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Categorical models of Linear Logic with fixed points of formulas

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Abstract. We develop a denotational semantics of Linear Logic with least and greatest fixed points in a general categorical setting based on the notion of Seely category and on strong functors acting on them. We exhibit a simple instance of this setting in the category of sets and relations, where least and greatest fixed points are interpreted in the same way, and in a category of sets equipped with a notion of totality (non-uniform totality spaces) and relations preserving them, where least and greatest fixed points have distinct interpretations.

1 Introduction

Propositional Linear Logic is a well-established logical system introduced by Girard in [13]. It provides a fine-grain analysis of proofs in intuitionistic and classical logic, and more specifically of their cut-elimination. LL features a logical account of the structural rules (weakening, contraction) which are handled implicitly in intuitionistic and classical logic. For this reason, LL has many useful outcomes in the Curry-Howard based approach to the theory of programming: logical understanding of evaluation strategies, new syntax of proofs/programs (proof-nets), connections with other branches of mathematics (linear algebra, differential calculus etc), new operational semantics (geometry of interaction)...

However propositional LL is not a reasonable programming language, by lack of data-types and iteration or recursion principles. This is usually remedied by extending propositional LL to the 2nd order, thus defining a logical system in which Girard's System F [15] can be embedded. However even if such a system is very expressive in terms of computable functions, its algorithmic expressiveness is poor: it is not possible to write a term $! \iota \multimap \iota$ (where $\iota = \forall \zeta !(\zeta \multimap \zeta) \multimap ! \zeta \multimap \zeta$ is the standard representation of integers) which computes the predecessor function in one (or a uniformly bounded) number of reduction steps³.

Another option to turn propositional LL into a programming language is to extend it with least and greatest fixed points of formulas as early suggested by Girard in an unpublished note [14], though the first comprehensive proof-theoretic investigation of such an extension of LL is recent [1]: Baelde considers an extension μ MALL of Multiplicative Additive LL sequent calculus with least

³ More precisely we know that such a proof does not exist in System F and it is very likely that it does not exist in second order LL either.

and greatest fixed points. His motivations arose from a proof-search and system verification perspective on LL and our purpose is to develop a more Curry-Howard oriented point of view on μMALL (or rather μLL ⁴) alternative to the “system F” approach to representing programs in LL. The ν -introduction rule of μLL (Park’s rule) leads to subtle cut-elimination rewrite rules for which Baelde could prove a restricted form of cut-elimination, sufficient for establishing for instance that a proof of the type of integers $\mu\zeta(1 \oplus \zeta)$ necessarily reduces to an integer. There are alternative proof-systems for the same logic, involving infinite or cyclic proofs, see [2], whose connections with the aforementioned finitary proof-system are not clear yet.

Since the proof-theory (and hence the “operational semantics”) of μLL is still under development, it is important to investigate its denotational semantics, whose definition does not rely on the rewrite system μLL is equipped with. We develop here a categorical semantics of μLL extending the standard notion of Seely category⁵ of classical LL. Such a model of μLL consists of a Seely category \mathcal{L} and of a class of functors $\mathcal{L}^n \rightarrow \mathcal{L}$ for all possible arities n which will be used for interpreting μLL formulas with free variables. These functors have to be equipped with a strength to deal properly with contexts in the ν rule.

Then we develop a simple instance of this setting which consists in taking for \mathcal{L} the category of sets and relations, a well-known Seely model of LL. The strong functors we consider on this category, that we call *variable sets*, are the pairs $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\widehat{\mathbb{F}}$ is the strength and $\overline{\mathbb{F}} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a functor which is Scott-continuous in the sense that it commutes with directed unions of morphisms which implies categorical cocontinuity on the category of sets and injections and maps inclusions to inclusions (this light additional requirement simplifies the presentation). There is no special requirement about the strength $\widehat{\mathbb{F}}$ beyond naturality and monoidality. Variable sets form a Seely model of μLL where linear negation is the identity on objects, the formulas $\mu\zeta F$ and $\nu\zeta F$ are interpreted as the same variable set, exactly as \otimes and \wp are interpreted in the same way (and similarly for additives and exponentials): this denotational “degeneracy” at the level of types is a well known feature of \mathbf{Rel} which doesn’t mean at all that the model is trivial; for instance normal multiplicative exponential LL proofs which have distinct relational interpretations have distinct associated proof-nets [9, 8].

Last we “enrich” this model \mathbf{Rel} by considering sets equipped with an additional structure of *totality: a non-uniform totality space* (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X)$ is a set of subsets which intuitively represent the total, that is, terminating computations of type X . This set $\mathcal{T}(X)$ is required to coincide with its bidual for a duality expressed in terms of non-empty intersections; one nice feature of this definition is that the bidual

⁴ Exponentials are not considered in μMALL because some form of exponential can be encoded using inductive/coinductive types, however these exponentials are not fully satisfactory from our point of view because their denotational semantics do not satisfy all required properties.

⁵ Sometimes called new-Seely category in the literature

of a set of subsets of $|X|$ is simply its upwards-closure (wrt. inclusion)⁶. Given two NUTS X and Y there is a natural notion of *total relation* $t \subseteq |X| \times |Y|$ giving rise to a category **Nuts** which is easily seen to be a Seely model of LL. To turn it into a categorical model of μ LL, we need a notion of strong functors $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. Rather than considering them directly as functors, we define *variable non-uniform totality spaces* (VNUTS) as pairs $\overline{\mathbb{X}} = (|\overline{\mathbb{X}}|, \mathcal{T}(\overline{\mathbb{X}}))$ where $|\overline{\mathbb{X}}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a variable set and, for each tuple $\vec{X} = (X_1, \dots, X_n)$ of VNUTS's, $\mathcal{T}(\overline{\mathbb{X}})(\vec{X})$ is a totality structure on the set $|\overline{\mathbb{X}}|(|\vec{X}|)$. It is also required that the action of the functor $|\overline{\mathbb{X}}|$ on **Nuts** morphisms and the strength $\overline{\mathbb{X}}$ respect this totality structures. Then it is easy to derive from such a VNUTS $\overline{\mathbb{X}}$ a strong functor $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ and we prove that, equipped with these strong functors, **Nuts** is a model of μ LL. Most proofs are given in an Appendix.

Related work. In [18], that we became aware of only recently, Loader extends the simply typed λ -calculus with inductive types and develops its denotational semantics. His models are cartesian closed category \mathcal{C} equipped with a class of strong functors and seem very close to ours (Section 2.3): one might think that any of our models yields a Loader model as its Kleisli category. This is not the case because in a Loader model the category \mathcal{C} is cocartesian⁷ whereas the Kleisli category of a Seely category is not in general: this would require an iso between $!(X \oplus Y)$ and $!X \oplus !Y$ which is usually absent. Loader studies two concrete instances of his models: one is based on recursion theory (partial equivalence relations) and the other on a notion of domains with totality described as a model of LL and might give rise to one of our Seely models, this point requires further studies. Our NUTS are quite different from Loader totality domains which feature a notion of “consistency” enforcing some kind of determinism and, combined with totality, allow the Kleisli category to be cocartesian as well. Our model is based on **Rel** and therefore is compatible with non-determinism [5] and PCF recursion. This is important for us because we would like to consider rules beyond Park’s rule for inductive and coinductive types, based on PCF fixed points with further guardedness conditions⁸ in the spirit of [20] or even on infinite terms in the spirit of [2]. We mention also the work of Clairambault [6, 7] who investigates the game with totality semantics of an extension of intuitionistic logic with least and greatest fixed points (without reference to [18]). A Kleisli-like connection with his work should be sought too.

Notations We use the following notational conventions: \vec{a} stands for a list (a_1, \dots, a_n) . An operation f is extended to lists of arguments in the obvious way: $f(\vec{a}) = (f(a_1), \dots, f(a_n))$. When we write natural transformations, we very often omit the objects where they are taken and prefer to keep these objects implicit for the sake of readability, because they can easily be retrieved

⁶ This is a major simplification wrt. notions of totality on coherence spaces [12] or Loader’s totality spaces [17] where biduality is much harder to deal with.

⁷ To account for the disjunction of his logical system which crucial for defining interesting data-types such as the integers.

⁸ For guaranteeing termination.

from the context. If \mathcal{A} is a category then $\text{Obj}(\mathcal{A})$ is its class of objects and if $A, B \in \text{Obj}(\mathcal{A})$ then $\mathcal{A}(A, B)$ is the set of morphisms from A to B in \mathcal{A} (all the categories we consider are locally small). If $\mathcal{F} : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is a functor and $A \in \text{Obj}(\mathcal{A})$ then $\mathcal{F}_A : \mathcal{B} \rightarrow \mathcal{C}$ is the functor defined by $\mathcal{F}_A(B) = \mathcal{F}(A, B)$ and $\mathcal{F}_A(f) = \mathcal{F}(\text{Id}_A, f)$.

2 Categorical models of LL

Seely categories. We define the basic notion of categorical model of LL (our main reference is the notion of a *Seely category* as presented in [19]. We refer to that survey for all the technical material that we do not record here).

A Seely category is a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \gamma)$ where $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X,Y,Z} \in \mathcal{L}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ and $\gamma_{X,Y} \in \mathcal{L}(X \otimes Y, Y \otimes X)$ are natural isomorphisms satisfying coherence diagrams that we do not record here. We use $X \multimap Y$ for the object of linear morphisms from X to Y , $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ for the evaluation morphism and cur for the linear curryfication map $\mathcal{L}(Z \otimes X, Y) \rightarrow \mathcal{L}(Z, X \multimap Y)$. We assume this SMCC to be $*$ -autonomous with dualizing object \perp (this object is part of the structure of a Seely category). We use X^\perp for the object $X \multimap \perp$ of \mathcal{L} (the dual, or linear negation, of X). It is also assumed that \mathcal{L} is cartesian with final object \top , product $X_1 \& X_2$ with projections π_1, π_2 . By $*$ -autonomy \mathcal{L} is cocartesian with initial object 0 , coproduct \oplus and injections $\bar{\pi}_i$.

We also assume to be given a comonad $!_ : \mathcal{L} \rightarrow \mathcal{L}$ with counit $\text{der}_X \in \mathcal{L}(!_X, X)$ (*dereliction*) and comultiplication $\text{dig}_X \in \mathcal{L}(!_X, !!X)$ (*digging*) together with a symmetric monoidal structure (Seely natural isos $\mathfrak{m}^0 : 1 \rightarrow !\top$ and \mathfrak{m}^2 with $\mathfrak{m}_{X_1, X_2}^2 : !X_1 \otimes !X_2 \rightarrow !(X_1 \& X_2)$ for the functor $!_$, from the symmetric monoidal category $(\mathcal{L}, \&)$ to the symmetric monoidal category (\mathcal{L}, \otimes) satisfying an additional coherence condition wrt. dig . This strong monoidal structure allows to define a lax monoidal structure (μ^0, μ^2) of $!_$ from (\mathcal{L}, \otimes) to itself. More precisely $\mu^0 \in \mathcal{L}(1, !1)$ and $\mu_{X_1, X_2}^2 \in \mathcal{L}(!_X \otimes !_Y, !(X_1 \otimes X_2))$ are defined using \mathfrak{m}^0 and \mathfrak{m}^2 (and are not isos in most cases). Also, for each object $X \in \text{Obj}(\mathcal{L})$, there is a canonical structure of commutative \otimes -comonoid on $!X$ given by $\text{w}_X \in \mathcal{L}(!_X, 1)$ and $\text{contr}_X \in \mathcal{L}(!_X, !_X \otimes !_X)$. The definition of these morphisms involves all the structure of $!_$ explained above, and in particular the Seely isos. We use $?_$ for the ‘‘De Morgan dual’’ of $!_$: $?X = (!X^\perp)^\perp$ and similarly for morphisms. It is a monad on \mathcal{L} .

2.1 Strong functors on \mathcal{L}

Given $n \in \mathbb{N}$, an *n-ary strong functor* on \mathcal{L} is a pair $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$ is a functor and $\widehat{\mathbb{F}}_{X, \vec{Y}} \in \mathcal{L}(!_X \otimes \overline{\mathbb{F}}(\vec{Y}), \overline{\mathbb{F}}(!_X \otimes \vec{Y}))$ is a natural transformation, called the *strength* of \mathbb{F} . We use the notation $Z \otimes (Y_1, \dots, Y_n) = (Z \otimes Y_1, \dots, Z \otimes Y_n)$. It is assumed moreover that the diagrams of Figure 1 commute, expressing a monoidality of this strength.

$$\begin{array}{ccc}
(!X_1 \otimes !X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) & \xrightarrow{m^2 \otimes \overline{\mathbb{F}}(\vec{Y})} & !(X_1 \& X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\downarrow !X_1 \otimes \widehat{\mathbb{F}}_{X_2, \vec{Y}} & & \downarrow \widehat{\mathbb{F}}_{X_1 \& X_2, \vec{Y}} \\
!X_1 \otimes \overline{\mathbb{F}}(!X_2 \otimes \vec{Y}) & & 1 \otimes \overline{\mathbb{F}}(\vec{Y}) \xrightarrow{m^0 \otimes \overline{\mathbb{F}}(\vec{Y})} !\top \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\widehat{\mathbb{F}}_{X_1, !X_2 \otimes \vec{Y}} \downarrow & & \wr \downarrow \widehat{\mathbb{F}}_{\top, \vec{Y}} \\
\overline{\mathbb{F}}(!X_1 \otimes !X_2 \otimes \overline{\mathbb{F}}(\vec{Y})) & \xrightarrow{\overline{\mathbb{F}}(m^2 \otimes \vec{Y})} & \overline{\mathbb{F}}(!\top \otimes \vec{Y}) \\
& & \overline{\mathbb{F}}(1 \otimes \vec{Y}) \xrightarrow{\overline{\mathbb{F}}(m^0 \otimes \vec{Y})} \overline{\mathbb{F}}(\overline{\mathbb{F}}(!\top \otimes \vec{Y}))
\end{array}$$

Fig. 1. Monoidality diagrams for strong functors

Operations on strong functors. There is an obvious unary identity strong functor \mathcal{I} and for each object Y of \mathcal{L} there is an n -ary Y -valued constant strong functor \mathcal{K}^Y ; in the first case the strength natural transformation is the identity morphism and in the second case, it is defined using $w_{!X}$. Let \mathbb{F} be an n -ary strong functor and $\mathbb{G}_1, \dots, \mathbb{G}_n$ be k -ary strong functors. Then one defines a k -ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n)$: the functorial component $\overline{\mathbb{H}}$ is defined in the obvious compositional way. The strength is defined as follows

$$!X \otimes \overline{\mathbb{H}}(\vec{Y}) \xrightarrow{\widehat{\mathbb{F}}} \overline{\mathbb{F}}((!X \otimes \overline{\mathbb{G}}_i(\vec{Y}))_{i=1}^n) \xrightarrow{\overline{\mathbb{F}}((\widehat{\mathbb{G}}_i)_{i=1}^k)} \overline{\mathbb{F}}((\overline{\mathbb{G}}_i(!X \otimes \vec{Y}))_{i=1}^n) = \overline{\mathbb{H}}(!X \otimes \vec{Y})$$

and is easily seen to satisfy the required monoidality commutations. Given an n -ary strong functor, we can define its *De Morgan dual* \mathbb{F}^\perp which is also an n -ary strong functor. On objects, we set $\overline{\mathbb{F}}^\perp(\vec{Y}) = \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp$ and similarly for morphisms. The strength of \mathbb{F}^\perp is defined as the Curry transpose of the following morphism (remember that $!X \multimap \vec{Y}^\perp = (!X \otimes \vec{Y})^\perp$ up to canonical iso):

$$\begin{array}{ccc}
!X \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \otimes \overline{\mathbb{F}}(!X \multimap \vec{Y}^\perp) & \xrightarrow{\sim} & !X \otimes \overline{\mathbb{F}}(!X \multimap \vec{Y}^\perp) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp \\
& & \downarrow \widehat{\mathbb{F}} \otimes \text{Id} \\
\perp \xleftarrow{\text{ev } \gamma} \overline{\mathbb{F}}(\vec{Y}^\perp) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp & \xleftarrow{\overline{\mathbb{F}}(\text{ev}) \otimes \text{Id}} & \overline{\mathbb{F}}(!X \otimes (!X \multimap \vec{Y}^\perp)) \otimes \overline{\mathbb{F}}(\vec{Y}^\perp)^\perp
\end{array}$$

Then it is possible to prove, using the $*$ -autonomy of \mathcal{L} , that $\mathbb{F}^{\perp\perp}$ and \mathbb{F} are canonically isomorphic (as strong functors)⁹. As a direct consequence of the definition of \mathbb{F}^\perp and of the canonical iso between $\mathbb{F}^{\perp\perp}$ and \mathbb{F} we get:

Lemma 1. $(\mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n))^\perp = \mathbb{F}^\perp \circ (\mathbb{G}_1^\perp, \dots, \mathbb{G}_n^\perp)$ up to canonical iso.

