



n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules

Mohammad Abobala

Faculty of Science, Tishreen University, Lattakia, Syria

e-mail: mohammadabobala777@gmail.com

Abstract

Refining the indeterminate I into many levels of indeterminacy is a way to explore many neutrosophic algebraic structures. This paper introduces the concept of n-cyclic refined algebraic system of sub-indeterminacies as a new way to refine a neutrosophic indeterminate I. This idea will be used to introduce the notion of n-cyclic refined neutrosophic ring and to study its AH-substructures. Also, this work presents the concept of n-cyclic refined neutrosophic modules with many related structures.

Keywords: n-cyclic refined neutrosophic system, n-cyclic refined neutrosophic ring, n-cyclic refined neutrosophic AH-substructures, n-cyclic refined neutrosophic modules, n-cyclic refined neutrosophic homomorphisms.

1. Introduction

Neutrosophy is one of the most important concepts in philosophy, while its derivative the notion of neutrosophic set has been used widely to define new algebraic structures and topologies such as neutrosophic groups, rings, spaces, and modules. See [1,3,4,5,7,8,9,13]. Recently, many generalizations were defined such as refined neutrosophic set [10]. It has been used in the study of refined neutrosophic rings [6], refined neutrosophic modules and spaces [12,14,15,16].

In [20], F. Smarandache refined the literal indeterminacy (I) into subindeterminacies I_1, I_2, \dots, I_n , and defined the multiplication law of subindeterminacies.

In [18], Smarandache et al. presented the concept of n-refined neutrosophic set by splitting the indeterminacy I into n-degrees of indeterminacy I_1, \dots, I_n . By defining an algebraic operation between sub-indeterminacies, Smarandache and Abobala introduced n-refined neutrosophic rings, n-refined neutrosophic vector spaces and modules, AH-structures in n-refined neutrosophic vector spaces [2,17,18,19].

The algebraic operation between sub-indeterminacies was defined as follows:

$I_i \cdot I_j = I_{\min(i,j)}$. Through this paper, we will define another closed algebraic operation between sub-indeterminacies, which allows us to build a new refined neutrosophic system and new n-refined neutrosophic algebraic structures.

We say that I_1, \dots, I_n is an n-cyclic refining system of indeterminacies if it has the following binary operation $I_i \cdot I_j = I_{(i+j \bmod n)}$. For example if $n = 4$, we have $\{I_1, I_2, I_3, I_4\}$ as 4-cyclic refining system with: $I_2 I_3 = I_1, I_3 I_3 = I_2, I_3 I_1 = I_4, I_4 I_3 = I_3$.

We remark that n-cyclic refining system I_1, \dots, I_n has a cyclic group structure with order n.

Another important remark is about that the elements of any n-refined neutrosophic ring are exactly equal to the elements of n-cyclic refined neutrosophic ring; but the products between elements are totally different according to the definition of multiplication between sub-indeterminacies, which means that we get a new kind of refined neutrosophic rings. Hence, there is a lot of similarity between this work and our paper [18], that is because we use the same symbols and ordering of phrases.

For easy expression, we assume that $aI_0 = a$ and $I_0 I_j = I_j$. For all possible values of j.

2. Preliminaries

Definition 2.1: [18]

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n subindeterminacies. We define $R_n(I) = \{a_0 + a_1 I + \dots + a_n I_n; a_i \in R\}$ to be n-refined neutrosophic ring.

Definition 2.2: [18]

(a) Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \{a_0 + a_1 I_1 + \dots + a_n I_n; a_i \in P_i\}$, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i. AHS-subring is defined by the condition $P_i = P_j$ for all i, j .

(b) P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j .

(c) The AH-ideal P is said to be null if $P_i = R \text{ or } P_i = \{0\}$ for all i.

Definition 2.3: [8]

Let $(M, +, \cdot)$ be a module over the ring R then $(M(I), +, \cdot)$ is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring $R(I)$.

Elements of $M(I)$ have the form $\mathbf{x} + \mathbf{y}I; \mathbf{x}, \mathbf{y} \in \mathbf{M}$, i.e $M(I)$ can be written as $\mathbf{M}(I) = \mathbf{M} + \mathbf{M}I$.

