

INVARIANCE PRINCIPLES FOR CLOCKS

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ABSTRACT. We show an invariance principle for rescaled clocks of positive semi-stable Markov processes, proving a conjecture presented in Remark 4 in Demni, Rouault, Zani [11], 2015.

1. INTRODUCTION - PSSMP AND OU

For $\alpha > 0$, a positive self-similar Markov process (pssMp) of index α , is a $[0, \infty)$ -valued strong Markov process (X, \mathbb{Q}_a) , $a > 0$ with càdlàg paths, fulfilling the scaling property

$$(\{bX_{b^{-\alpha}t}, t \geq 0\}, \mathbb{Q}_a) \stackrel{(d)}{=} (\{X_t, t \geq 0\}, \mathbb{Q}_{ba}) \quad (1.1)$$

for every $a, b > 0$.

The Lamperti transformation (see [17]) connects these processes to Lévy processes. Let us summarize this connection. We will follow the notations of [11].

Any pssMp X which never reaches the boundary state 0 may be expressed as the exponential of a Lévy process not drifting to $-\infty$, time changed by the inverse of its exponential functional. More formally, if $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0, set

$$T^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}, \quad (t \geq 0) \quad (1.2)$$

and let $A^{(X)}$ be its inverse, defined by

$$A^{(X)}(t) = \inf\{u \geq 0 : T^{(X)}(u) \geq t\}, \quad (t \geq 0), \quad (1.3)$$

and let ξ be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0, \quad (t \geq 0). \quad (1.4)$$

Then, for every $a > 0$, the distribution of $(\xi_t, t \geq 0)$ under \mathbb{Q}_a does not depend on a and is the distribution of a Lévy process starting from 0.

Conversely, let $(\xi_t, t \geq 0)$ be a Lévy process starting from 0 and let \mathbb{P} and \mathbb{E} denote the underlying probability and expectation, respectively.

Fix $\alpha > 0$. Set

$$\mathcal{A}^{(\xi)}(t) = \int_0^t e^{\alpha \xi_s} ds. \quad (1.5)$$

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Here, we assume that ξ does not drift to $-\infty$ i.e. satisfies, $\limsup_{t \uparrow \infty} \xi_t = \infty$. The inverse process $\tau^{(\xi)}$ of $\mathcal{A}^{(\xi)}$ is

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathcal{A}^{(\xi)}(u) \geq t\}, \quad (t \geq 0). \quad (1.6)$$

For every $a > 0$, let \mathbb{Q}_a be the law under \mathbb{P} of the time-changed process

$$X_t = a \exp \xi_{\tau^{(\xi)}(ta^{-\alpha})}, \quad (t \geq 0), \quad (1.7)$$

then $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0 and we have the fundamental relation

$$\tau^{(\xi)}(t) = T^{(X)}(tX_0^\alpha). \quad (1.8)$$

The process $(T^{(X)}(t), t \geq 0)$ is called the clock associated with the pssMp X . Some years ago, a particular interest was dedicated to the asymptotic behaviour of this process in long time ([24], [11]). To recall the Law of Large Numbers we need some notations.

Let ψ be the Laplace exponent of ξ , defined by

$$\mathbb{E} \exp(m\xi_t) = \exp(t\psi(m)), \quad (1.9)$$

and set $\text{dom } \psi = \{m : \psi(m) < \infty\}$. We assume

$$0 \in \text{int dom } \psi \text{ and } p := \mathbb{E}\xi_1 = \psi'(0) > 0. \quad (1.10)$$

Starting from $a > 0$ such a process X_t never hits 0. Nevertheless, a probability measure \mathbb{Q}_0 can be obtained as the weak limit of \mathbb{Q}_a when $a \downarrow 0$ and under \mathbb{Q}_0 the canonical process has the same transition semigroup as the one associated with $(X; (\mathbb{Q}_a)_{a>0})$ (see [3], [2], [7], [10], [20]). A sufficient condition is (1.10) plus

$$\text{the support of } \xi \text{ is not arithmetic}. \quad (1.11)$$

This latter measure is an entrance law for the semigroup $p_t f(x) = \mathbb{E}_x f(X_t)$ and satisfies

$$\mathbb{Q}_0(f(X_t)) = \frac{1}{\alpha \mathbb{E}\xi_1} \mathbb{E} \left[I_\infty^{-1} f \left((t/I_\infty)^{1/\alpha} \right) \right] \quad (1.12)$$

where

$$I_\infty = \int_0^\infty e^{-\alpha \xi_s} ds.$$

The Law of Large Numbers is the following.