The bifunctor \otimes can be turned into a strong functor: one defines the strength as $!X \otimes Y_1 \otimes Y_2 \xrightarrow{\text{contr}_{!X} \otimes \text{Id}} !X \otimes !X \otimes Y_1 \otimes Y_2 \xrightarrow{\sim} !X \otimes Y_1 \otimes !X \otimes Y_2$. By De Morgan duality, this endows \mathfrak{A} with a strength as well. The bifunctor \oplus is also endowed with a strength, simply using the distributivity of \otimes over \oplus (which

⁹ In the concrete settings considered in this paper, these canonical isos are actually identity maps.

in turn results from the fact that \mathcal{L} is symmetric monoidal closed). By duality again, $\&$ inherits a strength as well. Last the unary functor $!_-$ can be equipped with a strength as follows $!X \otimes !Y \xrightarrow{\text{dig}_X \otimes !Y} !!X \otimes !Y \xrightarrow{\mu^2} !(X \otimes Y)$.

2.2 Fixed Points of strong functors.

The following facts are standard in the literature on fixed points of functors.

Definition 1. *Let \mathcal{A} be a category and $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a functor. A coalgebra of \mathcal{F} is a pair (A, f) where A is an object of \mathcal{A} and $f \in \mathcal{A}(A, \mathcal{F}(A))$. Given two coalgebras (A, f) and (A', f') of \mathcal{F} , a coalgebra morphism from (A, f) to (A', f') is an $h \in \mathcal{A}(A, A')$ such that $f' h = \mathcal{F}(h) f$. The category of coalgebras of the functor \mathcal{F} will be denoted as $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$. The notion of algebra of an endofunctor is defined dually (reverse the directions of the arrows f and f') and the corresponding category is denoted as $\mathbf{Alg}_{\mathcal{A}}(\mathcal{F})$.*

By Lambek's Lemma, if (A, f) with $f \in \mathcal{A}(A, \mathcal{F}(A))$ is a final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$ then f is an iso. We assume that this iso is always the identity as this holds in our concrete models so that this final object $(\nu\mathcal{F}, \text{Id})$ satisfies $\mathcal{F}(\nu\mathcal{F}) = \nu\mathcal{F}$. We focus on coalgebras rather than algebras for reasons which will become clear when we deal with fixed points of strong functors. This universal property of $\nu\mathcal{F}$ gives us a powerful tool for proving equalities of morphisms.

Lemma 2. *Let $A \in \text{Obj}(\mathcal{A})$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

Lemma 3. *Let $\mathcal{F} : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{A}$ be a functor such that, for all $B \in \text{Obj}(\mathcal{B})$, the category $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ has a final object. Then there is a functor $\nu\mathcal{F}$ such that $(\nu\mathcal{F}(B), \text{Id})$ is the final object of $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ (so that $\mathcal{F}(B, \nu\mathcal{F}(B)) = \nu\mathcal{F}(B)$) for each $B \in \text{Obj}(\mathcal{B})$, and, for each $g \in \mathcal{B}(B, B')$, $\nu\mathcal{F}(g)$ is uniquely characterized by $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.*

We consider now the same $\nu\mathcal{F}$ operation applied to strong functors on a model \mathcal{L} of LL. Let $\overline{\mathbb{F}}$ be an $n + 1$ -ary strong functor on \mathcal{L} (so that $\overline{\mathbb{F}}$ is a functor $\mathcal{L}^{n+1} \rightarrow \mathcal{L}$). Assume that for each $\vec{X} \in \text{Obj}(\mathcal{L}^n)$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{X}})$ has a final object. We have defined a functor $\nu\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$ uniquely characterized by $\overline{\mathbb{F}}(\vec{X}, \nu\overline{\mathbb{F}}(\vec{X})) = \nu\overline{\mathbb{F}}(\vec{X})$ and $\overline{\mathbb{F}}(\vec{f}, \nu\overline{\mathbb{F}}(\vec{f})) = \nu\overline{\mathbb{F}}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ (Lemma 3). For each $Y, \vec{X} \in \mathcal{L}$, we define $\widehat{\nu\overline{\mathbb{F}}}_{Y, \vec{X}} \in \mathcal{L}(!Y \otimes \nu\overline{\mathbb{F}}(\vec{X}), \nu\overline{\mathbb{F}}(!Y \otimes \vec{X}))$.

We have $!Y \otimes \nu\overline{\mathbb{F}}(\vec{X}) = !Y \otimes \overline{\mathbb{F}}(\vec{X}, \nu\overline{\mathbb{F}}(\vec{X})) \xrightarrow{\overline{\mathbb{F}}_{Y, (\vec{X}, \nu\overline{\mathbb{F}}(\vec{X}))}} \overline{\mathbb{F}}(!Y \otimes \vec{X}, !Y \otimes \nu\overline{\mathbb{F}}(\vec{X}))$ exhibiting a $\overline{\mathbb{F}}_{!Y \otimes \vec{X}}$ -coalgebra structure on $!Y \otimes \nu\overline{\mathbb{F}}(\vec{X})$. Since $\nu\overline{\mathbb{F}}(!Y \otimes \vec{X})$ is the final coalgebra of the functor $\overline{\mathbb{F}}_{!Y \otimes \vec{X}}$, we define $\widehat{\nu\overline{\mathbb{F}}}_{Y, \vec{X}}$ as the unique morphism

$!Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X}) \rightarrow \overline{\nu\mathbb{F}}(!Y \otimes \overrightarrow{X})$ such that the following diagram commutes

$$\begin{array}{ccc}
!Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X}) & \xrightarrow{\widehat{\mathbb{F}}_{Y,(\overrightarrow{X},\overline{\nu\mathbb{F}}(\overrightarrow{X}))}} & \overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, !Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X})) \\
& \searrow^{\widehat{\nu\mathbb{F}}_{Y,\overrightarrow{X}}} & \downarrow \overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \widehat{\nu\mathbb{F}}_{Y,\overrightarrow{X}}) \\
& & \overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \overline{\nu\mathbb{F}}(!Y \otimes \overrightarrow{X})) = \overline{\nu\mathbb{F}}(!Y \otimes \overrightarrow{X})
\end{array} \tag{1}$$

Lemma 4. Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} such that for each $\overrightarrow{X} \in \text{Obj}(\mathcal{L}^n)$, the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\overrightarrow{X}})$ has a final object $\nu\overline{\mathbb{F}}_{\overrightarrow{X}}$. Then there is a unique n -ary strong functor $\nu\overline{\mathbb{F}}$ such that $\overline{\nu\mathbb{F}}(\overrightarrow{X}) = \nu\overline{\mathbb{F}}_{\overrightarrow{X}}$ (and hence $\overline{\mathbb{F}}(\overrightarrow{X}, \overline{\nu\mathbb{F}}(\overrightarrow{X})) = \overline{\nu\mathbb{F}}(\overrightarrow{X})$), $\overline{\mathbb{F}}(\overrightarrow{f}, \overline{\nu\mathbb{F}}(\overrightarrow{f})) = \overline{\nu\mathbb{F}}(\overrightarrow{f})$ for all $\overrightarrow{f} \in \mathcal{L}^n(\overrightarrow{X}, \overrightarrow{X}')$ and $\overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \widehat{\nu\overline{\mathbb{F}}}_{Y,\overrightarrow{X}}) \widehat{\mathbb{F}}_{Y,(\overrightarrow{X},\overline{\nu\mathbb{F}}(\overrightarrow{X}))} = \widehat{\nu\overline{\mathbb{F}}}_{Y,\overrightarrow{X}}$.

Lemma 5. Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} such that for each $\overrightarrow{X} \in \text{Obj}(\mathcal{L}^n)$, the category $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\overrightarrow{X}})$ has an initial object $\mu\overline{\mathbb{F}}_{\overrightarrow{X}}$. Then there is a unique n -ary strong functor $\mu\overline{\mathbb{F}}$ such that $\overline{\mu\mathbb{F}}(\overrightarrow{X}) = \mu\overline{\mathbb{F}}_{\overrightarrow{X}}$ (and hence $\overline{\mathbb{F}}(\overrightarrow{X}, \overline{\mu\mathbb{F}}(\overrightarrow{X})) = \overline{\mu\mathbb{F}}(\overrightarrow{X})$), $\overline{\mathbb{F}}(\overrightarrow{f}, \overline{\mu\mathbb{F}}(\overrightarrow{f})) = \overline{\mu\mathbb{F}}(\overrightarrow{f})$ for all $\overrightarrow{f} \in \mathcal{L}^n(\overrightarrow{X}, \overrightarrow{X}')$ and $\overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \widehat{\mu\overline{\mathbb{F}}}_{Y,\overrightarrow{X}}) \widehat{\mathbb{F}}_{Y,(\overrightarrow{X},\overline{\mu\mathbb{F}}(\overrightarrow{X}))} = \widehat{\mu\overline{\mathbb{F}}}_{Y,\overrightarrow{X}}$. Moreover $(\mu\overline{\mathbb{F}})^{\perp} = \nu(\overline{\mathbb{F}}^{\perp})$.

Proof. Apply Lemma 4 to the strong functor $\overline{\mathbb{F}}^{\perp}$.

2.3 A categorical axiomatization of models of μLL

Our general definition of Seely categorical model of μLL is based on the notions and results above. We refer in particular to Section 2.1 for the basic definitions of operations on strong functors in our LL categorical setting.

Definition 2. A categorical model or Seely model of μLL is a pair $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where

1. \mathcal{L} is a Seely category
2. $\overrightarrow{\mathcal{L}} = (\mathcal{L}_n)_{n \in \mathbb{N}}$ where \mathcal{L}_n is a class of strong functors $\mathcal{L}^n \rightarrow \mathcal{L}$, and $\mathcal{L}_0 = \text{Obj}(\mathcal{L})$
3. if $\mathbb{X} \in \mathcal{L}_n$ and $\mathbb{X}_i \in \mathcal{L}_k$ (for $i = 1, \dots, n$) then $\mathbb{X} \circ \overrightarrow{\mathbb{X}} \in \mathcal{L}_k$ and all k projection strong functors $\mathcal{L}^k \rightarrow \mathcal{L}$ belong to \mathcal{L}_k
4. the strong functors \otimes and $\&$ belong to \mathcal{L}_2 , the strong functor $!_-$ belongs to \mathcal{L}_1 and, if $\mathbb{X} \in \mathcal{L}_n$, then $\mathbb{X}^{\perp} \in \mathcal{L}_n$
5. and last, for all $\mathbb{X} \in \mathcal{L}_1$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{X}})$ (see Section 2.2) has a final object. Moreover, for any $\mathbb{X} \in \mathcal{L}_{k+1}$, the associated strong functor $\nu\mathbb{X} : \mathcal{L}^k \rightarrow \mathcal{L}$ (see Lemma 4) belongs to \mathcal{L}_k .

Our goal is now to outline the interpretation of μLL formulas and proofs in such a model. This requires first to describe the syntax of formulas and proofs.

Syntax of μLL . We assume to be given an infinite set of propositional variables \mathcal{V} (ranged over by Greek letters $\zeta, \xi \dots$). We introduce a language of propositional LL formulas with least and greatest fixed points.

$A, B, \dots := 1 \mid \perp \mid A \otimes B \mid A \wp B \mid 0 \mid \top \mid A \oplus B \mid A \& B \mid !A \mid ?A \mid \zeta \mid \mu\zeta A \mid \nu\zeta A$

The notion of closed types is defined as usual, the two last constructions being the only binders.

Remark 1. In contrast with second-order LL or dependent type systems where open formulas play a crucial role, in the case of fixed points, all formulas appearing in sequents and other syntactical devices allowing to give types to programs will be closed. In our setting, open types appear only locally, for allowing the expression of (least and greatest) fixed points.

We can define two basic operations on formulas.

- *Substitution:* $A[B/\zeta]$, taking care of not binding free variables (uses α -conversion).
- *Negation or dualization:* defined by induction on formulas $1^\perp = \perp, \perp^\perp = 1, (A \wp B)^\perp = A^\perp \otimes B^\perp, (A \otimes B)^\perp = A^\perp \wp B^\perp, 0^\perp = \top, \top^\perp = 0, (A \& B)^\perp = A^\perp \oplus B^\perp, (A \oplus B)^\perp = A^\perp \& B^\perp, (!A)^\perp = ?A^\perp, (?A)^\perp = !A^\perp, \zeta^\perp = \zeta, (\mu\zeta A)^\perp = \nu\zeta A^\perp$ and $(\nu\zeta A)^\perp = \mu\zeta A^\perp$. Obviously $A^{\perp\perp} = A$ for any formula A .

Remark 2. The only subtle point of this definition is negation of propositional variables: $\zeta^\perp = \zeta$ which entails $(B[A/\zeta])^\perp = B^\perp[A^\perp/\zeta]$. If we consider B as a compound connective with placeholders labeled by variables then B^\perp is its De Morgan dual. It is also a natural way of preventing the introduction of fixed points wrt. variables with negative occurrences as in $D = \mu\zeta(1 \& (!\zeta \multimap \zeta))$ which *is not* a formula of μLL (or has not the intended meaning).

The logical system is the usual unilateral sequent calculus of classical LL [13] extended with the fixed point fragment:

$$\frac{\vdash \Gamma, F[\mu\zeta F/\zeta]}{\vdash \Gamma, \mu\zeta F} \quad (\mu - \text{fold}) \qquad \frac{\vdash \Delta, A \quad \vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash \Delta, ?\Gamma, \nu\zeta F} \quad (\nu - \text{rec})$$

By taking, in the last rule, $\Delta = A^\perp$ and proving the left premise by an axiom, we obtain the following derived rule

$$\frac{\vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash ?\Gamma, A^\perp, \nu\zeta F} \quad (\nu - \text{rec}')$$

The reduction rules can be found in [1] which also provides a proof that this system admits cut-elimination¹⁰. A cut-free proof has not the sub-formula property in general because of rule $(\nu - \text{rec})$. Though, Baelde's theorem makes sure that a proof of a sequent *which does not contain any ν -formula* has a cut-free proof with the sub-formula property.

