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Frequency Design of Lossless Passive Electronic Filters: a state-space formulation of the direct synthesis approach

Arthur Perodou, Anton Korniienko, Gérard Scorletti, Mykhailo Zarudniev, Jean-Baptiste David and Ian O'Connor

***Abstract*—This paper deals with the frequency design of lossless passive electronic filters under magnitude constraints. With the huge increase in design complexity for mobile applications, new systematic and efficient methods are required. This paper focuses on the direct synthesis approach, an historical design approach that has not been recently updated. It consists in directly synthesizing the LC values of a pre-specified circuit until the spectral mask is satisfied. While beneficial in practice, this approach typically leads to an important computational time and requires an initial guess to reduce it. Based on recent developments of the System and Control community, that led to efficient methods for system design, the direct synthesis approach is revisited. To achieve this, the port-Hamiltonian Differential Algebraic Equation (pHDAE) representation, that particularly fits the design problem, is introduced. A synthesis method is then developed, leading to solve an optimization problem of moderate complexity. For particular cases, this complexity happens to be remarkably low. Based on this observation, a second method reveals how to obtain such complexity for the more general case, using an original combination between the pHDAE and the LFT representations. Finally, a numerical example shows the validity and illustrates the benefits of this work.**

***Index Terms*—Frequency filter design, LC circuits, passivity, port-Hamiltonian systems, pH DAE representation, LFT systems, BMI optimization, quadratic constraints.**

I. Introduction

The frequency design of passive electronic filters is one of the oldest topic of interest of Circuit and System theory. Started with the invention of the electrical filter at the beginning of last century, it especially drawn the attention during the inter-war period and accompanied the early development of certain research fields, such as Control theory, Signal Processing or Electronics [1], [2]. While having a venerable age at the engineering scale, and despite the rise of digital filters, the design of passive filters is still a research topic for electronic applications, such as mobile communications [3] or low-power circuits [4]. In particular, the deployment of the fourth generation (4G) of mobile standards and the development of the fifth generation (5G) have led the design problem to be highly challenging, mainly due to stringent filtering requirements, technological constraints and the emergence of advanced techniques such as carrier aggregation [5]. Traditional design methods appear limited to face this complexity and systematic methodologies need to be developed.

The design process of passive filter is usually split in several steps, where the complexity of the models is incremented step-by-step [6]. The first step, that is the subject of this paper, typically consists in synthesizing an equivalent circuit, defined as the electrical interconnection of ideal inductors L and capacitors C, such that the magnitude of the frequency response of the scattering parameter s_{21} satisfies given upper and lower bounds (see for instance Fig. 4 and Fig. 5 at the end of the paper). As this is the most repeated step, and that determines the remaining of the design flow, this step should be systematic and time- efficient [7].

To achieve this first step, two design approaches have historically competed [8], [9]. The first approach consists in synthesizing a transfer function that satisfies the frequency requirements and then exactly realizing it as a circuit. This approach usually leads to time-efficient methods, as the standard Butterworth and Chebyshev methods, but is limited to particular circuits such as LC-ladder filters [10]. It has been regularly updated in the last decade in order to synthesize more general filters, such as LC-resonator ladder filters [9], [11], [12] or Acoustic-Wave (AW) ladder filters [7], [9], [13], [14]. However, this approach does not enable to consider technological constraints and generally leads to introduce additional undesirable elements (see [14] for instance).

The second approach, called the direct synthesis approach in this paper, consists in directly synthesizing the element values of a pre-specified circuit until the given frequency requirements are fulfilled. While being flexible, able to consider technological constraints and closer to the design problem formulation, this approach is usually time-consuming. Unlike the first approach, the direct synthesis approach seems to have not been improved and is still mainly based on the intuition of designers and the use of generic solvers [6], [15].

In the past two decades, the System and Control community has obtained numerous time-efficient methods for system design problems [11], [12], [16]–[20]. Based on similar concepts, it is proposed in this paper to revisit the direct synthesis approach using modern Control and System tools.

II. Overview

From a System perspective, the direct synthesis approach can be viewed as the structured synthesis of a state-space

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representation. Unfortunately, this generally leads to *complex* optimization problems [21]. By complex, it is meant that they do not belong to the class of P-complex problems [22], i.e. they do not have polynomial-time resolution algorithm. A case in point of P-complex problems is the class of Linear Matrix Inequality (LMI) optimization problems [23], that are optimization problems where the constraints are inequalities being linear[1] in the optimization variables. This class has especially met a certain success in Control and System theory (see for instance [24]). A relevant extension of the LMI class is the class of Bilinear Matrix Inequality (BMI) optimization problems. While being generally complex (cf Discussion in [25]), BMI problems frequently appear in System Design [25], [26], including this work, and dedicated algorithms have been developed (see [27]–[29] for instance). A BMI problem can be viewed as an LMI problem that additionally contains inequality constraints with bilinear terms, i.e. products between optimization variables of order 2. Therefore, in this paper we use the following empirical rule: the more bilinear terms, the more complex is the optimization problem. Similarly, products between optimization variables of order 3, or higher, increase even more the complexity of the resulting optimization problem, and should be avoided.

In order to tackle the design problem and decrease its complexity, a structured state-space representation coming from the port-Hamiltonian (pH) formalism appears promising. The pH framework is a multi-physics approach for modeling, analyzing and controlling systems [30]. Surprisingly, while this framework seems particularly connected to electrical circuit theory [31]–[35], which is intrinsically linked with passive network theory and frequency filtering [1], [2], [36], the frequency design of passive filters has not been tackled using a pH approach yet, as far as the authors know. We then propose to use this framework and, more specifically, a state-space representation with a specific structure, namely the pH Differential-Algebraic Equations (pH DAE) representation [37], that is particularly adapted to address the design problem with the direct synthesis approach (Problem 2).

The first main contribution of this paper is to show how the design problem can be tackled by solving a BMI feasibility problem of moderate complexity (Theorem 1). This is obtained by developing a synthesis method that is adapted to the structure of the pH DAE representation (Theorem 1). For particular filters having a usual pH state-space representation, this synthesis method leads to a BMI feasibility problem with a remarkably-low complexity. Based on this observation, it is revealed how to obtain an optimization problem with similar complexity for any lossless passive electronic filters, leading to the second main contribution (Theorem 2). To achieve this, an original representation is especially introduced, based on the combination of the pH DAE representation and the so-called Linear Fractional Transformation (LFT) representation [38], that we denote as the pH LFT representation (Lemma 2). Furthermore, while the previous synthesis method is based on the so-called S-procedure lemma and existing quadratic constraints on input-output signals of integrators in the frequency domain (see for instance [17], [39]), the novelty of the pH LFT representation requires to derive new quadratic constraints on the input-output signals of a combination of integrators and derivators. These novel quadratic constraints are proposed in Appendix C since it is not the main contribution of the paper.

[1]actually affine

Structure: This article is organized as follows. In Section III, after providing some background about lossless passive electronic filters, the general statement of the design problem is given (Problem 1). In Section IV, the pH DAE representation is defined and adapted to our design purposes, while in Section V, Problem 1 is equivalently reformulated in a more explicit form, as the synthesis of a pH DAE representation (Problem 2), and sufficient conditions (Theorem 1) are provided, in the form of a BMI feasibility problem, to solve it. In Section VI, the pH LFT representation is introduced, the design problem is equivalently reformulated with this original representation (Problem 3), and sufficient conditions in the form of a BMI feasibility problem are obtained (Theorem 2). Finally, a numerical example illustrates the potentials of our synthesis method in Section VII, and Section VIII concludes this paper by providing certain perspectives of this work.

Notations: $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices, while $\mathbb{S}^n$ stands for the set of real symmetric $n\times n$ matrices and $\mathbb{D}^n$ the set of real, diagonal $n\times n$ matrices. The matrix I_n denotes the identity matrix of $\mathbb{R}^{n\times n}$ and $0_{n\times m}$ the zero matrix of $\mathbb{R}^{n\times m}$. The subscripts are omitted when obvious from the context. X^T and X^* respectively denote the transpose and transpose conjugate of the matrix X, while $X > (\geq)\ 0$ denotes positive (semi-) definiteness. The letter s is the Laplace variable, $\mathcal{L}(\cdot)$ the Laplace operator, and given a time-domain function $f(t)$, $f(s)$ will be its Laplace transformation, i.e. $f(s) = \mathcal{L}(f(\cdot))(s)$. Bold characters denote either explicit decision variables in a design problem or optimization variables in an optimization problem.

III. Background and general design problem

A. Background about lossless passive electronic filters

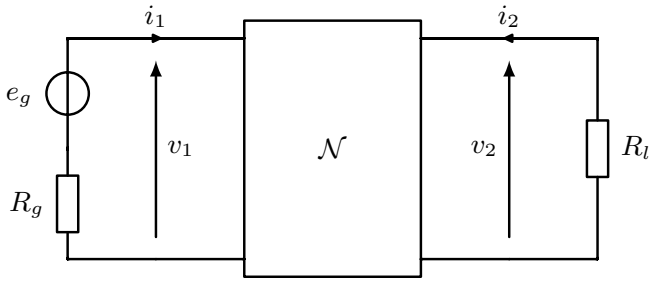


Fig. 1. Electronic filter $\mathcal{N}$

An electronic filter is a two-port $\mathcal{N}$ terminated on a resistive voltage source (e_g, R_g) and a resistive load R_l (Fig. 1). Its filtering performance is typically evaluated using the frequency-response of the so-called scattering matrix S defined by [40]:

$$\begin{bmatrix}\beta_1(s)\\ \beta_2(s)\end{bmatrix} = \underbrace{\begin{bmatrix}s_{11}(s) & s_{12}(s)\\ s_{21}(s) & s_{22}(s)\end{bmatrix}}_{S(s)} \begin{bmatrix}\alpha_1(s)\\ \alpha_2(s)\end{bmatrix}$$

where

$$\beta_1(s) = \frac{v_1(s) - R_g i_1(s)}{2\sqrt{R_g}} \qquad \alpha_1(s) = \frac{v_1(s) + R_g i_1(s)}{2\sqrt{R_g}}$$
$$\beta_2(s) = \frac{v_2(s) - R_l i_2(s)}{2\sqrt{R_l}} \qquad \alpha_2(s) = \frac{v_2(s) + R_l i_2(s)}{2\sqrt{R_l}}$$

Remark 1. Physically, β and α are interpreted as power waves and $|s_{21}(j\omega)|^2$ as the transducer power gain [40].

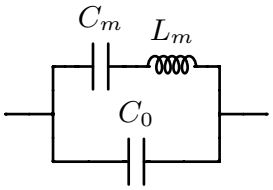


Fig. 2. An acoustic-wave (AW) element

Graphically, a two-port $\mathcal{N}$ is viewed as the ideal electrical interconnection of elements, where an element is a finite-dimension Linear Time-Invariant (LTI) model of a physical component. For instance, an element modeling an AW component is represented on Fig. 2 and has the impedance Z_{AW}:

$$Z_{AW}(j\omega) = \frac{1}{C_0 \cdot j\omega} \frac{\omega^2 - \omega_r^2}{\omega^2 - \omega_a^2}$$

where $\omega_r := \sqrt{\frac{1}{L_m C_m}}$ and $\omega_a := \omega_r \sqrt{1 + \frac{C_m}{C_0}}$ are respectively called resonance anti-resonance frequencies.

For electronic filters, the interconnection is usually referred to as the topology of the filter. For instance, an n-elements example of a two-port in a ladder topology, that is commonly used in mobile applications [6], is drawn on Fig. 3.