Theorem 1.1 ([11] Th.1). *Assume (1.10) and (1.11). As $t \rightarrow \infty$,*

(1) *For every $a > 0$,*

$$\frac{1}{\log t} \int_0^t \frac{ds}{X_s^\alpha} \rightarrow (\alpha p)^{-1}, \quad \mathbb{Q}_a - a.s. \quad (1.13)$$

(2)

$$\frac{1}{\log t} \int_1^t \frac{ds}{X_s^\alpha} \rightarrow (\alpha p)^{-1}, \quad \mathbb{Q}_0 - a.s. \quad (1.14)$$

In [11], the authors go on with the study of large deviations. A CLT for Bessel clocks was previously proved in Exercise X.3.20 in [22] and extended to an invariance principle in ([24]), using stochastic analysis. Following the result on LDP for clocks, a general CLT is conjectured in [11] Remark 4. In Section 2, we state an invariance principle (Functional CLT) for this kind of processes. proved in Section 3 and illustrated by examples in Section 4.

The main tool is the introduction of an ergodic process with nice asymptotic properties. With a pssMp (X_t) of index α , it is classical to associate a process called generalized Ornstein-Uhlenbeck (OU) by

$$U(t) = e^{-t/\alpha} X(e^t) \quad (1.15)$$

which is strictly stationary, Markovian, ergodic under \mathbb{Q}_0 , and its invariant measure is the law of X_1 under \mathbb{Q}_0 i.e.

$$\mu^U(f) = \frac{1}{\alpha \mathbb{E} \xi_1} \mathbb{E} \left[I_\infty^{-1} f \left(I_\infty^{-1/\alpha} \right) \right]. \quad (1.16)$$

The infinitesimal generators L^X and L^ξ are related by

$$L^X h(x) = x^{-\alpha} L^\xi (h \circ \exp)(\log x) \quad (1.17)$$

(see [9], [20]), and the generator of U is

$$L^U h(x) = L^X h(x) - \frac{x}{\alpha} h'(x). \quad (1.18)$$

Two examples of function f are simple and particularly useful.

If $m \in \text{dom } \psi$ the function $x \mapsto \exp(mx)$ is in the domain of L^ξ , so that

$$f_m : x \mapsto x^m,$$

is in the domain of L^X and we have

$$L^X f_m(x) = \psi(m) f_{m-\alpha}, \quad (1.19)$$

so that

$$L^U f_m = \psi(m) f_{m-\alpha} - \alpha m f_m. \quad (1.20)$$

The function $\mathbf{i}(x) = x$ is in the domain of L^ξ with $L^\xi \mathbf{i} = p$. Formula (1.17) tells us that the function $\varphi(x) = \log x$ is in the domain of L^X and

$$L^X \varphi(x) = \frac{1}{x^\alpha} L^\xi \mathbf{i}(\log x) = \frac{p}{x}, \quad (1.21)$$

and owing to (1.18)

$$L^U \varphi(x) = \frac{p}{x^\alpha} - \frac{1}{\alpha}. \quad (1.22)$$

Remark 1. There is a variant of U , defined by

$$\tilde{U}(t) = e^{-t/\alpha} X(e^t - 1)$$

which shares the same transition with U and begins at $X(0)$ at $t = 0$ ([1]).

Remark 2. If $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMa of index α , then the process $Y = (X^\alpha, (\mathbb{Q}_{a^\alpha})_{a>0})$, is a pssMp of index 1. Conversely if $(Y, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index 1 then, for any $\alpha > 0$, the process $(X = Y^{1/\alpha}, (\mathbb{Q}_{a^{1/\alpha}})_{a>0})$ is a pssMp of index α .

2. MAIN RESULT

The following theorem states an invariance principle (or functional central limit theorem FCLT) under two regimes, \mathbb{Q}_0 and (under conditions) $\mathbb{Q}_a, a > 0$.