Definition 2.4: [8]

Let $M(I)$ be a strong neutrosophic module over the neutrosophic ring $R(I)$ and $W(I)$ be a non empty subset of $M(I)$, then $W(I)$ is called a strong neutrosophic submodule if $W(I)$ itself is a strong neutrosophic module.

Definition 2.5: [8]

Let $U(I)$ and $W(I)$ be two strong neutrosophic submodules of $M(I)$ and let $f: U(I) \rightarrow W(I)$, we say that f is a neutrosophic module homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

3. n-Cyclic refined neutrosophic rings**Definition 3.1:**

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1I + \dots + a_nI_n; a_i \in R\}$ to be n -cyclic refined neutrosophic ring.

Operations on $R_n(I)$ are defined as:

$$\sum_{i=0}^n x_i I_i + \sum_{i=0}^n y_i I_i = \sum_{i=0}^n (x_i + y_i) I_i, \quad \sum_{i=0}^n x_i I_i \times \sum_{i=0}^n y_i I_i = \sum_{i,j=0}^n (x_i \times y_j) I_i I_j = \sum_{i,j=0}^n (x_i \times y_j) I_{(i+j \bmod n)}$$

Where \times is the multiplication on the ring R .

It is obvious that $R_n(I)$ is a ring in the algebraic ordinary concept.

Definition 3.2:

Let $R_n(I)$ be an n -cyclic refined neutrosophic ring, it is called commutative if $xy = yx$ for all $x, y \in R_n(I)$. If there is $1 \in R_n(I)$ such that $1 \cdot x = x \cdot 1 = x$, then it is called an n -cyclic refined neutrosophic ring with unity.

Theorem 3.3:

Let $R_n(I)$ be an n -cyclic refined neutrosophic ring. Then

(a) R is commutative if and only if $R_n(I)$ is commutative,

(b) R has unity if and only if $R_n(I)$ has unity,

(c) $R_n(I) = \sum_{i=0}^n R I_i = \{\sum_{i=0}^n x_i I_i : x_i \in R\}$.

Proof:

It is trivial, and similar to the classical case.

Definition 3.4:

(a) Let $R_n(I)$ be an n -cyclic refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i = \{a_0 + a_1 I + \dots + a_n I_n : a_i \in P_i\}$ where P_i is a subset of R , we define P to be an n -cyclic AH-subring if P_i is a subring of R for all i , n -cyclic AHS-subring is defined by the condition $P_i = P_j$ for all i, j .

(b) P is an n -cyclic AH-ideal if P_i is a two-sided ideal of R for all i , the n -cyclic AHS-ideal is defined by the condition $P_i = P_j$ for all i, j .

(c) The n -cyclic AH-ideal P is said to be null if $P_i = R \text{ or } P_i = \{0\}$ for all i .

Ideals and subrings by classical meaning need not to be defined, that is because they are well defined in classical studies.

Theorem 3.5:

Let $R_n(I)$ be an n -cyclic refined neutrosophic ring and P is an AH-ideal, $(P, +)$ is an abelian group with $k \leq n$ and $r.p \in P$ for all $p \in P$ and $r \in R$.

Proof :

Since P_i is abelian subgroup of $(R, +)$ and $r.x \in P_i$ for all $r \in R, x \in P_i$, the required result follows.

Definition 3.6:

Let $R_n(I)$ be an n -cyclic refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i, Q = \sum_{i=0}^n Q_i I_i$ be two AH-ideals, we define:

$$P+Q = \sum_{i=0}^n (P_i + Q_i) I_i, P \cap Q = \sum_{i=0}^n (P_i \cap Q_i) I_i.$$

Theorem 3.7:

Let $R_n(I)$ be any n -cyclic refined neutrosophic ring, $P = \sum_{i=0}^n P_i I_i, Q = \sum_{i=0}^n Q_i I_i$ be two n -cyclic AH- ideals, then $P+Q, P \cap Q$ are n -cyclic AH-ideals. If P, Q are n -cyclic AHS-ideals, then $P+Q, P \cap Q$ are n -cyclic AHS-ideals.

Proof :

The proof is similar to the case of n -refined neutrosophic rings in [18].

Definition 3.8:

Let $R_n(I)$ be an n -cyclic refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an n -cyclic AH-ideal, the AH- radical of P can be defined as $AH-rad(P) = \sum_{i=0}^n (\sqrt{P_i}) I_i$.