¹⁰ The system considered by Baelde is slightly different: no exponentials, no context in the $(\nu - \text{rec})$ rule. Though it seems quite clear that his proof can be adapted to the system presented here which is equivalent to Baelde's, in terms of provability at least.

Functoriality of formulas. The cut-elimination reduction rule for the $(\mu - \text{fold})/(\nu - \text{rec})$ cut requires the possibility of substituting a proof for a variable in a formula. More precisely, let $(\zeta, \xi_1, \dots, \xi_k)$ be a list of pairwise distinct variables containing all free variables of a formula F and let $\vec{C} = (C_1, \dots, C_k)$ be a sequence of closed formulas. Let π be a proof of $\vdash ?\Gamma, A^\perp, B$, then one can define a proof $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ of $\vdash ?\Gamma, F[A/\zeta, \vec{C}/\vec{\xi}]^\perp, F[B/\zeta, \vec{C}/\vec{\xi}]$ by induction on F , see [1]. As an example, assume that $F = \mu\xi G$ (so that $(\zeta, \xi, \xi_1, \dots, \xi_k)$ is a list of pairwise distinct variables containing all free variables G). The proof $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ is defined by (setting $G' = G[\vec{C}/\vec{\xi}]$)

$$\frac{\frac{G[\pi/\zeta, (\mu\xi G')[B/\zeta]/\xi, \vec{C}/\vec{\xi}]}{\vdash ?\Gamma, G'[A/\zeta, (\mu\xi G')[B/\zeta]/\xi]^\perp, G'[B/\zeta, (\mu\xi G')[B/\zeta]/\xi]} (\mu - \text{fold})}{\frac{\vdash ?\Gamma, G'[A/\zeta, (\mu\xi G')[B/\zeta]/\xi]^\perp, (\mu\xi G')[B/\zeta]}{\vdash ?\Gamma, (\mu\xi G')[A/\zeta]^\perp, (\mu\xi G')[B/\zeta]} (\nu - \text{rec}')}$$

Notice that this case uses the additional parameters \vec{C} is the definition of this substitution. Another example is $F = G_1 \otimes G_2$: $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ is defined as

$$\frac{\frac{G_1[\pi/\zeta, \vec{C}/\vec{\xi}]}{\vdash ?\Gamma, G'_1[A/\zeta]^\perp, G'_1[B/\zeta]} \quad \frac{G_2[\pi/\zeta, \vec{C}/\vec{\xi}]}{\vdash ?\Gamma, G'_2[A/\zeta]^\perp, G'_2[B/\zeta]} (\otimes)}{\frac{\vdash ?\Gamma, ?\Gamma, G'_1[A/\zeta]^\perp, G'_2[A/\zeta]^\perp, G'_1[B/\zeta] \otimes G'_2[B/\zeta]}{\vdash ?\Gamma, G'_1[A/\zeta]^\perp, G'_2[A/\zeta]^\perp, G'_1[B/\zeta] \otimes G'_2[B/\zeta]} (\text{c})} (\otimes)$$

Notice that it is essential that all formulas of the context are of shape $?H$ (even if F is exponential-free) since we use contraction rules on this context.

Interpreting formulas and proofs (outline). We assume to be given a μLL Seely model $(\mathcal{L}, \vec{\mathcal{L}})$, see Section 2.3. With any formula A and any repetition-free sequence $\vec{\zeta} = (\zeta_1, \dots, \zeta_k)$ of type variables containing all the free variables of A , we associate $\llbracket A \rrbracket_{\vec{\zeta}} \in \mathcal{L}_k$ in the obvious way, for instance $\llbracket A \otimes B \rrbracket_{\vec{\zeta}} = \otimes \circ (\llbracket A \rrbracket_{\vec{\zeta}}, \llbracket B \rrbracket_{\vec{\zeta}}) \in \mathcal{L}_k$ by conditions (4) and (3) in Definition 2 and $\llbracket \nu\zeta A \rrbracket_{\vec{\zeta}} = \nu(\llbracket A \rrbracket_{\vec{\zeta}, \zeta})$ using condition (5). Then $\llbracket A^\perp \rrbracket_{\vec{\zeta}} = \llbracket A \rrbracket_{\vec{\zeta}}^\perp$ up to a natural isomorphism. In this outline, we leave symmetric monoidality isomorphisms of \mathcal{L} and of $!_-$ implicit (see for instance [11] how *monoidal trees* allow to take them into account). With any $\Gamma = (A_1, \dots, A_n)$ we associate an object $\llbracket \Gamma \rrbracket$ of \mathcal{L} and with any proof π of $\vdash \Gamma$ we associate a morphism $\llbracket \pi \rrbracket \in \mathcal{L}(1, \llbracket \Gamma \rrbracket)$ using the categorical constructs of \mathcal{L} is a straightforward way, see [19]. Then one proves that if π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules, then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$. This is done by an inspection of the various cut-elimination rules.

In the case of $(\mu - \text{fold})/(\nu - \text{rec})$ cut-elimination, we need the following lemma that we state in a rough way (again, isos are kept implicit).

Lemma 6. *Let $\Gamma = (D_1, \dots, D_n)$ be a sequence of closed formulas, F be a formula and $\zeta, \xi_1, \dots, \xi_k$ be a repetition-free list of variables containing all the free variables of F . Let π be a proof of $\vdash ?\Gamma, A^\perp, B$ (so that, setting $X = \llbracket D_1^\perp \rrbracket \& \dots \& \llbracket D_n^\perp \rrbracket$, we can consider that $\llbracket \pi \rrbracket \in \mathcal{L}(!X \otimes \llbracket A \rrbracket, \llbracket B \rrbracket)$) and let $\vec{C} = (C_1, \dots, C_k)$ be a list of closed formulas. Then*

$$\llbracket F \left[\pi/\zeta, \vec{C}/\vec{\xi} \right] \rrbracket = \overline{\llbracket F \rrbracket}_{\zeta, \vec{\xi}}(\llbracket \pi \rrbracket, \mathbf{w}_X \otimes \overline{\llbracket C \rrbracket}) (\widehat{\llbracket F \rrbracket}_{\zeta, \vec{\xi}})_{X, (\llbracket A \rrbracket, \overline{\llbracket C \rrbracket})}.$$

To understand this formula notice that $\overline{\llbracket F \rrbracket}_{\zeta, \vec{\xi}}(\llbracket \pi \rrbracket, \mathbf{w}_X \otimes \overline{\llbracket C \rrbracket}) \in \mathcal{L}(\overline{\llbracket F \rrbracket}_{\zeta, \vec{\xi}}(!X \otimes \llbracket A \rrbracket, !X \otimes \overline{\llbracket C \rrbracket}), \overline{\llbracket F \rrbracket}_{\zeta, \vec{\xi}}(\llbracket B \rrbracket, \overline{\llbracket C \rrbracket}))$ and that $(\widehat{\llbracket F \rrbracket}_{\zeta, \vec{\xi}})_{X, (\llbracket A \rrbracket, \overline{\llbracket C \rrbracket})}$ belongs to $\mathcal{L}(!X \otimes \overline{\llbracket F \rrbracket}_{\zeta, \vec{\xi}}(\llbracket A \rrbracket, \overline{\llbracket C \rrbracket}), \llbracket F \rrbracket_{\zeta, \vec{\xi}}(!X \otimes \llbracket A \rrbracket, !X \otimes \overline{\llbracket C \rrbracket}))$.

3 Sets and relations

The category **Rel** has sets as objects, and given sets E and F , $\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$. Identity is the diagonal relation and composition is the usual composition of relations, denoted by simple juxtaposition. If $t \in \mathbf{Rel}(E, F)$ and $u \subseteq E$ then $t \cdot u = \{b \in F \mid \exists a \in u (a, b) \in t\}$.

Rel as a model of LL. This category is a well-known model of LL in which $1 = \perp = \{*\}$, $E \otimes F = (E \multimap F) = E \wp F = E \times F$ so that $E^\perp = E$. As to the additives, $0 = \top = \emptyset$ and $\&_{i \in I} E_i = \oplus_{i \in I} E_i = \bigcup_{i \in I} \{i\} \times E_i$. The exponentials are given by $!E = ?E = \mathcal{M}_{\text{fin}}(E)$.

For the additives and multiplicatives, the operations on morphisms are defined in the obvious way. Let us be more specific about the exponentials. Given $s \in \mathbf{Rel}(E, F)$, $!s \in \mathbf{Rel}(!E, !F)$ is $!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid \forall i (a_i, b_i) \in s\}$, $\text{der}_E \in \mathbf{Rel}(!E, E)$ is given by $\text{der}_E = \{([a], a) \mid a \in E\}$ and $\text{dig}_E \in \mathbf{Rel}(!E, !!E)$ is given by $\text{dig}_E = \{(m_1 + \dots + m_n, [m_1, \dots, m_n]) \mid \forall i m_i \in \mathcal{M}_{\text{fin}}(E)\}$. Last $\mathbf{m}^0 \in \mathbf{Rel}(1, !\top)$ is $\mathbf{m}^0 = \{(*, [])\}$ and $\mathbf{m}_{E, F}^2 \in \mathbf{Rel}(!E \otimes !F, !(E \& F))$ is given by

$$\mathbf{m}_{E, F}^2 = \{([a_1, \dots, a_k], [b_1, \dots, b_l]), [(1, a_1), \dots, (1, a_k), (2, b_1), \dots, (2, b_l)] \mid a_1, \dots, a_k \in E \text{ and } b_1, \dots, b_l \in F\}.$$

Weakening $\mathbf{w}_E \in \mathbf{Rel}(!E, 1)$ and $\text{contr}_E \in \mathbf{Rel}(!E, !E \otimes !E)$ are given by $\mathbf{w}_E = \{([], *)\}$ and $\text{contr}_E = \{(m_1 + m_2, (m_1, m_2)) \mid m_i \in \mathcal{M}_{\text{fin}}(E) \text{ for } i = 1, 2\}$.

3.1 Hom-continuous functors on Rel

A functor $\mathbb{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is hom-continuous¹¹ if, for all $\vec{E}, \vec{F} \in \mathbf{Rel}^n$ and all directed set $D \subseteq \mathbf{Rel}^n(\vec{E}, \vec{F})$, one has $\mathbb{F}(\bigcup D) = \bigcup \{\mathbb{F}(\vec{s}) \mid \vec{s} \in D\}$. This

¹¹ The right setting to express this property would be that of *cpo-enriched categories* but we don't really need this general concept here.

implies in particular that if $\vec{s} \subseteq \vec{t}$, one has $\mathbb{F}(\vec{s}) \subseteq \mathbb{F}(\vec{t})$ (taking $D = \{\vec{s}, \vec{t}\}$). To simplify notations assume that $n = 1$ (but what follows holds for all values of n).

Lemma 7. *Let E and F be sets and let $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, E)$. Assume that $ts = \text{Id}_E$ and that $st \subseteq \text{Id}_F$. Then s is an injective function and $t = \{(b, a) \in F \times E \mid (a, b) \in s\}$.*

Lemma 8. *Let $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ be a hom-continuous functor. Assume that $E \subseteq F$ and let $\eta_{E,F}^+ = \{(a, a) \mid a \in E\} \in \mathbf{Rel}(E, F)$ and $\eta_{E,F}^- = \{(a, a) \mid a \in E\} \in \mathbf{Rel}(F, E)$. Then $\mathbb{F}(\eta_{E,F}^+) \in \mathbf{Rel}(\mathbb{F}(E), \mathbb{F}(F))$ is an injective function.*

Proof. Notice first that $\eta_{E,F}^- \eta_{E,F}^+ = \text{Id}_E$ and $\eta_{E,F}^+ \eta_{E,F}^- \subseteq \text{Id}_F$ and therefore $\mathbb{F}(\eta_{E,F}^-) \mathbb{F}(\eta_{E,F}^+) = \text{Id}$ by functoriality and $\mathbb{F}(\eta_{E,F}^+) \mathbb{F}(\eta_{E,F}^-) \subseteq \text{Id}$ by hom-continuity. The announced property results from Lemma 7.

Let \mathbf{Rel}^{\subseteq} be the category whose objects are sets and morphisms are set inclusions (so that $\mathbf{Rel}^{\subseteq}(E, F)$ has $\eta_{E,F}^+$ as unique element if $E \subseteq F$ and is empty otherwise). Then η^+ can be thought of as the ‘‘inclusion functor’’ $\mathbf{Rel}^{\subseteq} \rightarrow \mathbf{Rel}$, acting as the identity on objects. Obviously, \mathbf{Rel}^{\subseteq} is cocomplete¹².

Proposition 1. *If $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ is hom-continuous then $\mathbb{F}\eta^+ : \mathbf{Rel}^{\subseteq} \rightarrow \mathbf{Rel}$ is directed-cocontinuous (that is, preserves the colimits of directed sets of sets).*

Remark 3. In the proof above, we use crucially that \mathcal{D} is directed. It is easy to find examples showing that this condition is necessary.

We know that a hom-continuous functor \mathbb{F} maps inclusions to injections, we shall say that \mathbb{F} is *strict* if it maps inclusions to inclusions, that is, if $E \subseteq F$ then $\mathbb{F}(E) \subseteq \mathbb{F}(F)$ and $\mathbb{F}(\eta_{E,F}^+) = \eta_{\mathbb{F}(E), \mathbb{F}(F)}^+$ (which implies $\mathbb{F}(\eta_{E,F}^-) = \eta_{\mathbb{F}(E), \mathbb{F}(F)}^-$). As a direct consequence of Proposition 1, we get:

Lemma 9. *If \mathbb{F} is strict hom-continuous then, for any directed set of sets \mathcal{D} , one has $\mathbb{F}(\bigcup \mathcal{D}) = \bigcup_{E \in \mathcal{D}} \mathbb{F}(E)$.*

3.2 Variable sets and basic constructions on them

Definition 3. *An n -ary variable set is a strong functor $\mathbb{V} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ such that $\overline{\mathbb{V}}$ is hom-continuous and strict.*

By the general considerations of Section 2.1, we know that there is a constant strong functor $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ with value E for each set E , that there are projection strong functors $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$, that \times (that is \otimes) and $+$ (that is \oplus) define strong functors $\mathbf{Rel}^2 \rightarrow \mathbf{Rel}$, that $\mathcal{M}_{\text{fin}}(_)$ (that is $!__$) defines a strong functor $\mathbf{Rel} \rightarrow \mathbf{Rel}$, that strong functors on \mathbf{Rel} are stable under composition, and that

¹² Notice that it is not complete, for instance it has no terminal object.

if \mathbb{V} is a strong functor $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ then there is a “dual” strong functor \mathbb{V}^\perp (which is actually identical to \mathbb{V} in this very simple model). We have only to check that for each of the strong functors \mathbb{V} defined in that way, the underlying functor $\overline{\mathbb{V}}$ is a strict hom-continuous functor.

We deal with $!_-$ and composition, the other cases are similar. The underlying functor of $!_-$ is $\mathcal{M} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ defined by $\mathcal{M}(E) = \mathcal{M}_{\text{fin}}(E)$, $\mathcal{M}(s) = \{([a_1, \dots, a_k], [b_1, \dots, b_k]) \mid (a_i, b_i) \in s \text{ for } i = 1, \dots, k\}$ if $s \in \mathbf{Rel}(E, F)$. First if $s \subseteq t \in \mathbf{Rel}(E, F)$, it follows from the definition that $\mathcal{M}(s) \subseteq \mathcal{M}(t)$. Let $D \subseteq \mathbf{Rel}(E, F)$ be directed, we prove $\mathcal{M}(\bigcup D) \subseteq \bigcup_{s \in D} \mathcal{M}(s)$: an element of $\mathcal{M}(\bigcup D)$ is a pair $([a_1, \dots, a_k], [b_1, \dots, b_k])$ with $(a_i, b_i) \in \bigcup D$ for $i = 1, \dots, k$. Since D is directed, there is an $s \in D$ such that $(a_i, b_i) \in s$ for $i = 1, \dots, k$ and the inclusion follows. Strictness is obvious.

