 4 in Demni, Rouault, Zani [11], 2015.

## 1. INTRODUCTION - PSSMP AND OU

For  $\alpha > 0$ , a positive self-similar Markov process (pssMp) of index  $\alpha$ , is a  $[0, \infty)$ -valued strong Markov process  $(X, \mathbb{Q}_a)$ ,  $a > 0$  with càdlàg paths, fulfilling the scaling property

$$(\{bX_{b^{-\alpha}t}, t \geq 0\}, \mathbb{Q}_a) \stackrel{(d)}{=} (\{X_t, t \geq 0\}, \mathbb{Q}_{ba}) \quad (1.1)$$

for every  $a, b > 0$ .

The Lamperti transformation (see [17]) connects these processes to Lévy processes. Let us summarize this connection. We will follow the notations of [11].

Any pssMp  $X$  which never reaches the boundary state 0 may be expressed as the exponential of a Lévy process not drifting to  $-\infty$ , time changed by the inverse of its exponential functional. More formally, if  $(X, (\mathbb{Q}_a)_{a>0})$  is a pssMp of index  $\alpha$  which never reaches 0, set

$$T^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}, \quad (t \geq 0) \quad (1.2)$$

and let  $A^{(X)}$  be its inverse, defined by

$$A^{(X)}(t) = \inf\{u \geq 0 : T^{(X)}(u) \geq t\}, \quad (t \geq 0), \quad (1.3)$$

and let  $\xi$  be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0, \quad (t \geq 0). \quad (1.4)$$

Then, for every  $a > 0$ , the distribution of  $(\xi_t, t \geq 0)$  under  $\mathbb{Q}_a$  does not depend on  $a$  and is the distribution of a Lévy process starting from 0.

Conversely, let  $(\xi_t, t \geq 0)$  be a Lévy process starting from 0 and let  $\mathbb{P}$  and  $\mathbb{E}$  denote the underlying probability and expectation, respectively.

Fix  $\alpha > 0$ . Set

$$\mathcal{A}^{(\xi)}(t) = \int_0^t e^{\alpha \xi_s} ds. \quad (1.5)$$

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Here, we assume that  $\xi$  does not drift to  $-\infty$  i.e. satisfies,  $\limsup_{t \uparrow \infty} \xi_t = \infty$ . The inverse process  $\tau^{(\xi)}$  of  $\mathcal{A}^{(\xi)}$  is

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathcal{A}^{(\xi)}(u) \geq t\}, \quad (t \geq 0). \quad (1.6)$$

For every  $a > 0$ , let  $\mathbb{Q}_a$  be the law under  $\mathbb{P}$  of the time-changed process

$$X_t = a \exp \xi_{\tau^{(\xi)}(ta^{-\alpha})}, \quad (t \geq 0), \quad (1.7)$$

then  $(X, (\mathbb{Q}_a)_{a>0})$  is a pssMp of index  $\alpha$  which never reaches 0 and we have the fundamental relation

$$\tau^{(\xi)}(t) = T^{(X)}(tX_0^\alpha). \quad (1.8)$$

The process  $(T^{(X)}(t), t \geq 0)$  is called the clock associated with the pssMp  $X$ . Some years ago, a particular interest was dedicated to the asymptotic behaviour of this process in long time ([24], [11]). To recall the Law of Large Numbers we need some notations.

Let  $\psi$  be the Laplace exponent of  $\xi$ , defined by

$$\mathbb{E} \exp(m\xi_t) = \exp(t\psi(m)), \quad (1.9)$$

and set  $\text{dom } \psi = \{m : \psi(m) < \infty\}$ . We assume

$$0 \in \text{int dom } \psi \text{ and } p := \mathbb{E}\xi_1 = \psi'(0) > 0. \quad (1.10)$$