Theorem 3.9:

The AH-radical of an AH-ideal is an AH-ideal in any n -cyclic refined neutrosophic ring.

Proof :

The proof is similar to the case of n-refined neutrosophic ring in [18].

Definition 3.10:

Let $R_n(I)$ be an n-cyclic refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an AH-ideal, we define the n-cyclic AH-factor $R(I)/P = \sum_{i=0}^n (R/P_i)I_i = \sum_{i=0}^n (x_i + P_i)I_i; x_i \in R$.

Theorem 3.11:

Let $R_n(I)$ be an n-cyclic refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an n-cyclic AH-ideal:

$R_n(I)/P$ is a ring with the following two binary operations

$$\sum_{i=0}^n (x_i + P_i)I_i + \sum_{i=0}^n (y_i + P_i)I_i = \sum_{i=0}^n (x_i + y_i + P_i)I_i,$$

$$\sum_{i=0}^n (x_i + P_i)I_i \times \sum_{i=0}^n (y_i + P_i)I_i = \sum_{i=0}^n (x_i \times y_i + P_i)I_i.$$

Proof :

The proof is similar to Theorem 3.12 in [18].

Definition 3.12:

(a) Let $R_n(I), T_n(I)$ be two n-cyclic refined neutrosophic rings respectively, and $f_R: R \rightarrow T$ be a ring homomorphism.

We define n-cyclic refined neutrosophic AHS-homomorphism as follows:

$$f: R_n(I) \rightarrow T_n(I); f(\sum_{i=0}^n x_i I_i) = \sum_{i=0}^n f_R(x_i) I_i.$$

(b) f is an n-cyclic refined neutrosophic AHS-isomorphism if it is a bijective n-cyclic refined neutrosophic AHS-homomorphism.

(c) $\text{AH-Ker } f = \sum_{i=0}^n \text{Ker}(f_R)I_i = \{\sum_{i=0}^n x_i I_i; x_i \in \text{Ker } f_R\}$.

Theorem 3.13:

Let $R_n(I), T_n(I)$ be two n-cyclic refined neutrosophic rings respectively and f be an n-cyclic refined neutrosophic AHS-homomorphism $f: R_n(I) \rightarrow T_n(I)$. Then

(a) If $P = \sum_{i=0}^n P_i I_i$ is an n-cyclic AH- subring of $R_n(I)$, then $f(P)$ is an n-cyclic AH- subring of $T_n(I)$,

(b) If $P = \sum_{i=0}^n P_i I_i$ is an n-cyclic AHS- subring of $R_n(I)$, then $f(P)$ is an n-cyclic AHS- subring of $T_n(I)$,

(c) If $P = \sum_{i=0}^n P_i I_i$ is an n-cyclic AH-ideal of $R_n(I)$, then $f(P)$ is an n-cyclic AH-ideal of $f(R_n(I))$,

(d) $P = \sum_{i=0}^n P_i I_i$ is an n-cyclic AHS-ideal of $R_n(I)$, then $f(P)$ is an n-cyclic AHS-ideal of $f(R_n(I))$,

(e) $R_n(I)/AH - Ker(f)$ is AHS – isomorphic to $f(R(I))$,

(f) The inverse image of an n-cyclic AH-ideal P in $T_n(I)$ is an n-cyclic AH-ideal in $R(I)$.

Proof :

(a) Since $f(P_i)$ is a subring of T , then $f(P)$ is an n-cyclic AH- subring of $T_n(I)$.

(b) Holds by a similar way to (a).

(c) Since $f(P_i)$ is an ideal of $f(R)$, then $f(P)$ is an n-cyclic AH- ideal of $f(R(I))$.

(d) It is similar to (c).

(e) We have $R/Ker(f_R) \cong f(R)$, by definition of n-cyclic AH-factor and $AH - Ker(f)$ we find that $R(I)/P \cong f(R(I))$.

(f) It is similar to the classical case.

Definition 3.14:

(a) Let $R(I)$ be a commutative n-cyclic refined neutrosophic ring, and $P = \sum_{i=0}^n P_i I_i$ be an n-cyclic AH- ideal, we define P to be a weak prime n-cyclic AH-ideal if P_i is a prime ideal of R for all i .