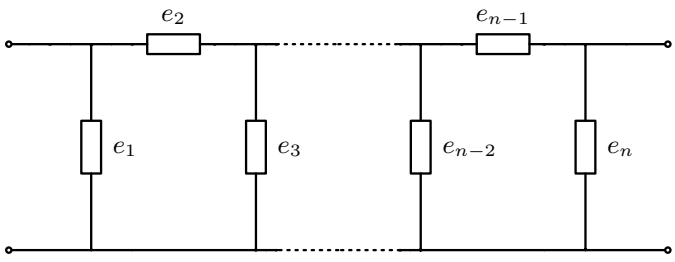


Fig. 3. An n-element example of a two-port in a ladder topology

In this article, the elements are assumed to be lossless passive. Passivity can be interpreted as considering components that can dissipate, store or release energy but can not release more energy than was supplied to them. Lossless passive components are then those that are passive but do not dissipate energy. For most of practical cases, this implies that the elements will be drawn as the ideal electrical interconnection of positive inductors $L > 0$ and capacitors $C > 0$ [36].

In this work, a lossless passive electronic filter is defined as the ideal electrical interconnection of lossless passive elements, i.e. the ideal electrical interconnection of positive inductors and capacitors.

B. General design problem formulation

The problem is to design electronic filters of which scattering matrix S satisfies magnitude constraints in the frequency-domain. More precisely, frequency filtering constraints are set on the magnitude of the frequency-response of s_{21} and consist of upper and lower bounds defined on frequency intervals. These constraints are referred to as a spectral mask.

Definition 1 (Spectral Mask $\mathcal{SM}$)**.**
Consider N_U upper bounds, defined by N_U positive real numbers U_u associated with N_U frequency intervals $\Omega_{U_u} \subseteq \mathbb{R}$. Consider N_L lower bounds, defined by N_L positive real numbers L_l associated with N_L frequency intervals $\Omega_{L_l} \subseteq \mathbb{R}$. Define the spectral mask $\mathcal{SM}$ as the union of these upper and lower bounds. A transfer function s_{21} will be said to satisfy $\mathcal{SM}$ if the following constraints hold:

$$\forall u \in \{1, \dots, N_U\}, \qquad \forall \omega \in \Omega_{U_u}, \quad |s_{21}(j\omega)|^2 \leq U_u^2 \quad (1)$$
$$\forall l \in \{1, \dots, N_L\}, \qquad \forall \omega \in \Omega_{L_l}, \quad |s_{21}(j\omega)|^2 \geq L_l^2 \quad (2)$$

In practice, the elements and the topology of the filter are chosen by designers using a circuit diagram through the interface of a CAD software. The problem is then to find the element values, for instance L_m, C_m and C_0 of Fig. 2, such that the resulting s_{21} satisfies a given spectral mask $\mathcal{SM}$, as stated in Problem 1.

Problem 1 (**Lossless passive electronic filter design**)**.**
GIVEN a spectral mask $\mathcal{SM}$,
GIVEN a circuit diagram of a lossless passive electronic filter,
FIND if there exist positive element values such that the resulting scattering parameter s_{21} is stable and satisfies the spectral mask $\mathcal{SM}$, i.e. such that (1)-(2) hold.
If such values exist, COMPUTE them.

From a System perspective, the state-space representation naturally appears when tackling Problem 1 with the direct synthesis approach. Indeed, when a circuit diagram is made by a designer, a mathematical model is implicitly built. The state-space representation provides a direct mean to make it explicit. By an adequate choice of the state vector, the dynamics of the elements are displayed, while the topology of the filter appears through structural constraints on the state-space matrices. Moreover, the elements values generally appear in a simpler form than, for instance, with a transfer function.

The design problem becomes then to synthesize the structured matrices of a state-space representation which defines a stable s_{21} such that (1)-(2) hold. Unfortunately, the structured state-space synthesis problem seems to be inherently complex [21]. Therefore, the objective of the next sections is to reduce this complexity. To achieve this, a suitable state-space representation is first introduced, without loss of generality. Indeed, among the infinite number of equivalent state-space representations, those leading to the lowest complexity should be preferred. Then an adapted synthesis approach is developed.

IV. pH DAE REPRESENTATIONS

A. On the use of the pH DAE representations

In order to solve Problem 1, the first step is to obtain a mathematical representation from the circuit diagram that is suitable

for design, i.e. a representation where the decision variables are explicitly exhibited and leading to an optimization problem with complexity as low as possible. The formalism of port-Hamiltonian (pH) systems appears for us appropriate to obtain such a representation for the reasons detailed below.
First, it provides a straightforward and systematic modeling[2] procedure to compute a structured state-space representation from any circuit diagram [32]. The result is generally not a usual state-space representation but is in a Differential-Algebraic Equations (DAE) form [37], called pH DAE representation in the sequel. Second, the resulting structure of the representation appears particularly simple from a design perspective. In fact, the design variables, i.e. the element values, are the entries of a certain *diagonal* matrix Q that compactly and linearly appears in the state-space matrices. Intuitively, these features should reduce the complexity of the design problem. Third, due to the structure of the pH representation, certain properties of the scattering parameter s_{21} can be easily ensured, such as stability and unitary gain.

B. pH DAE representations

A scattering description, i.e. linking the signals α_1, α_2, β_1 and β_2, may not be directly obtained from a given circuit diagram. Instead, an impedance description, i.e. linking the signals i_1, i_2, v_1 and v_2 (Fig. 1), is usually first obtained and then transformed to get a scattering description [36].

In order to obtain an impedance description, the modeling procedure of the pH framework is used [32]. It is a power-based approach that consists in writing the dynamics of each element and their interconnection scheme using the Kirchhoff's circuit laws, to obtain a state-space representation.

The result is generally not a usual state-space representation. This is due to two reasons. First, the impedance matrices may have both strictly proper and improper entries while a usual state-space representation engenders a proper transfer matrix [38]. A hybrid matrix, that only has proper entries, is sometimes used in Network theory [36]. Another option is to do not explicitly label the signals as inputs and outputs. This option is retained in the sequel as the input-output labeling of the impedance representation is not essential for solving our design problem. Second, due to the interconnection laws, algebraic constraints commonly appear between state variables. This leads to use an extended state-space representation for pH systems, named port-Hamiltonian Differential-Algebraic Equations (pH DAE) representation and defined below.

Definition 2 (**pH DAE representation** (Adapted from [37]))**.**

Given matrices E_s, $F_s \in \mathbb{R}^{(n+n_e)\times n}$, E_e, $F_e \in \mathbb{R}^{(n+n_e)\times n_e}$ such that:

$$E_S F_S^T + F_S E_S^T + E_e F_e^T + F_e E_e^T = 0 \tag{3}$$
$$\operatorname{rank}\begin{bmatrix} F_S & E_S & F_e & E_e \end{bmatrix} = n + n_e \tag{4}$$

[2]Rigorously, this is not a *modeling* procedure, as a circuit diagram is already a model of a physical system. This procedure enables to move from the level of circuit diagrams to the level of mathematical equations. However, for the sake of simplicity, the word modeling is preferred in this context.

a pH DAE representation is defined as:

$$F_S \dot{x}(t) = E_S Q x(t) + F_e f_e(t) + E_e e_e(t)$$

where $Q \in \mathbb{S}^n$ is a symmetric matrix, x is denoted as the state variables and (f_e, e_e) the external variables.

Remark 2. This representation is usually referred to as a matrix kernel representation of a Dirac structure [37].

Definition 2 provides a general representation that can be used for any power-preserving interconnection of lossless passive subsystems. In the context of lossless passive electronic filters, e_e and f_e are the electrical signals linking the circuit with the source and the load, i.e with conventions of Fig. 1:

$$f_e = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad e_e = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{5}$$

The state variables x and $\dot{x}$ contain the characteristic electrical quantities of the elements, while $Q \in \mathbb{D}^{n\times n}$ is a diagonal matrix with the inverse of the element values on its diagonal. Further details on the application of the modeling procedure to lossless passive electronic filters can be found in [9, Chap. 4].

The scattering pH DAE representation is now defined.

Definition 3 (**Scattering pH DAE representation** (Adapted from [41]))**.**
Given a pH DAE representation such as in Definition 2,
Given a positive-definite, diagonal matrix $\mathcal{R} > 0 \in \mathbb{D}^{n_e\times n_e}$, a *scattering* pH DAE representation associated to $\mathcal{R}$ is defined as:

$$F_S \dot{x}(t) = E_S Q x(t) + G z(t) + H w(t) \tag{6}$$

with $z(t)$, $w(t) \in \mathbb{R}^{n_e}$ and G, $H \in \mathbb{R}^{(n+n_e)\times n_e}$ given by:

$$z = \frac{1}{2}\left(\mathcal{R}^{-\frac{1}{2}} f_e - \mathcal{R}^{\frac{1}{2}} e_e\right) \qquad G = F_e \mathcal{R}^{\frac{1}{2}} - E_e \mathcal{R}^{-\frac{1}{2}}$$
$$w = \frac{1}{2}\left(\mathcal{R}^{-\frac{1}{2}} f_e + \mathcal{R}^{\frac{1}{2}} e_e\right) \qquad H = F_e \mathcal{R}^{\frac{1}{2}} + E_e \mathcal{R}^{-\frac{1}{2}}$$

The variables z and w are also denoted as *external* variables.

Remark 3. For this representation, (3) and (4) become:

$$E_S F_S^T + F_S E_S^T + \frac{1}{2}\left(H H^T - G G^T\right) = 0 \tag{7}$$
$$\operatorname{rank}\begin{bmatrix} F_S & E_S & G & H \end{bmatrix} = n + n_e \tag{8}$$

For lossless passive electronic filters, i.e. when f_e and e_e are given by (5), one gets $n_e = 2$ and

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

while $\mathcal{R}$ is provided by:

$$\mathcal{R} := \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix}$$

Unlike the impedance matrix, the scattering matrix of a passive electronic filter is known to have proper transfer functions as entries [36]. This implies that the external variables w and z can be explicitly split as inputs and outputs. This feature can be extended to the scattering pH DAE representation, as stated in the next property.

Property 1 (Descriptor form [9])**.**
Consider a scattering pH DAE representation such as in Definition 3. Then (6) can be re-written as follows:

$$\begin{cases} F_{S_1}\dot{x}(t) = E_{S_1}Qx(t) + H_1 w(t) \\ z(t) = E_{S_2}Qx(t) + H_2 w(t) \end{cases} \tag{9}$$

This representation is said to be in the descriptor *form.*

Proof. Cf [9, Property 4.1, p. 60] and the associate proof. □

When $F_{S_1} = I$, or F_{S_1} is invertible, the descriptor form reduces to a usual state-space representation. Due to (7)-(8), this is written using the pH formalism as follows [9], [41]:

$$\begin{cases} \dot{x}(t) = \left(J - BB^T\right)Qx(t) + \sqrt{2}Bw(t) \\ z(t) = -\sqrt{2}DB^TQx(t) + Dw(t) \end{cases} \tag{10}$$

where $J = -J^T$ and $D^TD = I$.

Furthermore, even when F_{S_1} is not invertible, the descriptor form (9) can be reduced, under a mere assumption, to a pH state-space representation such as (10) by eliminating the algebraic constraints between the state variables (refer to [37] and the references therein). However, this process breaks the diagonal structure of the matrix Q, that will be essential for limiting the complexity of the synthesis approach developed in the next section. In addition, the particular connection between the representation and the circuit diagram is lost.

C. Structural properties

The structure of pH representations enables to easily check certain system properties. In particular, an important result is that the condition $Q > 0$ is sufficient for ensuring stability and the all-pass property.

