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INVARIANCE PRINCIPLES FOR CLOCKS

MARIA-EMILIA CABALLERO\textsuperscript{1} AND ALAIN ROUAULT\textsuperscript{2}

Abstract. We show an invariance principle for rescaled clocks of positive semi-stable Markov processes, proving a conjecture presented in Remark 4 in Demni, Rouault, Zani [11], 2015.

1. Introduction - pssMp and OU

For $\alpha > 0$, a positive self-similar Markov process (pssMp) of index $\alpha$, is a $[0, \infty)$-valued strong Markov process $(X, Q_a)$, $a > 0$ with càdlàg paths, fulfilling the scaling property

$$(\{bX_{b^{-\alpha}t}, t \geq 0\}, Q_a) \overset{d}{=} (\{X_t, t \geq 0\}, Q_{ba})$$

for every $a, b > 0$.

The Lamperti transformation (see [17]) connects these processes to Lévy processes. Let us summarize this connection. We will follow the notations of [11].

Any pssMp $X$ which never reaches the boundary state 0 may be expressed as the exponential of a Lévy process not drifting to $-\infty$, time changed by the inverse of its exponential functional. More formally, if $(X, (Q_a)_{a>0})$ is a pssMp of index $\alpha$ which never reaches 0, set

$$T^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}, \ (t \geq 0)$$

and let $A^{(X)}$ be its inverse, defined by

$$A^{(X)}(t) = \inf\{u \geq 0 : T^{(X)}(u) \geq t\}, \ (t \geq 0)$$

and let $\xi$ be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0, \ (t \geq 0).$$

Then, for every $a > 0$, the distribution of $(\xi_t, t \geq 0)$ under $Q_a$ does not depend on $a$ and is the distribution of a Lévy process starting from 0.

Conversely, let $(\xi_t, t \geq 0)$ be a Lévy process starting from 0 and let $P$ and $E$ denote the underlying probability and expectation, respectively.

Fix $a > 0$. Set

$$A^{(c)}(t) = \int_0^t e^{a \xi_s} ds.$$
Here, we assume that $\xi$ does not drift to $-\infty$ i.e. satisfies, $\limsup_{t \to \infty} \xi_t = \infty$. The inverse process $\tau(\xi)$ of $\mathcal{A}(\xi)$ is

$$
\tau(\xi)(t) = \inf\{u \geq 0 : \mathcal{A}(\xi)(u) \geq t \}, \ (t \geq 0).
$$

(1.6)

For every $a > 0$, let $Q_a$ be the law under $P$ of the time-changed process

$$
X_t = a \exp \xi \tau(\xi)(ta^{-\alpha}), \ (t \geq 0),
$$

(1.7)

then $(X,(Q_a)_{a > 0})$ is a pssMp of index $\alpha$ which never reaches 0 and we have the fundamental relation

$$
\tau(\xi)(t) = T(X)(tX_0^\alpha).
$$

(1.8)

The process $(T(X)(t), t \geq 0)$ is called the clock associated with the pssMp $X$.

Some years ago, a particular interest was dedicated to the asymptotic behaviour of this process in long time ([24], [11]). To recall the Law of Large Numbers we need some notations.

Let $\psi$ be the Laplace exponent of $\xi$, defined by

$$
E \exp(\xi t) = \exp(\psi(t)),
$$

(1.9)

and set dom $\psi = \{m : \psi(m) < \infty\}$. We assume

$$
0 \in \text{int dom } \psi \text{ and } p := E\xi_1 = \psi'(0) > 0.
$$

(1.10)

Starting from $a > 0$ such a process $X_t$ never hits 0. Nevertheless, a probability measure $Q_0$ can be obtained as the weak limit of $Q_a$ when $a \downarrow 0$ and under $Q_0$ the canonical process has the same transition semigroup as the one associated with $(X; (Q_a)_{a > 0})$. (see [3], [2], [7], [10], [20]). A sufficient condition is (1.10) plus the support of $\xi$ is not arithmetic.

This latter measure is an entrance law for the semigroup $p_t f(x) = E_x f(X_t)$ and satisfies

$$
Q_0(f(X_t)) = \frac{1}{\alpha E\xi_1} E \left[ I_{\infty}^{1} f \left( \frac{t}{I_{\infty}} \right)^{1/\alpha} \right]
$$

(1.12)

where

$$
I_{\infty} = \int_0^\infty e^{-\alpha \xi_s} ds.
$$

The Law of Large Numbers is the following.

