A Finite Difference Method for Space Fractional Differential Equations with Variable Diffusivity Coefficient

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Abstract

Anomalous diffusion is a phenomenon that cannot be modeled accurately by second-order diffusion equations, but is better described by fractional diffusion models. The nonlocal nature of the fractional diffusion operators makes substantially more difficult the mathematical analysis of these models and the establishment of suitable numerical schemes. This paper proposes and analyzes the first finite difference method for solving *variable-coefficient* one-dimensional (steady state) fractional DEs, with two-sided fractional derivatives (FDs). The proposed scheme combines first-order forward and backward Euler methods for approximating the left-sided FD when the right-sided FD is approximated by two consecutive applications of the first-order backward Euler method. Our scheme reduces to the standard second-order central difference in the absence of FDs. The existence and uniqueness of the numerical solution are proved, and truncation errors of order h are demonstrated (h denotes the maximum space step size). The numerical tests illustrate the global $O(h)$ accuracy, except for nonsmooth cases which, as expected, have deteriorated convergence rates.

Keywords: Two sided fractional derivatives, Variable coefficients, Finite differences

1 Introduction

This work aims at constructing and analyzing a finite difference scheme for solving one-dimensional two-sided conservative fractional order differential equations with variable coefficient, κ, of the form:

$$\mathcal{A}^{\alpha,\theta}u(x) := -\partial_x\left(\kappa(x)\partial_x^{\alpha,\theta}u(x)\right) = f(x), \quad \text{for} \quad x \in \Omega := (a,b), \tag{1}$$

subject to absorbing boundary conditions $u = 0$ on $\mathbb{R}\backslash\Omega$ and so $u(a) = u(b) = 0$. Here, $\alpha \in (0,1)$ is the fractional order exponent, κ is the generalized diffusivity coefficient satisfying the positivity assumption $c_0 \le \kappa(x) \le c_1$ on Ω for some positive constants c_0 and c_1. For the truncation error analysis, we assume that $f \in C^1(\overline{\Omega})$ and $\kappa \in C^2(\overline{\Omega})$. In (1), ∂_x denotes the first-order derivative, and the two-sided fractional order differential operator $\partial_x^{\alpha,\theta} := \theta_a\mathrm{D}_x^\alpha + (1-\theta)_x\mathrm{D}_b^\alpha$. Here, $0 \le \theta \le 1$ is a parameter describing the relative probabilities of particles to travel ahead or behind the mean

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displacement, ${}_a\mathrm{D}_x^\alpha$ and ${}_x\mathrm{D}_b^\alpha$ are left-sided (LS) and right-sided (RS) Riemann-Liouville fractional derivatives, defined respectively as

$$
\begin{aligned}
{}_a\mathrm{D}_x^\alpha v(x) &:= \frac{\partial}{\partial x}\, {}_aI_x^{1-\alpha} v(x) = \frac{\partial}{\partial x}\int_a^x \omega_{1-\alpha}(x-z)v(z)\,dz,\\
{}_x\mathrm{D}_b^\alpha v(x) &:= \frac{\partial}{\partial x}\, {}_xI_b^{1-\alpha} v(x) = \frac{\partial}{\partial x}\int_x^b \omega_{1-\alpha}(z-x)v(z)dz\,.
\end{aligned}
$$

In the previous expressions, we denoted ${}_aI_x^{1-\alpha}$ and ${}_xI_b^{1-\alpha}$ the LS and RS Riemann-Liouville fractional integrals, respectively, with kernel $\omega_{1-\alpha}(x) := \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$.

In the limiting case $\alpha = 1$, the fractional derivative ∂_x^α reduces to ∂_x and the problem (1) reduces to the classical two-point elliptic boundary value problem, where $-\kappa\partial_x u$ is the ordinary diffusion flux from the Fick's law, Fourier's law, or Newtonian constitutive equation. An implied assumption is that the rate of diffusion at a certain location is independent of the global structure of the diffusing field. In the last few decades, an increasing number diffusion processes were found to be non-Fickian, and anomalous diffusion has been experimentally documented in many applications of interest [2, 31] (*e.g.*, viscoelastic materials, subsurface flows and plasma physics). In these situations, the mean square displacement grows in time faster (superdiffusion) or slower (subdiffusion) than that in a normal (Gaussian) diffusion process. This deviation from normal diffusion can be explained by non-Newtonian mechanics and Lévy processes. In such phenomena, the anomalous diffusion rate is affected not only by the local conditions (gradient) but also by the global state of the field. For instance, the time fractional derivative acting on the diffusion term (subdiffusion) [31] accommodates the existence of long-range correlations in the particle dynamics. Similarly, space fractional derivatives, which are suitable for the modeling of superdiffusion processes, account for anomalously large particle jumps at a rate inconsistent with the classical Brownian motion model. At the macroscopic level, these jumps give rise to a spatial fractional diffusion equation [2, 4]:

$$
\partial_t u - \partial_x(\kappa \partial_x^{\alpha,\theta} u) = g, \tag{2}
$$

where the numerical solution of this model was recently considered by several authors, see for example [1, 15, 24, 39] and related references therein.