Let $\mathbb{V}_i : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ be variable sets for $i = 1, \dots, k$ and let $\mathbb{W} : \mathbf{Rel}^k \rightarrow \mathbf{Rel}$ be a variable set. Then the functor $\overline{\mathbb{W}} \circ \overrightarrow{\mathbb{V}} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is clearly strict hom-continuous (since these conditions are preservation properties) from which it follows that the strong functor $\mathbb{U} = \mathbb{W} \circ \overrightarrow{\mathbb{V}}$ is a variable type.

Fixed point of a variable set. Let $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ be a strict hom-continuous functor. Since $\emptyset \subseteq \mathbb{F}(\emptyset)$ we have $\mathbb{F}^n(\emptyset) \subseteq \mathbb{F}^{n+1}(\emptyset)$ for all $n \in \mathbb{N}$, by induction on n and hence $\mathbb{F}(\bigcup_{n=0}^{\infty} \mathbb{F}^n(\emptyset)) = \bigcup \mathbb{F}^n(\emptyset)$ by Lemma 9 since $\{\mathbb{F}^n(\emptyset) \mid n \in \mathbb{N}\}$ is directed. Let $\sigma\mathbb{F} = \bigcup_{n=0}^{\infty} \mathbb{F}^n(\emptyset)$, then $(\sigma\mathbb{F}, \text{Id}_{\sigma\mathbb{F}})$ is an \mathbb{F} -coalgebra.

Lemma 10. *The coalgebra $(\sigma\mathbb{F}, \text{Id})$ is final in $\mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})$.*

Notice that $(\sigma\mathbb{F}, \text{Id})$ is also an initial object in $\mathbf{Alg}_{\mathbf{Rel}}(\mathbb{F})$. When we insist on considering $\sigma\mathbb{F}$ as a final coalgebra, we denote it as $\nu\mathbb{F}$.

Lemma 11. *Let $\mathbb{F} : \mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ be a strict hom-continuous functor. The functor $\nu\mathbb{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is strict hom-continuous.*

Let $\mathbb{V} : \mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ be a variable set, by Lemma 4, there is a unique strong functor $\nu\mathbb{V} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ which is characterized by: $\overline{\nu\mathbb{V}}(\overrightarrow{E}) = \nu\overline{\mathbb{V}}_{\overrightarrow{E}}$, for each $\overrightarrow{s} \in \mathbf{Rel}^n(\overrightarrow{E}, \overrightarrow{F})$, $\overline{\nu\mathbb{V}}(\overrightarrow{s}) = \overline{\mathbb{V}}(\overrightarrow{s}, \nu\overline{\mathbb{V}}(\overrightarrow{s}))$ and last $\overline{\mathbb{V}}(!E \otimes \overrightarrow{F}, \widehat{\mathbb{V}}_{E, \overrightarrow{F}}) = \widehat{\mathbb{V}}_{E, \overrightarrow{F}}$.

Lemma 12. *The functor $\nu\mathbb{V}$ is a variable set.*

Proof. By the conditions above satisfied by $\nu\mathbb{V}$ we have that $\overline{\nu\mathbb{V}} = \nu\overline{\mathbb{V}}$ and hence $\overline{\nu\mathbb{V}}$ is strict hom-continuous by Lemma 11.

A model of $\mu\mathbf{LL}$ based on variable sets. Let \mathbf{Rel}_n be the class of all n -ary variable sets, so that $\mathbf{Rel}_0 = \mathbf{Obj}(\mathbf{Rel})$. The fact that $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n \in \mathbb{N}})$ is a Seely model of $\mu\mathbf{LL}$ in the sense of Section 2.3 results mainly from the fact that we take *all* variable sets in the \mathbf{Rel}_n 's so that there is essentially nothing to check. More explicitly: (1) holds by Section 3, (2) holds by construction, (3) holds by the fact that variable sets compose as explained in Section 3.2 (notice that this condition refers to the general composition of strong functors defined in Section 2.1), (4) holds by Section 3.2 and by the fact that the De Morgan dual of a strong functor is strong, see Section 2.1 and (5) holds by Section 3.2.

4 Non-uniform totality spaces

Basic definitions. Let E be a set and let $\mathcal{T} \subseteq \mathcal{P}(E)$. We define $\mathcal{T}^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$. If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{P}(E)$ then $\mathcal{T}^\perp \subseteq \mathcal{S}^\perp$. We also have $\mathcal{T} \subseteq \mathcal{T}^{\perp\perp}$ and therefore $\mathcal{T}^{\perp\perp\perp} = \mathcal{T}^\perp$. One pleasant feature of this duality is that the biorthogonal closure admits a very simple characterization.

Lemma 13. *Let $\mathcal{T} \subseteq \mathcal{P}(E)$, then $\mathcal{T}^{\perp\perp} = \uparrow\mathcal{T} = \{v \subseteq E \mid \exists u \in \mathcal{T} \ u \subseteq v\}$.*

Proof. The \supseteq direction is obvious, let us prove the converse so let $u \subseteq E$ and assume that $u \notin \uparrow\mathcal{T}$. This means that for each $v \in \mathcal{T}$ there exist $a(v) \in v$ such that $a(v) \notin u$. Let $u' = \{a(v) \mid v \in \mathcal{T}\} \subseteq E$. By construction we have $u' \in \mathcal{T}^\perp$ and $u \cap u' = \emptyset$. This shows that $u \notin \mathcal{T}^{\perp\perp}$.

A *non-uniform totality space* (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathcal{T}(X) = \mathcal{T}(X)^{\perp\perp}$, that is $\mathcal{T}(X) = \uparrow\mathcal{T}(X)$. Of course we set $X^\perp = (|X|, \mathcal{T}(X)^\perp)$.

Example 1. Let $X = (\mathbb{N}, \mathcal{T}(X))$ where $\mathcal{T}(X)$ is the set of all infinite subsets of \mathbb{N} . It is a NUTS because a superset of an infinite set is infinite. Then $|X^\perp| = \mathbb{N}$ and $\mathcal{T}(X^\perp)$ is the set of all cofinite subsets of \mathbb{N} (the subsets u of \mathbb{N} such that $\mathbb{N} \setminus u$ is finite). If, with the same web \mathbb{N} , we take $\mathcal{T}(X) = \{u \subseteq \mathbb{N} \mid u \neq \emptyset\}$ (again $\mathcal{T}(X) = \uparrow\mathcal{T}(X)$ obviously), then $\mathcal{T}(X^\perp) = \{\mathbb{N}\}$.

We define four basic NUTS: $0 = (\emptyset, \emptyset)$, $\top = (\emptyset, \{\emptyset\})$ and $1 = \perp = (\{*\}, \{\{*\}\})$. Given NUTS X_1 and X_2 we define a NUTS $X_1 \otimes X_2$ by $|X_1 \otimes X_2| = |X_1| \times |X_2|$ and $\mathcal{T}(X_1 \otimes X_2) = \uparrow\{u_1 \times u_2 \mid u_i \in \mathcal{T}(X_i) \text{ for } i = 1, 2\}$. And then we define $X \multimap Y = (X \otimes Y^\perp)^\perp$.

Lemma 14. $t \in \mathcal{T}(X \multimap Y) \Leftrightarrow \forall u \in \mathcal{T}(X) \ t \cdot u \in \mathcal{T}(Y)$.

We define the category **Nuts** whose objects are the NUTS and $\mathbf{Nuts}(X, Y) = \mathcal{T}(X \multimap Y)$, composition being defined as the usual composition in **Rel** (relational composition) and identities as the diagonal relations. Lemma 14 shows that we have indeed defined a category.

Multiplicative structure

Lemma 15. *Let X and Y be NUTS and $t \in \mathbf{Nuts}(X, Y)$. Then t is an iso in **Nuts** iff t is (the graph of) a bijection $|X| \rightarrow |Y|$ such that $\forall u \subseteq |X| \ u \in \mathcal{T}(X) \Leftrightarrow t(u) \in \mathcal{T}(Y)$.*

Lemma 16. *Let $t \subseteq |X| \times |Y|$. One has $t \in \mathbf{Nuts}(X, Y)$ iff $t^\perp = \{(b, a) \mid (a, b) \in t\} \in \mathbf{Nuts}(Y^\perp, X^\perp)$.*

Proof. This is an obvious consequence of Lemma 14 and of the fact that $(X \multimap Y) = (X \otimes Y^\perp)^\perp$ and $(Y^\perp \multimap X^\perp) = (Y^\perp \otimes X)^\perp$.

Lemma 17. *Let $t \subseteq |X_1 \otimes X_2 \multimap Y|$. One has $t \in \mathbf{Nuts}(X_1 \otimes X_2, Y)$ iff for all $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$ one has $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$.*

Lemma 18. *The bijection $\alpha_{|X_1|, |X_2|, |Y|}$ is an isomorphism from $(X_1 \otimes X_2) \multimap Y$ to $X_1 \multimap (X_2 \multimap Y)$.*

We turn now \otimes into a functor, its action on morphisms being defined as in **Rel**. Let $t_i \in \mathbf{Nuts}(X_i, Y_i)$ for $i = 1, 2$, we have $t_1 \otimes t_2 \in \mathbf{Nuts}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ by Lemma 17 and by the equation $(t_1 \otimes t_2) \cdot (u_1 \otimes u_2) = (t_1 \cdot u_1) \otimes (t_2 \cdot u_2)$. This functor is monoidal, with unit 1 and symmetric monoidality isomorphisms λ, ρ, γ and α defined as in **Rel**. The only non-trivial thing to check is that α is indeed a morphism, namely $\alpha_{|X_1|, |X_2|, |X_3|} \in \mathbf{Nuts}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$. This results from Lemma 18 and from the observation that $((X_1 \otimes X_2) \otimes X_3)^\perp = ((X_1 \otimes X_2) \multimap X_3^\perp)$ and $(X_1 \otimes (X_2 \otimes X_3))^\perp = (X_1 \multimap (X_2 \multimap X_3^\perp))$.

The SMC category **Nuts** is closed, with $X \multimap Y$ as internal hom object from X to Y , and evaluation morphism $\text{ev} = \{((a, b), a), b \mid a \in |X| \text{ and } b \in |Y|\}$ which indeed belongs to $\mathbf{Nuts}((X \multimap Y) \otimes X, Y)$ by Lemma 17 since, for all $t \in \mathcal{T}(X \multimap Y)$ and $u \in \mathcal{T}(X)$ we have $\text{ev}(t \otimes u) = t u \in \mathcal{T}(Y)$. This category **Nuts** is also *-autonomous with dualizing object $\perp = 1$.

Additive structure. Let $(X_i)_{i \in I}$ be an at most countable family of objects of **Nuts**. We define $X = \&_{i \in I} X_i$ by: $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$ and $\mathcal{T}(X) = \{u \subseteq |X| \mid \forall i \in I \pi_i \cdot u \in \mathcal{T}(X_i)\}$. It is clear that $\mathcal{T}(X) = \uparrow \mathcal{T}(X)$ and hence X is an object of **Nuts**. By definition of X and by Lemma 14 we have $\forall i \in I \pi_i \in \mathbf{Nuts}(X, X_i)$. Given $\vec{t} = (t_i)_{i \in I}$ with $\forall i \in I t_i \in \mathbf{Nuts}(Y, X_i)$, we have $\langle \vec{t} \rangle \in \mathbf{Nuts}(Y, X)$ as easily checked (using Lemma 14 again). It follows that $(\&_{i \in I} X_i, (\pi_i)_{i \in I})$ is the cartesian product of the X_i 's in **Nuts**. This shows that the category **Nuts** has all countable products and hence is cartesian. Since it is *-autonomous, the category **Nuts** is also cocartesian, coproduct being given by $\oplus_{i \in I} X_i = (\&_{i \in I} X_i^\perp)^\perp$. It follows that $X = \oplus_{i \in I} X_i = (\&_{i \in I} X_i^\perp)^\perp$ satisfies $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$ and $\mathcal{T}(X) = \{v \subseteq \bigcup_{i \in I} \{i\} \times |X_i| \mid \exists i \in I \exists u \in \mathcal{T}(X_i) \{i\} \times u \subseteq v\}$ as easily checked.

Exponential. This exponential is an extension of the multiset exponential of **Rel** with totality. Remember that $u^{(!)} = \mathcal{M}_{\text{fin}}(u)$. We set $!X = \mathcal{M}_{\text{fin}}(|X|)$ and $\mathcal{T}(!X) = \{u^{(!)} \mid u \in \mathcal{T}(X)\}^{\perp\perp} = \uparrow \{u^{(!)} \mid u \in \mathcal{T}(X)\}$.

Lemma 19. *Let $t \subseteq \mathcal{M}_{\text{fin}}(|X|) \times |Y|$. One has $t \in \mathbf{Nuts}(!X, Y)$ iff for all $u \in \mathcal{T}(X)$ one has $t \cdot u^{(!)} \in \mathcal{T}(Y)$.*

Lemma 20. *Let $t \subseteq \mathcal{M}_{\text{fin}}(|X_1|) \times \mathcal{M}_{\text{fin}}(|X_2|) \times |Y|$. One has $t \in \mathbf{Nuts}(!X_1 \otimes !X_2, Y)$ iff for all $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$, one has $t \cdot (u_1^{(!)} \otimes u_2^{(!)}) \in \mathcal{T}(Y)$.*

Lemma 21. *For any $t \in \mathbf{Nuts}(X, Y)$, one has $!t \in \mathbf{Nuts}(!X, !Y)$.*

Proof. By Lemma 19 and the fact that $!t \cdot u^{(!)} = (t \cdot u)^{(!)}$.

To prove that **Nuts** is a categorical model of LL, it suffices to show that the various relational morphisms defining the strong symmetric monoidal monadic structure of $!_-$ in **Rel** (see Section 3) are actually morphisms in **Nuts**. This is essentially straightforward and based on Lemma 19.

Lemma 22. *Equipped with der , dig , m^0 and m^2 defined as in **Rel**, $!_-$ is a symmetric monoidal comonad which turns **Nuts** into a Seely model of LL.*

4.1 Variable non-uniform totality spaces (VNUTS)

Let E be a set, we use $\text{Tot}(E)$ for the set of all *totality candidates* on E , that is, of all subsets \mathcal{T} of $\mathcal{P}(E)$ such that $\mathcal{T} = \mathcal{T}^{\perp\perp}$ (remember that $\mathcal{T}^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$). In other words $\mathcal{T} \in \text{Tot}(E)$ means that $\uparrow\mathcal{T}$ by Lemma 13. Ordered by \subseteq , this set $\text{Tot}(E)$ is a complete lattice.