**Theorem 1.1** ([11] Th.1). Assume (1.10) and (1.11). As $t \to \infty$,

1. For every $a > 0$,

$$
\frac{1}{\log t} \int_0^t ds X_s^\alpha \to (ap)^{-1}, \ Q_a \text{ - a.s.}
$$

(1.13)

2. For every $a > 0$,

$$
\frac{1}{\log t} \int_1^t ds X_s^\alpha \to (ap)^{-1}, \ Q_0 \text{ - a.s.}
$$

(1.14)

In [11], the authors go on with the study of large deviations. A CLT for Bessel clocks was previously proved in Exercise X.3.20 in [22] and extended to an invariance principle in ([24]), using stochastic analysis. Following the result on LDP for clocks, a general CLT is conjectured in [11] Remark 4. In Section 2, we state an invariance principle (Functional CLT) for this kind of processes, proved in Section 3 and illustrated by examples in Section 4.
The main tool is the introduction of an ergodic process with nice asymptotic properties. With a pssMp \((X_t)\) of index \(\alpha\), it is classical to associate a process called generalized Ornstein-Uhlenbeck (OU) by

\[
U(t) = e^{-t/\alpha} X(e^t)
\]

which is strictly stationary, Markovian, ergodic under \(Q_0\), and its invariant measure is the law of \(X_1\) under \(Q_0\), i.e.

\[
\mu_U(f) = \frac{1}{\alpha E \xi} \mathbb{E} \left[ I_{\infty}^{-1} f \left( I_{\infty}^{-1/\alpha} \right) \right].
\]

The infinitesimal generators \(L_X\) and \(L^\xi\) are related by

\[
L_X h(x) = x^{-\alpha} L^\xi (h \circ \exp)(\log x)
\]

(see [9], [20]), and the generator of \(U\) is

\[
L^U h(x) = L_X h(x) - \frac{x}{\alpha} h'(x).
\]

Two examples of function \(f\) are simple and particularly useful.

If \(m \in \text{dom } \psi\) the function \(x \mapsto \exp(mx)\) is in the domain of \(L^\xi\), so that \(f_m : x \mapsto x^m\), is in the domain of \(L_X\) and we have

\[
L_X f_m(x) = \psi(m) f_{m-\alpha},
\]

so that

\[
L^U f_m = \psi(m) f_{m-\alpha} - \alpha m f_m.
\]

The function \(i(x) = x\) is in the domain of \(L^\xi\) with \(L^\xi i = p\). Formula (1.17) tells us that the function \(\varphi(x) = \log x\) is in the domain of \(L_X\) and

\[
L_X \varphi(x) = \frac{1}{x^\alpha} L^\xi (\log x) = \frac{p}{x},
\]

and owing to (1.18)

\[
L^U \varphi(x) = \frac{p}{x^\alpha} - \frac{1}{\alpha}.
\]

Remark 1. There is a variant of \(U\), defined by

\[
\tilde{U}(t) = e^{-t/\alpha} X(e^t - 1)
\]

which shares the same transition with \(U\) and begins at \(X(0)\) at \(t = 0\) ([1]).

Remark 2. If \((X, (Q_a)_{a>0})\) is a pssMa of index \(\alpha\), then the process \(Y = (X^\alpha, (Q_{\alpha^a})_{a>0})\), is a pssMp of index 1. Conversely if \((Y, (Q_a)_{a>0})\) is a pssMp if index 1 then, for any \(\alpha > 0\), the process \((X = Y^{1/\alpha}, (Q_{\alpha^{1/\alpha}})_{a>0})\) is a pssMp of index \(\alpha\).
2. Main result

The following theorem states an invariance principle (or functional central limit theorem FCLT) under two regimes, \( Q_0 \) and (under conditions) \( Q_a, a > 0 \).

**Theorem 2.1.**

1. Under \( Q_0 \), as \( T \to \infty \),
   \[
   \left( \log T \right)^{-1/2} \left( \int_1^T \frac{dr}{X^\alpha(r)} - \frac{t \log T}{\alpha p} \right) ; t \geq 0 \Rightarrow (vW(t); t \geq 0) .
   \]
   where
   \[
   v^2 = \frac{\sigma^2}{\alpha p}, \quad \sigma^2 = \psi''(0) .
   \]

2. If one of the following conditions
   (a) there exists \( m > 0 \) such that \( \psi(m) < 0 \)
   (b) there exists \( m > \alpha \) such that \( \psi(m) > 0 \),
   is satisfied, then for every \( a > 0 \), under \( Q_a \), as \( T \to \infty \), (2.1) holds true.

