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Universal Self-Similar Attractor in the Bending-Driven Leveling of Thin Viscous Films

Christian Pedersen¹ Thomas Salez^{2,3,*} Andreas Carlson¹

¹Mechanics Division, Department of Mathematics, University of Oslo, 0316 Oslo, Norway.

²Univ. Bordeaux, CNRS, LOMA, UMR 5798, F-33405 Talence, France.

³Global Station for Soft Matter, Global Institution for Collaborative Research and Education, Hokkaido University, Sapporo, Hokkaido 060-0808, Japan.

*thomas.salez@u-bordeaux.fr

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Abstract

We study theoretically and numerically the bending-driven leveling of thin viscous films within the lubrication approximation. We derive the Green's function of the linearized thin-film equation and further show that it represents a universal self-similar attractor at intermediate times. As such, the rescaled perturbation of the film profile converges in time towards the rescaled Green's function, for any summable initial perturbation profile. In addition, we characterize the convergence time in terms of the relevant physical and geometrical parameters. Finally, we numerically extend our analysis to the nonlinear thin-film equation, and we still observe the convergence to the universal attractor.

Introduction

Intermediate asymptotics and self-similar solutions [1] have proven to be essential tools for revealing a degree of generality in nonlinear problems within applied mathematics and physics. Even though analytical solutions are usually lacking for nonlinear problems, many phenomena seem to behave at intermediate times in ways that enable one to describe their dynamics using scaling methods. Examples are numerous, and include for instance the propagation of an explosive blast [2, 3], ground-water flows in porous rocks [4], formation of laccoliths [5], shaping of hard metals by means of electrolysis [6], and jet breakup [7].

A particular class of problems where scaling, intermediate asymptotics and self-similar solutions have been successfully employed is thin-film flow

within the lubrication theory [8]. In the latter framework, a spatial scale separation and a low Reynolds number lead to a notable simplification of the Navier-Stokes equations [9]. For instance, one is often interested in describing the temporal evolution of an interface between two phases, *e.g.* a thin viscous liquid film and a gaseous atmosphere, which can be described by lubrication theory through the so-called thin-film equation:

$$\partial_t h(x, y, t) = M \nabla \cdot [h^3(x, y, t) \nabla p(x, y, t)] \quad , \quad (1)$$

where t is the time, x and y are the planar spatial coordinates, $h(x, y, t)$ is the thickness of the liquid film in the normal direction z to the supporting wall, $p(x, y, t)$ is the pressure field in the liquid, ∇ is the partial spatial differential operator, and ∂_t is the partial temporal derivative. The coefficient M is inversely proportional to the dynamic shear viscosity μ of the liquid, with a numerical prefactor that depends on the specific boundary conditions for the flow. For example, a film with a no-shear-stress boundary condition at the free liquid-gas interface, flowing on a solid substrate with a no-slip boundary condition yields $M = 1/(3\mu)$.

Interestingly, in a variety of relevant physical phenomena, the pressure field often takes the form of an even-order spatial derivative of $h(x, y, t)$. An important particular case is the one of gravity-driven thin-film flows, where the pressure scales linearly with the film thickness, *i.e.* $p(x, y, t) = \rho g h(x, y, t)$, with ρ the liquid density and g the gravitational acceleration. In this case, Eq. (1) is a nonlinear diffusion-like equation that has been scrutinized by the physics [10, 11, 12, 13, 14] and applied mathematics [15, 16, 17] communities.