(b) P is called a weak maximal n-cyclic AH-ideal if P_i is a maximal ideal of R for all i .

(c) P is called a weak principal n-cyclic AH-ideal if P_i is a principal ideal of R for all i .

Theorem 3.15:

Let $R_n(I), T_n(I)$ be two commutative n-cyclic refined neutrosophic rings with an n-cyclic refined neutrosophic AHS-homomorphism $f: R_n(I) \rightarrow T_n(I)$:

(a) If $P = \sum_{i=0}^n P_i I_i$ is an n-cyclic AHS- ideal of $R_n(I)$ and $Ker(f_R) \leq P_i \neq R_n(I)$:

(a) P is a weak prime n-cyclic AHS-ideal if and only if $f(P)$ is a weak prime AHS-ideal in $f(R_n(I))$.

(b) P is a weak maximal n-cyclic AHS-ideal if and only if $f(P)$ is a weak maximal n-cyclic AHS-ideal in $f(R_n(I))$.

(c) If $Q = \sum_{i=0}^n Q_i I_i$ is an n-cyclic AHS-ideal of $T_n(I)$, then it is a weak prime n-cyclic AHS-ideal if and only if $f^{-1}(Q)$ is a weak prime in $R_n(I)$.

(d) if $Q = \sum_{i=0}^n Q_i I_i$ is an n -cyclic AHS-ideal of $T_n(I)$, then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in $R_n(I)$.

Proof :

Proof is similar to Theorem 16.3 in [18].

Example 3.16:

Let $R = Z$ be the ring of integers, $T = Z_6$ be the ring of integers modulo 6 with multiplication and addition modulo 6, we have:

(a) $f_R: R \rightarrow T; f(x) = x \text{ mod } 6$ is a ring homomorphism, $\ker(f_R) = 6Z$, the corresponding 4-cyclic AHS-homomorphism between $R_4(I), T_4(I)$ is:

$$f: R_4(I) \rightarrow T_4(I); f(a + bI_1 + cI_2 + dI_3 + eI_4) = (a \text{ mod } 6) + (b \text{ mod } 6)I_1 + (c \text{ mod } 6)I_2 + (d \text{ mod } 6)I_3 + (e \text{ mod } 6)I_4; a, b, c, d, e \in Z.$$

(b) $P = \langle 2 \rangle, Q = \langle 3 \rangle$ are two prime and maximal and principal ideals in R ,

$M = P + PI_1 + QI_2 + QI_3 + PI_4 = \{(2a + 2bI_1 + 3cI_2 + 3dI_3 + 2eI_4); a, b, c, d, e \in Z\}$ is a weak prime/ maximal 4-cyclic AH-ideal of $R_4(I)$.

(c) $\ker(f_R) = 6Z \leq P, Q, f_R(P) = \{0, 2, 4\}, f_R(Q) = \{0, 3\}$,

$f(M) = f(P) + f(P)I_1 + f(Q)I_2 + f(Q)I_3 + f(P)I_4$ which is a weak maximal/ prime/principal 4-cyclic AH-ideal of $T_4(I)$.

(d) $AH - \ker(f) = 6Z + 6ZI_1 + 6ZI_2 + 6ZI_3 + 6ZI_4$ which is an 4-cyclic AHS-ideal of $R_4(I)$.

(e) $R_4(I)/AH - \ker f = R/6Z + R/6Z I_1 + R/6Z I_2 + R/6Z I_3 + R/6Z I_4$ which is AHS-isomorphic to $f(R_4(I)) = T_4(I)$, since $R/6Z \cong T$.

Example 3.17:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-cyclic refined neutrosophic ring related with R is $Z_{8_3}(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8\}$.

(b) $P = \{0, 4\}$ is an ideal of $R, \sqrt{P} = \{0, 2, 4, 6\}, M = P + PI_1 + PI_2 + PI_3$ is a 3-cyclic AHS-ideal of $Z_{8_3}(I)$,

$AH - \text{Rad}(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$ which is a 3-cyclic AHS-ideal of $Z_{8_3}(I)$.