**Remark 3.** When the above criterion is not checked, the invariance principle holds true under \( Q_a \) for almost every \( a \) (see Theorem 2.8 in [5]).

3. Proof of the main result

3.1. FCLT under the invariant measure. Observe that, owing to (1.15)

\[
\int_1^T \frac{ds}{(X(s))^\alpha} - \frac{t \log T}{\alpha p} = \int_0^{t \log T} \left( \frac{1}{U(r)^\alpha} - \frac{1}{\alpha p} \right) dr
\]
which reduces the problem to an invariance principle for a functional of the process \( U \). We will use a classical result on weak convergence.

**Theorem 3.1** (Bhattacharya Th. 2.1 [5]). Let \( (Y_t) \) be a measurable stationary ergodic process with an invariant probability \( \pi \). If \( f \) is in the range \( \hat{A} \) of the extended generator of \( (Y_t) \), then, as \( n \to \infty \)

\[
(n^{-1/2} \int_0^n f(Y_s)ds)_{t \geq 0} \Rightarrow (\rho W(t))_{t \geq 0}
\]
where
\[
\rho^2 = -2 \int f(x)g(x)\pi(dx) , \quad \hat{A}g = f .
\]
Owing to (1.22) we see that the pair \( (f, g) \) with
\[
f(x) = \frac{1}{x^\alpha} - \frac{1}{\alpha p} , \quad g(x) = \frac{1}{p} \log x ,
\]
satisfies
\[
L^U g = f .
\]
Now the convergence in distribution comes from Th. 3.1 and formula (3.1), with
\[
v^2 = -2 \int f(x)g(x)\mu_U(dx)
\]
where \( \mu_U \) is the invariant distribution of the process \( U \).
It remains to compute the variance $v^2$. From (1.16) and (3.6) we have

\[ v^2 = -2(\alpha p)^{-1}E(I_{\infty}^{-1}f(I_{\infty}^{-1})g(I_{\infty}^{-1})) = 2(\alpha p)^{-1}E[I_{\infty}^{-1}(I_{\infty}-(\alpha p)^{-1})p^{-1}\log I_{\infty}] = 2(\alpha p)^{-2}E(\log I_{\infty}-(\alpha p)^{-1}I_{\infty}^{-1}\log I_{\infty}). \]  

(3.7)

The Mellin transform

\[ M(z) = E(I_{\infty}^{-z}) \]

may play a prominent role, since

\[ \mathbb{E}(\log I_{\infty}^{-1}) = M'(0) \quad \text{and} \quad \mathbb{E}(I_{\infty}^{-1}\log I_{\infty}^{-1}) = M'(1), \]

so that

\[ v^2 = 2(\alpha p)^{-2}(-M'(0) + (\alpha p)^{-1}M'(1)). \]  

(3.8)

We only know that $M$ satisfies the recurrence equation

\[ \psi(\alpha z)M(z) = zM(z+1). \]  

(3.9)

(see [4] Th. 2 i) and Th. 3. and apply scaling). Differentiating twice the above formula gives

\[ \alpha^2\psi''(\alpha z)M(z) + 2\alpha\psi'(\alpha z)M'(z) + \psi(\alpha z)M''(z) = 2M'(z+1) + zM''(z+1) \]

which, for $z = 0$, gives

\[ \alpha^2\psi''(0) + 2\alpha M'(0) = 2M'(1) \]

and then

\[ v^2 = \frac{\psi''(0)}{\alpha p^2} = \frac{\sigma^2}{\alpha p^3}. \]  

(3.10)

This fits exactly with the conjecture in [11] Rem. 4.

3.2. Quenched FCLT. We want to show the FCLT under $Q_a$ for every $a > 0$.

**Theorem 3.2** (Bhattacharya Th. 2.6 [5]). Let $(p_t)_{t \geq 0}$ be the semigroup of a Markov process $(Y_t)$. Assume that for every $x$, as $t \to \infty$

\[ \|p_t(x; \cdot) - \mu\|_{\text{var}} \to 0. \]

Then, with the notations of Theorem 3.1, the convergence (3.2) holds under $P_x$ for every $x$.