Another important particular case of Eq. (1) is the one of capillary-driven thin-film flows in the small-slope approximation $(\nabla h)^2 \ll 1$ [18, 19, 20, 21, 22], where the linearized capillary pressure reads $p(x, y, t) = -\gamma \nabla^2 h(x, y, t)$, with γ the liquid-gas surface tension. Inserting the capillary pressure into Eq. (1), the latter becomes a fourth-order partial differential equation. The capillary spreading of droplets in total wetting conditions was studied in the seminal work by Tanner [23], who found a power-law solution of the spreading dynamics, thus unveiling the central self-similar symmetry at intermediate times. Other similarity solutions to capillary-driven thin-film flows have been derived analytically [24, 25, 26, 27, 28, 29], and by performing numerical simulations [30, 31], allowing one to also address problems where finite-time singularities appear, such as *e.g.* thin-film rupture [32, 33, 34] and droplet coalescence [35, 36]. A recent study on the intermediate asymptotics of the capillary-driven thin-film equation has established that any

vanishingly small and summable initial perturbation profile must converge in time towards a universal self-similar attractor given by the Green’s function of the linearized problem [37]. Furthermore, this convergence was conjectured to hold for any finite-size perturbation, using numerical solutions with compact-support initial profiles. This latter prediction was corroborated experimentally using polymer films [38, 39, 40], and generalized to higher-order symmetries of the initial profiles [41].

Beyond the above second- and fourth-order partial differential thin-film equations associated with gravity and capillarity, respectively, a sixth-order realization is provided by elastic bending in the situation where an elastic plate is freely placed atop a thin viscous film – a widespread configuration in geophysical, physiological, biological and engineering settings. For small plate deformations, when elastic stretching effects can be neglected [42, 43, 44, 45, 46], the pressure is dominated by bending and is thus proportional to the bi-Laplacian of the film thickness [47], leading to the sixth-order version of Eq. (1). Intermediate asymptotics has been employed in such elastohydrodynamic flows [13, 48, 49], with in particular the use of asymptotic matching to obtain power-law solutions in non-linear geometries [50, 51]. In the latter geometries, as the profiles eventually approach their final flat equilibrium configuration, a crossover towards a different power law was predicted and tested numerically [52].

In the present article, we further investigate theoretically and numerically the intermediate asymptotics of the bending-driven thin-film equation. In particular, for any summable initial perturbation with respect to the flat equilibrium state, we show that the solutions converge towards a universal self-similar attractor provided by the Green’s function of the linearized problem. In addition, besides the obvious elastic and viscous physical parameters, we show that the convergence time is essentially set by the sixth power of the typical width of the initial perturbation. Finally, using a numerical method, we discuss the extension of these results towards the practically-relevant case of finite perturbations.

Mathematical model

We consider a free unconstrained elastic plate, with zero spontaneous curvature, resting on a thin liquid film. By “thin” we mean that the lateral extent of any variation of the film profile is much larger than the film thickness itself, as required for the lubrication approximation to be valid [9]. The liquid is described as an incompressible and Newtonian viscous fluid, that

is supported by a solid substrate at $z = 0$. The liquid-plate interface is located at $z = h(x, y, t)$. By continuity of the normal stress at the latter, the excess pressure field in the liquid (with respect to the atmospheric pressure) is set by the elastic stress, and thus reads $p(x, y, t) = B\nabla^4 h(x, y, t)$ [47] in the lubrication approximation, where B is the plate bending stiffness. The velocity field in the liquid is found by solving the Stokes equations within the lubrication approximation [8], under no-slip boundary conditions at the two solid-liquid interfaces. By integrating the continuity equation across the film thickness, we obtain the bending-driven thin-film equation [50, 52]:

$$\partial_t h(x, y, t) = \frac{B}{12\mu} \nabla \cdot [h^3(x, y, t) \nabla^5 h(x, y, t)] . \quad (2)$$

We nondimensionalize Eq. (2) with $h = \bar{h}h_0$, $x = \bar{x}h_0$, $y = \bar{y}h_0$ and $t = 12\bar{t}\mu h_0^3/B$, where the bar notation indicates dimensionless variables, and where h_0 is the liquid film thickness in the flat unperturbed state. By further omitting the bar notation for simplicity, we obtain the dimensionless version of Eq. (2):

$$\partial_t h(x, y, t) = \nabla \cdot [h^3(x, y, t) \nabla^5 h(x, y, t)] . \quad (3)$$