Theorem 2.1. (1) Under \mathbb{Q}_0 , as $T \rightarrow \infty$,

$$\left((\log T)^{-1/2} \left(\int_1^{T^t} \frac{dr}{X^\alpha(r)} - \frac{t \log T}{\alpha p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0) . \quad (2.1)$$

where

$$v^2 = \frac{\sigma^2}{\alpha p^3}, \quad \sigma^2 = \psi''(0) . \quad (2.2)$$

(2) If one of the following conditions

(a) there exists $m > 0$ such that $\psi(m) < 0$

(b) there exists $m > \alpha$ such that $\psi(m) > 0$,

is satisfied, then for every $a > 0$, under \mathbb{Q}_a , as $T \rightarrow \infty$, (2.1) holds true.

Remark 3. When the above criterion is not checked, the invariance principle holds true under \mathbb{Q}_a for almost every a (see Theorem 2.8 in [5]).

3. PROOF OF THE MAIN RESULT

3.1. FCLT under the invariant measure. Observe that, owing to (1.15)

$$\int_1^{T^t} \frac{ds}{(X(s))^\alpha} - \frac{t \log T}{\alpha p} = \int_0^{t \log T} \left(\frac{1}{U(r)^\alpha} - \frac{1}{\alpha p} \right) dr \quad (3.1)$$

which reduces the problem to an invariance principle for a functional of the process U . We will use a classical result on weak convergence.

Theorem 3.1 (Bhattacharya Th. 2.1 [5]). *Let (Y_t) be a measurable stationary ergodic process with an invariant probability π . If f is in the range \hat{A} of the extended generator of (Y_t) , then, as $n \rightarrow \infty$*

$$\left(n^{-1/2} \int_0^{nt} f(Y_s) ds \right)_{t \geq 0} \Rightarrow (\rho W(t))_{t \geq 0} \quad (3.2)$$

where

$$\rho^2 = -2 \int f(x) g(x) \pi(dx) , \quad \hat{A}g = f . \quad (3.3)$$

Owing to (1.22) we see that the pair (f, g) with

$$f(x) = \frac{1}{x^\alpha} - \frac{1}{\alpha p} , \quad g(x) = \frac{1}{p} \log x , \quad (3.4)$$

satisfies

$$L^U g = f . \quad (3.5)$$

Now the convergence in distribution comes from Th. 3.1 and formula (3.1), with

$$v^2 = -2 \int f(x) g(x) \mu^U(dx) \quad (3.6)$$

where μ^U is the invariant distribution of the process U .

It remains to compute the variance v^2 . From (1.16) and (3.6) we have

$$\begin{aligned} v^2 &= -2(\alpha p)^{-1} \mathbb{E} (I_\infty^{-1} f(I_\infty^{-1}) g(I_\infty^{-1})) \\ &= 2(\alpha p)^{-1} \mathbb{E} (I_\infty^{-1} (I_\infty - (\alpha p)^{-1}) p^{-1} \log I_\infty) \\ &= 2(\alpha p)^{-2} \mathbb{E} (\log I_\infty - (\alpha p)^{-1} I_\infty^{-1} \log I_\infty) . \end{aligned} \quad (3.7)$$

The Mellin transform

$$M(z) = \mathbb{E}(I_\infty^{-z})$$

may play a prominent role, since

$$\mathbb{E}(\log I_\infty^{-1}) = M'(0) ; \quad \mathbb{E}(I_\infty^{-1} \log I_\infty^{-1}) = M'(1) ,$$

so that

$$v^2 = 2(\alpha p)^{-2} (-M'(0) + (\alpha p)^{-1} M'(1)) . \quad (3.8)$$

We only know that M satisfies the recurrence equation

$$\psi(\alpha z) M(z) = z M(z+1) . \quad (3.9)$$

(see [4] Th. 2 i) and Th. 3. and apply scaling). Differentiating twice the above formula gives

$$\alpha^2 \psi''(\alpha z) M(z) + 2\alpha \psi'(\alpha z) M'(z) + \psi(\alpha z) M''(z) = 2M'(z+1) + z M''(z+1)$$

which, for $z = 0$, gives

$$\alpha^2 \psi''(0) + 2\alpha p M'(0) = 2M'(1)$$

and then

$$v^2 = \frac{\psi''(0)}{\alpha p^3} = \frac{\sigma^2}{\alpha p^3} . \quad (3.10)$$

This fits exactly with the conjecture in [11] Rem. 4.

3.2. Quenched FCLT. We want to show the FCLT under \mathbb{Q}_a for every $a > 0$.

Theorem 3.2 (Bhattacharya Th. 2.6 [5]). *Let $(p_t)_{t \geq 0}$ be the semigroup of a Markov process (Y_t) . Assume that for every x , as $t \rightarrow \infty$*

$$\|p_t(x; \cdot) - \mu\|_{var} \rightarrow 0 .$$

Then, with the notations of Theorem 3.1, the convergence (3.2) holds under P_x for every x .