For a constant diffusion coefficient κ and for $\theta = 1/2$ (i.e., the process to be symmetric [2]), (1) reduces to the Riesz fractional derivative of order $1+\alpha$, and many numerical methods have been proposed for its solution, see [3, 7, 9, 19, 22, 23, 25, 27, 33, 35, 37, 38, 43]. For constant κ, see also [29]. However, many practical problems require a model with variable diffusion coefficients κ [5], and the asymmetric diffusion process seems inherent in some physical systems [6, 34]. To the best of our knowledge, for a variable κ and with $\theta = 1/2$, the numerical solution of (2) was only considered recently in [12] assuming that κ is monotonically decreasing (a very restrictive condition) and $0.5546 < \alpha < 1$. These assumptions were used to show the unique solvability of the proposed numerical scheme. In relation, a different numerical scheme was considered in [13] but with the LS fractional derivative in place of the two-sided one.

Problem (1) is the steady state form of (2). For a constant κ, the operator $\partial_x(\kappa\partial_x^{\alpha,\theta})$ is a linear combination of the LS and RS fractional derivatives of order $\alpha+1$. In this case, Ervin and Roop [10] investigated the well-posedness of the Galerkin weak formulation of (1). They proved that the bilinear form associated with $\mathcal{A}^{\alpha,\theta}$ is coercive and continuous on $H_0^{1-\beta}(\Omega)\times H_0^{1-\beta}(\Omega) \to \mathbb{R}$, and hence, that (1) has a unique weak solution in $H_0^{1-\beta}(\Omega)$. For a rigorous study of the variational formulation of (1) when $\theta = 1$, see [20].

Unfortunately, it was shown in [40] that the Galerkin formulation loses coercivity in the variable κ case and the authors even propose a counterexample when $\theta = 1$, see [40, Lemma 3.2]. However, under restrictive assumptions on κ [16, Equation (3.16)], it is proved recently that this property can

be maintained [16]. As a result, the weak formulation is not an appropriate framework for variable coefficient κ, as the Galerkin finite element methods might fail to converge [41]. As an alternative, a Petrov-Galerkin method was investigated in [42] for the case of LS fractional derivatives. For the same setting, a finite difference method was proposed and analyzed in [36]. In [44], an intermediate unknown was introduced to rewrite problem (1) as a system of lower-order fractional differential equation with constant coefficients. Then, a spectral approximation scheme was developed to solve it.

It is worth to mention that extending existing numerical methods from constant to variable diffusivity is not straightforward, if feasible at all, because of the presence fractional order derivatives. Similarly, the analyses of the generic problem (1) remain scarce due to the mathematical difficulties induced by LS and RL nonlocal operators, that prevent reusing the results of classical elliptic equations. Therefore, the main motivation of the present work is to approximate the solution of (1) *via* finite difference methods, for variable diffusivity κ and allowing skewness parameter $0 \leq \theta \leq 1$. Specifically, we consider numerical schemes based on appropriate combinations of first-order backward and forward differences. For convenience, we first develop and analyze in Section 2 a finite difference scheme for (1) with $\theta = 1$, that is, we have to deal with the LS fractional derivative only. Then, in Section 3, the other limiting case $\theta = 0$ with RS fractional derivative only is considered. The contributions of both LS and RL fractional derivatives are subsequently combined in Section 4, to derive the generic finite difference scheme for (1) that reduces to the classical second-order central difference scheme in the limiting case $\alpha = 1$. For each case, we prove the existence and uniqueness of the finite difference solution and show $O(h)$ truncation errors for the resulting schemes, (h is the maximum space step size). We present several numerical experiments in Section 5 to support our theoretical convergence results in the case of smooth and nonsmooth solutions. Section 6 provides concluding remarks and recommendations for future works.