Property 2 ([31], [42])**.**
Given the scattering pH DAE representation (6), or the equivalent descriptor form (9), of a system Σ with transfer $U(s)$. If $Q = Q^T > 0$, then Σ is stable and is all-pass, i.e.

$$\forall \omega \in \mathbb{R}, \qquad U(j\omega)^*U(j\omega) = I$$

In our case, the standard result [36] that the scattering matrix S is stable and $\forall \omega \in \mathbb{R}$, $S(j\omega)^*S(j\omega) = I$ is found again. Consequently, the transfer function s_{21} is stable and

$$\forall \omega \in \mathbb{R}, \qquad |s_{21}(j\omega)|^2 \leq 1$$

D. Complex-domain representation of s_{21}

The modeling procedure of the pH framework and the previous considerations allow to obtain (9) as a time-domain scattering representation of a given circuit diagram. However, Problem 1 is formulated in the complex domain and on the scattering parameter s_{21}. In the sequel, it is shown how to obtain a complex-domain representation of s_{21}.

Assuming zero initial conditions, applying the Laplace transform to (9) and defining $p(s) := \mathcal{L}(x(\cdot))(s)$, one gets:

$$\begin{cases} p(s) = \left(\frac{1}{s} \cdot I_n\right) q(s) \\ F_{S_1}q(s) = E_{S_1}Qp(s) + H_1 w(s) \\ z(s) = E_{S_2}Qp(s) + H_2 w(s) \end{cases} \tag{11}$$

This provides a representation of the scattering matrix $S(s)$ and allows to compute a representation of $s_{21}(s)$, as presented in the next lemma.

Lemma 1.
Given *the complex-domain, scattering pH DAE representation in a descriptor form (11) of a lossless passive electronic filter,*
Define

$$\begin{bmatrix} E_{z_1} \\ E_{z_2} \end{bmatrix} := E_{S_2} \quad \begin{bmatrix} H_{w_1} & H_{w_2} \end{bmatrix} := H_1 \quad \begin{bmatrix} H_{s_{11}} & H_{s_{12}} \\ H_{s_{21}} & H_{s_{22}} \end{bmatrix} := H_2$$

Then a representation of $s_{21}(s)$ is given by:

$$\begin{cases} p(s) = \left(\frac{1}{s} \cdot I_n\right) q(s) \\ F_{S_1}q(s) = E_{S_1}Qp(s) + H_{w_1} w_1(s) \\ z_2(s) = E_{z_2}Qp(s) + H_{s_{21}} w_1(s) \end{cases} \tag{12}$$

Proof. This is straightly obtained from (11) by noticing that, with the convention of Fig 1, $w_2 = \alpha_2 = 0$, . □

V. Scattering pH DAE representation synthesis

The design problem is now tackled. First, using the systematic modeling procedure of the pH framework and applying Lemma 1, Problem 1 is equivalently restated (Problem 2). Then, an adapted synthesis approach to solve Problem 2 is provided (Theorem 1). Finally, the result is discussed and compared with a more traditional approach, highlighting then the benefits of our approach.

A. Equivalent design problem formulation

The general problem (Problem 1) is now equivalently restated using the scattering pH DAE representation (Problem 2). Unlike Problem 1, Problem 2 has the benefit of providing an explicit mathematical formulation of the design problem.

Problem 2 (Scattering pH DAE representation synthesis).

GIVEN a spectral mask $\mathcal{SM}$,
GIVEN a scattering pH DAE representation in the descriptor form (9) of a lossless passive electronic filter,
FIND if there exists a positive-definite, diagonal matrix $\boldsymbol{Q} > 0 \in \mathbb{D}^{n\times n}$ such that the scattering parameter s_{21}, given by (12), is stable and satisfies the spectral mask $\mathcal{SM}$, i.e. such that (1)-(2) hold.
If such matrix $\boldsymbol{Q}$ exists, COMPUTE it.

B. Adapted synthesis approach

Next theorem provides sufficient conditions, in the form of a feasibility problem, to solve Problem 2. This is obtained using a signal-based approach, based on the so-called S-procedure and the characterisation of the input-output signals of integrators by quadratic constraints (see Appendix A).

Theorem 1.
Given *a spectral mask $\mathcal{SM}$,*
Given *a scattering pH DAE representation in the descriptor form (9) of a lossless passive electronic filter,*
Given *real matrices $E_{z_2} \in \mathbb{R}^{1\times n}$, $H_{w_1} \in \mathbb{R}^{n\times 1}$, $H_{s_{21}} \in \mathbb{R}^{1\times 1}$*

as in Lemma 1,
A solution of statement (i) provides a solution to (ii).

(i) *There exists a solution to the following feasibility problem:* $\exists \boldsymbol{Q^{inv}} \in \mathbb{D}^{n\times n}$,
$\forall u \in \{1,\dots,N_U\}$, $\exists \boldsymbol{P_u}$, $\boldsymbol{D_u} \in \mathbb{S}^n$, $\exists \boldsymbol{\tau_u} \in \mathbb{R}$,
$\forall l \in \{1,\dots,N_L\}$, $\exists \boldsymbol{P_l}$, $\boldsymbol{D_l} \in \mathbb{S}^n$, $\exists \boldsymbol{\tau_l} \in \mathbb{R}$,

$$\boldsymbol{Q^{inv}} > 0$$

$$\boldsymbol{D_u} > 0 \qquad \boldsymbol{\tau_u} > 0$$

$$\mathcal{A}_{\Omega_{U_u}}(\boldsymbol{D_u},\boldsymbol{P_u},\boldsymbol{\tau_u}) + \begin{bmatrix} \boldsymbol{Q^{inv}} F_{S_1}^T F_{S_1} \boldsymbol{Q^{inv}} & \mathcal{A}\left(\boldsymbol{Q^{inv}}\right) \\ \mathcal{A}\left(\boldsymbol{Q^{inv}}\right)^T & \mathcal{C} \end{bmatrix} \geq 0$$

$$\boldsymbol{D_l} > 0 \qquad \boldsymbol{\tau_l} > 0$$

$$\mathcal{A}_{\Omega_{L_l}}(\boldsymbol{D_l},\boldsymbol{P_l},\boldsymbol{\tau_l}) + \begin{bmatrix} \boldsymbol{Q^{inv}} F_{S_1}^T F_{S_1} \boldsymbol{Q^{inv}} & \mathcal{A}\left(\boldsymbol{Q^{inv}}\right) \\ \mathcal{A}\left(\boldsymbol{Q^{inv}}\right)^T & \mathcal{C} \end{bmatrix} \geq 0$$

with

$$\mathcal{C} := \begin{bmatrix} E_{S_1}^T E_{S_1} + E_{z_2}^T E_{z_2} & -E_{z_2}^T & E_{S_1}^T H_{w_1} + E_{z_2}^T H_{s_{21}} \\ -E_{z_2} & 1 & -H_{s_{21}} \\ H_{w_1}^T E_{S_1} + H_{s_{21}}^T E_{z_2} & -H_{s_{21}}^T & H_{w_1}^T H_{w_1} + H_{s_{21}}^T H_{s_{21}} \end{bmatrix}$$

$$\mathcal{A}\left(\boldsymbol{Q^{inv}}\right) := \begin{bmatrix} -\boldsymbol{Q^{inv}} F_{S_1}^T E_{S_1} & 0 & -\boldsymbol{Q^{inv}} F_{S_1}^T H_{w_1} \end{bmatrix}$$

$$\mathcal{A}_{\Omega_{U_u}}(\boldsymbol{D_u},\boldsymbol{P_u},\boldsymbol{\tau_u}) := \begin{bmatrix} -z_{\Omega_{U_u}}\boldsymbol{D_u} & -y^*_{\Omega_{U_u}}\boldsymbol{D_u} - \boldsymbol{P_u} & 0 & 0 \\ -y_{\Omega_{U_u}}\boldsymbol{D_u} - \boldsymbol{P_u} & -x_{\Omega_{U_u}}\boldsymbol{D_u} & 0 & 0 \\ 0 & 0 & -\boldsymbol{\tau_u} & 0 \\ 0 & 0 & 0 & U_u^2\boldsymbol{\tau_u} \end{bmatrix}$$

$$\mathcal{A}_{\Omega_{L_l}}(\boldsymbol{D_l},\boldsymbol{P_l},\boldsymbol{\tau_l}) := \begin{bmatrix} -z_{\Omega_{L_l}}\boldsymbol{D_l} & -y^*_{\Omega_{L_l}}\boldsymbol{D_l} - \boldsymbol{P_l} & 0 & 0 \\ -y_{\Omega_{L_l}}\boldsymbol{D_l} - \boldsymbol{P_l} & -x_{\Omega_{L_l}}\boldsymbol{D_l} & 0 & 0 \\ 0 & 0 & \boldsymbol{\tau_l} & 0 \\ 0 & 0 & 0 & -L_l^2\boldsymbol{\tau_l} \end{bmatrix}$$

where $x_{\Omega_{U_u}}$, $y_{\Omega_{U_u}}$, $z_{\Omega_{U_u}}$, and $x_{\Omega_{L_l}}$, $y_{\Omega_{L_l}}$, $z_{\Omega_{L_l}}$, depend respectively on Ω_{U_u} and Ω_{L_l}, and are provided as follows:

$$\left.\begin{array}{ll} \begin{bmatrix} x_\Omega & y_\Omega \\ y^*_\Omega & z_\Omega \end{bmatrix} = \begin{bmatrix} \omega_0^2 & 0 \\ 0 & -1 \end{bmatrix} & \textit{for } \Omega = [0,\omega_0] \\ \begin{bmatrix} x_\Omega & y_\Omega \\ y^*_\Omega & z_\Omega \end{bmatrix} = \begin{bmatrix} -\omega_1\omega_2 & -j\frac{\omega_1+\omega_2}{2} \\ j\frac{\omega_1+\omega_2}{2} & -1 \end{bmatrix} & \textit{for } \Omega = [\omega_1,\omega_2] \\ \begin{bmatrix} x_\Omega & y_\Omega \\ y^*_\Omega & z_\Omega \end{bmatrix} = \begin{bmatrix} -\omega_0^2 & 0 \\ 0 & 1 \end{bmatrix} & \textit{for } \Omega = [\omega_0,+\infty) \end{array}\right\} \tag{13}$$

(ii) *Problem 2 is solved. Indeed, s_{21} given by (12), where $\boldsymbol{Q} := \left(\boldsymbol{Q^{inv}}\right)^{-1}$, is stable and verifies (1)-(2).*

Proof. Cf Appendix A. □

Using Theorem 1, Problem 2, and thus Problem 1, can be solved in the manner of the direct synthesis approach. It allows then to take benefits from this approach. In particular, affine constraints on the element values, such as upper Q_U^{inv} and lower Q_L^{inv} bounds, can be straightly included by adding LMI constraints on $\boldsymbol{Q^{inv}}$:

$$Q_L^{inv} \leq \boldsymbol{Q^{inv}} \leq Q_U^{inv}$$

This is of practical interest as it allows to include certain technological constraints that are expressed on the element values (see [43] for instance for AW elements).

However, it also inherits the inherent complexity of the direct synthesis approach. Indeed, the optimization problem of statement (i) contains bilinear terms and is then a BMI feasibility problem, known to be generally complex (see Discussion in [25]). Nonetheless, this complexity appears moderate as there is only a single bilinear term per constraint and it is the same for all the constraints, i.e. $\boldsymbol{Q^{inv}} F_{S_1}^T F_{S_1} \boldsymbol{Q^{inv}}$.

Furthermore, in the particular case where (12) is a usual pH state-space representation, i.e. when $F_{S_1} = I$, due to the diagonal structure of $\boldsymbol{Q}$ the shared bilinear matrix term degenerates to:

$$\boldsymbol{Q^{inv}}\boldsymbol{Q^{inv}} = \begin{bmatrix} \left(\boldsymbol{q_1^{inv}}\right)^2 & & \\ & \ddots & \\ & & \left(\boldsymbol{q_n^{inv}}\right)^2 \end{bmatrix}$$

That is, there is a particularly low number of bilinear terms equaled to n, the total number of inductors and capacitors used for representing the elements.