Such a process is called positive recurrent. Moreover, the exponential ergodicity is defined by the existence of a finite function h and a constant γ such that for every x

$$\|p_t(x, \cdot) - \pi\|_{var} \leq h(x) e^{-\gamma t} .$$

Let us first examine the possibility of a general criterion on the exponent ψ such that the assumptions of the latter theorem are fulfilled. A sufficient condition is given by the so-called Forster-Lyapounov drift criterion, due to [19]).

Theorem 3.3 (Wang Th. 2.1 [23]). *Let L the generator of a Markov process Y_t ,*

- (1) *If there exists a continuous function satisfying*

$$\lim_{|x| \rightarrow \infty} f(x) = \infty \quad (3.11)$$

and constants $K > 0, C > 0, D \in (-\infty, \infty)$ such that

$$Lf \leq -C + D1_{[-K, K]} \quad (3.12)$$

then the process (Y_t) is positively recurrent.

- (2) *If there exists a continuous function satisfying (3.11) and constants $K > 0, C > 0, D \in (-\infty, \infty)$ such that*

$$Lf \leq -Cf + D1_{[-K, K]} \quad (3.13)$$

then the process (Y_t) is exponentially ergodic.

We want to check these criteria for our models, using the function f_m and (1.20).

- (1) When $0 < m < \alpha$ and $\psi(m) > 0$, $f_{m-\alpha}$ is not bounded in the neighbouring of 0, and then is not convenient.
 (2) Let us look for K, C, D in the other cases. For every $C \in (0, \alpha m)$, let us define

$$h_m = L^U f_m + C f_m = \psi(m) f_{m-\alpha} + (C - \alpha m) f_m.$$

- (a) If $\psi(m) < 0$, then $h_m \leq 0$ and then for $D = 0$, (3.13) holds true for every K .
 (b) If $\psi(m) > 0$ and $m > \alpha$, then h_m is increasing for

$$0 < x < x_{\max} = \left(\frac{(m - \alpha)\psi(m)}{m(m - \alpha C)} \right)^{1/\alpha}$$

and decreasing after. It is 0 for $x = x_0 = (\psi(m)/(m - \alpha C))^{1/\alpha} > x_m$. Then choose $K = \psi(m)/(m - C)$ and $D = h_m(x_m)$ and (3.13) holds true.

4. EXAMPLES

In this section, we consider examples taken from [11].

When the function ψ is rational, the distribution of I_∞ may be found in [14]. Let us notice that in all these examples we compute the variance using the elementary formula

$$\psi''(0) = 2 \frac{d}{dm} \frac{\psi(m)}{m} \Big|_{m=0}.$$

4.1. Brownian motion with drift. This is also the Cox Ingersoll Ross model. Let us consider the Lévy process

$$\xi_t = 2B_t + 2\nu t,$$

where B_t is the standard linear Brownian motion and $\nu > 0$. In this case, X_t is the squared Bessel process of dimension $d = 2(1 + \nu)$. Its index is 1 and it is the only continuous pssMp (of index 1). We have

$$\psi(m) = 2m(m + \nu), \quad p = 2\nu, \quad \sigma^2 = 4, \quad v^2 = (4\nu^3)^{-1},$$

Condition (3.13) of Th. 3.3 is satisfied (see(2) b) above), we have exponential ergodicity, the FCLT holds under \mathbb{Q}_a for every $a \geq 0$.

The invariant measure for U is easy to determinate, since

$$I_\infty^{-1} \stackrel{(d)}{=} 2\mathcal{Z}_\nu$$

where \mathcal{Z}_ν is gamma with parameter ν (see [3] (6)), so that

$$\mathbb{Q}_0(\varphi) = \frac{1}{\mathbb{E}\xi_1} \int_0^\infty y\varphi(y) \frac{y^{\nu-1}}{2^\nu} e^{-y/2} dy$$

which means that the invariant measure is the distribution of $2\mathcal{Z}_{\nu+1}$.

