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Abstract In the paper, we discuss necessary and sufficient conditions to obtain the existence, uniqueness and stability of solutions of fractional impulsive functional differential equations towards the ψ-Caputo fractional derivative, through fixed point theorem, Arzela-Ascoli theorem and multivalued analysis theory.

Keywords Fractional Differential inclusions, existence and uniqueness, stability, Banach fixed point, multivalued analysis theory

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1 Introduction

Fractional differential equations and inclusions arise naturally in various fields of science, with numerous applications [1,35,8,36,12,25,18,21,23,24,39]. Over the years, the search for discussing properties of solutions of fractional differential equations, namely: existence, uniqueness, attractiveness, controllability, observability, among others, has been of great community attention [26–29,6,4,9,34,38]. Some authors dedicate to discuss more analytical properties of solutions, others already direct to applications.

In recent years, Benchohra et al. [24] and Chang et al. [11], have investigated the existence of solutions for impulsive functional inclusions Eq.(1) with $p(t) = 1$ and

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Eq.(1), respectively

$$\begin{cases} (p(t)y'(t))' \in F(t,u_t),\ t \in J=[0,a],\ t \neq t_k,\ k=1,2,\cdots,m, \\ \delta y(t)|_{t=t_k} = I_k(y(t_k^-)),\ k=1,2,\cdots,m, \\ \delta y'(t)|_{t=t_k} = J_k(y(t_k^-)),\ k=1,2,\cdots,m, \\ y(t)=\phi(t),\ t\in[-r,0], y'(0)=\eta. \end{cases} \tag{1}$$

Other works on existence, uniqueness, controllability involving differential equations of a integer order, can be obtained at [17,3,5,7,11,14,20].

Em 2018 Wang et al. [35] discuss the controllability of non-instantaneous impulsive semilinear differential inclusions involving integer and fractional order derivatives i.e.,

$$\begin{cases} x'(t) & \in A(t)x(t)+F(t,x_t)+B(u(t)) & ,a.e.t \in \overset{i=m}{\underset{i=1}{\cup}}(s_i,t_{i+1}] \\ x\left(t_i^+\right) & = \quad g_i\left(t_i,x\left(t_i^-\right)\right) & i=1,...,m \\ x(t) & = \quad g_i\left(t,x\left(t_i^-\right)\right) & ,t \in \overset{i=m}{\underset{i=1}{\cup}}(t_i,s_i] \\ x(t) & = \quad \Psi(t) & ,t\in[-r,0] \end{cases}$$

and

$$\begin{cases} {}^{C}D^{\alpha}_{s_i,t}x(t) & \in A(t)x(t)+F(t,x_t)+B(z(t)) & ,a.e.t \in \overset{i=m}{\underset{i=1}{\cup}}(s_i,t_{i+1}] \\ x\left(t_i^+\right) & = \quad g_i\left(t_i,x\left(t_i^-\right)\right) & i=1,...,m \\ x(t) & = \quad g_i\left(t,x\left(t_i^-\right)\right) & ,t \in \overset{i=m}{\underset{i=1}{\cup}}(t_i,s_i] \\ x(t) & = \quad \Psi(t) & ,t\in(-r,0] \end{cases}$$

where $\alpha \in (0,1)$, ${}^{C}D^{\alpha}_{s_i,t}(\cdot)$ is the Caputo fractional derivative with lower limit s_i of order α, A is the infinitesimal generator of C_0-semigroup $\{T(t) : t \leq 0\}$, and the control function z is given is given in $L^p(J,X)$, $p > \frac{1}{\alpha}$, a Banach space of admissible control functions.

In the present paper, we highlight the advances in the theory of fractional differential equations involving fractional derivatives with respect to another function, since they are more general and global operators that allow the discussion and analysis of a wide class of particular cases. The discussions of these problems are motivated by the fractional operators, ψ-Hilfer and ψ-Caputo. Some works in this segment can be obtained at [19,22,15,16,2]. Although there is an interesting range of published works, many open questions and problems, as well as new tools need to be presented, in this sense, one of the main purpose of this work, is to provide new results for the area and consequently contribute to its strengthening.

Let $0 = t_1 < t_1 < \cdots < t_p < t_{p+1} = T$, where T is a positive constant. Consider $F : [0,T] \times D \to \mathscr{P}(\mathbb{R}^n)$ is a multivalued map, where $D = \{\xi : [-r,0] \to \mathbb{R}^n$, ξ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\xi(\bar{t}^-)$ and $\xi(\bar{t}^+)$ exist with $\xi(\bar{t}^-) = \xi(\bar{t})$ $\}$, and $\mathscr{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$.