C. Discussion on the synthesis approach

Instead of the approach developed to obtain Theorem 1, a natural choice from a System perspective would be to directly extend the standard analysis approach [36], based on the Kalman-Yakubovich-Popov (KYP) lemma [44], to a design purpose. In the analysis case, and for particular system design cases (see for instance the design applications of [17]), this approach leads to solve an LMI feasibility problem, known to be efficiently solvable. Unfortunately, in our case, this approach leads to an optimization problem much more complex. For instance, assume that the state-space representation $\left[\begin{array}{c|c} (J-BB^T)\boldsymbol{Q} & \sqrt{2}B_1 \\ \hline \sqrt{2}B_2^T\boldsymbol{Q} & 0 \end{array}\right]$ of s_{21} is given and consider the following upper bound:

$$\forall \omega \in [0,\omega_0], \quad |s_{21}(j\omega)|^2 \leq U^2$$

Applying the appropriate KYP Lemma [17] leads to the equivalent feasibility problem:
$\exists \boldsymbol{Q} > 0 \in \mathbb{D}^{n\times n}$, $\exists \boldsymbol{P} \in \mathbb{S}^n$, $\exists \boldsymbol{D} > 0 \in \mathbb{S}^n$,

$$M^T \begin{bmatrix} \boldsymbol{QDQ} & \boldsymbol{QP} & 0 & 0 \\ \boldsymbol{PQ} & -\omega_0^2\boldsymbol{D} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & U^2 I \end{bmatrix} M \geq 0$$

where

$$M := \begin{bmatrix} J - BB^T & \sqrt{2}B_1 \\ I & 0 \\ \sqrt{2}B_2^T & 0 \\ 0 & I \end{bmatrix}$$

This approach leads then to solve a feasibility problem with constraints involving products between optimization variables of order 3. Moreover, these products are constraint-dependent, and then a spectral mask with several bounds will involve an important number of such non-linear terms.

Therefore, this implies that this seemingly-natural approach leads to an optimization problem more complex than the synthesis approach developed to establish Theorem 1. While

both are substantially underlying on the same tools, the S-procedure and quadratic constraints on signal, our contribution is to have adapted this approach to the pH representation and our design objective in order to reduce the complexity of the resulting optimization problem.

VI. pH LFT representation synthesis

The synthesis approach developed in the last section leads to a remarkably-low number of bilinear terms for lossless passive electronic filters that have a pH state-space representation such as in (10), with Q being a diagonal matrix. It is now revealed how to extend this feature to filters that have the more general representation (9), with Q still being diagonal.

As already mentioned, the process to obtain a usual state-space representation (10) from a descriptor form (9) is not suitable, as it breaks the diagonal structure of Q. Instead, we propose to combine the pH DAE representation with the Linear Fractional Transformation (LFT) representation. The LFT representation was introduced for modeling systems $z(s) = G(s)w(s)$ which are defined as the interconnection of N sub-systems defined by the transfer functions $\Delta_i(s)$ [38]:

$$\begin{cases} p(s) = \begin{pmatrix} \Delta_1(s) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_N(s) \end{pmatrix} q(s) \\ q(s) = M_{11}p(s) + M_{12}w(s) \\ z(s) = M_{21}p(s) + M_{22}w(s) \end{cases}$$

with M_{11}, M_{12}, M_{21}, M_{22} matrices of appropriate dimension. The state-space representation is a particular case of the LFT representation with $\Delta_i(s) = \frac{1}{s}$, $i = 1, \ldots, N$. The interconnection of more complex subsystems can be represented, such as multi-agent [20] or uncertain systems [45].

The combination of the pH and the LFT representations leads to an original representation, that will be denoted as the pH LFT representation. In the sequel, after exhibiting the pH LFT representation (Lemma 2), Problem 2 is equivalently reformulated to suit this new representation (Problem 3). Then, our synthesis approach is adapted in order to tackle Problem 3, leading to an optimization problem with similar complexity than in the previous case of a pH state-space representation.

A. pH LFT representation

Next lemma states that, using the LFT paradigm, the pH DAE representation is equivalent to a new representation, denoted as the pH LFT representation. An important feature is that the elements values, or their inverse, will still be the entries of a *diagonal* matrix $\check{Q}$ (cf proof in Appendix B).

Lemma 2 (pH LFT representation)**.**
Given *the complex-domain, scattering pH DAE representation in a descriptor form (11) with Q being a diagonal matrix,*
Define r *as the rank of* F_{S_1}, $r := \operatorname{rank} F_{S_1}$, *and* n_{AE} *such that* $r + n_{AE} = n$.
An equivalent representation of (11) is given by:

$$\begin{cases} \check{p}(s) = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \check{q}(s) \\ \check{q}(s) = \left(J - BB^T\right) \check{Q}\check{p}(s) + \sqrt{2}Bw(s) \\ z(s) = -\sqrt{2}DB^T\check{Q}\check{p}(s) + Dw(s) \end{cases} \tag{14}$$

where $\check{Q}$ *is diagonal,* $J = -J^T$ *and* $DD^T = I$.
We denote this representation as a pH LFT representation.

Proof. Cf Appendix B. □

When specified to lossless passive electronic filters, the pH LFT representation enables to obtain an expression of s_{21} with a similar structure.

Lemma 3.
Given *a pH LFT representation (14) of a lossless passive electronic filter,* Define

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] := \left[\begin{array}{c|c} J - BB^T & \sqrt{2}B \\ \hline -\sqrt{2}DB^T & D \end{array}\right]$$

Then a representation of $s_{21}(s)$ *is given by:*

$$\begin{cases} \check{p}(s) = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \check{q}(s) \\ \check{q}(s) = A\check{Q}\check{p}(s) + B_1 w_1(s) \\ z_2(s) = C_2\check{Q}\check{p}(s) + D_{21}w_1(s) \end{cases} \tag{15}$$

B. Equivalent design problem formulation

Problem 2 is now equivalently reformulated.

Problem 3 (pH LFT representation synthesis).
Given a spectral mask $\mathcal{SM}$,
Given a pH LFT representation (14) of a lossless passive electronic filter,
Find if there exists a positive-definite, diagonal matrix $\check{\boldsymbol{Q}} > 0 \in \mathbb{D}^{n\times n}$ such that the scattering parameter s_{21}, given by (15), is stable and satisfies the spectral mask $\mathcal{SM}$, i.e. such that (1)-(2) hold.
If such matrix $\check{\boldsymbol{Q}}$ exists, Compute it.

C. Adapted synthesis approach

The pH LFT representation enables to take advantages of both pH and LFT representations. Nonetheless, it requires to adapt the previous synthesis method. While it is based on a signal approach and can be directly applied, quadratic constraints have to be obtained in order to characterise the signals of the top relation of (14). However, this relation is particularly original and was not found in the literature by the authors. We propose then to use new quadratic constraints presented in App. C, leading to the next theorem.

Theorem 2.
Given *a spectral mask* $\mathcal{SM}$,
Given *a pH LFT representation (14) of a lossless passive electronic filter,*
Given *real matrices* $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times 1}$, $C_2 \in \mathbb{R}^{1\times 1}$,

$D_{21} \in \mathbb{R}$ *as in Lemma 3,*
A solution of statement (i) *provides a solution to* (ii).

(i) *There exist a solution to the following feasibility problem:*
$\exists \check{\boldsymbol{Q}}^{\boldsymbol{inv}} \in \mathbb{D}^{n\times n}$,
$\forall u \in \{1,\dots,N_U\}$,
$\exists \check{\boldsymbol{P}}_{\boldsymbol{u}},\ \check{\boldsymbol{D}}_{\boldsymbol{u}} \in \mathbb{M},\ \exists \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} \in \mathbb{M}_{int},\ \exists \boldsymbol{\tau_u} \in \mathbb{R}$,
$\forall l \in \{1,\dots,N_L\}$,
$\exists \check{\boldsymbol{P}}_{\boldsymbol{l}},\ \check{\boldsymbol{D}}_{\boldsymbol{l}} \in \mathbb{M},\ \exists \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} \in \mathbb{M}_{int},\ \exists \boldsymbol{\tau_l} \in \mathbb{R}$,

$$\check{\boldsymbol{Q}}^{\boldsymbol{inv}} > 0 \tag{16}$$

$$\check{\boldsymbol{D}}_{\boldsymbol{u}} + \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} > 0 \qquad \boldsymbol{\tau_u} > 0 \tag{17}$$

$$\check{\mathcal{A}}_{\Omega_{U_u}}\left(\check{\boldsymbol{D}}_{\boldsymbol{u}}, \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{u}}, \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}}, \boldsymbol{\tau_u}\right) + \begin{bmatrix} \check{\boldsymbol{Q}}^{\boldsymbol{inv}}\check{\boldsymbol{Q}}^{\boldsymbol{inv}} & \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right) \\ \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

$$\check{\boldsymbol{D}}_{\boldsymbol{l}} + \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} > 0 \qquad \boldsymbol{\tau_l} > 0 \tag{18}$$

$$\check{\mathcal{A}}_{\Omega_{L_l}}\left(\check{\boldsymbol{D}}_{\boldsymbol{l}}, \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{l}}, \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}}, \boldsymbol{\tau_l}\right) + \begin{bmatrix} \check{\boldsymbol{Q}}^{\boldsymbol{inv}}\check{\boldsymbol{Q}}^{\boldsymbol{inv}} & \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right) \\ \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

with

$$\check{\mathcal{C}} := \begin{bmatrix} A^TA + C_2^TC_2 & -C_2^T & A^TB_1 + C_2^TD_{21} \\ -C_2 & 1 & -D_{21} \\ B_1^TA + D_{21}^TC_2 & -D_{21}^T & B_1^TB_1 + D_{21}^TD_{21} \end{bmatrix}$$

$$\check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right) := \begin{bmatrix} -\check{\boldsymbol{Q}}^{\boldsymbol{inv}}A & 0 & -\check{\boldsymbol{Q}}^{\boldsymbol{inv}}B_1 \end{bmatrix}$$

$$\check{\mathcal{A}}_{\Omega_{U_u}}\left(\check{\boldsymbol{D}}_{\boldsymbol{u}}, \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{u}}, \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}}, \boldsymbol{\tau_u}\right)$$
$$:= \begin{bmatrix} -\check{z}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}^*_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}} & -\check{y}^*_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{u}} & 0 & 0 \\ -\check{y}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{u}} & -\check{x}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}_{\Omega_{U_u}}\check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}} & 0 & 0 \\ 0 & 0 & -\boldsymbol{\tau_u} & 0 \\ 0 & 0 & 0 & U_u^2\boldsymbol{\tau_u} \end{bmatrix}$$

$$\check{\mathcal{A}}_{\Omega_{L_l}}\left(\check{\boldsymbol{D}}_{\boldsymbol{l}}, \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{l}}, \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}}, \boldsymbol{\tau_l}\right)$$
$$:= \begin{bmatrix} -\check{z}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l}} - \check{y}^*_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}} & -\check{y}^*_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l}} - \check{z}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{l}} & 0 & 0 \\ -\check{y}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l}} - \check{z}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{l}} & -\check{x}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l}} - \check{y}_{\Omega_{L_l}}\check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} - \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}} & 0 & 0 \\ 0 & 0 & \boldsymbol{\tau_l} & 0 \\ 0 & 0 & 0 & -L_l^2\boldsymbol{\tau_l} \end{bmatrix}$$

where $\check{x}_{\Omega_{U_u}}$, $\check{y}_{\Omega_{U_u}}$, $\check{z}_{\Omega_{U_u}}$, *and* $\check{x}_{\Omega_{L_l}}$, $\check{y}_{\Omega_{L_l}}$, $\check{z}_{\Omega_{L_l}}$ *are defined such as:*

$$\left[\begin{array}{c|c} \check{x}_\Omega & \check{y}_\Omega \\ \hline \check{y}^*_\Omega & \check{z}_\Omega \end{array}\right] := \left[\begin{array}{cc|cc} x_\Omega \cdot I_r & 0 & y_\Omega \cdot I_r & 0 \\ 0 & z_\Omega \cdot I_{n_{AE}} & 0 & y^*_\Omega \cdot I_{n_{AE}} \\ \hline y^*_\Omega \cdot I_r & 0 & z_\Omega \cdot I_r & 0 \\ 0 & y_\Omega \cdot I_{n_{AE}} & 0 & x_\Omega \cdot I_{n_{AE}} \end{array}\right]$$

where x_Ω, y_Ω *and* z_Ω *depend on the frequency interval* Ω *and are provided by (13), and*

$$\mathbb{M} := \left\{ M \middle| \exists M_r \in \mathbb{S}^r, \exists M_{AE} \in \mathbb{S}^{n_{AE}}, M = \begin{bmatrix} M_r & 0 \\ 0 & M_{AE} \end{bmatrix} \right\}$$

$$\mathbb{M}_{int} := \left\{ M_{int} \middle| \exists M_{rAE} \in \mathbb{R}^{r\times n_{AE}}, M_{int} = \begin{bmatrix} 0 & M_{rAE} \\ M_{rAE}^T & 0 \end{bmatrix} \right\}$$