Remark 4. Actually it is proved in [18] Rem. 1.2 (2) that U is exponentially ergodic but not strongly ergodic, where strong ergodicity is defined by the existence of $\gamma > 0$ such that

$$\sup_x \|P_t(x, \cdot) - \pi\|_{var} \leq e^{-\gamma t}.$$

4.2. Poissonian examples. Let $\text{Pois}(a, b)_t$ be the compound Poisson process of parameter a whose jumps are exponential of parameter b . We will consider three models : $\xi_t = \mathbf{d}t + \text{Pois}(a, b)_t$ with $\mathbf{d} > 0$, $\xi_t = -t + \text{Pois}(a, b)_t$ and $\xi_t = t - \text{Pois}(a, b)_t$.

4.2.1. $\xi_t = \mathbf{d}t + \text{Pois}(a, b)_t$.

$$\psi(m) = m \left(\mathbf{d} + \frac{a}{b-m} \right) \quad (m \in (-\infty, b)),$$

$$p = \mathbf{d} + \frac{a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab^3}{(a + \mathbf{d}b)^3}.$$

4.2.2. $\xi_t = -t + \text{Pois}(a, b)_t$ with $b < a$.

$$\psi(m) = m \left(-1 + \frac{a}{b-m} \right) \quad (m \leq b),$$

$$p = \frac{a-b}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(a-b)^3}.$$

When $\delta = \mathbf{d} > 0$, $I_\infty^{-1} \stackrel{(d)}{=} \alpha^{-1} B(1 + b, a\alpha^{-1})$ where $B(u, v)$ is the Beta distribution of parameters (u, v) (see [13] Th. 2.1 i)). The invariant measure is then the distribution of W^{-1} where $W \stackrel{(d)}{=} \alpha^{-1} B(b, a\alpha^{-1})$.

When $\delta = -1$, $I_\infty^{-1} \stackrel{(d)}{=} B_2(1 + b, a - b)$ (see [13] Th. 2.1 j)), where $B_2(u, v)$ is the Beta distribution of the second order of parameters (u, v) . The invariant measure is then the $B_2(a - b + 1, b)$ distribution.

4.2.3. $\xi_t = t - \text{Pois}(a, b)_t$ with $b > a$. It is the so called spectrally negative saw-tooth process.

$$\psi(m) = m \left(1 - \frac{a}{b+m} \right), \quad (m \in (-b, \infty)),$$

$$p = \frac{b-a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(b-a)^3}.$$

We have $I_\infty^{-1} \stackrel{(d)}{=} B(b - a, a)$ (see [14] Th. 1), so that the invariant measure is $B(b - a + 1, a)$.

For examples 4.2.1 and 4.2.2, f_m is in the domain of the generator iff $m < b$; for example 4.2.3, f_m is always in the domain.

Looking at our above criterion, we see that we have the exponential ergodicity in the first two cases when $b > 1$, and completely for the third one.

4.3. Spectrally negative process conditioned to stay positive. For $\alpha \in (1, 2)$, let X^\uparrow be the spectrally α -stable process conditioned to stay positive as defined in [6] Sect. 3.2. and [21] Sect. 3. Its corresponding Lévy process has Laplace exponent

$$\psi(m) = \frac{\Gamma(m + \alpha)}{\Gamma(m)}, \quad (m \in (-\alpha, \infty)).$$

We have

$$p = \Gamma(\alpha), \quad \sigma^2 = 2(\Gamma'(\alpha) + \gamma\Gamma(\alpha))$$

where $\gamma = -\Gamma'(1)$ is the Euler constant, and then

$$v^2 = 2 \frac{\Gamma'(\alpha) + \gamma\Gamma(\alpha)}{\alpha\Gamma(\alpha)^3}.$$

Using (3.9) one sees directly that $M(z) = \Gamma(\alpha z + 1)/\Gamma(z + 1)$, so that $I_\infty \stackrel{(d)}{=} S_{1/\alpha}(1)$, the stable subordinator of index $1/\alpha$ evaluated at 1.

Condition (2) b) of Th. 2.1 is satisfied.

4.4. Hypergeometric stable process. The modulus of a Cauchy process in \mathbb{R}^d for $d > 1$ is a 1-pssMp with infinite lifetime. The associated Lévy process is a particular case of hypergeometric stable process of index α as defined in [8], with $\alpha < d$. The characteristic exponent given therein by Th. 7 yields the Laplace exponent :

$$\psi(m) = -2^\alpha \frac{\Gamma((-m + \alpha)/2)}{\Gamma(-m/2)} \frac{\Gamma((m + d)/2)}{\Gamma((m + d - \alpha)/2)}, \quad (m \in (-d, \alpha)).$$

We have

$$p = 2^{\alpha-1} \frac{\Gamma(\alpha/2)\Gamma(d/2)}{\Gamma(d - \alpha)/2}, \quad \sigma^2 = p[1 - \gamma - \Psi((d - \alpha)/2) - \Psi(\alpha/2)],$$

where Ψ is the Digamma function.