Motivated by the above work and the innumerable open questions, in this paper, we consider the α^{th} order impulsive fractional functional differential equations

$$\begin{cases} {}^{C}\mathbb{D}_{0^+}^{\alpha,\psi}u(t) \in F(t,u_t),\ t \in J=[0,T],\ t \neq t_k,\ k=1,2,\cdots,p, \\ \delta u^{(i)}(t_k) = \mathscr{I}_{ik}(u(t_k^-)),\ i=0,1,\cdots,m-1,\ k=1,2,\cdots,p, \\ u(t)=\phi(t),\ t\in[-r,0], \\ u^{(i)}(0)=\eta_i,\ i=1,2,\cdots,m-1, \end{cases} \tag{2}$$

where ${}^{C}\mathbb{D}_{0^+}^{\alpha,\psi}(\cdot)$ is the ψ-Caputo fractional derivative of order $m-1<\alpha\leq m$, $m\in\mathbb{N}$, $\phi\in D$ and $\mathscr{I}_{ik}\in C(\mathbb{R}^n,\mathbb{R}^n)$ $(i=0,1,...,m-1,k=1,2,...,p)$. Moreover, $u^{(i)}:[0,T]\to\mathbb{R}^n$ which are piecewise continuous in $[0,T]$ with points of discontinuity of the first kind at the points $t_k\in[0,T]$, i.e., there exists the limits $u^{(i)}(t_k^+)<\infty$ and $u^{(i)}(t_k^-)=u^{(i)}(t_k)<\infty$, $u^{(m)}:[0,T]\to\mathbb{R}^n$, and $\delta u^{(i)}(t_k)=u^{(i)}(t_k^+)-u^{(i)}(t_k)$ $(i=0,1,...,m-1,k=1,2,...,p)$. For any continuous function u defined on $[-r,T]/\{t_1,...,t_p\}$ and any $t\in[0,T]$, we denote by u_t the element of D defined by $u_t(\theta)=u(t+\theta)$, $\theta\in[-r,0]$. Here $u_t(\cdot)$ represents the history of the state form $t-r$, up to the present time t.

For the discussion of the main results of this paper, we will consider the following conditions:

(**A1**) $F:[0,T]\times D\to\mathscr{P}(\mathbb{R}^n)$ has the property that $F(\cdot,\xi):[0,T]\to P_{cp}(\mathbb{R}^n)$ is measurable for each $\xi\in D$.

(**A2**) There exists nonnegative constants c_{ik} $(\mathrm{i}=0,1,...,\mathrm{m}-1,\mathrm{k}=1,...,\mathrm{p})$ such that $|I_{ik}(u(t_k))-I_{ik}(v(t_k))|$, $I_{ik}(0)=0$, $i=0,1,...,m-1$, $k=1,...,p$ and for all $u,v\in\Omega$.

(**A3**) There exists a function $l\in L^1([0,T],\mathbb{R}_+)$ such that $H_d(F(t,\xi),F(t,\varphi))\leq l(t)||\xi-\varphi||_D$, for a.e. $t\in[0,T]$ and any $\xi,\varphi\in D$, and $E_d(0,F(t,0))\leq l(t)$, for a.e. $t\in[0,T]$.

(**A4**) Let $F:[0,T]\times D\to\mathscr{P}(\mathbb{R}^n)$ be a nonempty, compact valued multivalued map such that $(t,\xi)\to F(t,\xi)$ is $\mathscr{L}\times\mathscr{B}$ measurable, and $\xi\to F(t,\xi)$ is l.s.c for a.e. $t\in[0,T]$.

(**A5**) There exists a function $M\in L^1([0,T],\mathbb{R}_+)$ such that $||F(t,\xi)||=sup\{|v(t)|:v(t)\in F(t\xi)\}\leq M(t)$ for each $t\in[0,T]$.

We will here highlight the main contributions of this article explicitly and clearly. The first main contribution is to discuss the existence of solutions for the Eq.(2), that is, the following results:

Theorem 1 *Assume that* **(A1)**, **(A2)** *and* **(A3)** *are satisfied. Then* Eq.(2) *has at least one solution on* $[-r,T]$*, provided*

$$\mathscr{L}=\frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi(T)\|l\|_{L^1}+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(0))^i}{i!}C_{ik}<1.$$

Theorem 2 *In addition to* **(A4)** *and* **(A5)***, assume that the following condition hold:*

(A6) *There exists constants* $d_{ik}(i=0,1,\cdots,m-1,\ k=1,2,\cdots,p)$*, such that* $|\mathscr{I}_{ik}(u(t_k))|\leq d_{ik}|u(t_k)|$ *for each* $u\in\Omega$*. Then* Eq.(2) *has at least one solution on* $[-r,T]$*, provided*

$$\mathscr{L} = \sum_{k=1}^{p} \sum_{i=0}^{m-1} \frac{(\psi(T) - \psi(t_k))^i}{i!} d_{ik} < 1.$$

The second and third main contributions, is to guarantee the uniqueness and stability of solutions for Eq.(2), in other words, is to discuss the following results:

Theorem 3 *In addition to* **(A2)**, **(A4)** *and* **(A5)***, assume that the following condition hold:*

(A7) *There exists non-negative constant b, such that* $|f(u)(t) - f(v)(t)| \leq b\|u - v\|_{\Omega}$ *for any* $u, v \in \Omega$ *where f is deduced in the proof of* Theorem 2. *Then* Eq.(2) *has a unique solution on* $[-r, T]$*, provided*

$$\mathscr{L} = \frac{b(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=1}^{p} \sum_{i=0}^{m-1} C_{ik} \frac{\psi(T)^i}{i!} < 1.$$

Theorem 4 *Assume that all the conditions of* Theorem 3 *hold. Then the solution of the* Eq.(2) *is uniformly stable.*

In the rest, the paper is organized as follows. In section 2, we present definitions and essential results in obtaining the main results of this paper. Section 3 is divided into 3 subsections, each of which discusses a contribution from this paper, that is, the existence, uniqueness and uniform stability of solutions from Eq.(2).