Definition 4. *Let $n \in \mathbb{N}$, an n -ary VNUTS is a pair $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ where $|\mathbb{X}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a variable set $|\mathbb{X}| = (\overline{|\mathbb{X}|}, \widehat{|\mathbb{X}|})$ (see Section 3.2) and $\mathcal{T}(\mathbb{X})$ is an operation which with each n -tuple \vec{X} of objects of **Nuts** associates an element $\mathcal{T}(\mathbb{X})(\vec{X})$ of $\text{Tot}(\overline{|\mathbb{X}|}(|\vec{X}|))$ in such a way that*

1. *for any $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$, the element $\overline{|\mathbb{X}|}(\vec{t})$ of $\mathbf{Rel}(\overline{|\mathbb{X}|}(|\vec{X}|), |\mathbb{X}|(|\vec{Y}|))$ belongs actually to $\mathbf{Nuts}(\overline{|\mathbb{X}|}(\vec{X}), \overline{|\mathbb{X}|}(\vec{Y}))$ (where $\overline{|\mathbb{X}|}(\vec{X})$ denotes the NUTS $(\overline{|\mathbb{X}|}(|\vec{X}|), \mathcal{T}(\mathbb{X})(\vec{X}))$)*
2. *and for any $\vec{Y} \in \text{Obj}(\mathbf{Nuts}^n)$ and any $X \in \text{Obj}(\mathbf{Nuts})$ one has $\widehat{|\mathbb{X}|}_{|X|, |\vec{Y}|} \in \mathbf{Nuts}(!X \otimes \overline{|\mathbb{X}|}(\vec{Y}), \overline{|\mathbb{X}|}(!X \otimes \vec{Y}))$. In other words, for an $u \in \mathcal{T}(X)$ and $v \in \text{Tot}(\mathbb{X})(\vec{Y})$, one has $\widehat{|\mathbb{X}|}_{|X|, |\vec{Y}|} \cdot (u^{(!)} \otimes v) \in \text{Tot}(\mathbb{X})(!X \otimes \vec{Y})$.*

Lemma 23. *Any VNUTS $\mathbb{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ induces a strong functor $\mathcal{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ which satisfies*

- $|\mathcal{X}(\vec{X})| = \overline{|\mathbb{X}|}(|\vec{X}|)$,
- $\mathcal{T}(\mathcal{X}(\vec{X})) = \mathcal{T}(\mathbb{X})(\vec{X})$,
- $\mathcal{X}(\vec{t}) = \overline{|\mathbb{X}|}(\vec{t}) \in \mathbf{Nuts}(\overline{|\mathbb{X}|}(\vec{X}), \overline{|\mathbb{X}|}(\vec{Y}))$ for $\vec{t} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$,
- and $\mathcal{X}_{X, \vec{Y}} = \widehat{|\mathbb{X}|}_{|X|, |\vec{Y}|}$

and \mathbb{X} can be retrieved from \mathcal{X} .

For this reason we use \mathbb{X} to denote the functor \mathcal{X} . Given $n \in \mathbb{N}$ let \mathbf{Vnuts}_n be the class of strong n -ary VNUTS. We identify \mathbf{Vnuts}_0 with the class of objects of the Seely category **Nuts**. The following refers to Definition 2

Theorem 1. *$(\mathbf{Nuts}, (\mathbf{Vnuts}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL .*

Proof (partial). We deal with Condition (5). Let first $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ be a unary VNUTS. Let $E = \sigma|\overline{\mathbb{X}}|$ which is the least set such that $|\overline{\mathbb{X}}|(E) = E$, that is $E = \bigcup_{n=0}^{\infty} |\overline{\mathbb{X}}|^n(\emptyset)$. Let $\Phi : \text{Tot}(E) \rightarrow \text{Tot}(E)$ be defined as follows: given $\mathcal{T} \in \text{Tot}(E)$, then (E, \mathcal{T}) is a NUTS, and we set $\Phi(\mathcal{T}) = \mathcal{T}(\mathbb{X})(E, \mathcal{T}) \in \text{Tot}(|\overline{\mathbb{X}}|(E)) = \text{Tot}(E)$. This function Φ is monotone. Let indeed $\mathcal{S}, \mathcal{T} \in \text{Tot}(E)$ with $\mathcal{S} \subseteq \mathcal{T}$. Then we have $\text{ld} \in \mathbf{Nuts}((E, \mathcal{S}), (E, \mathcal{T}))$ and therefore, by Condition (1) satisfied by \mathbb{X} , we have $\text{ld} = |\overline{\mathbb{X}}|(\text{ld}) \in \mathbf{Nuts}(\overline{\mathbb{X}}(E, \mathcal{S}), \overline{\mathbb{X}}(E, \mathcal{T})) = \mathbf{Nuts}((E, \Phi(\mathcal{S})), (E, \Phi(\mathcal{T})))$ which means that $\Phi(\mathcal{S}) \subseteq \Phi(\mathcal{T})$. By the Knaster Tarski Theorem (remember that $\text{Tot}(E)$ is a complete lattice), Φ has a greatest fixpoint \mathcal{T} that we can describe as follows. Let $(\mathcal{T}_\alpha)_{\alpha \in \mathbb{O}}$, where \mathbb{O} is the class of ordinals, be defined by: $\mathcal{T}_0 = \mathcal{P}(E)$ (the largest possible notion of totality on E), $\mathcal{T}_{\alpha+1} = \Phi(\mathcal{T}_\alpha)$ and $\mathcal{T}_\lambda = \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha$ when λ is a limit ordinal. This sequence is decreasing (easy induction on ordinals using the monotonicity of Φ) and there is an ordinal θ such that $\mathcal{T}_{\theta+1} = \mathcal{T}_\theta$ (by a cardinality argument; we can assume that θ is the least such ordinal). The greatest fixpoint of Φ is then \mathcal{T}_θ as easily checked.

By construction $((E, \mathcal{T}_\theta), \text{ld})$ is an object of $\mathbf{Coalg}_{\mathbf{Nuts}}(\overline{\mathbb{X}})$, we prove that it is the terminal object. So let (Y, t) be another object of the same category. Since $(|Y|, t)$ is an object of $\mathbf{Coalg}_{\mathbf{Rel}}(|\overline{\mathbb{X}}|)$ and since (E, ld) is the terminal object in that category, we know by Lemma 10 that there is exactly one $e \in \mathbf{Rel}(|Y|, E)$ such that $|\overline{\mathbb{X}}|(e)t = e$. We prove that actually $e \in \mathbf{Nuts}(Y, (E, \mathcal{T}_\theta))$ so let $v \in \mathcal{T}(Y)$. We prove by induction on the ordinal α that $e \cdot v \in \mathcal{T}_\alpha$. For $\alpha = 0$ it is obvious since $\mathcal{T}_0 = \mathcal{P}(E)$. Assume that the property holds for α and let us prove it for $\alpha + 1$. We have $t \cdot v \in \mathcal{T}(\mathbb{X})(Y) = \mathcal{T}(\overline{\mathbb{X}}(Y))$ since $t \in \mathbf{Nuts}(Y, \overline{\mathbb{X}}(Y))$. Since $\overline{\mathbb{X}}(e) \in \mathbf{Nuts}(\overline{\mathbb{X}}(Y), \overline{\mathbb{X}}(E, \mathcal{T}_\alpha))$ and $\overline{\mathbb{X}}(E, \mathcal{T}_\alpha) = (E, \mathcal{T}_{\alpha+1})$ we have $(\overline{\mathbb{X}}(e)t) \cdot v \in \mathcal{T}_{\alpha+1}$, that is $e \cdot v \in \mathcal{T}_{\alpha+1}$. Last if λ is a limit ordinal and if we assume $\forall \alpha < \lambda e \cdot v \in \mathcal{T}_\alpha$ we have $e \cdot v \in \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha = \mathcal{T}_\lambda$. Therefore $e \cdot v \in \mathcal{T}_\theta$. We use $\nu\overline{\mathbb{X}}$ to denote this final coalgebra (E, \mathcal{T}_θ) (its definition depends only on $\overline{\mathbb{X}}$ and does not involve the strength $\widehat{\mathbb{X}}$).

So we have proven the first part of Condition (5) in the definition of a Seely model of μLL (see Section 2). As to the second part, let \mathbb{X} be an $n + 1$ -ary VNUTS. We know by the general Lemma 4 that there is a uniquely defined strong functor $\nu\mathbb{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ such that

- $\overline{\nu\mathbb{X}}(\overrightarrow{X}) = \nu(\overline{\mathbb{X}}_{\overrightarrow{X}})$, so that $\overline{\mathbb{X}}(\overrightarrow{X}, \overline{\nu\mathbb{X}}(\overrightarrow{X})) = \overline{\nu\mathbb{X}}(\overrightarrow{X})$, for all $\overrightarrow{X} \in \text{Obj}(\mathbf{Nuts}^n)$,
- $\overline{\mathbb{X}}(\overrightarrow{t}, \overline{\nu\mathbb{X}}(\overrightarrow{t})) = \overline{\nu\mathbb{X}}(\overrightarrow{t})$ for all $\overrightarrow{t} \in \mathbf{Nuts}(\overrightarrow{X}, \overrightarrow{Y})$
- and $\overline{\mathbb{X}}(Y \otimes \overrightarrow{X}, \overline{\nu\mathbb{X}}_{Y, \overrightarrow{X}} \widehat{\mathbb{X}}_{Y, (\overrightarrow{X}, \overline{\nu\mathbb{X}}(\overrightarrow{X}))}) = \widehat{\nu\mathbb{X}}_{Y, \overrightarrow{X}}$ for all $Y \in \text{Obj}(\mathbf{Nuts})$ and $\overrightarrow{X} \in \text{Obj}(\mathbf{Nuts}^n)$.

To end the proof, it will be enough to exhibit an n -ary VNUTS $\mathbb{Y} = (|Y|, \mathcal{T}(Y))$ whose associated strong functor coincides with $\nu\mathbb{X}$. The construction of \mathbb{Y} is essentially straightforward, using the constructions available in \mathbf{Rel} .

Remark 4. For any closed formula A , the web of its interpretation $\llbracket A \rrbracket^{\mathbf{Nuts}}$ in \mathbf{Nuts} coincides with its interpretation $\llbracket A \rrbracket^{\mathbf{Rel}}$ in \mathbf{Rel} . It is also easy to check that for any proof π of $\vdash A$, one has $\llbracket \pi \rrbracket^{\mathbf{Nuts}} = \llbracket \pi \rrbracket^{\mathbf{Rel}}$ (this can be formalized by a structure preserving functor $\mathbf{Nuts} \rightarrow \mathbf{Rel}$ which acts trivially on morphisms).

4.2 Examples of data-types

Integers. The type of “flat integers” is defined by $\iota = \mu\zeta(1 \oplus \zeta)$. In **Rel**, $1 \oplus \zeta$ is interpreted as the unary variable set $\llbracket 1 \oplus \zeta \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ which maps a set E to $1 \oplus E = \{(1, *)\} \cup (\{2\} \times E)$. Hence $\llbracket \iota \rrbracket^{\mathbf{Rel}}$ is the least set such that $\llbracket \iota \rrbracket = \{(1, *)\} \cup (\{2\} \times \llbracket \iota \rrbracket)$ that is, the set of all tuples $\bar{n} = (2, (2, (\dots (1, *) \dots)))$ where n is the number of occurrence of 2, that is $\llbracket \iota \rrbracket^{\mathbf{Rel}} = \mathbb{N}$ up to renaming. We have $|\llbracket \iota \rrbracket^{\mathbf{Nuts}}| = |\llbracket \iota \rrbracket^{\mathbf{Rel}}| = \mathbb{N}$ and we compute $\mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$ dually wrt. the proof of Theorem 1: it is the least fixed point of the operator $\Phi : \mathbf{Tot}(\mathbb{N}) \rightarrow \mathbf{Tot}(\mathbb{N})$ (remember that $\mathbf{Tot}(\mathbb{N})$ is just the set of all \subseteq -upwards-closed subsets of \mathbb{N}) such that, if $\mathcal{T} \in \mathbf{Tot}(\mathbb{N})$ then $\Phi(\mathcal{T}) = \{u \subseteq \mathbb{N} \mid 0 \in u \text{ or } \{n \in \mathbb{N} \mid n + 1 \in u\} \in \mathcal{T}\}$. Therefore $\mathbf{Tot}(\llbracket \iota \rrbracket^{\mathbf{Nuts}}) = \{u \subseteq \mathbb{N} \mid u \neq \emptyset\}$. So if π is a proof of $\vdash \iota$, we know that $\llbracket \pi \rrbracket^{\mathbf{Rel}} = \llbracket \pi \rrbracket^{\mathbf{Nuts}} \in \mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$, and hence is a non-empty set. Using an additional notion of coherence (which can be fully compatible with **Rel** as in the non-uniform coherence space models of [4, 3]) we can also prove that $\llbracket \pi \rrbracket^{\mathbf{Rel}}$ has at most one element, and hence is a singleton $\{n\}$. This is a denotational version of normalization expressing that indeed π “has a value” (and actually exactly one, which expresses a weak form of confluence).

Binary trees with integer leaves. This type can be defined as $\tau = \mu\zeta(\iota \oplus (\zeta \otimes \zeta))$. Then an element of $\llbracket \tau \rrbracket^{\mathbf{Rel}} = |\llbracket \tau \rrbracket^{\mathbf{Nuts}}|$ is an element of the set described by the following syntax: $\alpha, \beta, \dots := \langle n \rangle \mid \langle \alpha, \beta \rangle$. A computation similar to the previous one shows that $\mathbf{Tot}(\llbracket \tau \rrbracket^{\mathbf{Nuts}}) = \{u \subseteq \llbracket \tau \rrbracket^{\mathbf{Rel}} \mid u \neq \emptyset\}$.

A disappointing type of streams of integers. After reading [2], one could be tempted to define the type of streams of integers as $\sigma = \nu\zeta(\iota \otimes \zeta)$. The variable set $\llbracket \iota \otimes \zeta \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ maps a set E to $\mathbb{N} \times E$. The least fixed point of this operation on sets is \emptyset and hence $|\llbracket \sigma \rrbracket^{\mathbf{Nuts}}| = \emptyset$ and notice that $\mathbf{Tot}(\emptyset) = \{\emptyset, \{\emptyset\}\}$. In that case, the operation $\Phi : \mathbf{Tot}(\emptyset) \rightarrow \mathbf{Tot}(\emptyset)$ maps \mathcal{T} to $\{u \times v \mid v \in \mathcal{T} \text{ and } u \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}\}$ and hence $\{\emptyset\}$ to itself. It follows that $\mathcal{T}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}) = \{\emptyset\}$. What is the meaning of this trivial interpretation? It simply reflects that, though σ has a lot of non trivial proofs in $\mu\mathbf{LL}$, it is impossible to extract any finite information from these proofs within $\mu\mathbf{LL}$, and accordingly all these proofs are interpreted as \emptyset . For instance, there is no proof in $\mu\mathbf{LL}$ of $\vdash \sigma^{\perp}, \iota$ (such as a function extracting the first element of a stream). Indeed if π were such a proof, we would have $\llbracket \pi \rrbracket \in \mathbf{Nuts}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}, \llbracket \iota \rrbracket^{\mathbf{Nuts}})$ and hence $\emptyset = \llbracket \pi \rrbracket \cdot \emptyset \in \mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$ which is not the case. If such infinite types are meaningful in a proof-search perspective, their significance as data-types in a curry-Howard approach to $\mu\mathbf{LL}$ is dubious.

A type of streams of integers. Let $\sigma = \nu\zeta(1 \& (\iota \otimes \zeta))$. This type looks like the previous one, but the type 1 leaves space for some kind of “empty stream”. The $\&$ means that this empty stream will not be a total element: it will have to be complemented by some total element from the right argument of the $\&$.