Such a process is called positive recurrent. Moreover, the exponential ergodicity is defined by the existence of a finite function $h$ and a constant $\gamma$ such that for every $x$

\[ \|p_t(x, \cdot) - \pi\|_{\text{var}} \leq h(x)e^{-\gamma t}. \]

Let us first examine the possibility of a general criterion on the exponent $\psi$ such that the assumptions of the latter theorem are fulfilled. A sufficient condition is given by the so-called Forster-Lyapounov drift criterion, due to [19]).

**Theorem 3.3** (Wang Th. 2.1 [23]). Let $L$ the generator of a Markov process $Y_t$,
(1) If there exists a continuous function satisfying
\[ \lim_{|x| \to \infty} f(x) = \infty \]  
and constants \( K > 0, C > 0, D \in (-\infty, \infty) \) such that
\[ Lf \leq -C + D1_{[-K,K]} \]
then the process \((Y_t)\) is positively recurrent.

(2) If there exists a continuous function satisfying (3.11) and constants \( K > 0, C > 0, D \in (-\infty, \infty) \) such that
\[ Lf \leq -Cf + D1_{[-K,K]} \]  
then the process \((Y_t)\) is exponentially ergodic.

We want to check these criteria for our models, using the function \( f_m \) and (1.20).

(1) When \( 0 < m < \alpha \) and \( \psi(m) > 0 \), \( f_{m-\alpha} \) is not bounded in the neighbourhood of 0, and then is not convenient.

(2) Let us look for \( K, C, D \) in the other cases. For every \( C \in (0, \alpha m) \), let us define
\[ h_m = Lf_m + Cf_m = \psi(m) f_{m-\alpha} + (C - \alpha m) f_m. \]

(a) If \( \psi(m) < 0 \), then \( h_m \leq 0 \) and then for \( D = 0 \), (3.13) holds true for every \( K \).

(b) If \( \psi(m) > 0 \) and \( m > \alpha \), then \( h_m \) is increasing for
\[ 0 < x < x_{\text{max}} = \left( \frac{(m-\alpha)\psi(m)}{m(m-\alpha C)} \right)^{1/\alpha} \]
and decreasing after. It is 0 for \( x = x_0 = (\psi(m)/(m-\alpha C))^{1/\alpha} > x_m \). Then choose \( K = \psi(m)/(m-C) \) and \( D = h_m(x_m) \) and (3.13) holds true.

4. Examples

In this section, we consider examples taken from [11].

When the function \( \psi \) is rational, the distribution of \( I_\infty \) may be found in [14]. Let us notice that in all these examples we compute the variance using the elementary formula
\[ \psi''(0) = 2 \frac{d}{dm} \frac{\psi(m)}{m} \bigg|_{m=0}. \]

4.1. Brownian motion with drift. This is also the Cox Ingersoll Ross model. Let us consider the Lévy process
\[ \xi_t = 2B_t + 2\nu t, \]
where \( B_t \) is the standard linear Brownian motion and \( \nu > 0 \). In this case, \( X_t \) is the squared Bessel process of dimension \( d = 2(1 + \nu) \). Its index is 1 and it is the only continuous pssMp (of index 1). We have
\[ \psi(m) = 2m(m + \nu), \quad p = 2\nu, \quad \sigma^2 = 4, \quad \nu^2 = (4\nu^3)^{-1}. \]

Condition (3.13) of Th. 3.3 is satisfied (see(2) b) above), we have exponential ergodicity, the FCLT holds under \( Q_\alpha \) for every \( a \geq 0 \).
The invariant measure for $U$ is easy to determinate, since

$$I^{-1}_\infty \overset{(d)}{=} 2Z_\nu$$

where $Z_\nu$ is gamma with parameter $\nu$ (see [3] (6)), so that

$$Q_0(\varphi) = \int_0^\infty e^{-y/2} dy$$

which means that the invariant measure is the distribution of $2Z_\nu+1$.

**Remark 4.** Actually it is proved in [18] Rem. 1.2 (2) that $U$ is exponentially ergodic but not strongly ergodic, where strong ergodicity is defined by the existence of $\gamma > 0$ such that

$$\sup_x \|P_t(x,\cdot) - \pi\|_{var} \leq e^{-\gamma t}.$$

**4.2. Poissonian examples.** Let Pois$(a,b)_t$ be the compound Poisson process of parameter $a$ whose jumps are exponential of parameter $b$. We will consider three models : $\xi_t = d + \text{Pois}(a,b)_t$ with $d > 0$, $\xi_t = -t + \text{Pois}(a,b)_t$ and $\xi_t = t - \text{Pois}(a,b)_t$.