Linearized problem

We consider small perturbations, *i.e.* $h(x, y, t) = 1 + \epsilon(x, y, t)$ with $\epsilon \ll 1$, so that Eq. (3) can be linearized into:

$$\partial_t \epsilon(x, y, t) = \nabla^6 \epsilon(x, y, t) . \quad (4)$$

Green's function and symmetries

The Green's function $G(x, y, t)$ is defined as the solution of the following partial differential equation:

$$\mathcal{L}G(x, y, t) = \delta(x, y, t) , \quad (5)$$

where $\mathcal{L} = \partial_t - (\partial_x^2 + \partial_y^2)^3$ is the linear differential operator of Eq. (4), and $\delta(x, y, t)$ is the Dirac delta function in two-dimensional space and time. The solution $\epsilon(x, y, t)$, at any position (x, y) and time t , is then obtained from a convolution between the Green's function and the initial profile $\epsilon_0(x, y) = \epsilon(x, y, 0)$, as:

$$\epsilon(x, y, t) = \int dx' dy' G(x - x', y - y', t) \epsilon_0(x', y') . \quad (6)$$

We invoke the Fourier transform:

$$\hat{G}(k_x, k_y, \omega) = \int dx dy dt G(x, y, t) e^{-i(k_x x + k_y y + \omega t)} , \quad (7)$$

with k_x and k_y the angular spatial frequencies in the x and y directions, respectively, and ω the angular temporal frequency. By taking the Fourier transform of Eq. (5), we find:

$$\hat{G}(k_x, k_y, \omega) = \frac{1}{i\omega + (k_x^2 + k_y^2)^3} . \quad (8)$$

Expressing the inverse Fourier transform, and invoking the residue theorem for the integral over the angular temporal frequency, we obtain the Green's function in a general integral form:

$$G(x, y, t) = \frac{\mathcal{H}(t)}{(2\pi)^2} \int dk_x dk_y e^{-t(k_x^2 + k_y^2)^3} e^{i(k_x x + k_y y)} , \quad (9)$$

where $\mathcal{H}(t)$ is the Heaviside step function.

Let us now perform a change of variables towards polar coordinates, through $x = r \cos(\theta)$, $y = r \sin(\theta)$, $k_x = \rho \cos(\psi)$ and $k_y = \rho \sin(\psi)$. When inserted into Eq. (9), this change of variables leads to:

$$\begin{aligned} G(r, t) &= \frac{\mathcal{H}(t)}{(2\pi)^2} \int d\rho \rho e^{-\rho^6 t} \int d\psi e^{i\rho r \cos(\psi - \theta)} , \\ &= \frac{\mathcal{H}(t)}{2\pi} \int d\rho \rho e^{-\rho^6 t} J_0(\rho r) , \end{aligned} \quad (10)$$

with J_0 the zeroth-order Bessel function. As a consequence, the Green's function is axisymmetric. Furthermore, the last integral has an exact expression, leading to:

$$\begin{aligned} G(r, t) &= \frac{\mathcal{H}(t)\Gamma(\frac{1}{3})}{12\pi t^{1/3}} {}_0F_4 \left[\left\{ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1 \right\}, - \left(\frac{1}{6} \frac{r}{t^{1/6}} \right)^6 \right] \\ &\quad + \frac{\mathcal{H}(t)r^2\Gamma(-\frac{1}{3})}{144\pi t^{2/3}} {}_0F_4 \left[\left\{ \frac{2}{3}, 1, \frac{4}{3}, \frac{4}{3} \right\}, - \left(\frac{1}{6} \frac{r}{t^{1/6}} \right)^6 \right] \\ &\quad + \frac{\mathcal{H}(t)r^4}{768\pi t} {}_0F_4 \left[\left\{ \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \right\}, - \left(\frac{1}{6} \frac{r}{t^{1/6}} \right)^6 \right] , \end{aligned} \quad (11)$$

where Γ is the gamma function and ${}_0F_4$ is the (0,4)-hypergeometric function.