Starting from  $a > 0$  such a process  $X_t$  never hits 0. Nevertheless, a probability measure  $\mathbb{Q}_0$  can be obtained as the weak limit of  $\mathbb{Q}_a$  when  $a \downarrow 0$  and under  $\mathbb{Q}_0$  the canonical process has the same transition semigroup as the one associated with  $(X; (\mathbb{Q}_a)_{a>0})$ . (see [3], [2], [7], [10], [20]). A sufficient condition is (1.10) plus

$$\text{the support of } \xi \text{ is not arithmetic.} \quad (1.11)$$

This latter measure is an entrance law for the semigroup  $p_t f(x) = \mathbb{E}_x f(X_t)$  and satisfies

$$\mathbb{Q}_0(f(X_t)) = \frac{1}{\alpha \mathbb{E}\xi_1} \mathbb{E} \left[ I_\infty^{-1} f \left( (t/I_\infty)^{1/\alpha} \right) \right] \quad (1.12)$$

where

$$I_\infty = \int_0^\infty e^{-\alpha \xi_s} ds.$$

The Law of Large Numbers is the following.

**Theorem 1.1** ([11] Th.1). *Assume (1.10) and (1.11). As  $t \rightarrow \infty$ ,*

(1) *For every  $a > 0$ ,*

$$\frac{1}{\log t} \int_0^t \frac{ds}{X_s^\alpha} \rightarrow (\alpha p)^{-1}, \quad \mathbb{Q}_a - a.s. \quad (1.13)$$

(2)

$$\frac{1}{\log t} \int_1^t \frac{ds}{X_s^\alpha} \rightarrow (\alpha p)^{-1}, \quad \mathbb{Q}_0 - a.s. \quad (1.14)$$

In [11], the authors go on with the study of large deviations. A CLT for Bessel clocks was previously proved in Exercise X.3.20 in [22] and extended to an invariance principle in ([24]), using stochastic analysis. Following the result on LDP for clocks, a general CLT is conjectured in [11] Remark 4. In Section 2, we state an invariance principle (Functional CLT) for this kind of processes. proved in Section 3 and illustrated by examples in Section 4.

The main tool is the introduction of an ergodic process with nice asymptotic properties. With a pssMp  $(X_t)$  of index  $\alpha$ , it is classical to associate a process called generalized Ornstein-Uhlenbeck (OU) by

$$U(t) = e^{-t/\alpha} X(e^t) \quad (1.15)$$

which is strictly stationary, Markovian, ergodic under  $\mathbb{Q}_0$ , and its invariant measure is the law of  $X_1$  under  $\mathbb{Q}_0$  i.e.

$$\mu^U(f) = \frac{1}{\alpha \mathbb{E}\xi_1} \mathbb{E} \left[ I_\infty^{-1} f \left( I_\infty^{-1/\alpha} \right) \right]. \quad (1.16)$$

The infinitesimal generators  $L^X$  and  $L^\xi$  are related by

$$L^X h(x) = x^{-\alpha} L^\xi (h \circ \exp)(\log x) \quad (1.17)$$

(see [9], [20]), and the generator of  $U$  is

$$L^U h(x) = L^X h(x) - \frac{x}{\alpha} h'(x). \quad (1.18)$$

Two examples of function  $f$  are simple and particularly useful.

If  $m \in \text{dom } \psi$  the function  $x \mapsto \exp(mx)$  is in the domain of  $L^\xi$ , so that

$$f_m : x \mapsto x^m,$$

is in the domain of  $L^X$  and we have

$$L^X f_m(x) = \psi(m) f_{m-\alpha}, \quad (1.19)$$

so that

$$L^U f_m = \psi(m) f_{m-\alpha} - \alpha m f_m. \quad (1.20)$$

The function  $\mathfrak{i}(x) = x$  is in the domain of  $L^\xi$  with  $L^\xi \mathfrak{i} = p$ . Formula (1.17) tells us that the function  $\varphi(x) = \log x$  is in the domain of  $L^X$  and

$$L^X \varphi(x) = \frac{1}{x^\alpha} L^\xi \mathfrak{i}(\log x) = \frac{p}{x}, \quad (1.21)$$

and owing to (1.18)

$$L^U \varphi(x) = \frac{p}{x^\alpha} - \frac{1}{\alpha}. \quad (1.22)$$

*Remark 1.* There is a variant of  $U$ , defined by

$$\tilde{U}(t) = e^{-t/\alpha} X(e^t - 1)$$

which shares the same transition with  $U$  and begins at  $X(0)$  at  $t = 0$  ([1]).