**4.2.1.** $\xi_t = d + \text{Pois}(a,b)_t$.

$$\psi(m) = m \left( d + \frac{a}{b - m} \right) \quad (m \in (-\infty, b)),$$

$$p = d + \frac{a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab^3}{a + db^3}.$$

**4.2.2.** $\xi_t = -t + \text{Pois}(a,b)_t$ with $b < a$.

$$\psi(m) = m \left( -1 + \frac{a}{b - m} \right) \quad (m \leq b),$$

$$p = \frac{a - b}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(a - b)^3}.$$

When $\delta = d > 0$, $I^{-1}_\infty \overset{(d)}{=} \alpha^{-1}B(1 + b, a\alpha^{-1})$ where $B(u,v)$ is the Beta distribution of parameters $(u,v)$ (see [13] Th. 2.1 i)). The invariant measure is then the distribution of $W^{-1}$ where $W \overset{(d)}{=} \alpha^{-1}B(b, a\alpha^{-1})$.

When $\delta = -1$, $I^{-1}_\infty \overset{(d)}{=} B_2(1 + b, a - b)$ (see [13] Th. 2.1 j)), where $B_2(u,v)$ is the Beta distribution of the second order of parameters $(u,v)$. The invariant measure is then the $B_2(b - a + 1, b)$ distribution.

**4.2.3.** $\xi_t = t - \text{Pois}(a,b)_t$ with $b > a$. It is the so called spectrally negative saw-tooth process.

$$\psi(m) = m \left( 1 - \frac{a}{b + m} \right) \quad (m \in (-b, \infty)),$$

$$p = \frac{b - a}{b}, \quad \sigma^2 = \frac{a}{b^2}, \quad v^2 = \frac{ab}{(b - a)^3}.$$

We have $I^{-1}_\infty \overset{(d)}{=} B(b - a, a)$ (see [14] Th. 1), so that the invariant measure is $B(b - a + 1, a)$.

For examples 4.2.1 and 4.2.2, $f_m$ is in the domain of the generator iff $m < b$; for example 4.2.3 , $f_m$ is always in the domain.
Looking at our above criterion, we see that we have the exponential ergodicity in the first two cases when \( b > 1 \), and completely for the third one.

4.3. Spectrally negative process conditioned to stay positive. For \( \alpha \in (1, 2) \), let \( X^\uparrow \) be the spectrally \( \alpha \)-stable process conditioned to stay positive as defined in [6] Sect. 3.2. and [21] Sect. 3. Its corresponding Lévy process has Laplace exponent
\[
\psi(m) = \frac{\Gamma(m + \alpha)}{\Gamma(m)} , \quad (m \in (-\alpha, \infty)).
\]
We have
\[
p = \Gamma(\alpha) , \quad \sigma^2 = 2 \left( \Gamma'(\alpha) + \gamma \Gamma(\alpha) \right)
\]
where \( \gamma = -\Gamma'(1) \) is the Euler constant, and then
\[
v^2 = 2 \frac{\Gamma'(\alpha) + \gamma \Gamma(\alpha)}{\alpha \Gamma(\alpha)^2}.
\]
Using (3.9) one sees directly that \( M(z) = \Gamma(\alpha z + 1)/\Gamma(z + 1) \), so that \( I_\infty \overset{(d)}{=} S_{1/\alpha}(1) \), the stable subordinator of index \( 1/\alpha \) evaluated at 1.

Condition (2) b) of Th. 2.1 is satisfied.

4.4. Hypergeometric stable process. The modulus of a Cauchy process in \( \mathbb{R}^d \) for \( d > 1 \) is a 1-pssMp with infinite lifetime. The associated Lévy process is a particular case of hypergeometric stable process of index \( \alpha \) as defined in [8], with \( \alpha < d \). The characteristic exponent given therein by Th. 7 yields the Laplace exponent:
\[
\psi(m) = -2^\alpha \frac{\Gamma((-m + \alpha)/2) \Gamma((m + d)/2)}{\Gamma(-m/2) \Gamma((m + d - \alpha)/2)} , \quad (m \in (-d, \alpha)).
\]
We have
\[
p = 2^\alpha - \Gamma(\alpha/2) \Gamma(d/2) \Gamma(d - \alpha/2) , \quad \sigma^2 = p \left[ 1 - \gamma - \Psi((d - \alpha)/2) - \Psi(\alpha/2) \right],
\]
where \( \Psi \) is the Digamma function.