(ii) *Problem 3 is solved. Indeed,* s_{21} *given by (15), where* $\check{\boldsymbol{Q}} := \left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right)^{-1}$, *is stable and verifies (1)-(2).*

Proof. Cf Appendix C. □

Theorem 2 provides sufficient conditions in the form of a BMI feasibility problem to solve Problem 3, and thus Problem 1 and Problem 2. This BMI feasibility problem has a particularly reduced complexity as it involves a single bilinear matrix term, i.e. $\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\check{\boldsymbol{Q}}^{\boldsymbol{inv}}$, which is the same for all the constraints. In addition, due to the diagonal property of $\check{\boldsymbol{Q}}^{\boldsymbol{inv}}$, the number of bilinear terms is remarkably low as, again, equaled to n, the total number of inductors and capacitors.

VII. Numerical illustration

In order to provide an insight on the potentials of this work, our synthesis approach is applied on an illustrative example. To achieve this, the next spectral mask $\mathcal{SM}$, represented by the grey blocks on Fig. 4, is considered:

$$\begin{aligned} \forall \omega \in [0.01\cdot\omega_{U_1}, \omega_{U_1}], &\quad |s_{21}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], &\quad |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_{U_2}, 100\cdot\omega_{U_2}], &\quad |s_{21}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

with $\omega_{U_1} = 0.9$ rad/s, $\omega_{L_1} = 1.35$ rad/s, $\omega_{L_2} = 2.3$ rad/s, $\omega_{U_2} = 3.5$ rad/s and $U_1^2 = 0.15$, $L^2 = 0.81$, $U_2^2 = 0.2$.

In addition, assume that an initial guess is provided as the 3-elements, AW ladder filter of Fig. 5 with the elements values of Table I. The squared modulus of the frequency-response of the resulting scattering parameter s_{21} is plotted in black-dotted line on Fig. 4. This initial guess can be obtained by tuning the zeros of s_{21} and s_{11}, using their particular link with the resonance and anti-resonance frequencies of the resonators (see for instance [6] for more details).

TABLE I
AW-element values (in F and H)

	C_{0_1}	L_{m_1}	C_{m_1}	C_{0_2}	L_{m_2}	C_{m_2}
Initial guess	0.300	1.764	0.567	0.300	0.610	0.567
pH LFT	0.321	1.768	0.600	0.139	0.601	0.396
	C_{0_3}	L_{m_3}	C_{m_3}			
Initial guess	0.100	5.291	0.189			
pH LFT	0.100	5.291	0.206			

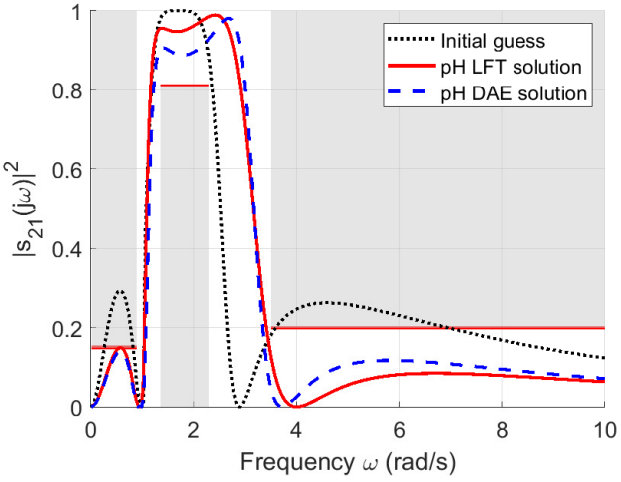


Fig. 4. Plot of $|s_{21}(j\omega)|^2$ of the initial guess and the solutions obtained

In the sequel, the following scheme is made. First, the matrices and signals of the pH LFT representation are computed. Then, by applying Theorem 2, a BMI problem is obtained.

An iterative algorithm based on LMI relaxations is then run to solve it. This requires an initial point and the previous initial guess is used for this purpose.

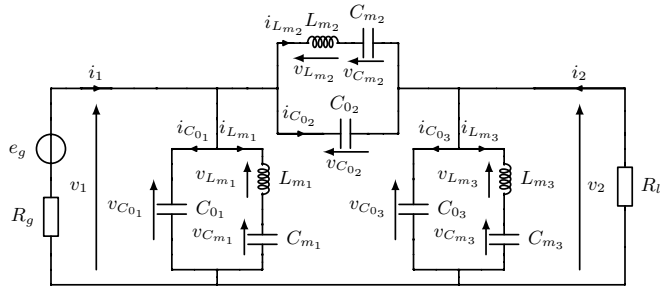


Fig. 5. A ladder filter with three AW elements

A. Computation of the pH LFT representation

First, using the systematic procedure of the pH framework (see [9, Chap. 4, Sec. 4.2.2.2] for further details) and applying the Laplace transform, the scattering pH DAE representation (11) is obtained, with $n_e = 2$, $n = 9$ and:

$$\left[\begin{array}{c|c|c} F_{S_1} & E_{S_1} & H_1 \\ \hline 0_{n_e \times n} & E_{S_2} & H_2 \end{array}\right] =$$

$$\left[\begin{array}{ccccccccc|cc|cc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{array}\right]$$

$$Q := \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}, \quad Q_k := \begin{bmatrix} \frac{1}{C_{0_k}} & 0 & 0 \\ 0 & \frac{1}{L_{m_k}} & 0 \\ 0 & 0 & \frac{1}{C_{m_k}} \end{bmatrix}$$

In addition, using notations of Fig. 5, q and p are defined by:

$$q := \begin{bmatrix} i_{C_{0_1}} & v_{L_{m_1}} & i_{C_{m_1}} & i_{C_{0_2}} & v_{L_{m_2}} & i_{C_{m_2}} & i_{C_{0_3}} & v_{L_{m_3}} & i_{C_{m_3}} \end{bmatrix}^T$$
$$p := \left(Q^{-1} \begin{bmatrix} v_{C_{0_1}} & i_{L_{m_1}} & v_{C_{m_1}} & v_{C_{0_2}} & i_{L_{m_2}} & v_{C_{m_2}} & v_{C_{0_3}} & i_{L_{m_3}} & v_{C_{m_3}} \end{bmatrix}\right)^T$$

Then, applying the procedure given in the proof of Lemma 2, the pH LFT representation (14) of the scattering matrix S is obtained, with $D := H_2$ and

$$J := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\check{Q} := \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \check{Q}_3 \end{bmatrix}, \quad \check{Q}_3 := \begin{bmatrix} \frac{1}{C_{m_3}} & 0 & 0 \\ 0 & \frac{1}{L_{m_3}} & 0 \\ 0 & 0 & C_{0_3} \end{bmatrix}$$

$$\check{q} := \frac{1}{\sqrt{2}} \begin{bmatrix} i_{C_{0_1}} & v_{L_{m_1}} & i_{C_{m_1}} & i_{C_{0_2}} & v_{L_{m_2}} & i_{C_{m_2}} & i_{C_{m_3}} & v_{L_{m_3}} & v_{C_{0_3}} \end{bmatrix}^T$$
$$\check{p} := \frac{1}{\sqrt{2}} \left(\check{Q}^{-1} \begin{bmatrix} v_{C_{0_1}} & i_{L_{m_1}} & v_{C_{m_1}} & v_{C_{0_2}} & i_{L_{m_2}} & v_{C_{m_2}} & v_{C_{m_3}} & i_{L_{m_3}} & i_{C_{0_3}} \end{bmatrix}\right)^T$$

B. Iterative algorithm based on LMI relaxations

Applying Theorem 2 leads to solve the BMI feasibility problem of its statement (i). To achieve this, an adapted version of the algorithm proposed in [28] is used (Algorithm 1). Starting from an arbitrary initial point $\check{Q}_{init}^{inv}$, chosen here as the initial guess, it proceeds by solving a sequence of LMI optimization problems. At each iteration k, an LMI relaxation of the BMI problem is solved (`Relaxation_LMI`), based on the result of the previous iteration $\check{Q}_{k-1}^{inv}$. Then, $\check{Q}_k^{inv}$ is tested (`Test_Satisfaction_SM`) to check if the resulting s_{21} satisfies the spectral mask $\mathcal{SM}$ or not. Again, this leads to solve an LMI problem. Finally, $\check{Q}_{k-1}^{inv}$ is updated in preparation of the next iteration (line 8 of Algorithm 1), based on the Nesterov's acceleration parameter $\eta_k := \frac{k-1}{k+2}$ (cf [28] and the reference therein for further details). Notice that the algorithm is stopped if a feasible solution of the original BMI problem is found or if the number of iterations exceeds max_iter.

One of the main benefits of Algorithm 1 is that each iteration only requires to solve LMI problems, that are efficiently solvable. The computational cost depends then mainly on the total number of iterations, that should be as low as possible. In order to decrease this number, a first possibility is to provide the initial guess as close as possible to the feasibility set of the original BMI problem. This may be achieved by improving the first synthesis approach. In a complementary manner, one may decrease the complexity of the BMI problem, and then get a closer LMI relaxation. This was achieved in this paper.

C. Results

Algorithm 1 is run on the example with $max_iter = 500$. The computation is performed using MATLAB 2020a, the semidefinite programming mode of the CVX framework and the solver MOSEK[3], on a computer with a 4-core 2.30 GHz CPU and 6GB RAM. After $nb_iter_{pHLFT} = 12$ iterations, lasting 13.5 s, the algorithm converges to a solution. The resulting element values are provided in Table I, where the values of the capacitors are given in Farad (F) and the inductors in Henry (H), and the associate $|s_{21}(j\omega)|^2$ is plotted in red-solid line on Fig. 4.

Moreover, Theorem 1 is applied and Algorithm 1 is adapted to solve the BMI feasibility problem resulting from the statement (i). This adapted algorithms converges in $nb_iter_{pHDAE} = 21$ iterations, for a computational time of 19.5 s, to a solution, from which the resulting $|s_{21}(j\omega)|^2$ is plotted in blue-dashed line on Fig. 4. Therefore, compared with Theorem 2, the application of Theorem 1 results in an increase of the total number of iterations, and thus the computational time. This is consistent with the empirical rule that aims to get as least bilinear terms as possible, and motivates the use of our synthesis approach.

Finally, for an illustrative purpose, a more direct approach, based on the use of generic solvers, is applied on this example.