*Remark 2.* If  $(X, (\mathbb{Q}_a)_{a>0})$  is a pssMa of index  $\alpha$ , then the process  $Y = (X^\alpha, (\mathbb{Q}_{a^\alpha})_{a>0})$ , is a pssMp of index 1. Conversely if  $(Y, (\mathbb{Q}_a)_{a>0})$  is a pssMp of index 1 then, for any  $\alpha > 0$ , the process  $(X = Y^{1/\alpha}, (\mathbb{Q}_{a^{1/\alpha}})_{a>0})$  is a pssMp of index  $\alpha$ .

## 2. MAIN RESULT

The following theorem states an invariance principle ( or functional central limit theorem FCLT) under two regimes,  $\mathbb{Q}_0$  and (under conditions)  $\mathbb{Q}_a, a > 0$ .

**Theorem 2.1.** (1) Under  $\mathbb{Q}_0$ , as  $T \rightarrow \infty$ ,

$$\left( (\log T)^{-1/2} \left( \int_1^{T^t} \frac{dr}{X^\alpha(r)} - \frac{t \log T}{\alpha p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \quad (2.1)$$

where

$$v^2 = \frac{\sigma^2}{\alpha p^3}, \quad \sigma^2 = \psi''(0). \quad (2.2)$$

(2) If one of the following conditions

(a) there exists  $m > 0$  such that  $\psi(m) < 0$

(b) there exists  $m > \alpha$  such that  $\psi(m) > 0$ ,

is satisfied, then for every  $a > 0$ , under  $\mathbb{Q}_a$ , as  $T \rightarrow \infty$ , (2.1) holds true.

*Remark 3.* When the above criterion is not checked, the invariance principle holds true under  $\mathbb{Q}_a$  for almost every  $a$  (see Theorem 2.8 in [5]).

## 3. PROOF OF THE MAIN RESULT

**3.1. FCLT under the invariant measure.** Observe that, owing to (1.15)

$$\int_1^{T^t} \frac{ds}{(X(s))^\alpha} - \frac{t \log T}{\alpha p} = \int_0^{t \log T} \left( \frac{1}{U(r)^\alpha} - \frac{1}{\alpha p} \right) dr \quad (3.1)$$

which reduces the problem to an invariance principle for a functional of the process  $U$ . We will use a classical result on weak convergence.

**Theorem 3.1** (Bhattacharya Th. 2.1 [5]). *Let  $(Y_t)$  be a measurable stationary ergodic process with an invariant probability  $\pi$ . If  $f$  is in the range  $\hat{A}$  of the extended generator of  $(Y_t)$ , then, as  $n \rightarrow \infty$*

$$\left( n^{-1/2} \int_0^{nt} f(Y_s) ds \right)_{t \geq 0} \Rightarrow (\rho W(t))_{t \geq 0} \quad (3.2)$$

where

$$\rho^2 = -2 \int f(x)g(x)\pi(dx), \quad \hat{A}g = f. \quad (3.3)$$

Owing to (1.22) we see that the pair  $(f, g)$  with

$$f(x) = \frac{1}{x^\alpha} - \frac{1}{\alpha p}, \quad g(x) = \frac{1}{p} \log x, \quad (3.4)$$

satisfies

$$L^U g = f. \quad (3.5)$$

Now the convergence in distribution comes from Th. 3.1 and formula (3.1), with

$$v^2 = -2 \int f(x)g(x)\mu^U(dx) \quad (3.6)$$

where  $\mu^U$  is the invariant distribution of the process  $U$ .