2 LS fractional derivative

We consider a partition of Ω with P subintervals $I_{1\leq n\leq P}$ constructed using a sequence of $(P+1)$ points such that $a = x_0 < x_1 < x_2 < \cdots < x_P = b$. Unless stated otherwise, we shall restrict ourselves to the case of uniform partitions with spatial step size $h = x_n - x_{n-1} = (b-a)/P$. We shall denote $x_{n+1/2} = \frac{x_n + x_{n+1}}{2}$ the center of interval I_{n+1}. Denoting $v^n := v(x_n)$, we use the symbol δv^n to denote the backward difference defined as $\delta v(x) = \delta v^n := v^n - v^{n-1}$ for $x \in I_n$, and the symbol $\overline{\delta} v^n := v^{n+1/2} - v^{n-1/2}$, to denote the central difference.

For the case of LS fractional derivative, that is, $\theta = 1$, Equation (1) reduces to

$$\mathcal{A}^{\alpha,1}u(x) = -\partial_x \left(\kappa(x)\, {}_a\mathrm{D}_x^{\alpha} u\right)(x) = f(x). \tag{3}$$

Using first a forward type difference treatment of the operator ∂_x, we propose the following approximation:

$$\partial_x \left(\kappa\, {}_a\mathrm{D}_x^{\alpha} u(x_n)\right) \approx h^{-1}\left[\kappa^{n+1/2}\, {}_a\mathrm{D}_x^{\alpha} u(x_{n+1}) - \kappa^{n-1/2}\, {}_a\mathrm{D}_x^{\alpha} u(x_n)\right], \quad \text{with} \quad \kappa^{n+1/2} := \kappa(x_{n+1/2}). \tag{4}$$

Observe that the proposed scheme involves a half-cell shift in the localization of the values of κ, resembling the case of the classical second-order elliptic equation.

Remarking that ${}_a\mathrm{D}_x^{\alpha} u = {}_aI_x^{1-\alpha} u'$, because $u(a) = 0$, equation (4) can be recast as

$$\partial_x(\kappa\, {}_a\mathrm{D}_x^{\alpha} u)(x_n) \approx h^{-1}[\kappa^{n+1/2}\, {}_aI_x^{1-\alpha} u'(x_{n+1}) - \kappa^{n-1/2}\, {}_aI_x^{1-\alpha} u'(x_n)].$$

Applying now the backward difference approximation to the derivatives inside the integrals, results in

$$-\partial_x(\kappa {}_a\mathrm{D}_x^{\alpha} u)(x_n) \approx \mathcal{A}_h^{\alpha,1} u(x_n), \quad \text{for} \quad n = 1, \ldots, P-1,$$

where the discrete fractional LS operator

$$\mathcal{A}_h^{\alpha,1}u(x_n) = -\frac{1}{h^2}\Big(\kappa^{n+1/2}({}_aI_x^{1-\alpha}\delta u)(x_{n+1}) - \kappa^{n-1/2}({}_aI_x^{1-\alpha}\delta u)(x_n)\Big).$$

Noting that

$${}_aI_x^{1-\alpha}\delta u(x_n) = \sum_{j=1}^{n}\int_{I_j}\omega_{1-\alpha}(x_n - s)\delta u^j\, ds = \omega_{2-\alpha}(h)\sum_{j=1}^{n} w_{n,j}\delta u^j, \tag{5}$$

with the weights defined as

$$w_{n,j} := (n+1-j)^{1-\alpha} - (n-j)^{1-\alpha} \quad \text{for } n \geq j \geq 1. \tag{6}$$

We denote by $U^n \approx u^n$ the finite difference solution, which for the model problem in (3) is required to satisfy

$$\mathcal{A}_h^{\alpha,1}U(x_n) = f^n, \quad n = 1, \cdots, P-1, \quad \text{with } \; U^0 = U^P = 0. \tag{7}$$

Using (5), the finite difference scheme can be recast as

$$\kappa^{n-1/2}\sum_{j=1}^{n} w_{n,j}\delta U^j - \kappa^{n+1/2}\sum_{j=1}^{n+1} w_{n+1,j}\delta U^j = \tilde{f}_h^n, \tag{8}$$

with the modified right-hand-side

$$\tilde{f}_h^n := h^2 f^n/\omega_{2-\alpha}(h). \tag{9}$$

For computational convenience, (7) can be expressed in a compact form as

$$\sum_{j=1}^{n}(a_{n,j} - a_{n+1,j})U^j - \kappa^{n+1/2}U^{n+1} = \tilde{f}_h^n, \;\; \text{for } \; n = 1, \cdots, P-1, \;\; \text{with } \; U^0 = U^P = 0,$$

where $a_{n,n} = \kappa^{n-1/2}$ and $a_{n,j} = \kappa^{n-1/2}[w_{n,j} - w_{n-1,j}]$ for $j < n$.