[3]www.cvxr.com, www.mosek.com

Algorithm 1: Iterative LMI-relaxations algorithm

Input: max_iter, $\check{Q}^{inv}_{init}$
Output: $\check{Q}^{inv}$

$\check{Q}^{inv}_{k-1} \leftarrow \check{Q}^{inv}_{init}$
$t_k \leftarrow 1$
$k \leftarrow 1$
while $t_k \geq 0 \parallel k \leq max_iter$ **do**
  $\check{Q}^{inv}_{k} \leftarrow$ `Relaxation_LMI(`$\check{Q}^{inv}_{k-1}$`)`
  $t_k \leftarrow$ `Test_Satisfaction_SM(`$\check{Q}^{inv}_{k}$`)`
  $k \leftarrow k+1$
  $\check{Q}^{inv}_{k-1} \leftarrow \check{Q}^{inv}_{k} + \frac{k-1}{k+2}\left(\check{Q}^{inv}_{k} - \check{Q}^{inv}_{k-1}\right)$
$nb_iter \leftarrow k-1$
$\check{Q}^{inv} \leftarrow \check{Q}^{inv}_{k}$

Function `Relaxation_LMI(`$\check{Q}^{inv}_{k-1}$`)`
  DEFINE $\quad \check{\boldsymbol{Q}}^{\boldsymbol{inv}},\ \boldsymbol{X} \in \mathbb{D}^{n\times n}$,
  $\left\{\check{\boldsymbol{P}}_{\boldsymbol{u}},\ \check{\boldsymbol{D}}_{\boldsymbol{u}} \in \mathbb{M},\ \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} \in \mathbb{M}_{int}, \boldsymbol{\tau_u} \in \mathbb{R}\right\}_{u=1}^{N_U}$,
  $\left\{\check{\boldsymbol{P}}_{\boldsymbol{l}},\ \check{\boldsymbol{D}}_{\boldsymbol{l}} \in \mathbb{M},\ \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} \in \mathbb{M}_{int}, \boldsymbol{\tau_l} \in \mathbb{R}\right\}_{l=1}^{N_L}$,
  MINIMISE $\quad \text{trace}\left(\boldsymbol{X} - 2\check{Q}^{inv}_{k-1}\check{\boldsymbol{Q}}^{\boldsymbol{inv}} + \check{Q}^{inv}_{k-1}\right)$
  subject to

$$(16),\ (17),\ (18)$$

$$\begin{bmatrix} \boldsymbol{X} & \check{\boldsymbol{Q}}^{\boldsymbol{inv}} \\ \check{\boldsymbol{Q}}^{\boldsymbol{inv}} & I_n \end{bmatrix} \geq 0$$

$$\check{\mathcal{A}}_{\Omega_{U_u}}\left(\check{\boldsymbol{D}}_{\boldsymbol{u}}, \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{u}}, \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}}, \boldsymbol{\tau_u}\right) + \begin{bmatrix} \boldsymbol{X} & \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right) \\ \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

$$\check{\mathcal{A}}_{\Omega_{L_l}}\left(\check{\boldsymbol{D}}_{\boldsymbol{l}}, \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{l}}, \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}}, \boldsymbol{\tau_l}\right) + \begin{bmatrix} \boldsymbol{X} & \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right) \\ \check{\mathcal{A}}\left(\check{\boldsymbol{Q}}^{\boldsymbol{inv}}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

  return $\check{Q}^{inv}$

Function `Test_Satisfaction_SM(`$\check{Q}^{inv}$`)`
  if *there exist*
  $\left\{\check{\boldsymbol{P}}_{\boldsymbol{u}},\ \check{\boldsymbol{D}}_{\boldsymbol{u}} \in \mathbb{M},\ \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} \in \mathbb{M}_{int}, \boldsymbol{\tau_u} \in \mathbb{R}\right\}_{u=1}^{N_U}$,
  $\left\{\check{\boldsymbol{P}}_{\boldsymbol{l}},\ \check{\boldsymbol{D}}_{\boldsymbol{l}} \in \mathbb{M},\ \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}},\ \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}} \in \mathbb{M}_{int}, \boldsymbol{\tau_l} \in \mathbb{R}\right\}_{l=1}^{N_L}$,
  such that

$$(17),\ (18),$$

$$\check{\mathcal{A}}_{\Omega_{U_u}}\left(\check{\boldsymbol{D}}_{\boldsymbol{u}}, \check{\boldsymbol{D}}_{\boldsymbol{u_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{u}}, \check{\boldsymbol{P}}_{\boldsymbol{u_{int}}}, \boldsymbol{\tau_u}\right) + \begin{bmatrix} \check{Q}^{inv}\check{Q}^{inv} & \check{\mathcal{A}}\left(\check{Q}^{inv}\right) \\ \check{\mathcal{A}}\left(\check{Q}^{inv}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

$$\check{\mathcal{A}}_{\Omega_{L_l}}\left(\check{\boldsymbol{D}}_{\boldsymbol{l}}, \check{\boldsymbol{D}}_{\boldsymbol{l_{int}}}, \check{\boldsymbol{P}}_{\boldsymbol{l}}, \check{\boldsymbol{P}}_{\boldsymbol{l_{int}}}, \boldsymbol{\tau_l}\right) + \begin{bmatrix} \check{Q}^{inv}\check{Q}^{inv} & \check{\mathcal{A}}\left(\check{Q}^{inv}\right) \\ \check{\mathcal{A}}\left(\check{Q}^{inv}\right)^T & \check{\mathcal{C}} \end{bmatrix} \geq 0$$

  hold
  then
    $t \leftarrow -1$
  else
    $t \leftarrow 1$
  return t

To achieve this, it is proposed to use the generic $fmincon$[4] solver, with the default interior-point algorithm, to tackle the nonlinear optimization problem of finding the elements values such that the resulting s_{21} satisfies the spectral mask at some *frequency samples* $\{\omega_i,\ i = 1, \ldots, N_f\}$. This requires to provide these samples and to explicitly calculate s_{21}. Generating $N_f = 100$ evenly spaced samples per frequency intervals Ω_{U_1}, Ω_L and Ω_{U_2}, and calculating s_{21} using the dss and tf functions, a solution is obtained in 23.0 s. Thus, this increase in computational time illustrates a benefit of our approach, but has to be related to the simplicity of the example. It is expected that the time difference will be higher for more complex design examples that appear in practice. Furthermore, the sampling of the frequency intervals may lead to the following issue: the solution given by the solver satisfies the spectral mask at the frequency samples but not on the whole frequency intervals. This may be mitigated by considering more frequency samples, but at the risk of increasing significantly the computational cost. For instance, considering here five times more frequency samples, i.e. $N_f = 500$, leads to a computational time of 245.5 s, that is more than a 10-fold increase. This issue does not appear in the approach developed in this paper as the obtained optimization problems are independent from the frequency variable ω.

[4] www.mathworks.com/help/optim/ug/fmincon.html

VIII. Conclusions and Perspectives

In this paper, the direct synthesis approach for the frequency design of lossless passive electronic filters was revisited from a System perspective. This leads to solve a synthesis problem of a structured state-space representation, that is known to be complex. First, the pH DAE representation, coming from the port-Hamiltonian framework, was introduced as it provides a state-space representation that appears suitable to tackle the design problem. Then, a method was developed using System tools to transform the design problem into a BMI optimization problem of moderate complexity. This method is particularly adapted to the structure of the pH DAE representation, leading to the first main contribution of this paper (Theorem 1). The second main contribution is the introduction of an original representation, the pH LFT representation, and the development of another method in order to get a BMI optimization problem with a remarkably low complexity. Finally, a numerical example was provided to show the validity and illustrate the benefits of this work.

In future works, the developed methods will be first applied to more complex design examples of practical use. This may require to lower the computational cost, that may be achieved by choosing suitably the Nesterov's acceleration parameter η_k, empirically or with Control theory concepts [46], [47], or by developing an algorithm that explicitly takes into account the particular diagonal structure of the bilinear term. Another perspective of this work is to extend it to the design of passive electronic filters, i.e. with RLC-models of components, using the more general pH representation provided in [41, Chap. 6]. Furthermore, both approaches mentioned in the Introduction are in fact complementary as the first approach, when unsuccessful, provides a result that can constitute an initial point to the second approach [8], [9]. Therefore, it would be interesting to combine them in the future to decrease even more the global computational cost.

APPENDIX A
PROOF OF THEOREM 1

In order to prove Theorem 1, a signal-based approach is used. This requires first to transform the representation (12) and the spectral mask (1)-(2) as quadratic constraints on signals. Then a signal-free, finite-dimensional optimization problem is obtained by applying the so-called S-procedure lemma. For our purpose, the most adapted version of this lemma is the following [17, Sec. III]. Given $\Theta = \Theta^T$,

$$\begin{aligned} & \xi^*\Theta\xi \geq 0 \qquad \forall \xi = \begin{bmatrix} q \\ p \end{bmatrix},\ p = \left(\frac{1}{j\omega}\cdot I\right) q,\ \omega \in \Omega \\ \Leftrightarrow \quad & \exists \boldsymbol{P} = \boldsymbol{P}^T, \exists \boldsymbol{D} = \boldsymbol{D}^T > 0, \\ & \qquad \Theta + \underbrace{\begin{bmatrix} -z_\Omega \boldsymbol{D} & -y_\Omega^* \boldsymbol{D} - \boldsymbol{P} \\ -y_\Omega \boldsymbol{D} - \boldsymbol{P} & -z_\Omega \boldsymbol{D} \end{bmatrix}}_{\Pi} \geq 0 \end{aligned}$$

where the scalars x_Ω, y_Ω, z_Ω are constant and depend on Ω as in (13) (see [17, Sec. IV] for more details). The matrix Π actually defines a quadratic constraint on the signal p and q in the frequency domain when $\omega \in \Omega$. The S-procedure lemma equivalently converts then an inequality condition with signal constraints to an unconstrained inequality condition. In the analysis case, i.e. when Θ is a constant matrix, the last condition happens to be an LMI feasibility problem and is then efficiently solvable. The newly introduced matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ are called *multipliers* and will be used as optimization matrices in the optimization problem coming from the S-procedure.

For the purpose of this paper, we introduce the following modified version of the S-procedure (Lemma 4), in the spirit of [48, Th. 2.3.3, p. 54]. This allows to consider an extra equality constraint in the first statement but losses the equivalence. Only the $\Leftarrow$ part remain, which is direct to prove.

Lemma 4 (S-Procedure with equalities)**.**
Given $\Theta \in \mathbb{S}^m$, $\Phi \in \mathbb{S}^m$, the following implication holds:

$$\begin{aligned} & \xi^*\Theta\xi \geq 0 \quad \forall \xi = \begin{bmatrix} q \\ p \\ z \\ w \end{bmatrix} \in \mathbb{C}^m,\ \begin{cases} \xi^*\Phi\xi = 0, \\ p = \left(\frac{1}{j\omega}\cdot I\right) q,\ \omega \in \Omega \end{cases} \\ \Leftarrow \quad & \exists \boldsymbol{P} \in \mathbb{S}^n, \exists \boldsymbol{D} > 0 \in \mathbb{S}^n, \exists \boldsymbol{\tau} \in \mathbb{R}, \\ & \Theta + \boldsymbol{\tau}\cdot\Phi + \begin{bmatrix} -z_\Omega \boldsymbol{D} & -y_\Omega^* \boldsymbol{D} - \boldsymbol{P} & 0 & 0 \\ -y_\Omega \boldsymbol{D} - \boldsymbol{P} & -x_\Omega \boldsymbol{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0 \end{aligned}$$

Using Lemma 4, Theorem 1 is now proved.