It remains to compute the variance  $v^2$ . From (1.16) and (3.6) we have

$$\begin{aligned} v^2 &= -2(\alpha p)^{-1} \mathbb{E} (I_\infty^{-1} f(I_\infty^{-1}) g(I_\infty^{-1})) \\ &= 2(\alpha p)^{-1} \mathbb{E} (I_\infty^{-1} (I_\infty - (\alpha p)^{-1}) p^{-1} \log I_\infty) \\ &= 2(\alpha p)^{-2} \mathbb{E} (\log I_\infty - (\alpha p)^{-1} I_\infty^{-1} \log I_\infty) . \end{aligned} \quad (3.7)$$

The Mellin transform

$$M(z) = \mathbb{E}(I_\infty^{-z})$$

may play a prominent role, since

$$\mathbb{E}(\log I_\infty^{-1}) = M'(0) ; \quad \mathbb{E} (I_\infty^{-1} \log I_\infty^{-1}) = M'(1) ,$$

so that

$$v^2 = 2(\alpha p)^{-2} (-M'(0) + (\alpha p)^{-1} M'(1)) . \quad (3.8)$$

We only know that  $M$  satisfies the recurrence equation

$$\psi(\alpha z) M(z) = z M(z+1) . \quad (3.9)$$

(see [4] Th. 2 i) and Th. 3. and apply scaling). Differentiating twice the above formula gives

$$\alpha^2 \psi''(\alpha z) M(z) + 2\alpha \psi'(\alpha z) M'(z) + \psi(\alpha z) M''(z) = 2M'(z+1) + z M''(z+1)$$

which, for  $z = 0$ , gives

$$\alpha^2 \psi''(0) + 2\alpha p M'(0) = 2M'(1)$$

and then

$$v^2 = \frac{\psi''(0)}{\alpha p^3} = \frac{\sigma^2}{\alpha p^3} . \quad (3.10)$$

This fits exactly with the conjecture in [11] Rem. 4.

**3.2. Quenched FCLT.** We want to show the FCLT under  $\mathbb{Q}_a$  for every  $a > 0$ .

**Theorem 3.2** (Bhattacharya Th. 2.6 [5]). *Let  $(p_t)_{t \geq 0}$  be the semigroup of a Markov process  $(Y_t)$ . Assume that for every  $x$ , as  $t \rightarrow \infty$*

$$\|p_t(x; \cdot) - \mu\|_{var} \rightarrow 0 .$$

*Then, with the notations of Theorem 3.1, the convergence (3.2) holds under  $P_x$  for every  $x$ .*

Such a process is called positive recurrent. Moreover, the exponential ergodicity is defined by the existence of a finite function  $h$  and a constant  $\gamma$  such that for every  $x$

$$\|p_t(x, \cdot) - \pi\|_{var} \leq h(x) e^{-\gamma t} .$$

Let us first examine the possibility of a general criterion on the exponent  $\psi$  such that the assumptions of the latter theorem are fulfilled. A sufficient condition is given by the so-called Forster-Lyapounov drift criterion, due to [19]).

**Theorem 3.3** (Wang Th. 2.1 [23]). *Let  $L$  the generator of a Markov process  $Y_t$ ,*

(1) *If there exists a continuous function satisfying*

$$\lim_{|x| \rightarrow \infty} f(x) = \infty \quad (3.11)$$

*and constants  $K > 0, C > 0, D \in (-\infty, \infty)$  such that*

$$Lf \leq -C + D1_{[-K, K]} \quad (3.12)$$

*then the process  $(Y_t)$  is positively recurrent.*

(2) *If there exists a continuous function satisfying (3.11) and constants  $K > 0, C > 0, D \in (-\infty, \infty)$  such that*

$$Lf \leq -Cf + D1_{[-K, K]} \quad (3.13)$$

*then the process  $(Y_t)$  is exponentially ergodic.*

We want to check these criteria for our models, using the function  $f_m$  and (1.20).