The finite difference solution is then obtained solving the $(P-1)$-by-$(P-1)$ linear system $\mathbf{B}_L\mathbf{U} = \mathbf{F}$, where $\mathbf{U} = [U^1, U^2, \cdots, U^{P-1}]^T$, $\mathbf{F} = [\tilde{f}_h^1, \tilde{f}_h^2, \cdots, \tilde{f}_h^{P-1}]^T$, and the matrix $\mathbf{B}_L = [c_{n,j}]$ having lower-triagonal entries

$$c_{n,j} = \begin{cases} \kappa^{n-1/2} + \kappa^{n+1/2}[2 - 2^{1-\alpha}] & j = n, \\ a_{n,j} - a_{n+1,j} & j < n, \end{cases}$$

while $c_{n,n+1} = -\kappa^{n+1/2}$ and all other entries are zeros. Note that for the case of a constant diffusivity, the matrix $\mathbf{B}_L$ reduces to the Toeplitz form.

As mentioned earlier, in the limiting case $\alpha = 1$, equation (1) reduces to $-\partial_x(\kappa\partial_x u) = f$. Furthermore, the numerical scheme (7) reduces to the classical second order difference scheme for elliptic problems .

Lemma 1. *For $1 \leq n \leq P$, the finite difference solution U^n of (7) exists and is unique.*

Proof. Since the finite difference solution U^n satisfies a square linear system of equations, the existence of U^n follows from its uniqueness. To prove uniqueness, we need to show that the finite difference solution is identically zero when $f = 0$, that is when the system right-hand-side is zero, that is $f^j = 0$ for $j = 1, \cdots, P-1$ in the finite difference scheme (7). To do so, sum (8) over index n, leading to

$$\sum_{n=1}^{m}\kappa^{n-1/2}\sum_{j=1}^{n} w_{n,j}\delta U^j - \sum_{n=1}^{m}\kappa^{n+1/2}\sum_{j=1}^{n+1} w_{n+1,j}\delta U^j = 0,$$

and consequently,

$$\sum_{n=0}^{m-1} \kappa^{n+1/2} \sum_{j=1}^{n+1} w_{n+1,j}\delta U^j - \sum_{n=1}^{m} \kappa^{n+1/2} \sum_{j=1}^{n+1} w_{n+1,j}\delta U^j = 0.$$

After simplifying, we conclude that

$$\kappa^{m+1/2} \sum_{j=1}^{m+1} w_{m+1,j}\delta U^j = \kappa^{1/2}\delta U^1, \quad \text{for } \ 1 \le m \le P-1, \tag{10}$$

which can alternatively be expressed as

$$\mathbf{W}^\alpha \Phi = \delta U^1 \mathbf{K}, \tag{11}$$

where $\Phi = [\delta U^1, \delta U^2, \cdots, \delta U^P]^T$, $\mathbf{K} = [k_1, k_2, \cdots, k_P]^T$ with $k_j = \kappa^{1/2}/\kappa^{j-1/2}$, and

$$\mathbf{W}^\alpha = \begin{bmatrix} b_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & 0 & 0 & \cdots & 0 \\ b_2 & b_1 & b_0 & 0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{P-1} & b_{P-2} & w_{P-3} & b_{P-4} & \cdots & b_1 & b_0 \end{bmatrix}, \tag{12}$$

with $b_0 = 1$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha} > 0$ for $j \ge 1$. Since $\mathbf{W}^\alpha$ is a nonsingular lower triangular Toeplitz matrix, its inverse, denoted by $\mathbf{W}^\alpha_{inv}$, is also a lower triangular Toeplitz matrix with elements

$$e_0 = \frac{1}{b_0} = 1, \quad \text{and} \quad e_j = -\sum_{i=0}^{j-1} b_{j-i}\, e_i, \quad \text{for } \ j \ge 1.$$

Now, from (11), $\Phi = \mathbf{W}^\alpha_{inv}\mathbf{K}\delta U^1$ and thus $\delta U^j = \delta U^1 \sum_{i=1}^{j} e_{j-i}\, k_i$. Since $\sum_{j=1}^{P} \delta U^j = 0$ (because $U^0 = U^P = 0$),

$$\delta U^1 \sum_{j=1}^{P}\sum_{i=1}^{j} e_{j-i}\, k_i = \delta U^1 \sum_{i=1}^{P} k_i \sum_{j=i}^{P} e_{j-i} = \delta U^1 \sum_{i=1}^{P} k_i \sum_{j=0}^{P-i} e_j = 0. \tag{13}$$