The distribution of the limiting variable I_∞ is studied in [15] and [14].

Condition (2) b) of Th. 2.1 is never satisfied. Condition (2) a) can be satisfied if $\alpha > 2$, taking $m \in (2, \min(\alpha, 4))$ since $\Gamma(-m/2) > 0$ hence $\psi(m) < 0$.

4.5. Continuous State branching process with immigration (CBI). Let $\kappa \in [0, 1)$ and $\delta > \kappa/(\kappa + 1)$. Let X be the continuous state branching process with immigration ([16] Sec. 13.5) whose branching mechanism is

$$\phi(\lambda) = \frac{1}{\kappa} \lambda^{\kappa+1}$$

and immigration mechanism is

$$\chi(\lambda) = \delta\phi'(\lambda).$$

We have the representation

$$\phi(\lambda) = \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \mu(dz), \quad \mu(dz) = \frac{\kappa + 1}{\Gamma(1 - \kappa)} \frac{dz}{z^{\kappa+2}}.$$

This process is self-similar of index κ (see [21], lemma 4.8)¹ and the corresponding Laplace exponent is

$$\psi(m) = c(\kappa - (\kappa + 1)\delta - m) \frac{\Gamma(-m + \kappa)}{\Gamma(-m)}, \quad (m \in (-\infty, \kappa))$$

and

$$\begin{aligned} p &= c((\kappa + 1)\delta - \kappa)\Gamma(\kappa) > 0, \quad \sigma^2 = c(\Gamma(\kappa) + (\kappa - (\kappa + 1)\delta)(\Gamma'(\kappa) + \gamma\Gamma(\kappa))) \\ v^2 &= \frac{\sigma^2}{\kappa p^3}. \end{aligned} \quad (4.1)$$

We can then apply Th. 2.1(1) and conclude that under \mathbb{Q}_0 , as $T \rightarrow \infty$

$$\left((\log T)^{-1/2} \left(\int_1^{T^t} \frac{dr}{X^\kappa(r)} - \frac{t \log T}{\kappa p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \quad (4.2)$$

The entrance law is given in Remark 4.9 (2) in [21]. Let us notice that the case $\delta = 1$ corresponds to a critical continuous state branching process conditioned never to be extinct as mentioned in Remark 4.9 (1) in [21].

Now, to get an invariance principle under \mathbb{Q}_a for $a > 0$, we have a problem since we cannot choose m such that f_m satisfies (3.12). Nevertheless there is another way to get an invariance principle under \mathbb{Q}_a for $a > 0$.

We introduce the OU process defined by

$$\tilde{U}(t) := e^{-\kappa^{-1}t} X(e^t - 1) \quad (4.3)$$

which is a CBI with immigration mechanism χ and branching mechanism

$$\tilde{\phi}(\lambda) = \phi(\lambda) + \kappa^{-1}\lambda,$$

(see [21] section 5.1). Let us stress that it is not stationary. We observe that

$$\int_1^\infty \log z \, \mu(dz) < \infty,$$

so that applying [12] Th. 7.7 and Cor. 5.10, we conclude that \tilde{U} is exponentially ergodic and the convergence (3.2) holds under the conditions (3.3).

Pushing forward this result to the process X we obtain the following Proposition which is an invariance principle for the clock of CBI.

Proposition 4.1. *For $a > 0$, under \mathbb{Q}_a as $T \rightarrow \infty$,*

$$\left((\log T)^{-1/2} \left(\int_0^{T^t-1} \frac{dr}{X^\kappa(r)} - \frac{t \log T}{\kappa p} \right); t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \quad (4.4)$$

Remark 5. In [12] Cor. 5.10, the authors mentioned that if one starts from a general test function, it is unlikely to find an explicit formula for the asymptotic variance in terms of its admissible parameters, except when $f(x) = \exp(\lambda x)$. Our result provides one more example, $f(x) = x^{-\kappa} - (\kappa p)^{-1}$ of a test function with explicit asymptotic variance.

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¹Beware, our ϕ is $-\varphi$ therein

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