- (1) When  $0 < m < \alpha$  and  $\psi(m) > 0$ ,  $f_{m-\alpha}$  is not bounded in the neighbouring of 0, and then is not convenient.
- (2) Let us look for  $K, C, D$  in the other cases. For every  $C \in (0, \alpha m)$ , let us define

$$h_m = L^U f_m + C f_m = \psi(m) f_{m-\alpha} + (C - \alpha m) f_m.$$

- (a) If  $\psi(m) < 0$ , then  $h_m \leq 0$  and then for  $D = 0$ , (3.13) holds true for every  $K$ .
- (b) If  $\psi(m) > 0$  and  $m > \alpha$ , then  $h_m$  is increasing for

$$0 < x < x_{\max} = \left( \frac{(m - \alpha)\psi(m)}{m(m - \alpha C)} \right)^{1/\alpha}$$

and decreasing after. It is 0 for  $x = x_0 = (\psi(m)/(m - \alpha C))^{1/\alpha} > x_m$ . Then choose  $K = \psi(m)/(m - C)$  and  $D = h_m(x_m)$  and (3.13) holds true.

#### 4. EXAMPLES

In this section, we consider examples taken from [11].

When the function  $\psi$  is rational, the distribution of  $I_\infty$  may be found in [14]. Let us notice that in all these examples we compute the variance using the elementary formula

$$\psi''(0) = 2 \frac{d}{dm} \frac{\psi(m)}{m} \Big|_{m=0}.$$

**4.1. Brownian motion with drift.** This is also the Cox Ingersoll Ross model. Let us consider the Lévy process

$$\xi_t = 2B_t + 2\nu t,$$

where  $B_t$  is the standard linear Brownian motion and  $\nu > 0$ . In this case,  $X_t$  is the squared Bessel process of dimension  $d = 2(1 + \nu)$ . Its index is 1 and it is the only continuous pssMp (of index 1). We have

$$\psi(m) = 2m(m + \nu), \quad p = 2\nu, \quad \sigma^2 = 4, \quad v^2 = (4\nu^3)^{-1},$$

Condition (3.13) of Th. 3.3 is satisfied (see(2) b) above), we have exponential ergodicity, the FCLT holds under  $\mathbb{Q}_a$  for every  $a \geq 0$ .

The invariant measure for  $U$  is easy to determinate, since

$$I_\infty^{-1} \stackrel{(d)}{=} 2\mathcal{Z}_\nu$$

where  $\mathcal{Z}_\nu$  is gamma with parameter  $\nu$  (see [3] (6)), so that

$$\mathbb{Q}_0(\varphi) = \frac{1}{\mathbb{E}\xi_1} \int_0^\infty y\varphi(y) \frac{y^{\nu-1}}{2^\nu} e^{-y/2} dy$$

which means that the invariant measure is the distribution of  $2\mathcal{Z}_{\nu+1}$ .

*Remark 4.* Actually it is proved in [18] Rem. 1.2 (2) that  $U$  is exponentially ergodic but not strongly ergodic, where strong ergodicity is defined by the existence of  $\gamma > 0$  such that

$$\sup_x \|P_t(x, \cdot) - \pi\|_{var} \leq e^{-\gamma t}.$$

**4.2. Poissonian examples.** Let  $\text{Pois}(a, b)_t$  be the compound Poisson process of parameter  $a$  whose jumps are exponential of parameter  $b$ . We will consider three models :  $\xi_t = dt + \text{Pois}(a, b)_t$  with  $d > 0$ ,  $\xi_t = -t + \text{Pois}(a, b)_t$  and  $\xi_t = t - \text{Pois}(a, b)_t$ .