2 Preliminaries

In this section, we present some fundamental concepts and results for the development of this paper.

Let (X, d) be a metric space and $N : X \to \mathscr{P}(X)$ be a multivalued map with nonempty values. We use the notations $P(X) = \{Y \in \mathscr{P}(X) : Y \neq 0\}$, $P_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ bounded}\}$, $P_c = \{Y \in \mathscr{P}(X) : Y \text{ convex}\}$, and $P_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ compact}\}$.

Let $\alpha \in (0,1)$ and $f \in L^1[a,b]$, $[a,b]$ be a finite or infinite interval of the real line $\mathbb{R}$ $(-\infty \leq a < b \leq \infty)$. Also let $\psi(\cdot)$ be an increasing and positive continuous function on $[a,b]$, having a continuous derivatives $\psi'(x) \neq 0$ on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$, is defined [30–32]

$$\mathscr{I}_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s)\, ds. \tag{3}$$

On the other hand, let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $I = [a,b]$, is the interval such that $-\infty \leq a < b \leq \infty$ and there exist two function $f, \psi \in C^n([a,b], \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivatives left-sided and right-sided ${}^{H}\mathbb{D}_{a+,x}^{\alpha,\beta;\psi}(\cdot)$ function or order α and type $0 \leq \beta \leq 1$ is defined by [30–32]

$${}^{H}\mathbb{D}_{a+,x}^{\alpha,\beta;\psi} f(x) = \mathscr{I}_{a+,x}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathscr{I}_{a+,x}^{(1-\beta)(n-\alpha);\psi} f(x). \tag{4}$$

The ψ-Hilfer fractional derivatives defined as above can we written in the following form

$$ {}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+,x} f(x) = \mathscr{I}^{\gamma-\alpha;\psi}_{a+,x} \mathscr{D}^{\gamma;\psi}_{a+,x} f(x) $$

where $\mathscr{D}^{\gamma;\psi}_{a+,x}(\cdot)$ is left-sided ψ-Riemann-Liouville fractional derivative.

Taking limit $\beta \to 1$ on both sides of Eq.(4), we have the Caputo fractional derivative with respect to another function, given by [30]

$$ {}^{C}\mathbb{D}^{\alpha;\psi}_{a+,x} f(x) = \mathscr{I}^{(n-\alpha);\psi}_{a+,x} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} f(x). \tag{5} $$

For the formulation of the main result of this paper, we use the Caputo fractional derivative with respect to another function given by Eq.(5).

For the definitions and lemmas presented below, we use the following references [37,1,8,10,13,33].

Definition 1 A multivalued map $N : [0,T] \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $g : [0,T] \to \mathbb{R}_+$, defined by

$$ g(t) = E_d(x, N(t)) = \inf\{|x-z| : z \in N(t)\} $$

belongs to $L^1([0,T],\mathbb{R})$.

Definition 2 Let X be a nonempty closed subset of $\mathbb{R}^n$, and $N : X \to \mathscr{P}(\mathbb{R}^n)$ be a multivalued map with nonempty closed values. N is lower semicontinuous (l.s.c) on X if the set $\{x \in X : N(x) \cap C \neq \emptyset\}$ is open for each open set C in $\mathbb{R}^n$.

Definition 3 Let $A_{12} \subseteq [0,T] \times D$. A_{12} is $\mathscr{L} \otimes \mathscr{B}$ if A_{12} belongs to the σ-algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$ where $\mathscr{J}$ is Lebesgue measurable in $[0,T]$ and $\mathscr{D}$ is Borel measurable in D.

Definition 4 Let X de a separable metric space and $\mathscr{N} : X \to \mathscr{P}(L^1([0,Y],\mathbb{R}^n))$ be a multivalued operator. We say $\mathscr{N}$ has property (BC) if:

1. $\mathscr{N}$ is (l.s.c).
2. $\mathscr{N}$ has nonempty closed and decomposable values.

In order to define the solutions of Eq.(2), we consider the following spaces:

$$ PC = \left\{ \begin{array}{c} u : [0,T] \to \mathbb{R}^n | u_k \in C((t_k, t_{k+1}], \mathbb{R}^n), k = 0, ..., p \\ \text{and there exist } u(t_k^-) \text{ and } u(t_k^+) \text{ with } u(t_k^-) = u(t_k), k = 0, ..., p \end{array} \right\} $$

,which is a Banach space with the norm $||u||_{PC} = \max\left\{ ||u_k||_{(t_k,t_{k+1}]},\ k = 0, ..., p \right\}$ where u_k is the restriction of u to $(t_k, t_{k+1}]$, $k = 0, ..., p$.