On the other hand, the sequence $\{b_j\}_{j\ge 0}$ is positive, slowly decaying ($\lim_{j\to\infty} b_j = 0$ and $\sum_{j=1}^{\infty} b_k = \infty$) and is strictly *log-convex* ($b_j^2 < b_{j-1}b_{j+1}$ for $j \ge 1$). Then, we deduce that $e_n < 0$ and $\sum_{j=0}^{n} e_j > 0$ for $n \ge 1$, see [17, Theorem 22] or [14, Theorem 2.2 and Lemma 2.4]. Using this in (13) and also using the fact that $k_i > 0$ for $i \ge 1$, yield $\delta U^1 = 0$. Therefore, by (11), $\Phi \equiv \mathbf{0}$ ($\mathbf{W}^\alpha$ is nonsingular). Consequently, the finite difference solution U^n is identically zero, for $1 \le n \le P-1$, because $U^0 = U^P = 0$. This completes the proof of the uniqueness of the numerical solution U. $\square$

We now turn to establishing the truncation error T_h^n of the proposed scheme. From (3) and (7),

$$T_h^n = \mathcal{A}_h^{\alpha,1} u(x_n) - \mathcal{A}^{\alpha,1} u(x_n), \quad \text{for } \ 1 \le n \le P-1\,.$$

Since $\partial_x(\kappa_a \mathrm{D}_x^\alpha u)(x_n) = [f(x_n) - f(x)] + \partial_x(\kappa_a \mathrm{D}_x^\alpha u)(x)$,

$$\int_{I_{n+1}} \partial_x(\kappa_a \mathrm{D}_x^\alpha u)(x_n)\, dx = -\frac{h^2}{2} f'(\zeta_n) + \kappa^{n+1}\, {}_aI_x^{1-\alpha} u'(x_{n+1}) - \kappa^n\, {}_aI_x^{1-\alpha} u'(x_n),$$

for some $\zeta_n \in I_{n+1}$, and thus,

$$T_h^n = \mathcal{A}_h^{\alpha,1} u(x_n) - \frac{h}{2} f'(\zeta_n) + \frac{1}{h}\Big[\kappa^{n+1}\, {}_aI_x^{1-\alpha} u'(x_{n+1}) - \kappa^n\, {}_aI_x^{1-\alpha} u'(x_n)\Big].$$

For the truncation error in the next theorem, we assume that the solution u satisfies certain regularity properties. To guarantee this, certain compatibility assumptions on the diffusion coefficient κ and on the source term f are required. Similar assumptions (or even heavier) were assumed in almost all papers in the literature that dealt with the computational solutions of problems with space fractional derivatives. Noting that, for the case of a nonsmooth u, we illustrate numerically that $O(h)$ can still be observed by grading the mesh near the singularity, see Table 3.

Theorem 2. *The truncation error is of order h for x_n not too close to the left boundary. More precisely,*

$$T_h^n = O(h)(1 + (x_n - a)^{-\alpha}), \quad \textit{for } 1 \le n \le P-1.$$

Proof. Using the change of variable $s = q + h$, we observe that

$$\begin{aligned} {}_aI_x^{1-\alpha} u'(x_{n+1}) &= \sum_{j=1}^{n+1} \int_{I_j} \omega_{1-\alpha}(x_{n+1} - s) u'(s)\, ds \\ &= \int_{I_1} \omega_{1-\alpha}(x_{n+1} - s) u'(s)\, ds + \sum_{j=1}^{n} \int_{I_j} \omega_{1-\alpha}(x_n - q) u'(q+h)\, dq. \end{aligned}$$

Similarly, for the backward difference we have

$$\begin{aligned} {}_aI_x^{1-\alpha} \delta u(x_{n+1}) &= \sum_{j=1}^{n+1} \int_{I_j} \omega_{1-\alpha}(x_{n+1} - s) \delta u^j\, ds \\ &= \delta u^1 \int_{I_1} \omega_{1-\alpha}(x_{n+1} - s)\, ds + \sum_{j=1}^{n} \delta u^{j+1} \int_{I_j} \omega_{1-\alpha}(x_n - q)\, dq. \end{aligned}$$

Therefore, the truncation error can be rewritten as

$$T_h^n = -\frac{h}{2} f'(\zeta_n) + E_1^n + \sum_{j=1}^{n} \int_{I_j} \omega_{1-\alpha}(x_n - q) E_2^{n,j}(q)\, dq, \quad \text{for } \ n \ge 1,$$

where

$$E_1^n := h^{-1} \int_{I_1} \omega_{1-\alpha}(x_{n+1} - s)[\kappa^{n+1} u'(s) - h^{-1} \kappa^{n+1/2} u^1] ds, \tag{14}$$

$$E_2^{n,j}(q) := \frac{\kappa^{n+1} u'(q+h) - \kappa^n u'(q)}{h} - \frac{\kappa^{n+1/2} \delta u^{j+1} - \kappa^{n-1/2} \delta u^j}{h^2}\,. \tag{15}$$