4.2.1.  $\xi_t = dt + \text{Pois}(a, b)_t$ .

$$\begin{aligned} \psi(m) &= m \left( d + \frac{a}{b-m} \right) \quad (m \in (-\infty, b)), \\ p &= d + \frac{a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab^3}{(a+db)^3}. \end{aligned}$$

4.2.2.  $\xi_t = -t + \text{Pois}(a, b)_t$  with  $b < a$ .

$$\begin{aligned} \psi(m) &= m \left( -1 + \frac{a}{b-m} \right) \quad (m \leq b), \\ p &= \frac{a-b}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(a-b)^3}. \end{aligned}$$

When  $\delta = d > 0$ ,  $I_\infty \stackrel{(d)}{=} \alpha^{-1} B(1+b, a\alpha^{-1})$  where  $B(u, v)$  is the Beta distribution of parameters  $(u, v)$  (see [13] Th. 2.1 i)). The invariant measure is then the distribution of  $W^{-1}$  where  $W \stackrel{(d)}{=} \alpha^{-1} B(b, a\alpha^{-1})$ .

When  $\delta = -1$ ,  $I_\infty \stackrel{(d)}{=} B_2(1+b, a-b)$  (see [13] Th. 2.1 j)), where  $B_2(u, v)$  is the Beta distribution of the second order of parameters  $(u, v)$ . The invariant measure is then the  $B_2(a-b+1, b)$  distribution.

4.2.3.  $\xi_t = t - \text{Pois}(a, b)_t$  with  $b > a$ . It is the so called spectrally negative saw-tooth process.

$$\begin{aligned} \psi(m) &= m \left( 1 - \frac{a}{b+m} \right), \quad (m \in (-b, \infty)), \\ p &= \frac{b-a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(b-a)^3}. \end{aligned}$$

We have  $I_\infty^{-1} \stackrel{(d)}{=} B(b-a, a)$  (see [14] Th. 1), so that the invariant measure is  $B(b-a+1, a)$ .

For examples 4.2.1 and 4.2.2,  $f_m$  is in the domain of the generator iff  $m < b$ ; for example 4.2.3,  $f_m$  is always in the domain.



Looking at our above criterion, we see that we have the exponential ergodicity in the first two cases when  $b > 1$ , and completely for the third one.

**4.3. Spectrally negative process conditioned to stay positive.** For  $\alpha \in (1, 2)$ , let  $X^\uparrow$  be the spectrally  $\alpha$ -stable process conditioned to stay positive as defined in [6] Sect. 3.2. and [21] Sect. 3. Its corresponding Lévy process has Laplace exponent

$$\psi(m) = \frac{\Gamma(m + \alpha)}{\Gamma(m)}, \quad (m \in (-\alpha, \infty)).$$

We have

$$p = \Gamma(\alpha), \quad \sigma^2 = 2(\Gamma'(\alpha) + \gamma\Gamma(\alpha))$$

where  $\gamma = -\Gamma'(1)$  is the Euler constant, and then

$$v^2 = 2 \frac{\Gamma'(\alpha) + \gamma\Gamma(\alpha)}{\alpha\Gamma(\alpha)^3}.$$

Using (3.9) one sees directly that  $M(z) = \Gamma(\alpha z + 1)/\Gamma(z + 1)$ , so that  $I_\infty \stackrel{(d)}{=} S_{1/\alpha}(1)$ , the stable subordinator of index  $1/\alpha$  evaluated at 1.

Condition (2) b) of Th. 2.1 is satisfied.

**4.4. Hypergeometric stable process.** The modulus of a Cauchy process in  $\mathbb{R}^d$  for  $d > 1$  is a 1-pssMp with infinite lifetime. The associated Lévy process is a particular case of hypergeometric stable process of index  $\alpha$  as defined in [8], with  $\alpha < d$ . The characteristic exponent given therein by Th. 7 yields the Laplace exponent :

$$\psi(m) = -2^\alpha \frac{\Gamma((-m + \alpha)/2)}{\Gamma(-m/2)} \frac{\Gamma((m + d)/2)}{\Gamma((m + d - \alpha)/2)}, \quad (m \in (-d, \alpha)).$$

We have

$$p = 2^{\alpha-1} \frac{\Gamma(\alpha/2)\Gamma(d/2)}{\Gamma(d - \alpha)/2}, \quad \sigma^2 = p[1 - \gamma - \Psi((d - \alpha)/2) - \Psi(\alpha/2)],$$

where  $\Psi$  is the Digamma function.