Example 3.18:

DOI: 10.5281/zenodo.4278999

Let $R=Z_2$ be the ring of integers modulo 2, let $n = 3$. The corresponding 3-cyclic refined neutrosophic ring is

$$Z_{2,3}(I)=\{0,1,I_1,I_2,I_3,1+I_1,1+I_2,1+I_3,I_1+I_2,I_1+I_3,I_1+I_2+I_3,I_2+I_3,1+I_1+I_2+I_3,1+I_2+I_3,1+I_1+I_3,1+I_1+I_2\}.$$

We can remark that elements in the n-cyclic refined neutrosophic ring are exactly equal to elements in the corresponding n-refined neutrosophic ring; but the products of them are different.

4. n-Cyclic refined neutrosophic polynomial rings

Definition 4.1:

Let $R_n(I)$ be a commutative n-cyclic refined neutrosophic ring and $P: R_n(I) \rightarrow R_n(I)$ is a function defined as $P(x) = \sum_{i=0}^m a_i x^i$ such that $a_i \in R_n(I)$, we call P an n-cyclic refined neutrosophic polynomial on $R_n(I)$.

We denote by $R_n(I)[x]$ to be the ring of n-cyclic refined neutrosophic polynomials over $R_n(I)$.

Since $R_n(I)$ is a classical ring, then $R_n(I)[x]$ is a classical ring.

Theorem 4.2:

Let $R_n(I)$ be a commutative n-cyclic refined neutrosophic ring. Then $R_n(I)[x] = \sum_{i=0}^n R[x]I_i$.

Proof :

Let $P(x) = \sum_{i=0}^n P_i(x)I^i \in \sum_{i=0}^n R[x]I^i$, by rearranging the previous sum we can write it as $P(x) = \sum_{i=0}^m a_i x^i \in R_n(I)[x]$.

Conversely, if $P(x) = \sum_{i=0}^n a_i x^i \in R_n(I)[x]$, then we can write it as

$P(x) = \sum_{i=0}^n P_i(x)I_i \in \sum_{i=0}^n R[x]I_i$, by the previous argument we find the proof.

Example 4.3:

Let $Z_3(I)$ be a 3-cyclic refined neutrosophic ring and $P(x) = I_1 + (2+I_1)x + (I_1+I_3)x^2$ a polynomial over $Z_{3n}(I)$, then we can write $P(x) = 2x + I_1(1+x+x^2) + I_2x^2$.

It is obvious that $R_n(I) \leq R_n(I)[x]$.

Definition 4.4:

Let $P(x) = \sum_{i=0}^n P_i(x)I^i$ a neutrosophic polynomial over $R_n(I)$ we define the degree of P by $\deg P = \max(\deg P_i)$.

5. n-Cyclic refined neutrosophic modules

Definition 5.1 :

Let $(M, +, \cdot)$ be a module over the ring R , we say that $M_n(I) = M + MI_1 + \dots + MI_n = \{x_0 + x_1I_1 + \dots + x_nI_n; x_i \in M\}$ is a weak n -cyclic refined neutrosophic module over the ring R . Elements of $M_n(I)$ are called n -cyclic refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n -cyclic refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n -cyclic refined neutrosophic module over the n -cyclic refined neutrosophic ring $R_n(I)$. Elements of $M_n(I)$ are called n -refined neutrosophic scalars.

Remark 5.2:

Addition on $M_n(I)$ is defined as:

$$\sum_{i=0}^n a_i I_i + \sum_{i=0}^n b_i I_i = \sum_{i=0}^n (a_i + b_i) I_i.$$

Multiplication by a scalar $m \in R$ is defined as:

$$m \cdot \sum_{i=0}^n a_i I_i = \sum_{i=0}^n (m \cdot a_i) I_i.$$

Multiplication by an n -cyclic refined neutrosophic scalar $m = \sum_{i=0}^n m_i I_i \in R_n(I)$ is defined as:

$$\sum_{i=0}^n m_i I_i \cdot \sum_{i=0}^n a_i I_i = \sum_{i=0}^n (m_i \cdot a_i) I_i I_j.$$

Where $a_i \in M, m_i \in R, I_i I_j = I_{(i+j \bmod n)}$.

Theorem 5.3 :

Let $(M, +, \cdot)$ be a module over the ring R . Then a weak n -cyclic refined neutrosophic module $M_n(I)$ is a module over the ring R . A strong n -cyclic refined neutrosophic module is a module over the n -cyclic refined neutrosophic ring $R_n(I)$.