Focusing on the second error contribution, E_1^n, we observe that for $\kappa \in C^1(I_{n+1})$ and $u \in C^2(a, x_1]$, we have

$$\begin{aligned} \kappa^{n+1} u'(s) - h^{-1} \kappa^{n+1/2} u^1 &= \kappa^{n+1} u'(s) - h^{-1}[\kappa^{n+1} + O(h)][h\, u'(x_1) + O(h^2)] \\ &= \kappa^{n+1}[u'(s) - u'(x_1)] + O(h) = O(h). \end{aligned}$$

Consequently, an application of the mean value theorem for integral yields

$$E_1^n = O(1)\int_{I_1} \omega_{1-\alpha}(x_{n+1}-s)ds = O(h)(x_{n+1}-\xi)^{-\alpha}, \quad \text{for some } \ \xi \in I_1. \tag{16}$$

Regarding the last error contribution in T_h^n above, we first remark that for any $q \in (x_{j-1}, x_j)$, one has

$$\kappa^{n+1}u'(q+h) - \kappa^n u'(q) = \kappa^n[u'(q+h) - u'(q)] + \delta\kappa^{n+1}u'(q+h),$$

and that, for $\kappa \in C^2[x_{n-1}, x_{n+1}]$ and $u \in C^3[x_{j-1}, x_{j+1}]$, Taylor series expansions give

$$\begin{aligned}\kappa^{n+1/2}\delta u^{j+1} - \kappa^{n-1/2}\delta u^j &= [(\kappa^{n+1/2} - \kappa^n) + \kappa^n][\delta u^{j+1} - \delta u^j] + \overline{\delta}\kappa^n \delta u^j \\ &= h^2[\frac{h}{2}\kappa'(x_n) + \kappa^n]u''(x_j) + h^2\kappa'(x_n)u'(x_j) + O(h^3)\,.\end{aligned}$$

Gathering the previous results, we obtain for $E_2^{n,j}$

$$\begin{aligned}E_2^{n,j}(q) &= h^{-1}\kappa^n[u'(q+h) - u'(q) - hu''(x_j)] + h^{-1}[\delta\kappa^{n+1} - h\kappa'(x_n)]u'(q+h) + \kappa'(x_n)[u'(q+h) - u'(x_j)] \\ &= -h^{-1}\kappa^n \int_q^{q+h}\int_t^{x_j} u'''(x)\,dx\,dt + \frac{h}{2}\kappa''(\xi^n)u'(q+h) + \kappa'(x_n)\int_{x_j}^{q+h} u''(x)\,dx,\end{aligned}$$

for some $\xi^n \in I_{n+1}$. The double integral term is $O(h^2)$, whereas the second and third terms are $O(h)$. This leads to the conclusion that the last error contribution to T_h^n is $O(h)$. Putting all these estimates together, we get the desired result. □

3 RS fractional derivative

Here, we focus on the finite difference approximation of problem (1) when $\theta = 0$, i.e., the problem:

$$\mathcal{A}^{\alpha,0}u(x) = -\partial_x(\kappa\, {}_x\mathrm{D}_b^\alpha u)(x) = f(x). \tag{17}$$

We shall rely on the same notations as in the previous section. Contrary to the case of the LS fractional derivative, we propose a backward difference type treatment for the operator ∂_x, and consider the approximation

$$\partial_x(\kappa\, {}_x\mathrm{D}_b^\alpha u)(x_n) \approx h^{-1}[\kappa^{n+1/2}\, {}_x\mathrm{D}_b^\alpha u(x_n) - \kappa^{n-1/2}\, {}_x\mathrm{D}_b^\alpha u(x_{n-1})].$$

Again, observe the shift in the evaluation points for κ (at the cell centers) compared to fractional differential operator (at the mesh point), which is crucial to ensure the recovery of the classical second order scheme when $\alpha \to 1$. Noting that ${}_x\mathrm{D}_b^\alpha u = {}_xI_b^{1-\alpha}u'$, because $u(b) = 0$, we have

$$\partial_x(\kappa\, {}_x\mathrm{D}_b^\alpha u)(x_n) \approx h^{-1}\left[\kappa^{n+1/2}\, {}_xI_b^{1-\alpha}u'(x_n) - \kappa^{n-1/2}\, {}_xI_b^{1-\alpha}u'(x_{n-1})\right].$$