Definition 5 Let $F : [0,T] \times D \to \mathscr{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator $\mathscr{F} : \Omega \to \mathscr{P}(L^1([0,T],\mathbb{R}^n))$ by letting $\mathscr{F}(u) = \left\{ v \in L^1([0,T],\mathbb{R}^n) | v(t) \in F(t,u_t) \text{ for a.e } t \in [0,T] \right\}$. The operator $\mathscr{F}$ is called the Niemytzki operator associated with F.

Definition 6 A function $u \in \Omega$ is said to be a solution of Eq.(2) if u satisfies Eq.(2).

Definition 7 A multivalued operator $N : X \to P_{cl}(X)$ is called:

1. γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x),N(y)) \leq \gamma d(x,y)$ for each $x,y \in X$.
2. contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Definition 8 Let $F : [0,T] \times D \to \mathscr{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. We say F is of lower semicontinuous type (l.s.c type) if its associated Niemytzki operator $\mathscr{F}$ is l.s.c and has nonempty closed and decomposable values.

Definition 9 The muultivalued map N has a fixed point if there exists $x \in X$ such that $x \in N(x)$. The set of fixed points of the multivalued map N will be denoted by Fix N.

Definition 10 For a function $u : [-r,T] \to \mathbb{R}^n$, the set

$$S_{F,u} = \left\{v \in L^1([0,T],\mathbb{R}^n) | v(t) \in F(t,u_t)\right\}$$

is known as the set of selection functions.

Definition 11 F has a measurable selection if there exists a measurable function (single-valued) $h : [0,T] \to \mathbb{R}^n$ such that $h(t) \in S_{F,u}$ for each $t \in [0,T]$.

We will present some essential results from [37,8,10,13,33] to discuss the main results of this paper.

Lemma 1 *Let $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ by*

$$H_d(A,B) = \max\{sup_{a\in A}E_d(a,B), sup_{b\in B}E_d(A,b)\}$$

. Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a complete metric space.

Lemma 2 *Let $\Omega = D \cup PC$. Then Ω is a Banach space with norm*

$$||u||_\Omega = \max\{||u||_D, ||u||_{PC}\}$$

.

Lemma 3 *Let X be a separable metric space and $N : X \to \mathscr{P}(L^1([0,T],\mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e., there exists a continuous function $f : X \to L^1([0,T],\mathbb{R}^n)$ such that $f(x) \in N(x)$ for each $x \in X$.*

Lemma 4 *Let X be a normed linear space with $S \subset X$ convex and $0 \in S$. Assume $H : S \to S$ is a completely continuous operator. If the set*

$$\varepsilon(H) = \{x \in S | x = \lambda H(x) \text{ for some } \lambda \in (0,1) \text{ is bounded}\}$$

is bounded, then H has at least one fixed point in S.

Lemma 5 *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then Fix $N \neq \emptyset$.*

Lemma 6 *$H \subseteq \Omega$ is a relatively compact set if and only if $H \subseteq \Omega$ is uniformly bounded and equicontinuous on each J_k $(\mathrm{k} = 0,...,\mathrm{p})$, where $J_0 = [-0,r]$, $J_k = (t_k,t_{k+1}]$ $(\mathrm{k} = 0,...,\mathrm{p})$.*

Lemma 7 *Let $F : [0,T] \times D \to \mathscr{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume* (**A4**) *and* (**A5**) *hold. Then F is of l.s.c type.*

3 Mains results

In this section, we will discuss the main results of this paper, i.e., first we will investigate the existence of solutions through the fixed point technique and the Arzela-Ascoli theorem and through the necessary and sufficient conditions **(A1)**-**(A6)**, as presented in the introduction. In this sense, assuming the conditions **(A2)**, **(A4)** and **(A5)** imposing the condition a new condition **(A7)**, we obtain the uniqueness of solutions via the Banach fixed point theorem. Finally, we discuss the uniform stability of solutions.

3.1 Existence of solutions

Theorem 5 *Assume that* **(A1)**, **(A2)** *and* **(A3)** *are satisfied. Then* Eq.(2) *has at least one solution on* $[-r,T]$*, provided*

$$\mathscr{L} = \frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi(T)\|l\|_{L^1} + \sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(0))^i}{i!}C_{ik} < 1.$$

Proof First, we consider multivalued map $\mathscr{G}:\Omega\to\mathscr{P}(\Omega)$ defined by $\mathscr{G}(u)=\{g\in\Omega\}$, where

$$g(t)=\begin{cases}\phi(t),\ t\in[-r,0]\\ \phi(0)+\displaystyle\sum_{i=1}^{m-1}\frac{\eta_i}{i!}(\psi(t)-\psi(0))^i+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds\\ +\displaystyle\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(u(t_k)),\ t\in[0,T]\text{ and }h\in S_{F,u}.\end{cases}$$

It is clear that the fixed points of $\mathscr{G}$ are solutions of Eq.(2). For each $u\in\Omega$, the set $S_{F,u}$ is nonempty since by **(A1)**, f has measurable selection.

We shall show that, $\mathscr{G}$ satisfies the assumptions of Lemma 5. The proof will be divided into two steps.

Step 1: $\mathscr{G}(u)\subseteq\mathscr{P}_{cl}(\Omega)$ for each $u\in\Omega$.