The distribution of the limiting variable  $I_\infty$  is studied in [15] and [14].

Condition (2) b) of Th. 2.1 is never satisfied. Condition (2) a) can be satisfied if  $\alpha > 2$ , taking  $m \in (2, \min(\alpha, 4))$  since  $\Gamma(-m/2) > 0$  hence  $\psi(m) < 0$ .

**4.5. Continuous State branching process with immigration (CBI).** Let  $\kappa \in [0, 1)$  and  $\delta > \kappa/(\kappa + 1)$ . Let  $X$  be the continuous state branching process with immigration ([16] Sec. 13.5) whose branching mechanism is

$$\phi(\lambda) = \frac{1}{\kappa} \lambda^{\kappa+1}$$

and immigration mechanism is

$$\chi(\lambda) = \delta\phi'(\lambda).$$

We have the representation

$$\phi(\lambda) = \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)\mu(dz), \quad \mu(dz) = \frac{\kappa + 1}{\Gamma(1 - \kappa)} \frac{dz}{z^{\kappa+2}}.$$

This process is self-similar of index  $\kappa$  (see [21], lemma 4.8)<sup>1</sup> and the corresponding Laplace exponent is

$$\psi(m) = c(\kappa - (\kappa + 1)\delta - m) \frac{\Gamma(-m + \kappa)}{\Gamma(-m)}, \quad (m \in (-\infty, \kappa))$$

and

$$\begin{aligned} p &= c((\kappa + 1)\delta - \kappa)\Gamma(\kappa) > 0, \quad \sigma^2 = c(\Gamma(\kappa) + (\kappa - (\kappa + 1)\delta)(\Gamma'(\kappa) + \gamma\Gamma(\kappa))) \\ v^2 &= \frac{\sigma^2}{\kappa p^3}. \end{aligned} \quad (4.1)$$

We can then apply Th. 2.1(1) and conclude that under  $\mathbb{Q}_0$ , as  $T \rightarrow \infty$

$$\left( (\log T)^{-1/2} \left( \int_1^{T^t} \frac{dr}{X^{\kappa}(r)} - \frac{t \log T}{\kappa p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \quad (4.2)$$

The entrance law is given in Remark 4.9 (2) in [21]. Let us notice that the case  $\delta = 1$  corresponds to a critical continuous state branching process conditioned never to be extinct as mentioned in Remark 4.9 (1) in [21].

Now, to get an invariance principle under  $\mathbb{Q}_a$  for  $a > 0$ , we have a problem since we cannot choose  $m$  such that  $f_m$  satisfies (3.12). Nevertheless there is another way to get an invariance principle under  $\mathbb{Q}_a$  for  $a > 0$ .

We introduce the OU process defined by

$$\tilde{U}(t) := e^{-\kappa^{-1}t} X(e^t - 1) \quad (4.3)$$

which is a CBI with immigration mechanism  $\chi$  and branching mechanism

$$\tilde{\phi}(\lambda) = \phi(\lambda) + \kappa^{-1}\lambda,$$

(see [21] section 5.1). Let us stress that it is not stationary. We observe that

$$\int_1^\infty \log z \mu(dz) < \infty,$$

so that applying [12] Th. 7.7 and Cor. 5.10, we conclude that  $\tilde{U}$  is exponentially ergodic and the convergence (3.2) holds under the conditions (3.3).

Pushing forward this result to the process  $X$  we obtain the following Proposition which is an invariance principle for the clock of CBI.