Proof of Theorem 1. Rewrite (12) in the kernel form:

$$\begin{bmatrix} -F_{S_1} & E_{S_1}\boldsymbol{Q} & 0 & H_{w_1} \\ 0 & E_{z_2}\boldsymbol{Q} & -1 & H_{s_{21}} \end{bmatrix} \begin{bmatrix} q(s) \\ p(s) \\ z_2(s) \\ w_1(s) \end{bmatrix} = 0$$

This is equivalent to the quadratic form:

$$\begin{bmatrix} q(s) \\ p(s) \\ z_2(s) \\ w_1(s) \end{bmatrix}^* \begin{bmatrix} -F_{S_1}^T & 0 \\ \boldsymbol{Q}E_{S_1}^T & \boldsymbol{Q}E_{z_2}^T \\ 0 & -1 \\ H_{w_1}^T & H_{s_{21}}^T \end{bmatrix} \begin{bmatrix} -F_{S_1} & E_{S_1}\boldsymbol{Q} & 0 & H_{w_1} \\ 0 & E_{z_2}\boldsymbol{Q} & -1 & H_{s_{21}} \end{bmatrix} \begin{bmatrix} q(s) \\ p(s) \\ z_2(s) \\ w_1(s) \end{bmatrix} = 0 \tag{19}$$

Constraint (1) can be expressed using two constraints: one quadratic constraint to ensure that $s_{21}(j\omega)$ is upper bounded, and $\omega \in \Omega$. Using the relation $z_2(j\omega) = s_{21}(j\omega)w_1(j\omega)$, the first constraint can be stated as follows:

$$\begin{bmatrix} z_2(j\omega) \\ w_1(j\omega) \end{bmatrix}^* \begin{bmatrix} -1 & 0 \\ 0 & U_u^2 \end{bmatrix} \begin{bmatrix} z_2(j\omega) \\ w_1(j\omega) \end{bmatrix} \geq 0 \tag{20}$$

Constraint (1) can then be restated as follows: Find if (20) holds for all $p(s)$, $q(s)$, $z_2(s)$ and $w_1(s)$ that satisfy (19) for $s = j\omega$, $\omega \in \Omega$, and $p(j\omega) = \left(\frac{1}{j\omega}\cdot I\right) q(j\omega)$. Applying Lemma 4 leads to check:

$$\begin{aligned} & \exists \overline{\boldsymbol{P}}_{\boldsymbol{u}} \in \mathbb{S}^n,\ \exists \overline{\boldsymbol{D}}_{\boldsymbol{u}} > 0 \in \mathbb{S}^n,\ \exists \overline{\boldsymbol{\tau}}_{\boldsymbol{u}} \in \mathbb{R}, \\ & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & U_u^2 \end{bmatrix} + \overline{\boldsymbol{\tau}}_{\boldsymbol{u}} \begin{bmatrix} -F_{S_1}^T & 0 \\ \boldsymbol{Q}E_{S_1}^T & \boldsymbol{Q}E_{z_2}^T \\ 0 & -1 \\ H_{w_1}^T & H_{s_{21}}^T \end{bmatrix} \begin{bmatrix} -F_{S_1} & E_{S_1}\boldsymbol{Q} & 0 & H_{w_1} \\ 0 & E_{z_2}\boldsymbol{Q} & -1 & H_{s_{21}} \end{bmatrix} \\ & \qquad + \begin{bmatrix} -z_{\Omega_u^U}\overline{\boldsymbol{D}}_{\boldsymbol{u}} & -y_{\Omega_u^U}^*\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & 0 & 0 \\ -y_{\Omega_u^U}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & -x_{\Omega_u^U}\overline{\boldsymbol{D}}_{\boldsymbol{u}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0 \end{aligned}$$

Without loss of generality, one can impose $\overline{\boldsymbol{\tau}}_{\boldsymbol{u}} > 0$ as the term behind $\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}$ is semi-definite positive. Moreover, pre- and post-multiplying last inequality by

$$\begin{bmatrix} \boldsymbol{Q}^{-1} & 0 & 0 & 0 \\ 0 & \boldsymbol{Q}^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and defining $\boldsymbol{Q^{inv}} := \boldsymbol{Q}^{-1}$, $\boldsymbol{D_u} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\boldsymbol{Q}^{-1}\overline{\boldsymbol{D}}_{\boldsymbol{u}}\boldsymbol{Q}^{-1}$, $\boldsymbol{P_u} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\boldsymbol{Q}^{-1}\overline{\boldsymbol{P}}_{\boldsymbol{u}}\boldsymbol{Q}^{-1}$ and $\boldsymbol{\tau_u} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}$ leads to the result. Same reasoning applies for (2). □

APPENDIX B
PROOF OF LEMMA 2 (PH LFT REPRESENTATION)

The following proof is based on the ideas for establishing the hybrid input-output representation [30] in the pH framework (refer to [49, App. B] for the associate proof).

Proof of Lemma 2. The proof is made of four steps.
First, using a singular value decomposition [38] of F_{S_1}, there exist a non-singular matrix $V \in \mathbb{R}^{n\times n}$, a permutation matrix $T \in \mathbb{R}^{n\times n}$ and a matrix $F_{S_{12_r}} \in \mathbb{R}^{r\times n_{AE}}$ such that:

$$VF_{S_1} = \begin{bmatrix} I_r & F_{S_{12_r}} \\ 0_{n_{AE}\times r} & 0_{n_{AE}\times n_{AE}} \end{bmatrix} T$$

Hence, (11) can be equivalently re-written as:

$$\begin{cases} \begin{bmatrix} p_r \\ p_{AE} \end{bmatrix} = \left(\frac{1}{s}\cdot I_n\right) \begin{bmatrix} q_r \\ q_{AE} \end{bmatrix} \\ \begin{bmatrix} I_r & F_{S_{12_r}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_r \\ q_{AE} \end{bmatrix} = \begin{bmatrix} E_{S_{11_r}} & E_{S_{12_r}} \\ E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \end{bmatrix} \begin{bmatrix} Q_r p_r \\ Q_{AE} p_{AE} \end{bmatrix} + \begin{bmatrix} H_{1_r} \\ H_{1_{AE}} \end{bmatrix} w \\ z = \begin{bmatrix} E_{S_{21}} & E_{S_{22}} \end{bmatrix} \begin{bmatrix} Q_r p_r \\ Q_{AE} p_{AE} \end{bmatrix} + H_2 w \end{cases} \tag{21}$$

where $\begin{bmatrix} q_r \\ q_{AE} \end{bmatrix} := Tq$, $\begin{bmatrix} p_r \\ p_{AE} \end{bmatrix} := Tp$ and

$$\begin{bmatrix} E_{S_{11_r}} & E_{S_{12_r}} \\ E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \end{bmatrix} := VE_{S_1}T^T \qquad \begin{bmatrix} H_{1_r} \\ H_{1_{AE}} \end{bmatrix} := VH_1$$

$$\begin{bmatrix} E_{S_{21}} & E_{S_{22}} \end{bmatrix} := E_{S_2} T^T \quad \begin{bmatrix} Q_r & 0 \\ 0 & Q_{AE} \end{bmatrix} := TQT^T$$

One may note that (7) and (8) become:

$$\begin{bmatrix} E_{S_{11_r}} & E_{S_{12_r}} \\ E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \\ E_{S_{21}} & E_{S_{22}} \end{bmatrix} \begin{bmatrix} I_r & F_{S_{12_r}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T + \begin{bmatrix} I_r & F_{S_{12_r}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{S_{11_r}} & E_{S_{12_r}} \\ E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \\ E_{S_{21}} & E_{S_{22}} \end{bmatrix}^T + \frac{1}{2} \left(\begin{bmatrix} H_{1_r} \\ H_{1_{AE}} \\ H_2 \end{bmatrix} \begin{bmatrix} H_{1_r} \\ H_{1_{AE}} \\ H_2 \end{bmatrix}^T - \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}^T \right) = 0 \tag{22}$$

$$\operatorname{rank} \begin{bmatrix} I_r & F_{S_{12_r}} & E_{S_{11_r}} & E_{S_{12_r}} & 0 & H_{1_r} \\ 0 & 0 & E_{S_{11_{AE}}} & E_{S_{12_{AE}}} & 0 & H_{1_{AE}} \\ 0 & 0 & E_{S_{21}} & E_{S_{22}} & I_{n_e} & H_2 \end{bmatrix} = r + n_{AE} + n_e \tag{23}$$

Second, (21) can be straightly resorted as:

$$\begin{cases} \begin{bmatrix} p_r \\ q_{AE} \end{bmatrix} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \begin{bmatrix} q_r \\ p_{AE} \end{bmatrix} \\ \begin{bmatrix} I_r & -E_{S_{12_r}} \\ 0 & -E_{S_{12_{AE}}} \end{bmatrix} \begin{bmatrix} q_r \\ Q_{AE} p_{AE} \end{bmatrix} = \begin{bmatrix} E_{S_{11_r}} & -F_{S_{12_r}} \\ E_{S_{11_{AE}}} & 0 \end{bmatrix} \begin{bmatrix} Q_r p_r \\ q_{AE} \end{bmatrix} + \begin{bmatrix} H_{1_r} \\ H_{1_{AE}} \end{bmatrix} w \\ z = \begin{bmatrix} E_{S_{21}} & E_{S_{22}} \end{bmatrix} \begin{bmatrix} Q_r p_r \\ Q_{AE} p_{AE} \end{bmatrix} + H_2 w \end{cases} \tag{24}$$

Let us demonstrate that $\begin{bmatrix} I_r & -E_{S_{12_r}} \\ 0 & -E_{S_{12_{AE}}} \end{bmatrix}$ is invertible, that is, $E_{S_{12_{AE}}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$ is invertible, i.e. $E_{S_{12_{AE}}}$ is full rank. To achieve this, one may notice that (22) implies that

$$H_{1_{AE}} = 0 \quad \text{and} \quad E_{S_{11_{AE}}} = -E_{S_{12_{AE}}} F_{S_{12_r}}^T$$

Thus, it comes that:

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \end{bmatrix} &= \operatorname{rank} \left(E_{S_{12_{AE}}} \begin{bmatrix} -F_{S_{12_r}}^T & I \end{bmatrix} \right) \\ &= \operatorname{rank} \left(E_{S_{12_{AE}}} \right) \end{aligned}$$

Now, from (23), $\operatorname{rank} \begin{bmatrix} E_{S_{11_{AE}}} & E_{S_{12_{AE}}} \end{bmatrix} = n_{AE}$. Therefore, $E_{S_{12_{AE}}}$ is invertible and so is $\begin{bmatrix} I_r & -E_{S_{12_r}} \\ 0 & -E_{S_{12_{AE}}} \end{bmatrix}$.
Third, little bit of calculation shows that (24) is equivalent to:

$$\begin{cases} \begin{bmatrix} p_r \\ q_{AE} \end{bmatrix} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \begin{bmatrix} q_r \\ p_{AE} \end{bmatrix} \\ \begin{bmatrix} q_r \\ Q_{AE} p_{AE} \end{bmatrix} = \widetilde{A} \begin{bmatrix} Q_r p_r \\ q_{AE} \end{bmatrix} + \widetilde{B} w \\ z = \widetilde{C} \begin{bmatrix} Q_r p_r \\ q_{AE} \end{bmatrix} + D w \end{cases}$$

where

$$\widetilde{A} = \begin{bmatrix} I_r & -E_{S_{12_r}} \\ 0 & -E_{S_{12_{AE}}} \end{bmatrix}^{-1} \begin{bmatrix} E_{S_{11_r}} & -F_{S_{12_r}} \\ E_{S_{11_{AE}}} & 0 \end{bmatrix} \qquad \widetilde{B} = \begin{bmatrix} I_r & -E_{S_{12_r}} \\ 0 & -E_{S_{12_{AE}}} \end{bmatrix}^{-1} \begin{bmatrix} H_{1_r} \\ 0 \end{bmatrix}$$
$$\widetilde{C} = \begin{bmatrix} E_{S_{21}} + E_{S_{22}} F_{S_{12_r}}^T & 0 \end{bmatrix} \qquad D = H_2$$

Fourth, and finally, (22) implies that:

$$\frac{1}{2} \left(\widetilde{A} + \widetilde{A}^T \right) = -\frac{1}{4} \widetilde{B} \widetilde{B}^T \quad \widetilde{C} = -\frac{1}{2} D \widetilde{B}^T \quad D D^T = I$$

Therefore, (14) is obtained by defining:

$$\check{q} := \frac{1}{\sqrt{2}} \begin{bmatrix} q_r \\ Q_{AE} p_{AE} \end{bmatrix} \quad \check{p} := \frac{1}{\sqrt{2}} \begin{bmatrix} p_r \\ Q_{AE} q_{AE} \end{bmatrix}$$

$$J := \frac{1}{2} \left(\widetilde{A} - \widetilde{A}^T \right) \quad B := \frac{\widetilde{B}}{2} \quad \check{Q} := \begin{bmatrix} Q_r & 0 \\ 0 & Q_{AE}^{-1} \end{bmatrix}$$

□

APPENDIX C
PROOF OF THEOREM 2

To prove Theorem 2, same reasoning than for Theorem 1 is used. However, due to the originality of the pH LFT representation (14), the Π matrix introduced in Appendix A needs to be modified. This is achieved in the next proof.