Proof:

It is similar to the classical case.

Example 5.4:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-cyclic refined neutrosophic module over the ring Z_2 is

$$M_n(I) = \{0, 1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}.$$

Definition 5.5:

Let $M_n(I)$ be a weak n -cyclic refined neutrosophic module over the ring R , a nonempty subset $W_n(I)$ is called a weak n -cyclic refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 5.6:

Let $M_n(I)$ be a strong n -cyclic refined neutrosophic module over the n -cyclic refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n -cyclic refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 5.7:

Let $M_n(I)$ be a weak n -cyclic refined neutrosophic module over the ring R , $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n -cyclic refined neutrosophic submodule if and only if:

$$x + y \in W_n(I), m \cdot x \in W_n(I) \text{ for all } x, y \in W_n(I), m \in R.$$

Proof:

It holds directly from the fact that $W_n(I)$ is a submodule of $M_n(I)$.

Theorem 5.8:

Let $M_n(I)$ be a strong n -cyclic refined neutrosophic module over the n -cyclic refined neutrosophic ring $R_n(I)$, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a strong n -cyclic refined neutrosophic submodule if and only if:

$$x + y \in W_n(I), m \cdot x \in W_n(I) \text{ for all } x, y \in W_n(I), m \in R_n(I).$$

Proof:

It holds directly from the fact that $W_n(I)$ is a submodule of $M_n(I)$ over the n -cyclic refined neutrosophic ring $R_n(I)$.

Example 5.9:

$M = R^2$ is a module over the ring of real numbers R , $W = \langle (0,1) \rangle$ is a submodule of M , $R_2^2(I) = \{(a, b) + (m, s)I_1 + (k, t)I_2; a, b, m, s, k, t \in R\}$ is the corresponding weak/strong 2-cyclic refined neutrosophic module.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0, x) + (0, y)I_1 + (0, z)I_2; x, y, z \in R\}$ is a weak 2-cyclic refined neutrosophic submodule of the weak 2-cyclic refined neutrosophic module $R_2^2(I)$ over the ring R .

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0, x) + (0, y)I_1 + (0, z)I_2; x, y, z \in R\}$ is a strong 2-cyclic refined neutrosophic submodule of the strong 2-cyclic refined neutrosophic module $R_2^2(I)$ over the n-cyclic refined neutrosophic ring $R_2(I)$.

Definition 5.10:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R , x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$; $a_i \in R, x_i \in M_n(I)$.

Example 5.11:

Consider the weak 2-cyclic refined neutrosophic module in Example 3.9,

$x = (0, 2) + (1, 3)I_1 \in R_2^2(I)$, $x = 2(0, 1) + 1 \cdot (1, 0)I_1 + 3(0, 1)I_1$, i.e x is a linear combination of the set $\{(0, 1), (1, 0)I_1, (0, 1)I_1\}$ over the ring R .

Definition 5.12:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$; $a_i \in R_n(I), x_i \in M_n(I)$.

Example 5.13:

Consider the strong 2-cyclic refined neutrosophic module $R_2^2(I) = \{(a, b) + (m, s)I_1 + (k, t)I_2; a, b, m, s, k, t \in R\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$,

$x = (0, 2) + (2, 3)I_1 = (2 + I_1) \cdot (0, 1) + (1 + I_2) \cdot (1, 1)I_1 = (0, 2) + (0, 1)I_1 + (1, 1)I_1 + (1, 1)I_{(3 \bmod 2)} = x$, hence x is a linear combination of the set $\{(0, 1), (1, 1)I_1\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$.

Definition 5.14:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-cyclic refined neutrosophic module $M_n(I)$ over the ring R , X is a weak linearly independent set if $\sum_{i=0}^m a_i x_i = 0$ implies $a_i = 0$; $a_i \in R$.

Definition 5.15:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-cyclic refined neutrosophic module $M_n(I)$ over the n-cyclic refined neutrosophic ring $R_n(I)$, X is a strong linearly independent set if $\sum_{i=0}^m a_i x_i = 0$ implies $a_i = 0$; $a_i \in R_n(I)$.