Finally, introducing the similarity variable $\xi = rt^{-1/6}$, Eq. (11) can be recast into:

$$G(\xi, t) = \frac{\mathcal{H}(t)}{t^{1/3}} f(\xi), \quad (12)$$

where:

$$\begin{aligned} f(\xi) = & \frac{\Gamma(\frac{1}{3})}{12\pi} {}_0F_4 \left(\left\{ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1 \right\}, - \left(\frac{\xi}{6} \right)^6 \right) \\ & + \frac{\xi^2 \Gamma(-\frac{1}{3})}{144\pi} {}_0F_4 \left(\left\{ \frac{2}{3}, 1, \frac{4}{3}, \frac{4}{3} \right\}, - \left(\frac{\xi}{6} \right)^6 \right) \\ & + \frac{\xi^4}{768\pi} {}_0F_4 \left(\left\{ \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \right\}, - \left(\frac{\xi}{6} \right)^6 \right). \end{aligned} \quad (13)$$

As a consequence, one has:

$$\frac{G(\xi, t)}{G(0, t)} = \frac{f(\xi)}{f(0)}, \quad (14)$$

which means that, when properly normalized, the Green's function is essentially a function of the self-similar variable ξ only.

General solution and long-term behaviour

In general, the double integral of Eq. (9) can be evaluated numerically, as well as the solution of Eq. (4) for any initial profile, using Eq. (6). Moreover, in the particular case where the initial profile is axisymmetric, *i.e.* $\epsilon_0(x, y) = \epsilon_0(r)$, the spatial convolution defined in Eq. (6) reads in polar coordinates:

$$\epsilon(r, t) = \int dr' r' \epsilon_0(r') \int d\theta G \left(\sqrt{r^2 + r'^2 - 2rr' \cos(\theta)}, t \right). \quad (15)$$

Therefore, the solution is axisymmetric at any time, as expected.

In Fig. 1a, we compare the solution of a finite-element numerical integration (FENI) [52] of Eq. (4) and the numerical evaluation of the convolution in Eq. (15), for a stepped axisymmetric initial profile $\epsilon_0(r) = \mathcal{H}(1 - r)$, at three different times t . We observe an excellent agreement, which confirms the validity of both the Green's function and the convolution.

As the magnitude of the axisymmetric solution $\epsilon(r, t)$ above decays over time (see Fig. 1a), to study the long-term behaviour of the solution we rescale $\epsilon(r, t)$ by its amplitude $\epsilon(0, t)$ at $r = 0$. Furthermore, guided by the self-similarity of the Green's function (see Eq. (12)), we introduce the similarity