Indeed, let $\{u_n\}\in\mathscr{G}(u)$ such that $u_n\to u_*$. Then there exists $h_n\in S_{F,u}$, such that for each $t\in[0,T]$,

$$u_n(t)=\phi(0)+\sum_{i=1}^{m-1}\frac{\eta_i}{i!}(\psi(t)-\psi(0))^i+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_n(s)ds$$
$$+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(u(t_k)).$$

Since $F(0,\psi)$ has compact values and **(A5)** hold, we may pass to a subsequence if necessary to get that h_n converges to h in $L^1([0,T],\mathbb{R}^n)$ and hence $h\in S_{F,u}$. Then, for each $t\in[0,T]$

$$u_n(t)\to u_*(t)=\phi(0)+\sum_{i=1}^{m-1}\frac{\eta_i}{i!}(\psi(t)-\psi(0))^i+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds$$

$$+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(u(t_k)).$$

So $u_*\in\mathscr{G}(u)$, and in particular, $\mathscr{G}(u)\subseteq\mathscr{P}_{cl}(\Omega)$.
Step 2. It can be shown that there exists $\mathscr{L}<1$ such that $H_d(\mathscr{G}(u),\mathscr{G}(\bar{u}))\leq\mathscr{L}\|u-\bar{u}\|_\Omega$ for all $u,\bar{u}\in\Omega$.

Let $u,\bar{u}\in\Omega$ and $g\in\mathscr{G}(u)$. Then there exists $h(t)\in F(t,u_t)$ such that for each $t\in[0,T]$

$$g(t)=\phi(0)+\sum_{i=1}^{m-1}\frac{\eta_i}{i!}(\psi(t)-\psi(0))^i+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds$$
$$+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(u(t_k)).$$

From **(A3)** it follows that, for each $t\in[0,T)$,

$$H_d(F(t,u_t),F(t,\bar{u}_t))\leq l(t)\|u_t-\bar{u}_t\|_D.$$

Hence there exists $w(t)\in F(t,\bar{u}_t)$ such that

$$|h(t)-w(t)|\leq l(t)\|u_t-\bar{u}_t\|_D,\ t\in[0,T].$$

Consider $U:[0,T]\to\mathscr{P}_{cl}(\mathbb{R}^n)$, given by $U(t)=\{w(t):|h(t)-w(t)|\leq l(t)\|u_t-\bar{u}_t\|_D\}$. Since, the multivalued operator $V(t)=U(t)\cap F(t,\bar{u}_t)$ is measurable, there exists a function $\bar{h}(t)$ which is a measurable selection for V. So, $\bar{h}(t)\in F(t,\bar{u}_t)$ and $|h(t)-\bar{h}(t)|\leq l(t)\|u_t-\bar{u}_t\|_D$ for each $t\in[0,T]$.

We define, for each $t\in[0,T]$,

$$g(t)=\phi(0)+\sum_{i=1}^{m-1}\frac{\eta_i}{i!}(\psi(t)-\psi(0))^i+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\bar{h}(s)ds$$
$$+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(\bar{u}(t_k)).$$

Then, we have

$$\begin{aligned}
&|g(t)-\bar{g}(t)|\\
&=\left|\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\bar{h}(s)ds\right.\\
&\quad\left.+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(u(t_k))-\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}\mathscr{I}_{ik}(\bar{u}(t_k))\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|h(s)-\bar{h}(s)|ds\\
&\quad+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(t)-\psi(t_k))^i}{i!}|\mathscr{I}_{ik}(u(t_k))-\mathscr{I}_{ik}\bar{u}(t_k)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi'(T)\int_0^t |h(s)-\bar{h}(s)|ds \\
&\quad + \sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(t_k))^i}{i!}C_{ik}\,|u(t_k)-\bar{u}(t_k)| \\
&\leq \frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi'(T)\int_0^t l(s)\|u_s-\bar{u}_s\|_D ds \\
&\quad + \sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(t_k))^i}{i!}C_{ik}\,|u(t_k)-\bar{u}(t_k)| \\
&\leq \frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi'(T)\|u-\bar{u}\|_\Omega\int_0^t l(s)ds \\
&\quad + \sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(t_k))^i}{i!}C_{ik}\|u-\bar{u}\|_\Omega \\
&\leq \left\{\frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)}\psi'(T)\|l\|_{L^1} + \sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(t_k))^i}{i!}C_{ik}\right\}\|u-\bar{u}\|_\Omega. \qquad (6)
\end{aligned}
$$

So, $\|g(t)-\bar{g}(t)\|_\Omega \leq \mathscr{L}\|u-\bar{u}\|_\Omega$. By an analogous reasoning, obtained by interchanging the roles of u and $\bar{u}$, it follows that $H_d\,(G(u),G(\bar{u})) \leq L\|u-\bar{u}\|_\Omega$. Therefore $\mathscr{G}$ is contraction . By Lemma 5, $\mathscr{G}$ has a fixed point which is solution of Eq.(2).