Applying the backward difference to the derivatives inside the integrals, one gets

$$\mathcal{A}^{\alpha,0}u(x_n) \approx \mathcal{A}_h^{\alpha,0}u(x_n), \quad \text{for } \ n = 1, \cdots, P-1,$$

where discrete fractional RS operator

$$\mathcal{A}_h^{\alpha,0}u(x_n) = -\frac{1}{h^2}\Big(\kappa^{n+1/2}({}_xI_b^{1-\alpha}\delta u)(x_n) - \kappa^{n-1/2}({}_xI_b^{1-\alpha}\delta u)(x_{n-1})\Big).$$

The finite difference solution $U^n \approx u^n$ of the (RS) fractional model problem (17) satisfies the system:

$$\mathcal{A}_h^{\alpha,0} U(x_n) = f^n, \quad n = 1, \cdots, P-1, \quad \text{with } U^0 = U^P = 0. \tag{18}$$

Further, application of the integral form of the RS Riemann-Liouville fractional derivative to the finite difference, δv, yields:

$${}_xI_b^{1-\alpha} \delta v(x_{n-1}) = \sum_{j=n}^{P} \int_{I_j} \omega_{1-\alpha}(s - x_{n-1}) \delta v^j \, ds = \omega_{2-\alpha}(h) \sum_{j=n}^{P} w_{j,n} \delta v^j,$$

such that the numerical scheme (18) can be expressed as

$$\kappa^{n-1/2} \sum_{j=n}^{P} w_{j,n} \delta U^j - \kappa^{n+1/2} \sum_{j=n+1}^{P} w_{j,n+1} \delta U^j = \tilde{f}_h^n. \tag{19}$$

In (18), the weights $w_{n,j}$ and modified right-hand side $\tilde{f}_h^n$ follow the definitions of the previous section, see equations (6) and (9) respectively. Making use of the equality

$$\sum_{j=n}^{P} w_{j,n} \delta v^j = \sum_{j=n}^{P-1} [w_{j,n} - w_{j+1,n}] v^j - w_{n,n} v^{n-1},$$

the finite difference scheme (18) can be rewritten as

$$\sum_{j=n}^{P-1} \Big(b_{jn} - b_{j,n+1} \Big) U^j - \kappa^{n-1/2} U^{n-1} = \tilde{f}_h^n, \quad n = 1, \cdots, P-1,$$

where $b_{n,n+1} = -\kappa^{n+1/2}$ and $b_{j,n} = \kappa^{n-1/2}[w_{j,n} - w_{j,n-1}]$ for $j \geq n$.

The finite difference solution of the RS fractional diffusion problem is thus obtained by solving the $(P-1)$-by-$(P-1)$ linear system $\mathbf{B}_R \mathbf{U} = \mathbf{F}$, with the system matrix $\mathbf{B}_R = [d_{n,j}]$ having upper-triagonal entries

$$d_{n,j} = \begin{cases} -\kappa^{n-1/2} w_{j,n-1} + (\kappa^{n-1/2} + \kappa^{n+1/2}) w_{j,n} - \kappa^{n+1/2} w_{j,n+1}, & j > n, \\ \kappa^{n+1/2} + \kappa^{n-1/2}[2 - 2^{1-\alpha}], & j = n, \end{cases}$$

while $d_{n+1,n} = -\kappa^{n+1/2}$ and all other entries are zeros.

Lemma 3. *The finite difference solution U^n to the RS scheme* (18) *exists and is unique.*

Proof. As in the case of the LS fractional derivative, the existence of the solution U^n of (18) follows from its uniqueness, and it is sufficient to show that the finite difference solution is identically zero when $f^n = 0$ for $n = 1, \cdots, P-1$. To do so, we follow the same path as in Lemma 1. Summing (19) over the index n, we get

$$\sum_{n=m}^{P-1} \kappa^{n-1/2} \sum_{j=n}^{P} w_{j,n} \delta U^j - \sum_{n=m}^{P-1} \kappa^{n+1/2} \sum_{j=n+1}^{P} w_{j,n+1} \delta U^j = 0.$$

The second sum equals $\sum_{n=m+1}^{P} \kappa^{n-1/2} \sum_{j=n}^{P} w_{j,n} \delta U^j$, and so,

$$\kappa^{m-1/2} \sum_{j=m}^{P} w_{j,m} \delta U^j - \kappa^{P-1/2} \delta U^P = 0.$$

and it ensues that

$$\kappa^{n-1/2}\sum_{j=n}^{P} w_{j,n}\delta U^j = \kappa^{P-1/2}\delta U^P, \quad \text{for } \ 1 \le n \le P.$$