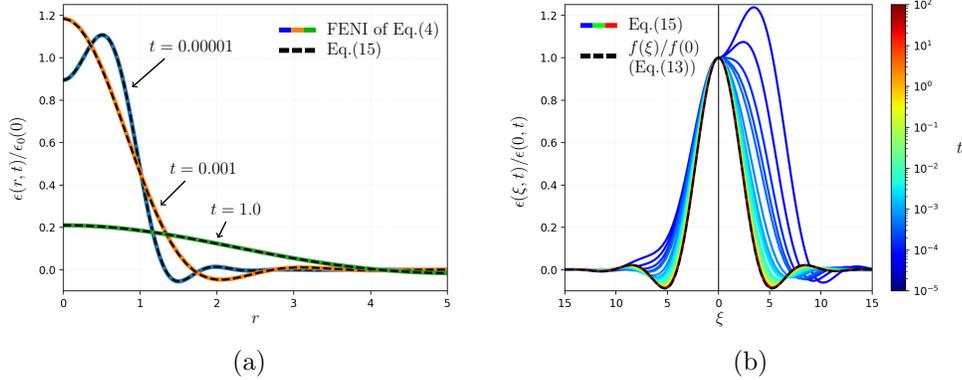


Figure 1: (a) Normalized solutions $\epsilon(r, t)/\epsilon_0(0)$ as a function of the radial coordinate r , at three different times t as indicated, for a stepped axisymmetric initial profile $\epsilon_0(r) = \mathcal{H}(1-r)$. These solutions were obtained from: i) finite-element numerical integration (FENI) [52] of Eq. (4) (dashed lines); ii) numerical evaluation of the convolution in Eq. (15) (solid lines). (b) Rescaled solutions $\epsilon(r, t)/\epsilon(0, t)$ as a function of the similarity variable $\xi = rt^{-1/6}$ (solid lines), for various times t (color bar), as numerically computed from Eq. (15), for two different axisymmetric initial profiles: i) an homogeneous polynomial $\epsilon_0(r) = (1-r^2)^2 \mathcal{H}(1-r)$ (left) ; ii) a stepped axisymmetric function $\epsilon_0(r) = \mathcal{H}(1-r)$ (right). For comparison, $f(\xi)/f(0)$ (see Eq. (13)) is shown (dashed line).

variable $\xi = rt^{-1/6}$ and study $\epsilon(\xi, t)/\epsilon(0, t)$. The latter rescaled solution is numerically computed from Eq. (15) for two different axisymmetric initial profiles, and plotted in Fig. 1b as a function of ξ for various times t . In the left panel, we have used an homogeneous polynomial, $\epsilon_0(r) = (1-r^2)^2 \mathcal{H}(1-r)$, as the initial profile; and in the right panel, we have used the stepped axisymmetric initial profile $\epsilon_0(r) = \mathcal{H}(1-r)$ previously employed in Fig. 1a. In both cases, the rescaled solutions appear not to depend on time t anymore at long times, which suggests their late-time self-similarity. Moreover, they both seem to converge towards $f(\xi)/f(0)$, suggesting the existence of a universal self-similar attractor.

Universal attractor and convergence time

Invoking the self-similarity of the Green's function (see Eq. (12)), the solution (see Eq. (15)) of Eq. (4) for an axisymmetric initial profile becomes:

$$\epsilon(\xi, t) = \frac{\mathcal{H}(t)}{t^{1/3}} \int dr' r' \epsilon_0(r') \int d\theta f \left(\sqrt{\xi^2 + (r't^{-1/6})^2 - 2\xi r't^{-1/6} \cos(\theta)} \right) . \quad (16)$$

At long positive times, this expression is equivalent to:

$$\epsilon(\xi, t) \simeq \frac{V_0}{t^{1/3}} f(\xi) , \quad (17)$$

where $V_0 = 2\pi \int dr r \epsilon_0(r)$ is the dimensionless volume of the initial perturbation profile. Therefore, the rescaled solution $\epsilon(\xi, t)/\epsilon(0, t)$ converges towards the self-similar attractor $f(\xi)/f(0)$, no matter the axisymmetric initial perturbation profile (provided it is summable), as previously suggested by Fig. 1b. For the sixth-order bending-driven thin-film equation, we thus find the intermediate asymptotic solution [1] of the linearized problem to be the rescaled Green's function, which is reminiscent of the fourth-order capillary case [37].

As seen in Fig. 1b, the time it takes for an arbitrary axisymmetric initial profile to converge to the self-similar attractor seems not to be unique. To investigate in particular the role of the dimensionless initial volume V_0 on the convergence dynamics, we numerically evaluate from Eq. (15) the solution $\epsilon(r, 0.01)$ at a given time $t = 0.01$, for axisymmetric initial profiles $\epsilon_0(r) = [1 - (r/r_0)^2]^2 \mathcal{H}(r_0 - r)$, with $r_0 = 1, 2, 4$ giving $V_0 = 0.26, 1.05, 4.19$, respectively. The rescaled results are plotted in the left panel of Fig. 2, where we see that the three profiles do not collapse with one another. This indicates that the dynamics is influenced by V_0 . In contrast, when we numerically evaluate from Eq. (15) the solutions $\epsilon(r, 0.1V_0^3)$ at times $t = 0.1V_0^3$, for the three same initial profiles as above, the rescaled profiles now seem to collapse with one another. This suggests that the dimensionless convergence time t_c is proportional to V_0^3 .