Theorem 6 *In addition to* **(A4)** *and* **(A5)***, assume that the following condition holds:*

(A6) *There exists constants $d_{ik}(i=0,1,\cdots,m-1,\ k=1,2,\cdots,p)$, such that $|\mathscr{I}_{ik}(u(t_k))| \leq d_{ik}|u(t_k)|$ for each $u\in\Omega$. Then* Eq.(2) *has at least one solution on $[-r,T]$, provided*

$$\mathscr{L} = \sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(T)-\psi(t_k))^i}{i!}d_{ik} < 1.$$

Proof Note that **(A4)**, **(A5)** and Lemma 7 imply that f is of L. S. C. type. Then, from Lemma 3, there exists a continuous function $f:\Omega\to L^1([0,T],\mathbb{R}^n)$ such that $f(u)\in F(u)$, for each $u\in\Omega$.

We consider

$$
\begin{cases}
{}^{C}\mathbb{D}_{0^+}^{\alpha,\psi}u(t) = f(u)(t),\ t\in J=[0,T],\ t\neq t_k,\ k=1,2,\cdots,p,\\
\delta u^{(i)}(t_k) = \mathscr{I}_{ik}(u(t_k)),\ i=0,1,\cdots,m-1,\ k=1,2,\cdots,p,\\
u(t)=\phi(t),\ t\in[-r,0],\\
u^{(i)}(0)=\eta_i,\ i=1,2,\cdots,m-1.
\end{cases} \qquad (7)
$$

We shall show that T satisfies all assumptions of Lemma 4. The proof will be given in four steps.

Step 1: T is continuous.

Since the functions f and $\mathscr{I}_{ik}$ are continuous, this conclusion can be easily obtained.

Step 2: T maps arbitrary bounded subset of Ω into on bounded set in Ω.

Let $B_a = \{u \in \Omega : \|u\|_\Omega \leq a\}$ be arbitrary bounded subset of Ω and $u \in B_a$, then there exists $f \in F(u)$ such that for $t \in [0,T]$,

$$T(u)(t) = \phi(0) + \sum_{i=1}^{m-1} \frac{\eta_i}{i!} (\psi(t) - \psi(0))^i + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(u)(s) ds$$
$$+ \sum_{0<t_k<t} \sum_{i=0}^{m-1} \frac{(\psi(t) - \psi(t_k))^i}{i!} \mathscr{I}_{ik}(u(t_k)). \tag{8}$$

Using the conditions **(A5)** and **(A6)**, we have

$$\begin{aligned}
|T(u)(t)| &\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \int_0^T |f(u)(s)| ds \\
&\quad + \sum_{k=1}^{p} \sum_{i=0}^{m-1} \frac{(\psi(T) - \psi(0))^i}{i!} |\mathscr{I}_{ik}(u(t_k))| \\
&\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \int_0^T |M(s)| ds \\
&\quad + \sum_{k=1}^{p} \sum_{i=0}^{m-1} \frac{(\psi(t) - \psi(t_k))^i}{i!} d_{ik} |u(t_k)| \\
&\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \|M\|_{L^1} \\
&\quad + \sum_{k=1}^{p} \sum_{i=0}^{m-1} \frac{(\psi(t) - \psi(t_k))^i}{i!} d_{ik} \|u\|_\Omega
\end{aligned}$$

for each $t \in [0,T]$.

Therefore, for each $u \in B_a$, we have

$$\begin{aligned}
\|T(u)\|_\Omega &\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \|M\|_{L^1} + \mathscr{L} \|u\|_\Omega \\
&\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \|M\|_{L^1} + \mathscr{L} a.
\end{aligned} \tag{9}$$

Therefore, $T(B_a)$ is bounded.

Step 3. T maps arbitrary bounded set of points into equicontinuous set in Ω.

Let $\tau_1, \tau_2 \in (t_k, t_{k+1})$, $k = 1, 2, \cdots, p$, $\tau_1 < \tau_2$, and $u \in B_a$ be arbitrary bounded subset of Ω. By (8) we get