More precisely $\llbracket 1 \& (\mathbb{N} \otimes \zeta) \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ is the variable set which maps a set E to $\{(1, *)\} \cup \{2\} \times \mathbb{N} \times E$ so that up to renaming $\llbracket \sigma \rrbracket^{\mathbf{Nuts}} = \mathbb{N}^{<\omega}$ (all *finite sequences* of integers). In this case, the operator $\Phi : \mathbf{Tot}(\mathbb{N}^{<\omega}) \rightarrow \mathbf{Tot}(\mathbb{N}^{<\omega})$ maps \mathcal{T} to $\{v \subseteq \mathbb{N}^{<\omega} \mid () \in v \text{ and } \exists n \in \mathbb{N}, u \in \mathcal{T} \{n\} \times u \subseteq v\}$ where we use $()$ for the empty sequence. So $\Phi^0(\mathcal{P}(\mathbb{N}^{<\omega})) = \mathcal{P}(\mathbb{N}^{<\omega})$, $\Phi^1(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid () \in u\}$, $\Phi^2(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid () \in u \text{ and } \exists n (n) \in u\}$, $\Phi^3(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid \exists n_1, n_2 (), (n_1), (n_1, n_2) \in u\}$ etc. The greatest fixed point is reached in ω steps, explicitly $\mathbf{Tot}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}) = \bigcap_{n < \omega} \Phi^n(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \subseteq \mathbb{N}^{<\omega} \mid \exists f \in \mathbb{N}^\omega \forall k < \omega (f(1), \dots, f(k)) \in u\}$. So a total subset of $\llbracket \sigma \rrbracket^{\mathbf{Nuts}}$ must contain (at least) an infinite stream of integer. It is easy to build a proof of $\vdash \sigma^\perp, \iota$ extracting the first element of a stream, interpreted as $\{((n), n) \mid n \in \mathbb{N}\}$.

5 Conclusion and further work

One of the main goals of this work is to develop syntax-independent tools to study new proof-systems for $\mu\mathbf{LL}$ and more specifically its infinite proof-systems as in [2]; denotational semantics is clearly a natural framework for such tools. A crucial step will be to prove that these infinite proofs can be interpreted as total sets in **Nuts**. We base the interpretation of such infinite proofs on the interpretation of their finite approximations and this requires denotational models which also host partial objects as our **Rel** model (in contrast with **Nuts**, but remember that the interpretations of a $\mu\mathbf{LL}$ proof in **Nuts** and **Rel** are *exactly the same set*).

This denotational semantics will also serve as a guideline for the design of a functional language based on $\mu\mathbf{LL}$, generalizing Gödel's System T in the spirit of [18]. However, as explained in the Introduction, Loader's syntax does not seem fully compatible with **LL** as it is based on *cocartesian* cartesian closed categories which, as explained in [16] (based on earlier observations by Lawvere), are not compatible with PCF fixpoints – accounting for recursive function definitions in functional languages – which are available in many models of **LL** such as **Rel**, (probabilistic, hyper-) coherence spaces etc and can be discarded by an additional totality structure as in **Nuts**.

We base the development of this syntax on the idea of representing data-types as *positive* formulas of $\mu\mathbf{LL}$ that we can interpret in the Eilenberg-Moore category $\mathcal{L}^!$ of the underlying categorical model \mathcal{L} of **LL** (Seely category), in a setting similar to [10] and therefore have built-in structural primitives (weakening and contraction in particular). In $\mathcal{L}^!$, the \oplus of **LL** is a coproduct and the \otimes is a cartesian product as expected, the price to pay being that the targeted λ -calculus will have to feature a notion of *value* accounting syntactically for the morphisms of $\mathcal{L}^!$, substitution in terms being allowed only for values because only them can be safely discarded and duplicated thanks to their structural structure.

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6 Appendix

6.1 Proof of Lemma 3

Proof. We have $\mathcal{F}(g, \nu\mathcal{F}(B)) \in \mathcal{A}(\nu\mathcal{F}(B), \mathcal{F}(B', \nu\mathcal{F}(B)))$ thus defining a $\mathcal{F}_{B'}$ -coalgebra structure on $\nu\mathcal{F}(B)$ and hence there exists a unique morphism $\nu\mathcal{F}(g)$ such that

$$\mathcal{F}(B', \nu\mathcal{F}(g)) \mathcal{F}(g, \nu\mathcal{F}(B)) = \nu\mathcal{F}(g),$$

that is $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.

Functoriality follows: consider also $g' \in \mathcal{B}(B', B'')$, then we know that $h = \nu\mathcal{F}(g'g)$ satisfies $\mathcal{F}(g'g, h) = h$ by the definition above. Now $h' = \nu\mathcal{F}(g')\nu\mathcal{F}(g)$ satisfies the same equation by functoriality of \mathcal{F} and because $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$ and $\mathcal{F}(g', \nu\mathcal{F}(g')) = \nu\mathcal{F}(g')$, and hence $h' = h$ by Lemma 2, taking $l = \mathcal{F}(g'g, \nu\mathcal{F}(B))$. In the same way one proves that $\nu\mathcal{F}(\text{Id}) = \text{Id}$.

6.2 Proof of Lemma 4

Proof. The part of the statement which concerns the functor $\overline{\nu\mathbb{F}}$ is a direct application of Lemma 3 so we only have to deal with the strength. Let us prove naturality so let $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ and $g \in \mathcal{L}(Y, Y')$, we must prove that the following diagram commutes

$$\begin{array}{ccc} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y, \vec{X}}} & \overline{\nu\mathbb{F}}(!Y \otimes \vec{X}) \\ !g \otimes \overline{\nu\mathbb{F}}(\vec{f}) \downarrow & & \downarrow \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \\ !Y' \otimes \overline{\nu\mathbb{F}}(\vec{X}') & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y', \vec{X}'}} & \overline{\nu\mathbb{F}}(!Y' \otimes \vec{X}') \end{array}$$

Let $h_1 = \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}(!g \otimes \overline{\nu\mathbb{F}}(\vec{f}))$ and $h_2 = \overline{\nu\mathbb{F}}(!g \otimes \vec{f})\widehat{\nu\mathbb{F}}_{Y, \vec{X}}$ be the two morphisms we must prove equal. We use Lemma 2, taking the following morphism l .

$$\begin{array}{c} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) = !Y \otimes \mathbb{F}(\vec{X}, \overline{\nu\mathbb{F}}(\vec{X})) \\ \downarrow \mathbb{F}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ \mathbb{F}(!Y \otimes \vec{X}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \\ \downarrow \mathbb{F}(!g \otimes \vec{f}, \text{Id}) \\ \mathbb{F}(!Y' \otimes \vec{X}', !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \end{array}$$

With these notations we have

$$\begin{aligned}
\overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, h_1) l &= \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \widehat{\nu\mathbb{F}}_{Y', \overrightarrow{X'}}) \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, !g \otimes \overline{\nu\mathbb{F}}(\overrightarrow{f})) \\
&\quad \overline{\mathbb{F}}(!g \otimes \overrightarrow{f}, !Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X})) \widehat{\mathbb{F}}_{Y, (\overrightarrow{X}, \overline{\nu\mathbb{F}}(\overrightarrow{X}))} \\
&= \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \widehat{\nu\mathbb{F}}_{Y', \overrightarrow{X'}}) \overline{\mathbb{F}}(!g \otimes \overrightarrow{f}, !g \otimes \overline{\nu\mathbb{F}}(\overrightarrow{f})) \widehat{\mathbb{F}}_{Y, (\overrightarrow{X}, \overline{\nu\mathbb{F}}(\overrightarrow{X}))} \\
&= \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \widehat{\nu\mathbb{F}}_{Y', \overrightarrow{X'}}) \widehat{\mathbb{F}}_{Y', (\overrightarrow{X'}, \overline{\nu\mathbb{F}}(\overrightarrow{X}'))} (!g \otimes \overline{\mathbb{F}}(\overrightarrow{f}, \overline{\nu\mathbb{F}}(\overrightarrow{f}))) \\
&\quad \text{by naturality of } \widehat{\mathbb{F}} \\
&= \widehat{\nu\mathbb{F}}_{Y', \overrightarrow{X'}} (!g \otimes \overline{\mathbb{F}}(\overrightarrow{f}, \overline{\nu\mathbb{F}}(\overrightarrow{f}))) \text{ by (1)} \\
&= \widehat{\nu\mathbb{F}}_{Y', \overrightarrow{X'}} (!g \otimes \overline{\nu\mathbb{F}}(\overrightarrow{f})) \text{ by Lemma 3}
\end{aligned}$$

so that $\overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, h_1) l = h_1$ as required. On the other hand we have

$$\begin{aligned}
\overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, h_2) l &= \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \overline{\nu\mathbb{F}}(!g \otimes \overrightarrow{f})) \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \widehat{\nu\mathbb{F}}_{Y, \overrightarrow{X}}) \\
&\quad \overline{\mathbb{F}}(!g \otimes \overrightarrow{f}, !Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X})) \widehat{\mathbb{F}}_{Y, (\overrightarrow{X}, \overline{\nu\mathbb{F}}(\overrightarrow{X}))} \\
&= \overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, \overline{\nu\mathbb{F}}(!g \otimes \overrightarrow{f})) \overline{\mathbb{F}}(!g \otimes \overrightarrow{f}, !Y \otimes \overline{\nu\mathbb{F}}(\overrightarrow{X})) \\
&\quad \overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \widehat{\nu\mathbb{F}}_{Y, \overrightarrow{X}}) \widehat{\mathbb{F}}_{Y, (\overrightarrow{X}, \overline{\nu\mathbb{F}}(\overrightarrow{X}))} \\
&= \overline{\mathbb{F}}(!g \otimes \overrightarrow{f}, \overline{\nu\mathbb{F}}(!g \otimes \overrightarrow{f})) \widehat{\nu\mathbb{F}}_{Y, \overrightarrow{X}} \text{ by (1)} \\
&= \overline{\nu\mathbb{F}}(!g \otimes \overrightarrow{f}) \widehat{\nu\mathbb{F}}_{Y, \overrightarrow{X}} \text{ by Lemma 3}
\end{aligned}$$

so that $\overline{\mathbb{F}}(!Y' \otimes \overrightarrow{X'}, h_2) l = h_2$ which proves our contention. The monoidality condition on $\widehat{\nu\mathbb{F}}$ is proved similarly.

6.3 Proof of Lemma 7

Proof. Let $a \in E$, since $(a, a) \in \text{Id}_E = ts$, there must exist $b \in F$ such that $(a, b) \in s$ and $(b, a) \in t$. If $(a, b') \in s$ then $(b, b') \in st \subseteq \text{Id}_F$ and hence $b' = b$. It follows that s is a total function $E \rightarrow F$. Let $(a, b) \in s$ (that is $a \in E$ and $b = s(a)$). Since $ts = \text{Id}_E$, we must have $(b, a) \in t$. Conversely let $(b, a) \in t$, we have $(b, s(a)) \in st$ and hence $b = s(a)$. We have proven that $t = \{(s(a), a) \mid a \in E\}$. If $a, a' \in E$ satisfy $s(a) = s(a')$ we have therefore $(a, a') \in ts = \text{Id}_E$ and hence $a = a'$; this shows that s is injective.

6.4 Proof of Proposition 1

Proof. Let \mathcal{D} be a directed set of sets and let H be a set. For each $E \in \mathcal{D}$ let $s_E \in \mathbf{Rel}(\mathbb{F}(E), H)$ so that $(s_E)_{E \in \mathcal{D}}$ defines a cocone, that is, for each $E, F \in \mathcal{D}$ such that $E \subseteq F$, one has $s_E = s_F \mathbb{F}(\eta_{E, F}^+)$. Let $L = \bigcup \mathcal{D}$. Let $s \in$

$\mathbf{Rel}(\mathbb{F}(L), H)$ be given by $s = \bigcup_{E \in \mathcal{D}} s_E \mathbb{F}(\eta_{E,L}^-)$. Let $E \in \mathcal{D}$, we have $s \mathbb{F}(\eta_{E,L}^+) = \bigcup_{F \in \mathcal{D}} s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+)$ so that $s_E \subseteq s \mathbb{F}(\eta_{E,L}^+)$ (since $s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) = s_E$ when $F = E$).

We prove the converse inclusion. Let $F \in \mathcal{D}$ and let $G \in \mathcal{D}$ be such that $E, F \subseteq G$. We have

$$\begin{aligned} s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) &= s_F \mathbb{F}(\eta_{F,G}^- \eta_{G,L}^- \eta_{E,L}^+ \eta_{E,G}^+) = s_F \mathbb{F}(\eta_{F,G}^- \eta_{E,G}^+) \\ &= s_G \mathbb{F}(\eta_{F,G}^+) \mathbb{F}(\eta_{F,G}^- \eta_{E,G}^+) \\ &\subseteq s_G \mathbb{F}(\eta_{E,G}^+) = s_E \end{aligned}$$

where we have used the fact that $\eta_{F,G}^+ \eta_{F,G}^- \subseteq \mathbf{Id}_G$ and hence $\mathbb{F}(\eta_{F,G}^+ \eta_{F,G}^-) \subseteq \mathbf{Id}_{\mathbb{F}(G)}$ by hom-continuity of \mathbb{F} . So $s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) \subseteq s_E$ for all $F \in \mathcal{D}$ and hence $s \mathbb{F}(\eta_{E,L}^+) \subseteq s_E$ as contended.

Let now $s' \in \mathbf{Rel}(\mathbb{F}(L), H)$ be such that $s' \mathbb{F}(\eta_{E,L}^+) = s_E$ for each $E \in \mathcal{D}$, we show that $s' = s$ thus proving the uniqueness part of the universal property. For $E \in \mathcal{D}$, let $\theta_E = \eta_{E,L}^+ \eta_{E,L}^- \in \mathbf{Rel}(L, L)$. Then $(\theta_E)_{E \in \mathcal{D}}$ is a directed family (for \subseteq) and $\bigcup_{E \in \mathcal{D}} \theta_E = \mathbf{Id}_L$. By hom-continuity of \mathbb{F} , we have

$$s' = s' \mathbf{Id}_{\mathbb{F}(L)} = s' \bigcup_{E \in \mathcal{D}} \mathbb{F}(\theta_E) = \bigcup_{E \in \mathcal{D}} s' \mathbb{F}(\eta_{E,L}^+) \mathbb{F}(\eta_{E,L}^-) = \bigcup_{E \in \mathcal{D}} s_E \mathbb{F}(\eta_{E,L}^-) = s$$

by our assumption on s' and by definition of s . This shows that the cocone $(\mathbb{F}(\eta_{E,L}^+))_{E \in \mathcal{D}}$ on $\mathbb{F} \eta^+$ is colimiting, thus proving that $\mathbb{F} \eta^+$ is directed cocontinuous.

6.5 Proof of Lemma 10

Proof. Let (E, t) be an \mathbb{F} -coalgebra. Let $e = \emptyset \in \mathbf{Rel}(E, \emptyset)$ (this is the unique morphism to the terminal object of \mathbf{Rel}). We define a sequence $e_n \in \mathbf{Rel}(E, \sigma \mathbb{F})$ as follows: $e_0 = \emptyset$ and $e_{n+1} = \mathbb{F}(e_n) t$. Then $e_n \subseteq e_{n+1}$ for all n by an easy induction, using the fact that \mathbb{F} is hom-continuous. Let $e = \bigcup_{n=0}^{\infty} e_n \in \mathbf{Rel}(E, \sigma \mathbb{F})$, by hom-continuity of \mathbb{F} we have $\mathbb{F}(e) t = (\bigcup_{n=0}^{\infty} \mathbb{F}(e_n)) t = \bigcup_{n=0}^{\infty} (\mathbb{F}(e_n) t) = \bigcup_{n=0}^{\infty} e_{n+1} = e$ which means that $e \in \mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})((E, t), (\sigma \mathbb{F}, \mathbf{Id}))$. We end the proof by showing that e is the unique such morphism, so let

$$e' \in \mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})((E, t), (\sigma \mathbb{F}, \mathbf{Id})),$$

which means that $e' \in \mathbf{Rel}(E, \sigma \mathbb{F})$ and $\mathbb{F}(e') t = e'$.