To quantitatively define t_c , we need a relevant criterion. Defining a relative mathematical distance between the solution and the attractor, and fixing some arbitrary but small upper bound to it is a natural approach. However, for regular axisymmetric initial profiles, convergence is typically occurring when the central height $\epsilon_0(0)$ of the initial perturbation profile matches the central height $\epsilon(0, t) \simeq V_0 f(0)/t^{1/3}$ of the asymptotic solution (see Eq. (17)). Using this criterion, combined with Eq. (13), allows us to

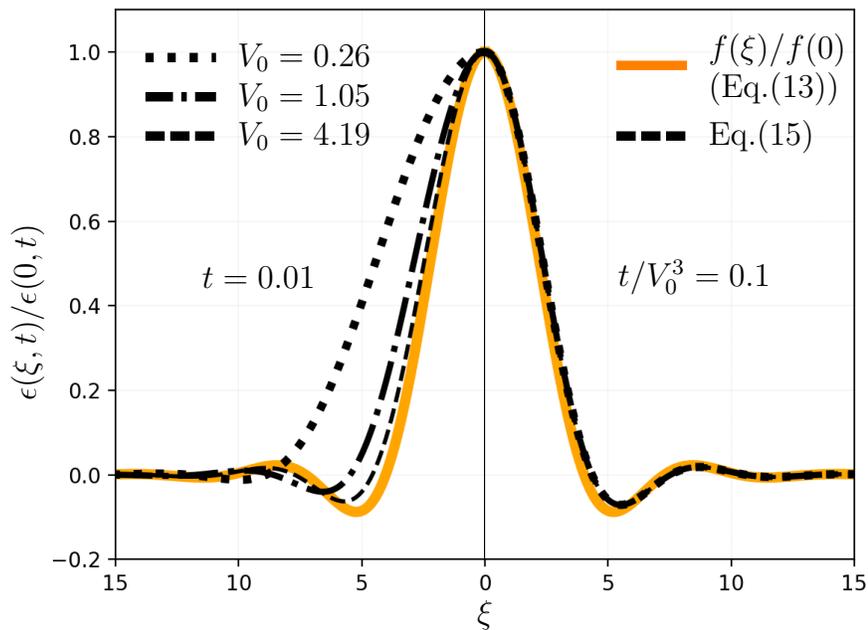


Figure 2: (left) Rescaled solutions $\epsilon(\xi, 0.01)/\epsilon(0, 0.01)$ at a given time $t = 0.01$ (dashed and dotted lines), for stepped axisymmetric initial profiles $\epsilon_0(r) = [1 - (r/r_0)^2]^2 \mathcal{H}(r_0 - r)$, with $r_0 = 1, 2, 4$ giving the indicated values of V_0 , as numerically evaluated from Eq. (15). (right) Rescaled solutions $\epsilon(\xi, 0.1V_0^3)/\epsilon(0, 0.1V_0^3)$ at times $t = 0.1V_0^3$ (dashed and dotted lines), for the three same initial profiles as in left panel, as numerically evaluated from Eq. (15). For comparison, on both the left and right panels, the self-similar attractor $f(\xi)/f(0)$ (see Eq. (13)) is shown.

define the convergence time as:

$$t_c = \left[\frac{V_0 \Gamma(1/3)}{12\pi \epsilon_0(0)} \right]^3. \quad (18)$$

Apart from numerical prefactors, the dimensionless convergence time is thus proportional to $[V_0/\epsilon_0(0)]^3$ only, confirming in particular the observation made in the right panel of Fig. 2 for the $\epsilon_0(0) = 1$ case.