Proof of Theorem 2. Rewrite (15) in the quadratic form:

$$\begin{bmatrix} \check{q}(s) \\ \check{p}(s) \\ z_2(s) \\ w_1(s) \end{bmatrix}^* \check{\boldsymbol{\Phi}} \begin{bmatrix} \check{q}(s) \\ \check{p}(s) \\ z_2(s) \\ w_1(s) \end{bmatrix} = 0 \tag{25}$$

where

$$\check{\boldsymbol{\Phi}} := \begin{bmatrix} -I & 0 \\ \check{\boldsymbol{Q}} A^T & \check{\boldsymbol{Q}} C_2^T \\ 0 & -1 \\ B_1^T & D_{21}^T \end{bmatrix} \begin{bmatrix} -I & A\check{\boldsymbol{Q}} & 0 & B_1 \\ 0 & C_2 \check{\boldsymbol{Q}} & -1 & D_{21} \end{bmatrix}$$

Define $\begin{bmatrix} \check{q}_r \\ \check{q}_{AE} \end{bmatrix} := \check{q}$ and $\begin{bmatrix} \check{p}_r \\ \check{p}_{AE} \end{bmatrix} := \check{p}$ of appropriate dimensions such that:

$$\begin{bmatrix} \check{p}_r(s) \\ \check{q}_{AE}(s) \end{bmatrix} = \left(\frac{1}{s} \cdot I_n \right) \begin{bmatrix} \check{q}_r(s) \\ \check{p}_{AE}(s) \end{bmatrix}$$

Defining $q := \begin{bmatrix} \check{q}_r \\ \check{p}_{AE} \end{bmatrix}$ and $p := \begin{bmatrix} \check{p}_r \\ \check{q}_{AE} \end{bmatrix}$, it comes $\begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix} = \check{T} \begin{bmatrix} q \\ p \end{bmatrix}$ where $\check{T}$ is a permutation matrix given by:

$$\check{T} := \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{AE}} \\ 0 & 0 & I_r & 0 \\ 0 & I_{n_{AE}} & 0 & 0 \end{bmatrix}$$

Therefore, (25) can be equivalently re-written as follows:

$$\left[\begin{array}{c} q(s) \\ p(s) \\ \hline z_2(s) \\ w_1(s) \end{array}\right]^* \left[\begin{array}{c|c} \check{T} & 0 \\ \hline 0 & I \end{array}\right] \check{\boldsymbol{\Phi}} \left[\begin{array}{c|c} \check{T} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} q(s) \\ p(s) \\ \hline z_2(s) \\ w_1(s) \end{array}\right] = 0 \tag{26}$$

Constraint (1) can then be restated as follows: Find if (20) holds for all $p(s)$, $q(s)$, $z_2(s)$, and $w_1(s)$ that satisfy (26) for $s = j\omega$, $\omega \in \Omega$, and $p(j\omega) = \left(\frac{1}{j\omega} \cdot I \right) q(j\omega)$. Applying Lemma 4 leads to check:

$\exists \overline{\boldsymbol{P}}_{\boldsymbol{u}} \in \mathbb{S}^n$, $\exists \overline{\boldsymbol{D}}_{\boldsymbol{u}} > 0 \in \mathbb{S}^n$, $\exists \overline{\boldsymbol{\tau}}_{\boldsymbol{u}} \in \mathbb{R}$,

$$\left[\begin{array}{cc|cc} -z_{\Omega_{U_u}} \overline{\boldsymbol{D}}_{\boldsymbol{u}} & -y^*_{\Omega_{U_u}} \overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & 0 & 0 \\ -y_{\Omega_{U_u}} \overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & -x_{\Omega_{U_u}} \overline{\boldsymbol{D}}_{\boldsymbol{u}} & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & U_u^2 \end{array}\right] + \overline{\boldsymbol{\tau}}_{\boldsymbol{u}} \left[\begin{array}{c|c} \check{T} & 0 \\ \hline 0 & I \end{array}\right] \check{\boldsymbol{\Phi}} \left[\begin{array}{c|c} \check{T} & 0 \\ \hline 0 & I \end{array}\right] \geq 0 \tag{27}$$

Define $\overline{\boldsymbol{P}}_{\boldsymbol{u_r}} \in \mathbb{R}^{r \times r}$, $\overline{\boldsymbol{P}}_{\boldsymbol{u_{rAE}}} \in \mathbb{R}^{r \times n_{AE}}$, $\overline{\boldsymbol{P}}_{\boldsymbol{u_{AE}}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{u_r}} \in \mathbb{R}^{r \times r}$, $\overline{\boldsymbol{D}}_{\boldsymbol{u_{rAE}}} \in \mathbb{R}^{r \times n_{AE}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{u_{AE}}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$ as:

$$\begin{bmatrix} \overline{\boldsymbol{P}}_{\boldsymbol{u_r}} & \overline{\boldsymbol{P}}_{\boldsymbol{u_{rAE}}} \\ \overline{\boldsymbol{P}}_{\boldsymbol{u_{rAE}}}^{\boldsymbol{T}} & \overline{\boldsymbol{P}}_{\boldsymbol{u_{AE}}} \end{bmatrix} := \overline{\boldsymbol{P}}_{\boldsymbol{u}} \quad \begin{bmatrix} \overline{\boldsymbol{D}}_{\boldsymbol{u_r}} & \overline{\boldsymbol{D}}_{\boldsymbol{u_{rAE}}} \\ \overline{\boldsymbol{D}}_{\boldsymbol{u_{rAE}}}^{\boldsymbol{T}} & \overline{\boldsymbol{D}}_{\boldsymbol{u_{AE}}} \end{bmatrix} := \overline{\boldsymbol{D}}_{\boldsymbol{u}}$$

Moreover, define $\overline{\check{\boldsymbol{P}}}_{\boldsymbol{u}}$, $\overline{\check{\boldsymbol{P}}}_{\boldsymbol{u_{int}}}$, $\overline{\check{\boldsymbol{D}}}_{\boldsymbol{u}}$, $\overline{\check{\boldsymbol{D}}}_{\boldsymbol{u_{int}}}$ as follows:

$$\overline{\check{\boldsymbol{P}}}_{\boldsymbol{u}} := \begin{bmatrix} \overline{\boldsymbol{P}}_{\boldsymbol{u_r}} & 0 \\ 0 & \overline{\boldsymbol{P}}_{\boldsymbol{u_{AE}}} \end{bmatrix} \qquad \overline{\check{\boldsymbol{P}}}_{\boldsymbol{u_{int}}} := \begin{bmatrix} 0 & \overline{\boldsymbol{P}}_{\boldsymbol{u_{rAE}}} \\ \overline{\boldsymbol{P}}_{\boldsymbol{u_{rAE}}}^{\boldsymbol{T}} & 0 \end{bmatrix}$$

$$\overline{\check{\boldsymbol{D}}}_{\boldsymbol{u}} := \begin{bmatrix} \overline{\boldsymbol{D}}_{\boldsymbol{u_r}} & 0 \\ 0 & \overline{\boldsymbol{D}}_{\boldsymbol{u_{AE}}} \end{bmatrix} \qquad \overline{\check{\boldsymbol{D}}}_{\boldsymbol{u_{int}}} := \begin{bmatrix} 0 & \overline{\boldsymbol{D}}_{\boldsymbol{u_{rAE}}} \\ \overline{\boldsymbol{D}}_{\boldsymbol{u_{rAE}}}^{\boldsymbol{T}} & 0 \end{bmatrix}$$

One may then note that:

$$\check{T}\begin{bmatrix} -z_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} & -y^*_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\check{\boldsymbol{P}}}_{\boldsymbol{u}} \\ -y_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \overline{\check{\boldsymbol{P}}}_{\boldsymbol{u}} & -x_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} \end{bmatrix}\check{T}$$
$$= \begin{bmatrix} -\check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}^*_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u_{int}}} & -\check{y}^*_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} \\ -\check{y}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & -\check{x}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u_{int}}} \end{bmatrix}$$

Pre- and post-multiplying (27) by $\left[\begin{array}{c|c} \check{T} & 0 \\ \hline 0 & I \end{array}\right]$ leads then to:

$$\left[\begin{array}{cc|cc} -\check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}^*_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u_{int}}} & -\check{y}^*_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & 0 & 0 \\ -\check{y}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{z}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u}} & -\check{x}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u}} - \check{y}_{\Omega_{U_u}}\overline{\boldsymbol{D}}_{\boldsymbol{u_{int}}} - \overline{\boldsymbol{P}}_{\boldsymbol{u_{int}}} & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & U_u^2 \end{array}\right] + \overline{\boldsymbol{\tau}}_{\boldsymbol{u}}\check{\boldsymbol{\Phi}} \geq 0$$

where $\check{x}_{\Omega_{U_u}}$, $\check{y}_{\Omega_{U_u}}$, $\check{z}_{\Omega_{U_u}}$, are defined as in Theorem 2.

Finally, similarly than in proof of Theorem 1, one can impose $\overline{\boldsymbol{\tau}}_{\boldsymbol{u}} > 0$ without loss of generality. The result is obtained by pre- and post-multiplying last inequality by

$$\begin{bmatrix} \check{\boldsymbol{Q}}^{-1} & 0 & 0 & 0 \\ 0 & \check{\boldsymbol{Q}}^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and defining $\check{\boldsymbol{Q}}^{\boldsymbol{inv}} := \check{\boldsymbol{Q}}^{-1}$, $\check{\boldsymbol{P}}_{\boldsymbol{u}} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\check{\boldsymbol{Q}}^{-1}\overline{\check{\boldsymbol{P}}}_{\boldsymbol{u}}\check{\boldsymbol{Q}}^{-1}$, $\check{\boldsymbol{P}}_{\boldsymbol{u_{int}}} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\check{\boldsymbol{Q}}^{-1}\overline{\check{\boldsymbol{P}}}_{\boldsymbol{u_{int}}}\check{\boldsymbol{Q}}^{-1}$, $\check{\boldsymbol{D}}_{\boldsymbol{u}} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\check{\boldsymbol{Q}}^{-1}\overline{\check{\boldsymbol{D}}}_{\boldsymbol{u}}\check{\boldsymbol{Q}}^{-1}$, $\check{\boldsymbol{D}}_{\boldsymbol{u_{int}}} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}\check{\boldsymbol{Q}}^{-1}\overline{\check{\boldsymbol{D}}}_{\boldsymbol{u_{int}}}\check{\boldsymbol{Q}}^{-1}$, $\boldsymbol{\tau}_{\boldsymbol{u}} := \frac{1}{\overline{\boldsymbol{\tau}}_{\boldsymbol{u}}}$. □

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