Let $i_n \in \mathbf{Rel}(\sigma \mathbb{F}, \sigma \mathbb{F})$ be defined by induction by $i_0 = \emptyset$ and $i_{n+1} = \mathbb{F}(i_n)$. Then $(i_n)_{n \in \mathbb{N}}$ is monotone and $\bigcup_{n=0}^{\infty} i_n = \mathbf{Id}$ by definition of $\sigma \mathbb{F}$. We prove by induction on n that $\forall n \in \mathbb{N} i_n e' = i_n e$. Clearly $i_0 e' = i_0 e = \emptyset$. Next

$$\begin{aligned} i_{n+1} e' &= \mathbb{F}(i_n) \mathbb{F}(e') t = \mathbb{F}(i_n e') t = \mathbb{F}(i_n e) t \quad \text{by inductive hypothesis} \\ &= i_{n+1} e. \end{aligned}$$

Therefore $e' = (\bigcup_{n \in \mathbb{N}} i_n) e' = \bigcup_{n \in \mathbb{N}} (i_n e') = \bigcup_{n \in \mathbb{N}} (i_n e) = e$.

6.6 Proof of Lemma 11

Proof. As usual we assume that $n = 1$ to increase readability. We need to prove first that $\nu\mathbb{F}$ is monotone on morphisms, so let $s, t \in \mathbf{Rel}(E, F)$ with $s \subseteq t$. We have $\nu\mathbb{F}(s) = \bigcup_{n \in \mathbb{N}} s_n$ and $\nu\mathbb{F}(t) = \bigcup_{n \in \mathbb{N}} t_n$ with $s_0 = t_0 = \emptyset$, $s_{n+1} = \mathbb{F}(s, s_n)$ and $t_{n+1} = \mathbb{F}(t, t_n)$. By induction and hom-monotonicity of \mathbb{F} we have $\forall n \in \mathbb{N} s_n \subseteq t_n$ and hence $\nu\mathbb{F}(s) \subseteq \nu\mathbb{F}(t)$. Let us prove now hom-continuity so let $D \subseteq \mathbf{Rel}(E, F)$ be directed and let $t = \bigcup D$, we prove that $\nu\mathbb{F}(t) = \bigcup_{s \in D} \nu\mathbb{F}(s) \in \mathbf{Rel}(\nu\mathbb{F}(E), \nu\mathbb{F}(F))$ using Lemma 2 (with the notations of that lemma, we take $l = \mathbb{F}(t, \nu\mathbb{F}(E))$). We have $\mathbb{F}_F(\nu\mathbb{F}(t)) \mathbb{F}(t, \nu\mathbb{F}(E)) = \nu\mathbb{F}(t)$ by definition of the functor $\nu\mathbb{F}$ and

$$\begin{aligned} \mathbb{F}_F\left(\bigcup_{s \in D} \nu\mathbb{F}(s)\right) \mathbb{F}(t, \nu\mathbb{F}(E)) &= \bigcup_{s \in D} \mathbb{F}(F, \nu\mathbb{F}(s)) \bigcup_{s \in D} \mathbb{F}(s, \nu\mathbb{F}(E)) \quad \text{by hom-cont.} \\ &= \bigcup_{s \in D} \mathbb{F}(s, \nu\mathbb{F}(s)) = \bigcup_{s \in D} \nu\mathbb{F}(s). \end{aligned}$$

In the second equation, we used the facts that D is directed and the monotonicity of \mathbb{F} and $\nu\mathbb{F}$ on morphisms.

Let $E \subseteq F$, we prove that $\nu\mathbb{F}(E) \subseteq \nu\mathbb{F}(F)$. This results from the observation that if $E' \subseteq F'$, then $\mathbb{F}_E(E') \subseteq \mathbb{F}_F(F')$ and hence $\forall n \in \mathbb{N} \mathbb{F}_E^n(\emptyset) \subseteq \mathbb{F}_F^n(\emptyset)$. Let us check that $\nu\mathbb{F}(\eta_{E,F}^+) = \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+ \in \mathbf{Rel}(\nu\mathbb{F}(E), \nu\mathbb{F}(F))$. We have $\mathbb{F}(F, \nu\mathbb{F}(\eta_{E,F}^+)) \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(E)) = \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(\eta_{E,F}^+)) = \nu\mathbb{F}(\eta_{E,F}^+)$ by definition of the functor $\nu\mathbb{F}$ and

$$\mathbb{F}(F, \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+) \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(E)) = \eta_{\mathbb{F}(E, \nu\mathbb{F}(E)), \mathbb{F}(F, \nu\mathbb{F}(F))}^+ = \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+$$

by strictness of \mathbb{F} . The equation follows by Lemma 2, so that the functor $\nu\mathbb{F}$ is strict.

6.7 Proof of Lemma 14

Proof. Let $t \in \mathcal{T}(X \multimap Y)$ and let $u \in \mathcal{T}(X)$. Let $v' \in \mathcal{T}(Y)^\perp$, since $u \times v' \in \mathcal{T}(X \otimes Y^\perp)$ we have $t \cap (u \times v') \neq \emptyset$ and hence $(t \cdot u) \cap v' \neq \emptyset$. Therefore $t \cdot u \in \mathcal{T}(Y)^{\perp\perp} = \mathcal{T}(Y)$. Conversely assume that $\forall u \in \mathcal{T}(X) t \cdot u \in \mathcal{T}(Y)$. Let $u \in \mathcal{T}(X)$ and $v' \in \mathcal{T}(Y^\perp) = \mathcal{T}(Y)^\perp$. Since $t \cdot u \in \mathcal{T}(Y)$ we have $(t \cdot u) \cap v' \neq \emptyset$ and hence $t \cap (u \times v') \neq \emptyset$ and this shows that $t \in \mathcal{T}(X \multimap Y)$.

6.8 Proof of Lemma 15

Proof. Assume that t is an iso in **Nuts** so that there is $t' \in \mathbf{Nuts}(Y, X)$ such that $t' t = \text{Id}_{|X|}$ and $t t' = \text{Id}_{|Y|}$ and since we know that the isos in **Rel** are the bijections we know that t is a bijection. The fact that $\forall u \subseteq |X| u \in \mathcal{T}(X) \Leftrightarrow t(u) \in \mathcal{T}(Y)$ results from the fact that both t and $t' = t^{-1}$ are morphisms in **Nuts**.

The converse implication is obvious.

6.9 Proof of Lemma 17

Proof. The condition is obviously necessary, let us prove that it is sufficient so assume that t fulfills it and let us prove that $t \in \mathcal{T}(X_1 \otimes X_2 \multimap Y)$. To this end it suffices to prove that $t^\perp \in \mathcal{T}(Y^\perp \multimap (X_1 \otimes X_2)^\perp)$. So let $v' \in \mathcal{T}(Y^\perp)$ and let us prove that $t^\perp \cdot v' \in \mathcal{T}((X_1 \otimes X_2)^\perp) = \{u_1 \otimes u_2 \mid u_1 \in \mathcal{T}(X_1) \text{ and } u_2 \in \mathcal{T}(X_2)\}^\perp$. So let $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$. We know that $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$ and hence $(t \cdot (u_1 \otimes u_2)) \cap v' \neq \emptyset$, that is $(u_1 \otimes u_2) \cap (t^\perp \cdot v') \neq \emptyset$, proving our contention.

6.10 Proof of Lemma 18

Proof. Let $t \in \mathcal{T}((X_1 \otimes X_2) \multimap Y)$ and let us prove that $s = \alpha \cdot t \in \mathcal{T}(X_1 \multimap (X_2 \multimap Y))$. Given $u_i \in \mathcal{T}(X_i)$ it suffices to prove that $(t' \cdot u_1) \cdot u_2 \in \mathcal{T}(Y)$ which results from the fact that $(s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)$. Conversely let $s \in \mathcal{T}(X_1 \multimap (X_2 \multimap Y))$ and let us prove that $t = \alpha^{-1} \cdot s \in \mathcal{T}((X_1 \otimes X_2) \multimap Y)$. This results from lemma 17 and from the equation $(s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)$.

6.11 Proof of Lemma 19

Proof. The condition is obviously necessary, so let us assume that it holds. By Lemma 16, it suffices to prove that $t^\perp \in \mathbf{Nuts}(Y^\perp, (!X)^\perp)$. Let $v' \in \mathcal{T}(Y^\perp)$, we prove that $t^\perp \cdot v' \in \mathcal{T}(!Y)^\perp$. So let $u \in \mathcal{T}(X)$, since $t \cdot u^{(!)} \in \mathcal{T}(Y)$ and hence $(t \cdot u^{(!)}) \cap v' \neq \emptyset$, that is $(t^\perp \cdot v') \cap u^{(!)} \neq \emptyset$.

6.12 Proof of Lemma 20

Proof. The condition is necessary since, if $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$, then $u_1^{(!)} \otimes u_2^{(!)} \in \mathcal{T}(!X_1 \otimes !X_2)$. So assume that it holds. Let $t' = \mathbf{curt} \in \mathbf{Rel}(|X_1| \multimap (|X_2| \multimap |Y|))$. Let $u_1 \in \mathcal{T}(X_1)$, we have $t' \cdot u_1^{(!)} \in \mathcal{P}(!X_2 \multimap Y)$. Let $u_2 \in \mathcal{T}(X_2)$, we have $(t' \cdot u_1^{(!)}) \cdot u_2^{(!)} = t \cdot (u_1^{(!)} \otimes u_2^{(!)}) \in \mathcal{T}(Y)$ by our assumption. It follows by Lemma 19 that $t' \cdot u_1^{(!)} \in \mathcal{T}(!X_2 \multimap Y)$ and since this holds for any $u_1 \in \mathcal{T}(X_1)$ we actually have $t' \in \mathbf{Nuts}(!X_1, !X_2 \multimap Y)$. It follows that $t = \mathbf{cur}^{-1}(t') \in \mathbf{Nuts}(!X_1 \otimes !X_2, Y)$ as contended.

6.13 Proof of Lemma 23

Proof. It is clear that \mathcal{X} so defined is a strong functor. Let us check that \mathbb{X} can be retrieved from \mathcal{X} . Given a set E , $(E, \mathcal{P}(E))$ is a NUTS that we denote as $\mathbf{p}(E)$. Notice that \mathbf{p} can be extended into a functor $\mathbf{Rel} \rightarrow \mathbf{Nuts}$ which acts as the identity on morphisms. There is also a forgetful functor $\mathbf{u} : \mathbf{Nuts} \rightarrow \mathbf{Rel}$ which maps X to $|X|$ and acts as the identity on morphisms (btw. \mathbf{p} is right adjoint to \mathbf{u}). Let \mathbb{X} be a unary VNUTS and let $\mathcal{X} : \mathbf{Nuts} \rightarrow \mathbf{Nuts}$ be the associated strong functor. Then we have $\overline{|\mathbb{X}|} = \mathbf{u} \circ \overline{\mathcal{X}} \circ \mathbf{p}$ and $\widehat{|\mathbb{X}|}_{E,F} = \widehat{\mathcal{X}}_{\mathbf{p}(E), \mathbf{p}(F)}$ for any sets E and F . Last, given a NUTS X , we have that $\overline{\mathcal{T}(\mathbb{X})(X)}$ is just the totality component of the NUTS $\mathcal{X}(X)$. This shows that \mathbb{X} is determined by \mathcal{X} as contended.

6.14 Proof of Lemma 22

Proof. Given an object X of \mathbf{Nuts} , we set $\mathbf{der}_X = \mathbf{der}_{|X|} \in \mathbf{Rel}(|!X|, |X|)$ and $\mathbf{dig}_X = \mathbf{dig}_{|X|} \in \mathbf{Rel}(|!X|, |!X|)$. Given $u \in \mathcal{T}(X)$, we have $\mathbf{der}_X \cdot u^{(!)} = u \in \mathcal{T}(X)$ and $\mathbf{dig}_X \cdot u^{(!)} = u^{(!)} \in \mathcal{T}(!X)$. It follows by Lemma 19 that $\mathbf{der}_X \in \mathbf{Nuts}(!X, X)$ and $\mathbf{dig}_X \in \mathbf{Nuts}(!X, !X)$.

Naturality and monadicity trivially hold because they hold in \mathbf{Rel} : we have an obvious faithful forgetful functor $\mathbf{Nuts} \rightarrow \mathbf{Rel}$ which commutes with all LL categorical constructs.

We are left with defining the strong monoidality structure of $!_-$ (Seely isomorphisms), for $\mathbf{m}^0 \in \mathbf{Nuts}(1, !\top)$ we take the same morphism as in \mathbf{Rel} . And we set $\mathbf{m}_{X_1, X_2}^2 = \mathbf{m}_{|X_1|, |X_2|}^2 \in \mathbf{Rel}(!X_1 \otimes !X_2, !(X_1 \& X_2))$. Let $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$. We have $\mathbf{m}_{X_1, X_2}^2 \cdot (u_1^{(!)} \otimes u_2^{(!)}) = (u_1 \& u_2)^{(!)} \in \mathcal{T}(!(X_1 \& X_2))$ since $u_1 \& u_2 \in \mathcal{T}(X_1 \& X_2)$, and hence by Lemma 20 we have $\mathbf{m}_{X_1, X_2}^2 \in \mathbf{Nuts}(!X_1 \otimes !X_2, !(X_1 \& X_2))$. Any element w of $\mathcal{T}(X_1 \& X_2)$ is of shape $w = u_1 \& u_2$ with $u_i \in \mathcal{T}(X_i)$, namely $u_i = \pi_i \cdot w$. We have $(\mathbf{m}_{X_1, X_2}^2)^{-1} \cdot w^{(!)} = u_1^{(!)} \otimes u_2^{(!)} \in \mathcal{T}(!X_1 \otimes !X_2)$ and hence by Lemma 19 we have $(\mathbf{m}_{X_1, X_2}^2)^{-1} \in \mathbf{Nuts}(!(X_1 \& X_2), (!X_1 \otimes !X_2))$. This ends the proof that \mathbf{Nuts} is a model of classical Linear Logic since the required commutations obviously hold because they hold in \mathbf{Rel} .