In dimensional units, and avoiding numerical prefactors, the convergence time scales as $t_c \sim \mu V_0^3 / [\epsilon_0(0)^3 B h_0^3]$, with V_0 and $\epsilon_0(0)$ being now the di-

dimensional volume and the central height of the initial perturbation. Introducing the typical lateral size λ of the initial perturbation, one gets $V_0 \sim \lambda^2 \epsilon_0(0)$ and $t_c \sim \mu \lambda^6 / (B h_0^3)$, as expected from the scaling analysis of Eq. (2). The convergence is slower for a larger liquid film viscosity μ and a larger lateral size λ of the perturbation, while it is faster for a larger bending rigidity B of the elastic plate and a larger liquid film thickness h_0 .

Extension to nonlinear dynamics

Being able to evaluate the typical time for convergence towards the self-similar attractor can be crucial when modelling natural, biological or engineering processes associated with the bending-driven thin-film equation. However, so far, we have limited the analysis to the linearized problem. In the following, we re-examine the convergence to the self-similar attractor in the nonlinear case described by Eq. (3).

We solve Eq. (3), using a finite-element numerical integration (FENI) [52]. The rescaled solution $\epsilon(\xi, t)/\epsilon(0, t) = [h(\xi, t) - 1]/[h(0, t) - 1]$, for a stepped initial axisymmetric profile $\epsilon_0(r) = h(r, 0) - 1 = \mathcal{H}(1 - r)$ is shown in Fig. 3 for various times t .

We observe that the rescaled nonlinear solution converges in time towards the self-similar attractor $f(\xi)/f(0)$ (see Eq. (13)) of the linear case. We have checked (not shown) that this statement holds for all the various compact-support axisymmetric initial profiles tested, and are thus led to conjecture its validity for any summable initial profile of arbitrary magnitude. The physical reason behind this phenomenon is rooted in the dissipative character of Eq. (3), that ensures the condition $\epsilon(r, t) \ll 1$ to be always reached eventually, at sufficiently long times.

Conclusion

We studied the sixth-order bending-driven thin-film equation, both theoretically and numerically. We derived the Green's function of the linearized problem, and showed that it represents a universal self-similar attractor. As such, the linear solution from any summable axisymmetric initial perturbation profile converges towards the rescaled Green's function at intermediate times. In addition, we characterized the convergence time in terms of the relevant physical and geometrical parameters. Finally, we extended numerically our analysis to the nonlinear case, and verified that the convergence towards the self-similar attractor is maintained.