Definition 5.16:

Let $M_n(I), W_n(I)$ be two strong n-cyclic refined neutrosophic modules over the n-cyclic refined neutrosophic ring $R_n(I)$, let $f: M_n(I) \rightarrow U_n(I)$ be a well defined map. It is called a strong n-cyclic refined neutrosophic homomorphism if:

$$f(a.x + b.y) = a.f(x) + b.f(y) \text{ for all } x, y \in M_n(I), a, b \in R_n(I).$$

A weak n-cyclic refined neutrosophic homomorphism can be defined by the same.

Definition 5.17:

Let $f: M_n(I) \rightarrow U_n(I)$ be a weak/strong n-cyclic refined neutrosophic homomorphism, we define:

$$(a) \text{ Ker}(f) = \{x \in M_n(I); f(x) = 0\}.$$

$$(b) \text{ Im}(f) = \{y \in U_n(I); \exists x \in M_n(I) \text{ and } y = f(x)\}.$$

Theorem 5.18:

Let $f: M_n(I) \rightarrow U_n(I)$ be a weak n-cyclic refined neutrosophic homomorphism. Then

$$(a) \text{ Ker}(f) \text{ is a weak n-cyclic refined neutrosophic submodule of } M_n(I).$$

$$(b) \text{ Im}(f) \text{ is a weak n-cyclic refined neutrosophic submodule of } U_n(I).$$

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules, hence $\text{Ker}(f)$ is a submodule of the module $M_n(I)$, thus $\text{Ker}(f)$ is a weak n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 5.19:

Let $f: M_n(I) \rightarrow U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

$$(a) \text{ Ker}(f) \text{ is a strong n-cyclic refined neutrosophic submodule of } M_n(I).$$

$$(b) \text{ Im}(f) \text{ is a strong n-cyclic refined neutrosophic submodule of } U_n(I).$$

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $\text{Ker}(f)$ is a submodule of the module $M_n(I)$, thus $\text{Ker}(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 5.20:

Let $f: M_n(I) \rightarrow U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $\text{Ker}(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) $\text{Im}(f)$ is a strong n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $\text{Ker}(f)$ is a submodule of the module $M_n(I)$, thus $\text{Ker}(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Example 5.21:

Let $R_2^2(I) = \{x_0 + x_1I_1 + x_2I_2; x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1I_1 + y_2I_2; y_0, y_1, y_2 \in R^3\}$ be two weak 2-cyclic refined neutrosophic modules over the ring of real numbers R . Consider $f: R_2^2(I) \rightarrow R_2^3(I)$, where

$f[(a, b) + (m, n)I_1 + (k, s)I_2] = (a, 0, 0) + (m, 0, 0)I_1 + (k, 0, 0)I_2$, f is a weak 2-cyclic refined neutrosophic homomorphism over the ring R .

$$\text{Ker}(f) = \{(0, b) + (0, n)I_1 + (0, s)I_2; b, n, s \in R\}.$$

$$\text{Im}(f) = \{(a, 0, 0) + (m, 0, 0)I_1 + (k, 0, 0)I_2; a, m, k \in R\}.$$

Example 5.22:

Let $W_2(I) = \langle (0, 0, 1)I_1 \rangle = \{q \cdot (0, 0, a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I) = \langle (0, 1, 0)I_1 \rangle = \{q \cdot (0, a, 0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-cyclic refined neutrosophic modules of the strong 2-cyclic refined neutrosophic module $R_2^3(I)$ over 2-refined neutrosophic ring $R_2(I)$. Define $f: W_2(I) \rightarrow U_2(I)$; $f[q(0, 0, a)I_1] = q(0, a, 0)I_1$; $q \in R_2(I)$.

f is a strong 2-cyclic refined neutrosophic homomorphism:

Let $A = q_1(0, 0, a)I_1, B = q_2(0, 0, b)I_1 \in W_2(I)$; $q_1, q_2 \in R_2(I)$, we have

$$A + B = (q_1 + q_2)(0, 0, a + b)I_1, f(A + B) = (q_1 + q_2) \cdot (0, a + b, 0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-cyclic refined neutrosophic scalar, we have

$f(m \cdot A) = m \cdot f(A)$ by the definition of f , hence f is a strong 2-cyclic refined neutrosophic homomorphism.