**Proposition 4.1.** *For  $a > 0$ , under  $\mathbb{Q}_a$  as  $T \rightarrow \infty$ ,*

$$\left( (\log T)^{-1/2} \left( \int_0^{T^t-1} \frac{dr}{X^{\kappa}(r)} - \frac{t \log T}{\kappa p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \quad (4.4)$$

*Remark 5.* In [12] Cor. 5.10, the authors mentioned that if one starts from a general test function, it is unlikely to find an explicit formula for the asymptotic variance in terms of its admissible parameters, except when  $f(x) = \exp(\lambda x)$ . Our result provides one more example,  $f(x) = x^{-\kappa} - (\kappa p)^{-1}$  of a test function with explicit asymptotic variance.

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<sup>1</sup>Beware, our  $\phi$  is  $-\varphi$  therein

## REFERENCES

- [1] J. Bertoin. Ergodic aspects of some Ornstein–Uhlenbeck type processes related to Lévy processes. *Stochastic Processes Appl.*, 129(4):1443–1454, 2019.
- [2] J. Bertoin and M.E. Caballero. Entrance from  $0+$  for increasing semi-stable Markov processes. *Bernoulli*, 8(2):195–205, 2002.
- [3] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002.
- [4] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005.
- [5] R.N. Bhattacharya. On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. Verw. Gebiete*, 60(2):185–201, 1982.
- [6] M.E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006.
- [7] M.E. Caballero and L. Chaumont. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.*, 34(3):1012–1034, 2006.
- [8] M.E. Caballero, J.C. Pardo, and J.L. Pérez. Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli*, 17(1):34–59, 2011.
- [9] Ph. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, pages 73–130. Bibl. Rev. Mat. Iberoam., 1997.
- [10] L. Chaumont, A. Kyprianou, J.C. Pardo, and V. Rivero. Fluctuation theory and exit systems for positive self-similar Markov processes. *Ann. Probab.*, 40(1):245–279, 2012.
- [11] N. Demni, A. Rouault, and M. Zani. Large deviations for clocks of self-similar processes. In *In Memoriam Marc Yor—Séminaire de Probabilités XLVII*, pages 443–466. Springer, 2015.
- [12] M. Friesen, P. Jin, J. Kremer, and B. Rüdiger. Exponential ergodicity for stochastic equations of nonnegative processes with jumps. *arXiv preprint arXiv:1902.02833*, 2019.
- [13] H. K Gjessing and J. Paulsen. Present value distributions with applications to ruin theory and stochastic equations. *Stochastic Processes Appl.*, 71(1):123–144, 1997.
- [14] A. Kuznetsov. On the distribution of exponential functionals for Lévy processes with jumps of rational transform. *Stochastic Processes Appl.*, 122(2):654–663, 2012.
- [15] A. Kuznetsov and J. C. Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Appl. Math.*, 123(1):113–139, 2013.
- [16] A.E. Kyprianou. *Fluctuations of Lévy processes with applications*. Universitext. Springer, Heidelberg, second edition, 2014. Introductory lectures.
- [17] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.
- [18] P.-S. Li and J. Wang. Exponential ergodicity for general continuous-state nonlinear branching processes. *Electron. J. Probab.*, 25, 2020.
- [19] S.P. Meyn and R.L. Tweedie. Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Adv. Appl. Probab.*, 25(3):518–548, 1993.
- [20] J.C. Pardo and V. Rivero. Self-similar Markov processes. *Bol. Soc. Mat. Mexicana (3)*, 19(2):201–235, 2013.
- [21] P. Patie. Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration. *Bull. Sci. Math.*, 133(4):355–382, 2009.
- [22] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [23] J. Wang. Criteria for ergodicity of Lévy type operators in dimension one. *Stochastic Processes Appl.*, 118(10):1909–1928, 2008.
- [24] M. Yor and M. Zani. Large deviations for the Bessel clock. *Bernoulli*, 7:351–362, 2001.