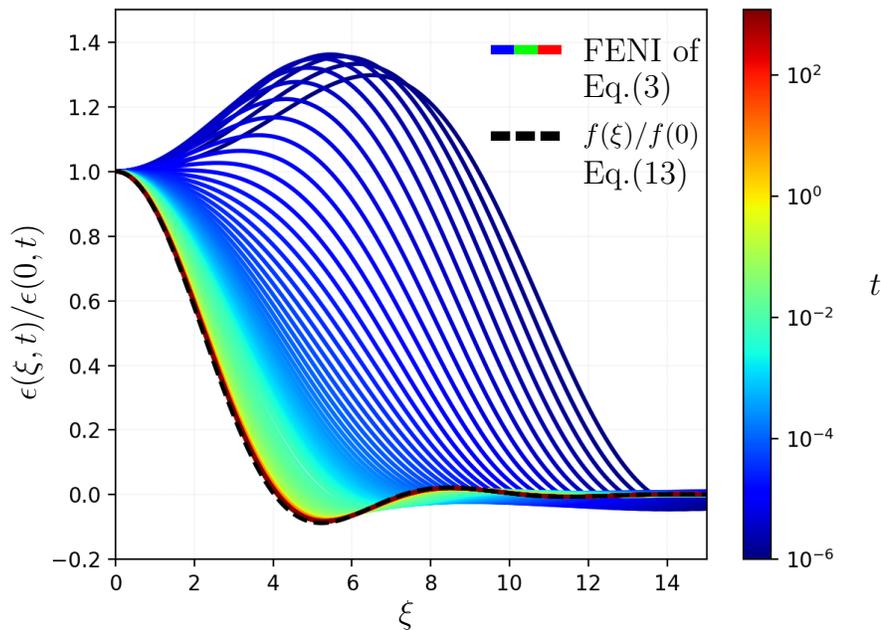


Figure 3: Rescaled solution $\epsilon(\xi, t)/\epsilon(0, t)$ as a function of similarity variable $\xi = rt^{-1/6}$ (solid lines), for a stepped axisymmetric initial profile $\epsilon_0(r) = \mathcal{H}(1 - r)$, at various times t (color bar), as obtained from finite-element numerical integration (FENI) of Eq. (3) [52]. For comparison, the self-similar attractor $f(\xi)/f(0)$ (see Eq. (13)) is shown (dashed line).

Acknowledgments

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Appendix: 1D linearized problem

We derive the Green's function, the associated self-similar attractor, and the convergence time for the 1D version of Eq. (4). The linear differential operator is now $\mathcal{L}_{1D} = \partial_t - \partial_x^6$, with x the single spatial coordinate. We

start by taking the Fourier transform of Eq. (5) in 1D, which yields:

$$\hat{G}_{1D}(k, \omega) = \frac{1}{i\omega + k^6} . \quad (19)$$

From the inverse Fourier transform and the residue theorem, we get:

$$G_{1D}(x, t) = \frac{\mathcal{H}(t)}{2\pi} \int dk e^{-k^6 t} e^{ikx} . \quad (20)$$

Using the similarity variable $\zeta = xt^{-1/6}$, it follows:

$$G_{1D}(\zeta, t) = \frac{\mathcal{H}(t)}{t^{1/6}} f(\zeta) , \quad (21)$$

with:

$$\begin{aligned} f_{1D}(\zeta) = & -\frac{2}{\Gamma(-\frac{1}{6})} {}_0F_4 \left(\left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right\}, -\left(\frac{\zeta}{6}\right)^6 \right) \\ & + \frac{\zeta^2}{12\sqrt{\pi}} {}_0F_4 \left(\left\{ \frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3} \right\}, -\left(\frac{\zeta}{6}\right)^6 \right) \\ & + \frac{\zeta^4}{432\Gamma(\frac{7}{6})} {}_0F_4 \left(\left\{ \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3} \right\}, -\left(\frac{\zeta}{6}\right)^6 \right) , \end{aligned} \quad (22)$$

where Γ is the gamma function and ${}_0F_4$ is the (0,4)-hypergeometric function.

The solution is then obtained from the 1D convolution with the initial profile:

$$\epsilon(x, t) = \int dx' G_{1D}(x - x', t) \epsilon_0(x') . \quad (23)$$

Using the similarity variable $\zeta = xt^{-1/6}$ and Eq. (21), Eq. (23) becomes:

$$\epsilon(x, t) = \frac{\mathcal{H}(t)}{t^{1/6}} \int dx' \epsilon_0(x') f_{1D}(\zeta - x' t^{-1/6}) . \quad (24)$$

At long positive times, Eq. (24) is equivalent to:

$$\epsilon(\zeta, t) \simeq \frac{A_0}{t^{1/6}} f_{1D}(\zeta) , \quad (25)$$

where $A_0 = \int dx' \epsilon_0(x')$ is the dimensionless area of the initial perturbation profile. Therefore, the rescaled solution $\epsilon(\zeta, t)/\epsilon(0, t)$ converges towards the universal self-similar attractor $f_{1D}(\zeta)/f_{1D}(0)$, no matter the initial perturbation profile (provided it is summable).

Assuming as a criterion that the convergence to the universal attractor is typically occurring when the central height $\epsilon_0(0)$ of the initial perturbation profile matches the central height $\epsilon(0, t) \simeq A_0 f_{1D}(0)/t^{1/6}$ of the asymptotic solution (see Eq. (25)), we find the 1D convergence time:

$$t_{c,1D} = \left[\frac{2A_0}{\epsilon_0(0)\Gamma(-1/6)} \right]^6. \quad (26)$$

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