The distribution of the limiting variable \( I_\infty \) is studied in [15] and [14].

Condition (2) b) of Th. 2.1 is never satisfied. Condition (2) a) can be satisfied if \( \alpha > 2 \), taking \( m \in (2, \min(\alpha, 4)) \) since \( \Gamma(-m/2) > 0 \) hence \( \psi(m) < 0 \).

4.5. Continuous State branching process with immigration (CBI). Let \( \kappa \in [0, 1) \) and \( \delta > \kappa/(\kappa + 1) \). Let \( X \) be the continuous state branching process with immigration ([16] Sec. 13.5) whose branching mechanism is
\[
\phi(\lambda) = \frac{1}{\kappa} \lambda^{\kappa+1}
\]
and immigration mechanism is
\[
\chi(\lambda) = \delta \phi'(\lambda).
\]
We have the representation
\[
\phi(\lambda) = \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \mu(dz) , \quad \mu(dz) = \frac{\kappa + 1}{\Gamma(1 - \kappa)} \frac{dz}{z^{\kappa+2}}.
\]
This process is self-similar of index $\kappa$ (see [21], lemma 4.8)\footnote{Beware, our $\phi$ is $-\varphi$ therein} and the corresponding Laplace exponent is

$$
\psi(m) = c(\kappa - (\kappa + 1)\delta - m)\frac{\Gamma(-m + \kappa)}{\Gamma(-m)}, \ (m \in (-\infty, \kappa))
$$

and

$$
p = c((\kappa + 1)\delta - \kappa)\Gamma(\kappa) > 0, \ \sigma^2 = c(\Gamma(\kappa) + (\kappa - (\kappa + 1)\delta)(\Gamma'(\kappa) + \gamma\Gamma(\kappa)))
$$

\begin{equation}
\nu^2 = \frac{\sigma^2}{\kappa p^3}.
\end{equation}

We can then apply Th. 2.1(1) and conclude that under $Q_0$, as $T \to \infty$

$$
\left( (\log T)^{-1/2} \left( \int_1^T \frac{dr}{X^\kappa(r)} - \frac{t\log T}{\kappa p} \right) : t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \tag{4.2}
$$

The entrance law is given in Remark 4.9 (2) in [21]. Let us notice that the case $\delta = 1$ corresponds to a critical continuous state branching process conditioned never to be extinct as mentioned in Remark 4.9 (1) in [21].

Now, to get an invariance principle under $Q_0$ for $a > 0$, we have a problem since we cannot choose $m$ such that $f_m$ satisfies (3.12). Nevertheless there is another way to get an invariance principle under $Q_a$ for $a > 0$.

We introduce the OU process defined by

$$
\tilde{U}(t) := e^{-\kappa^{-1} t}X(e^t - 1)
$$

which is a CBI with immigration mechanism $\chi$ and branching mechanism

$$
\tilde{\phi}(\lambda) = \phi(\lambda) + \kappa^{-1}\lambda,
$$

(see [21] section 5.1). Let us stress that it is not stationary. We observe that

$$
\int_1^\infty \log z \mu(dz) < \infty,
$$

so that applying [12] Th. 7.7 and Cor. 5.10, we conclude that $\tilde{U}$ is exponentially ergodic and the convergence (3.2) holds under the conditions (3.3).

Pushing forward this result to the process $X$ we obtain the following Proposition which is an invariance principle for the clock of CBI.

**Proposition 4.1.** For $a > 0$, under $Q_a$ as $T \to \infty$,

$$
\left( (\log T)^{-1/2} \left( \int_0^{Tt} \frac{dr}{X^\kappa(r)} - \frac{t\log T}{\kappa p} \right) : t \geq 0 \right) \Rightarrow (vW(t); t \geq 0). \tag{4.4}
$$

**Remark 5.** In [12] Cor. 5.10, the authors mentioned that if one starts from a general test function, it is unlikely to find an explicit formula for the asymptotic variance in terms of its admissible parameters, except when $f(x) = \exp(\lambda x)$. Our result provides one more example, $f(x) = x^{-\kappa} - (\kappa p)^{-1}$ of a test function with explicit asymptotic variance.

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