$$\begin{aligned}
&|T(u)(\tau_2) - T(u)(\tau_1)| \\
&\leq \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} \left[(\psi(\tau_2) - \psi(t_k))^i - (\psi(\tau_1) - \psi(t_k))^i \right] \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi'(s) (\psi(\tau_2) - \psi(s))^{\alpha-1} f(u)(s) ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\psi'(s)\left((\psi(\tau_2)-\psi(s))^{\alpha-1}+(\psi(\tau_1)-\psi(s))^{\alpha-1}\right)f(u)(s)ds\Bigg|\\
&+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(\tau_2)-\psi(t_k))^i}{i!}|\mathscr{I}_{ik}(u(t_k))|+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(\tau_1)-\psi(t_k))^i}{i!}|\mathscr{I}_{ik}(u(t_k))|\\
&\leq\sum_{i=1}^{m-1}\frac{|\eta_i|}{i!}\left(\psi(\tau_2)^i-\psi(\tau_1)^i\right)+\Bigg|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\sum_{i=0}^{m-1}C_{m-1}^i\psi'(s)\psi(s)^{\alpha-1-i}\psi(\tau_2)^i f(u)(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\sum_{i=0}^{m-1}C_{m-1}^i\psi'(s)\psi(s)^{\alpha-1-i}\left(\psi(\tau_2)^i-\psi(\tau_1)^i\right)f(u)(s)ds\Bigg|\\
&+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{(\psi(\tau_2)-\psi(t_k))^i+(\psi(\tau_1)-\psi(t_k))^i}{i!}|\mathscr{I}_{ik}(u(t_k))|\\
&\leq\sum_{i=1}^{m-1}\frac{|\eta_i|}{i!}\left(\psi(\tau_2)^i-\psi(\tau_1)^i\right)+\Bigg|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\sum_{i=0}^{m-1}C_{m-1}^i\psi'(s)\psi(s)^{\alpha-1-i}\psi(\tau_2)^i f(u)(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\sum_{i=0}^{m-1}C_{m-1}^i\psi'(s)\psi(s)^{\alpha-1-i}\left(\psi(\tau_2)^i-\psi(\tau_1)^i\right)f(u)(s)ds\Bigg|\\
&+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{1}{i!}\sum_{j=0}^{i}C_i^j d_{ik}\left(\psi(\tau_2)^j-\psi(\tau_1)^j\right)\psi(\tau_k)^{i-j}\\
&\leq(\psi(\tau_2)-\psi(\tau_1))\sum_{i=1}^{m-1}\frac{|\eta_i|}{i!}\frac{\psi(T)^{i-1}}{(i-1)!}+\frac{1}{\Gamma(\alpha)}\Bigg[\psi'(T)\psi(T)^{\alpha-1}2^{m-1}\int_{\tau_1}^{\tau_2}f(u)(s)ds\\
&+\psi'(T)\psi(T)^{\alpha-2}(\psi(\tau_2)-\psi(\tau_1))\int_0^{\tau_1}|f(u)(s)ds|\sum_{i=1}^{m-1}iC_{m-1}^i\Bigg]\\
&+a\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{1}{i!}\sum_{j=0}^{i}d_{ik}\left(\psi(\tau_2)^j-\psi(\tau_1)^j\right)\psi(\tau_k)^{i-j}\\
&\leq(\psi(\tau_2)-\psi(\tau_1))\sum_{i=1}^{m-1}\frac{|\eta_i|}{i!}\frac{\psi(T)^{i-1}}{(i-1)!}+\frac{1}{\Gamma(\alpha)}\Bigg[\psi'(T)\psi(T)^{\alpha-1}2^{m-1}\int_{\tau_1}^{\tau_2}M(s)ds\\
&+\psi'(T)\psi(T)^{\alpha-2}(\psi(\tau_2)-\psi(\tau_1))\|f\|_{L^1}\sum_{i=1}^{m-1}iC_{m-1}^i\Bigg]\\
&+a(\psi(\tau_2)-\psi(\tau_1))\sum_{i=1}^{m-1}\frac{2^i\psi(T)^{i-1}}{(i-1)!}d_{ik}\\
&\leq(\psi(\tau_2)-\psi(\tau_1))\sum_{i=1}^{m-1}\frac{|\eta_i|}{i!}\frac{\psi(T)^{i-1}}{(i-1)!}+\frac{1}{\Gamma(\alpha)}\Bigg[\psi'(T)\psi(T)^{\alpha-1}2^{m-1}\int_{\tau_1}^{\tau_2}M(s)ds\\
&+\psi'(T)\psi(T)^{\alpha-2}(\psi(\tau_2)-\psi(\tau_1))m2^{m-1}\|f\|_{L^1}\Big]+a(\psi(\tau_2)-\psi(\tau_1))\sum_{i=1}^{m-1}\frac{2^i\psi(T)^{i-1}}{(i-1)!}d_{ik}.
\end{aligned}
$$

According to the completely continuity of integrable function M, the right hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. The convergence for the uses $\tau_2, \tau_1 \in (0, t_1]$ and $[-r, 0]$ is obvious. Then $T(B_a)$ is equicontinuous set in Ω.

As a consequences of **Step 1** to **Step 3** together with Lemma 6 and the Ascoli-Arzela theorem, we conclude that $T : \Omega \to \Omega$ is completely continuous.

Step 4: The set $\varepsilon(T) = \{u \in \Omega : u = \lambda T(u), \text{ for some } 0 < \lambda < 1\}$ is bounded.

For each $u \in \varepsilon(T)$, by (9), we have

$$\begin{aligned}\|u\|_\Omega &= \lambda \|T(u)\|_\Omega \\ &\leq |\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \|M\|_{L^1} + \mathscr{L}\|u\|_\Omega\end{aligned}$$

Then

$$\|u\|_\Omega \leq \frac{|\phi(0)| + \sum_{i=1}^{m-1} \frac{|\eta_i|}{i!} (\psi(T) - \psi(0))^i + \frac{(\psi(T) - \psi(0))^{\alpha-1}}{\Gamma(\alpha)} \psi'(T) \|M\|_{L^1}}{1 - \mathscr{L}}$$

i.e. $\varepsilon(T)$ is bounded.

In view of Lemma 4, we deduce that T has a fixed which in turn is a solution of Eq.(2).