6.15 Full proof of Theorem 1

Proof. Concerning Condition (3), let $(\mathbb{X}_i)_{i=1}^k$ be elements of \mathbf{Vnuts}_n and let $\mathbb{X} \in \mathbf{Vnuts}_k$. Considering \mathbb{X} and the \mathbb{X}_i 's as strong functors, we know that $\mathbb{X} \circ \vec{\mathbb{X}}$ is a strong functor $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. We simply have to exhibit a VNUTS whose associated strong functor is $\mathbb{X} \circ \vec{\mathbb{X}}$. Let $\mathbb{F} = |\mathbb{X}| \circ |\vec{\mathbb{X}}|$ (composition of variable sets, Section 3.2). Let $\vec{X} \in \mathbf{Nuts}^n$, each $\overline{\mathbb{X}}_i(\vec{X})$ is an object of \mathbf{Nuts} and hence $(\overline{\mathbb{F}}(|\vec{X}|), \mathcal{T}(\mathbb{X})(\overline{\mathbb{X}}_1(\vec{X}), \dots, \overline{\mathbb{X}}_k(\vec{X})))$ is a NUTS. Moreover given $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$, we know that for each $i = 1, \dots, k$, one has $\overline{\mathbb{X}}_i(\vec{t}) \in \mathbf{Nuts}(\overline{\mathbb{X}}_i(\vec{X}), \overline{\mathbb{X}}_i(\vec{Y}))$ since \mathbb{X}_i is a VNUTS. Since \mathbb{X} is a VNUTS we have $\overline{\mathbb{F}}(\vec{t}) \in \mathbf{Nuts}(\overline{\mathbb{X}}(\overline{\mathbb{X}}_1(\vec{X}), \dots, \overline{\mathbb{X}}_k(\vec{X})), \overline{\mathbb{X}}(\overline{\mathbb{X}}_1(\vec{Y}), \dots, \overline{\mathbb{X}}_k(\vec{Y})))$.

Let $X \in \mathbf{Obj}(\mathbf{Nuts})$ and $\vec{Y} \in \mathbf{Obj}(\mathbf{Nuts}^k)$. For $i = 1, \dots, k$ we know that $\widehat{\mathbb{X}}_{X, \vec{Y}} \in \mathbf{Nuts}(!X \otimes \overline{\mathbb{X}}_i(\vec{Y}), \overline{\mathbb{X}}_i(!X \otimes \vec{Y}))$. Therefore

$$\overline{\mathbb{X}}(\widehat{\mathbb{X}}_{X, \vec{Y}})_{i=1}^k \in \mathbf{Nuts}(\overline{\mathbb{X}}(!X \otimes \overline{\mathbb{X}}_i(\vec{Y}))_{i=1}^k, \overline{\mathbb{X}}(\overline{\mathbb{X}}_i(!X \otimes \vec{Y}))_{i=1}^k)$$

and hence

$$\overline{\mathbb{X}}(\widehat{\mathbb{X}}_{X, \vec{Y}})_{i=1}^k \widehat{\mathbb{X}}_{X, (\overline{\mathbb{X}}_i(\vec{Y}))_{i=1}^k} \in \mathbf{Nuts}(!X \otimes \overline{\mathbb{X}}((\overline{\mathbb{X}}_i(\vec{Y}))_{i=1}^k), \overline{\mathbb{X}}(\overline{\mathbb{X}}_i(!X \otimes \vec{Y}))_{i=1}^k).$$

Moreover we have

$$\begin{aligned}
\widehat{\mathbb{F}}_{|X|,|\vec{Y}|} &= \overline{|\mathbb{X}|}(\widehat{(|\mathbb{X}_i|_{|X|,|\vec{Y}|})}_{i=1}^k) \widehat{|\mathbb{X}|}_{|X|,(\overline{|\mathbb{X}_i|(\vec{Y})})}_{i=1}^k \quad \text{by definition of } \mathbb{F} \\
&= \overline{|\mathbb{X}|}(\widehat{(|\mathbb{X}_i|_{|X|,|\vec{Y}|})}_{i=1}^k) \widehat{|\mathbb{X}|}_{|X|,(\overline{|\mathbb{X}_i|(|\vec{Y}|)})}_{i=1}^k \\
&= \overline{|\mathbb{X}|}(\widehat{|\mathbb{X}_i|_{X,\vec{Y}}}_{i=1}^k) \widehat{|\mathbb{X}|}_{X,(\overline{|\mathbb{X}_i|(\vec{Y})})}_{i=1}^k
\end{aligned}$$

using again the fact that \mathbb{X} and the \mathbb{X}_i 's are VNUTS. This shows that the pair $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ given by $|\mathbb{Y}| = \mathbb{F}$ and $\mathcal{T}(\mathbb{Y})(\vec{X}) = \mathcal{T}(\mathbb{X})(\overline{|\mathbb{X}_1|(\vec{X})}, \dots, \overline{|\mathbb{X}_k|(\vec{X})})$ is a VNUTS whose associated strong functor is $\mathbb{X} \circ \overline{\mathbb{X}}$ thus proving our contention.

Concerning Condition (4), let us deal only with the case of $!_-$, the others being similar. We have to exhibit a unary VNUTS \mathbb{X} whose associated strong functor $\mathbf{Nuts} \rightarrow \mathbf{Nuts}$ coincides with $!_-$ (which is known to be a strong functor $\mathbf{Nuts} \rightarrow \mathbf{Nuts}$ by Section 4 and by the general considerations of Section 2.1). For $|\mathbb{X}|$, which has to be a variable set $\mathbf{Rel} \rightarrow \mathbf{Rel}$, we take the interpretation \mathbb{E} of $!_-$ in the model \mathbf{Rel} (Section 3.2) which is an element of \mathbf{Rel}_1 , that is, a unary variable set. Next, given $X \in \mathbf{Obj}(\mathbf{Nuts})$, we take $\mathcal{T}(\mathbb{X})(X) = \mathcal{T}(!_X)$. Condition (1) in the definition of VNUTS holds by functoriality of $!_-$ on \mathbf{Nuts} . Condition (2) holds by the definition of $\widehat{\mathbb{F}}_{|X|,|\vec{Y}|}$ as described in Section 2.1 which coincides with $\mu^2(\mathbf{dig}_X \otimes !Y) \in \mathbf{Nuts}(!_X \otimes !Y, !(X \otimes Y))$.

Let us now turn to Condition (5) which is a bit more challenging.

Fixed Points of VNUTS. Let first $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ be a unary VNUTS. Let $E = \sigma \overline{|\mathbb{X}|}$ which is the least set such that $\overline{|\mathbb{X}|}(E) = E$, that is $E = \bigcup_{n=0}^{\infty} \overline{|\mathbb{X}|}^n(\emptyset)$. Let $\Phi : \mathbf{Tot}(E) \rightarrow \mathbf{Tot}(E)$ be defined as follows: given $\mathcal{T} \in \mathbf{Tot}(E)$, then (E, \mathcal{T}) is a NUTS, and we set $\Phi(\mathcal{T}) = \mathcal{T}(\mathbb{X})(E, \mathcal{T}) \in \mathbf{Tot}(\overline{|\mathbb{X}|}(E)) = \mathbf{Tot}(E)$. This function Φ is monotone. Let indeed $\mathcal{S}, \mathcal{T} \in \mathbf{Tot}(E)$ with $\mathcal{S} \subseteq \mathcal{T}$. Then we have $\mathbf{ld} \in \mathbf{Nuts}((E, \mathcal{S}), (E, \mathcal{T}))$ and therefore, by Condition (1) satisfied by \mathbb{X} , we have $\mathbf{ld} = \overline{|\mathbb{X}|}(\mathbf{ld}) \in \mathbf{Nuts}(\overline{|\mathbb{X}|}(E, \mathcal{S}), \overline{|\mathbb{X}|}(E, \mathcal{T})) = \mathbf{Nuts}((E, \Phi(\mathcal{S})), (E, \Phi(\mathcal{T})))$ which means that $\Phi(\mathcal{S}) \subseteq \Phi(\mathcal{T})$. By the Knaster Tarski Theorem (remember that $\mathbf{Tot}(E)$ is a complete lattice), Φ has a greatest fixpoint \mathcal{T} that we can describe as follows. Let $(\mathcal{T}_\alpha)_{\alpha \in \mathbb{O}}$, where \mathbb{O} is the class of ordinals, be defined by: $\mathcal{T}_0 = \mathcal{P}(E)$ (the largest possible notion of totality on E), $\mathcal{T}_{\alpha+1} = \Phi(\mathcal{T}_\alpha)$ and $\mathcal{T}_\lambda = \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha$ when λ is a limit ordinal. This sequence is decreasing (easy induction on ordinals using the monotonicity of Φ) and there is an ordinal θ such that $\mathcal{T}_{\theta+1} = \mathcal{T}_\theta$ (by a cardinality argument; we can assume that θ is the least such ordinal). The greatest fixpoint of Φ is then \mathcal{T}_θ as easily checked.

By construction $((E, \mathcal{T}_\theta), \mathbf{ld})$ is an object of $\mathbf{Coalg}_{\mathbf{Nuts}}(\overline{|\mathbb{X}|})$, we prove that it is the terminal object. So let (Y, t) be another object of the same category. Since $(|Y|, t)$ is an object of $\mathbf{Coalg}_{\mathbf{Rel}}(\overline{|\mathbb{X}|})$ and since (E, \mathbf{ld}) is the terminal object in that category, we know by Lemma 10 that there is exactly one $e \in \mathbf{Rel}(|Y|, E)$ such that $\overline{|\mathbb{X}|}(e) t = e$. We prove that actually $e \in \mathbf{Nuts}(Y, (E, \mathcal{T}_\theta))$ so let $v \in \mathcal{T}(Y)$. We prove by induction on the ordinal α that $e \cdot v \in \mathcal{T}_\alpha$. For $\alpha = 0$ it is obvious since $\mathcal{T}_0 = \mathcal{P}(E)$. Assume that the property holds for α and let us prove it for $\alpha + 1$. We have $t \cdot v \in \mathcal{T}(\mathbb{X})(Y) = \mathcal{T}(\overline{|\mathbb{X}|}(Y))$ since $t \in \mathbf{Nuts}(Y, \overline{|\mathbb{X}|}(Y))$.

Since $\overline{\mathbb{X}}(e) \in \mathbf{Nuts}(\overline{\mathbb{X}}(Y), \overline{\mathbb{X}}(E, \mathcal{T}_\alpha))$ and since $\overline{\mathbb{X}}(E, \mathcal{T}_\alpha) = (E, \mathcal{T}_{\alpha+1})$ we have $(\overline{\mathbb{X}}(e)t) \cdot v \in \mathcal{T}_{\alpha+1}$, that is $e \cdot v \in \mathcal{T}_{\alpha+1}$. Last if λ is a limit ordinal and if we assume $\forall \alpha < \lambda e \cdot v \in \mathcal{T}_\alpha$ we have $e \cdot v \in \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha = \mathcal{T}_\lambda$. Therefore $e \cdot v \in \mathcal{T}_\theta$. We use $\nu\overline{\mathbb{X}}$ to denote this final coalgebra (E, \mathcal{T}_θ) (its definition depends only on $\overline{\mathbb{X}}$ and does not involve the strength $\widehat{\mathbb{X}}$).

So we have proven the first part of Condition (5) in the definition of a Seely model of $\mu\mathbf{LL}$ (see Section 2). As to the second part, let \mathbb{X} be an $n + 1$ -ary VNUTS. We know by the general Lemma 4 that there is a uniquely defined strong functor $\nu\mathbb{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ such that

- $\nu\overline{\mathbb{X}}(\vec{X}) = \nu(\overline{\mathbb{X}}_{\vec{X}})$, so that $\overline{\mathbb{X}}(\vec{X}, \nu\overline{\mathbb{X}}(\vec{X})) = \overline{\nu\overline{\mathbb{X}}}(\vec{X})$, for all $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$,
- $\overline{\mathbb{X}}(\vec{t}, \nu\overline{\mathbb{X}}(\vec{t})) = \overline{\nu\overline{\mathbb{X}}}(\vec{t})$ for all $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$
- and $\overline{\mathbb{X}}(Y \otimes \vec{X}, \widehat{\nu\overline{\mathbb{X}}}_{Y, \vec{X}} \widehat{\mathbb{X}}_{Y, (\vec{X}, \nu\overline{\mathbb{X}}(\vec{X}))}) = \widehat{\nu\overline{\mathbb{X}}}_{Y, \vec{X}}$ for all $Y \in \mathbf{Obj}(\mathbf{Nuts})$ and $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$.

To end the proof, it will be enough to exhibit an n -ary VNUTS $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ whose associated strong functor coincides with $\nu\mathbb{X}$. We know that $|\mathbb{X}|$ is a variable set $\mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ so let $\mathbb{F} = \nu|\mathbb{X}| = \sigma|\mathbb{X}|$ which is a variable set $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ (see Section 3.2). Let $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$, we have $|\overline{\nu\overline{\mathbb{X}}}(\vec{X})| = |\nu(\overline{\mathbb{X}}_{\vec{X}})| = \bigcup_{n=0}^{\infty} |\overline{\mathbb{X}}_{\vec{X}}|^n(\emptyset) = \overline{\mathbb{F}}(|\vec{X}|)$. Let $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$, then $\overline{\nu\overline{\mathbb{X}}}(\vec{t})$ is the unique element s of $\mathbf{Nuts}(\overline{\nu\overline{\mathbb{X}}}(\vec{X}), \overline{\nu\overline{\mathbb{X}}}(\vec{Y})) \subseteq \mathbf{Rel}(\overline{\mathbb{F}}(|\vec{X}|), \overline{\mathbb{F}}(|\vec{Y}|))$ which satisfies $\overline{\mathbb{X}}(\vec{t}, s) = s$, that is $|\overline{\mathbb{X}}|(\vec{t}, s) = s$, which means that $\overline{\nu\overline{\mathbb{X}}}(\vec{t}) = s = \overline{\mathbb{F}}(\vec{t})$. By a completely similar uniqueness argument we have $\widehat{\nu\overline{\mathbb{X}}}_{X, \vec{Y}} = \widehat{\mathbb{F}}_{|\mathbb{X}|, |\vec{Y}|}$ for all $X \in \mathbf{Obj}(\mathbf{Nuts})$ and $\vec{Y} \in \mathbf{Obj}(\mathbf{Nuts}^n)$. So we set $|\mathbb{Y}| = \mathbb{F}$.

Next, given $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$ we set $\mathcal{T}(\mathbb{Y})(\vec{X}) = \mathcal{T}(\overline{\nu\overline{\mathbb{X}}}(\vec{X})) \in \mathbf{Tot}(|\overline{\nu\overline{\mathbb{X}}}(\vec{X})|) = \mathbf{Tot}(\overline{\mathbb{F}}(|\vec{X}|))$. Given $\vec{t} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$ we have

$$\overline{\mathbb{F}}(\vec{t}) = \overline{\nu\overline{\mathbb{X}}}(\vec{t}) \in \mathbf{Nuts}((\overline{\mathbb{F}}(|\vec{X}|), \mathcal{T}(\mathbb{Y})(\vec{X})), (\overline{\mathbb{F}}(|\vec{Y}|), \mathcal{T}(\mathbb{Y})(\vec{Y})))$$

since $(\overline{\mathbb{F}}(|\vec{X}|), \mathcal{T}(\mathbb{Y})(\vec{X})) = \overline{\nu\overline{\mathbb{X}}}(\vec{X})$ and similarly for \vec{Y} . Last since $\widehat{\mathbb{F}}_{|\mathbb{X}|, |\vec{Y}|} = \widehat{\nu\overline{\mathbb{X}}}_{X, \vec{Y}} \in \mathbf{Nuts}(!X \otimes \overline{\nu\overline{\mathbb{X}}}(\vec{Y}), \overline{\nu\overline{\mathbb{X}}}(X \otimes \vec{Y}))$ we know that $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ is a VNUTS whose associated strong functor is $\nu\mathbb{X}$. This ends the proof that $(\mathbf{Nuts}, (\mathbf{Vnuts}_n)_{n \in \mathbb{N}})$ is a Seely model of $\mu\mathbf{LL}$.