5. Conclusion

In this paper, we have defined the n-cyclic refined neutrosophic ring and n-cyclic refined neutrosophic polynomial ring as new generalizations of the concept of refined neutrosophic set, we have introduced and studied n-cyclic AH-structures such as:

n-cyclic AH-ideal, n-cyclic AHS-ideal, n-cyclic AH-weak principal ideal, n-cyclic AH-weak prime ideal.

Also, we have introduced the concept of n-cyclic refined neutrosophic modules as a direct application of n-cyclic refined neutrosophic rings. In the future, we aim to define and study some n-cyclic refined neutrosophic algebraic structures such as n-cyclic refined neutrosophic vector spaces, and groups. Another application of n-cyclic refined system can be in the theory of decision making and neutrosophic programming in a similar way of classical refined neutrosophic set.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

- [1] Abobala, M., "AH-Subspaces in Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6 , pp. 80-86, 2020.
- [2] Abobala, M., "A Study of AH-Substructures in n-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science", Vol. 9, pp.74-85, 2020.
- [3] Abobala, M., "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 72-75, 2020.
- [4] Agboola, A.A.A., Akwu, A.D., and Oyebo, Y.T., "Neutrosophic Groups and Subgroups", International J .J Math.Combin, Vol. 3, pp. 1-9, 2012.
- [5] Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol. 4 , pp. 72-81, 2020.
- [6] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 59-66, 2020.
- [7] Alhamido, R., and Gharibah, T., "Neutrosophic Crisp Tri-Topological Spaces", Journal of New Theory, Vol. 23 , pp.13-21, 2018.
- [8] Alhamido, R., and Abobala, M., "AH-Substructures in Neutrosophic Modules", International Journal of Neutrosophic Science, Vol. 7, pp. 79-86 , 2020.
- [9] Agboola, A.A.A., and Akinleye, S.A., "Neutrosophic Vector Spaces", Neutrosophic Sets and Systems, Vol. 4 , pp. 9-17, 2014.
- [10] Agboola, A.A.A., "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, Vol.10, pp. 99-101, 2015.
- [11] Anjrou, S., "Linear Algebra 1", Tishreen University Press, pp. 121-205, 2018.

- [12] Hatip, A., and Olgun, N., " On Refined Neutrosophic R-Module", *International Journal of Neutrosophic Science*, Vol. 7, pp.87-96, 2020.
- [13] Hatip, A., Alhamido, R., and Abobala, M., "A Contribution to Neutrosophic Groups", *International Journal of Neutrosophic Science*", Vol. 0, pp. 67-76 , 2019.
- [14] Hatip, A., and Abobala, M., "AH-Substructures In Strong Refined Neutrosophic Modules", *International Journal of Neutrosophic Science*, Vol. 9, pp. 110-116 , 2020.
- [15] Ibrahim, M.A., Agboola, A.A.A, Badmus, B.S., and Akinleye, S.A., "On Refined Neutrosophic Vector Spaces I", *International Journal of Neutrosophic Science*, Vol. 7, pp. 97-109, 2020.
- [16] Ibrahim, M.A., Agboola, A.A.A, Badmus, B.S., and Akinleye, S.A., "On Refined Neutrosophic Vector Spaces II", *International Journal of Neutrosophic Science*, Vol. 9, pp. 22-36, 2020.
- [17] Smarandache F., and Abobala, M., " n-Refined Neutrosophic Vector Spaces", *International Journal of Neutrosophic Science*, Vol. 7, pp. 47-54, 2020.
- [18] Smarandache, F., and Abobala, M., "n-Refined neutrosophic Rings", *International Journal of Neutrosophic Science*, Vol. 5, pp. 83-90, 2020.
- [19] Sankari, H., and Abobala, M., "n-Refined Neutrosophic Modules", *Neutrosophic Sets and Systems*, Vol. 36, pp. 1-11, 2020.
- [20] F. Smarandache, Refined Literal Indeterminacy and the Multiplication Law of Subindeterminacies, Chapter 5, pp. 133-160, in the book by the same author, *Symbolic Neutrosophic Theory*, EuropaNova asbl Bruxelles, 2015, <http://fs.unm.edu/SymbolicNeutrosophicTheory.pdf>