3.2 Uniqueness of solutions

Theorem 7 *In addition to* **(A2)**, **(A4)** *and* **(A5)**, *assume that the following condition hold:*

(A7) *There exists non-negative constant b, such that* $|f(u)(t) - f(v)(t)| \leq b\|u - v\|_\Omega$ *for any* $u, v \in \Omega$ *where f is deduced in the proof of* Theorem 2. *Then* Eq.(2) *has a unique solution on* $[-r, T]$, *provided*

$$\mathscr{L} = \frac{b(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \sum_{k=1}^{p} \sum_{i=0}^{m-1} C_{ik} \frac{\psi(T)^i}{i!} < 1.$$

Proof By (8), for any $u, v \in \Omega$, and each $t \in [0, T]$, we obtain

$$\begin{aligned}|T(u)(t) - T(v)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |f(u)(s) - f(v)(s)| ds \\ &\quad + \sum_{0 < t_k < t} \sum_{i=0}^{m-1} \frac{(\psi(t) - \psi(t_k))^i}{i!} |\mathscr{I}_{ik}(u(t_k)) - \mathscr{I}_{ik}(\bar{u}(t_k))| \\ &\leq \frac{b}{\Gamma(\alpha)} \|u - v\|_\Omega \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} ds \\ &\quad + \sum_{0 < t_k < t} \sum_{i=0}^{m-1} \frac{(\psi(t) - \psi(t_k))^i}{i!} C_{ik} |u(t_k) - \bar{u}(t_k)|\end{aligned}$$

$$\leq \frac{b}{\Gamma(\alpha+1)}\|u-v\|_{\Omega}\left(\psi(T)-\psi(0)\right)^{\alpha}+\|u-v\|_{\Omega}\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{\left(\psi(T)\right)^{i}}{i!}C_{ik}$$

$$\leq \left(\frac{b}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{\left(\psi(T)\right)^{i}}{i!}C_{ik}\right)\|u-v\|_{\Omega}.$$

So

$$\|T(u)-T(v)\|_{\Omega}\leq \mathscr{L}\|u-v\|_{\Omega}.$$

Thus all conditions of the fixed point theorem of Banach are satisfied. Then Eq.(2) has a unique solution on $[-r,T]$.

3.3 Uniformly stable of solutions

Theorem 8 *Assume that all the conditions of* Theorem 3 *hold. Then the solution of the* Eq.(2) *is uniformly stable.*

Proof Let $u(t)$ be a solution of Eq.(2), and let $\bar{u}(t)$ be a solution of Eq.(2) satisfying condition $\bar{X}(0)=\bar{\phi}(0)$, then

$$u(t)-\bar{u}(t)=\begin{cases}\phi(t)-\bar{\phi}(t),\ t\in[-r,0],\\ \phi(0)-\bar{\phi}(0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|f(u)(s)-f(\bar{u}(s))|ds\\ \quad+\displaystyle\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{\left(\psi(t)-\psi(t_k)\right)^{i}}{i!}\left[\mathscr{I}_{ik}(u(t_k))-\mathscr{I}_{ik}(\bar{u}(t_k))\right],\ t\in[0,T].\end{cases}$$

So we get for each $t\in(0,T]$

$$\begin{aligned}\|u(t)-\bar{u}(t)\|&\leq\|\phi(0)-\bar{\phi}(0)\|+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\|f(u)(s)-f(\bar{u})(s)\|ds\\&\quad+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{\left(\psi(t)-\psi(t_k)\right)^{i}}{i!}\|\mathscr{I}_{ik}(u(t_k))-\mathscr{I}_{ik}(\bar{u}(t_k))\|\\&\leq\|\phi-\bar{\phi}\|+\frac{b}{\Gamma(\alpha)}\|u-\bar{u}\|_{\infty}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}ds\\&\quad+\sum_{0<t_k<t}\sum_{i=0}^{m-1}\frac{\left(\psi(t)-\psi(t_k)\right)^{i}}{i!}C_{ik}\|u(t_k)-\bar{u}(t_k)\|\\&\leq\|\phi-\bar{\phi}\|+\frac{b}{\Gamma(\alpha+1)}\|u-\bar{u}\|_{\infty}\left(\psi(T)-\psi(0)\right)^{\alpha}\\&\quad+\|u-\bar{u}\|_{\infty}\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{\left(\psi(t)-\psi(t_k)\right)^{i}}{i!}C_{ik}\\&\leq\|\phi-\bar{\phi}\|+\left(\frac{b}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}+\sum_{k=1}^{p}\sum_{i=0}^{m-1}\frac{\left(\psi(T)\right)^{i}}{i!}C_{ik}\right)\|u-\bar{u}\|_{\infty}.\end{aligned}$$

So we have

$$\|u-\bar{u}\|_{\infty} \leq \|\phi-\bar{\phi}\| + \mathscr{L}\|u-\bar{u}\|_{\infty}.$$

This gives

$$\|u-\bar{u}\|_{\infty} \leq \frac{1}{1-\mathscr{L}}\|\phi-\bar{\phi}\|.$$

Therefore, if $\|\phi-\bar{\phi}\| < \delta(\varepsilon)$, then $\|u-\bar{u}\| < \varepsilon$ which implies that the solution of Eq.(2) is uniformly stable.

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