This equation can be cast in the matrix form,

$$\mathbf{W}^{\alpha T}\Phi = \delta U^P\,\hat{\mathbf{K}} \Longleftrightarrow \mathbf{W}^{\alpha}\Phi = \delta U^P\,\tilde{\mathbf{K}}, \tag{20}$$

with the same matrix $\mathbf{W}^\alpha$ as in equation (11), whereas $\hat{\mathbf{K}} = [\frac{\kappa^{P-1/2}}{\kappa^{1/2}}, \frac{\kappa^{P-1/2}}{\kappa^{3/2}}, \cdots, 1]^T$ and $\tilde{\mathbf{K}} = [1, \frac{\kappa^{P-1/2}}{\kappa^{P-3/2}}, \cdots, \frac{\kappa^{P-1/2}}{\kappa^{1/2}}]^T$. Since (11) and (20) have the same form, by following the derivation in Lemma 1, we deduce that $\delta U^P = 0$. It is again immediate to conclude from (20) that $\Phi \equiv \mathbf{0}$ because $\mathbf{W}^\alpha$ is nonsingular. Consequently, the finite difference solution $U^n = 0$ for $1 \le n \le P-1$ because $U^0 = U^P = 0$. Therefore, the solution to the RS scheme (18) exists and is unique. $\square$

Next, we study the truncation error T_h^n of the proposed finite difference discretization of problem (17). The truncation error in this case is

$$T_h^n = \mathcal{A}_h^{\alpha,0}u(x_n) - \mathcal{A}^{\alpha,0}u(x_n), \quad \text{for } \ 1 \le n \le P-1.$$

Regarding the continuous part, we proceed with a procedure similar to the LS case, to get

$$h\,\partial_x(\kappa\,{}_x\mathrm{D}_b^\alpha u)(x_n) = \int_{I_n}\partial_x(\kappa\,{}_x\mathrm{D}_b^\alpha u)(x_n)\,dx = \frac{h^2}{2}f'(\zeta_n) + hG_h^n, \quad \text{for some } \ \zeta_n \in I_n,$$

where

$$G_h^n = \frac{1}{h}\Big[\kappa^n\,{}_xI_b^{1-\alpha}u'(x_n) - \kappa^{n-1}\,{}_xI_b^{1-\alpha}u'(x_{n-1})\Big].$$

Consequently,

$$T_h^n = \mathcal{A}_h^{\alpha,0}u(x_n) + G_h^n + O(h)\,. \tag{21}$$

Theorem 4. *The truncation error is of order h for x_n not too close to the right boundary. Explicitly,*

$$T_h^n = O(h)(1 + (b - x_{n-1})^{-\alpha}), \quad \textit{for } 1 \le n \le P-1.$$

Proof. Noting first that

$$\begin{aligned} {}_xI_b^{1-\alpha}u'(x_{n-1}) &= \sum_{j=n}^{P}\int_{I_j}\omega_{1-\alpha}(s-x_{n-1})u'(s)\,ds \\ &= \int_{I_P}\omega_{1-\alpha}(s-x_{n-1})u'(s)\,ds + \sum_{j=n+1}^{P}\int_{I_j}\omega_{1-\alpha}(q-x_n)u'(q-h)\,dq, \end{aligned} \tag{22}$$

and

$$\begin{aligned} {}_xI_b^{1-\alpha}\delta u(x_{n-1}) &= \sum_{j=n}^{P}\int_{I_j}\omega_{1-\alpha}(s-x_{n-1})\delta u^j\,ds \\ &= \delta u^P\int_{I_P}\omega_{1-\alpha}(s-x_{n-1})\,ds + \sum_{j=n+1}^{P}\delta u^{j-1}\int_{I_j}\omega_{1-\alpha}(q-x_n)\,dq. \end{aligned} \tag{23}$$

On the one hand, the equality in (23) is used to obtain

$$h^2\mathcal{A}_h^{\alpha,0}u(x_n) = -\kappa^{n-1/2}u^{P-1}\int_{I_P}\omega_{1-\alpha}(s-x_{n-1})\,ds - \sum_{j=n+1}^{P}[\kappa^{n+1/2}\delta u^j - \kappa^{n-1/2}\delta u^{j-1}]\int_{I_j}\omega_{1-\alpha}(s-x_n)\,ds,$$

where for the second sum, one shows that

$$\begin{aligned}\kappa^{n+1/2}\delta u^j - \kappa^{n-1/2}\delta u^{j-1} &= [(\kappa^{n+1/2}-\kappa^n)+\kappa^n][\delta u^j - \delta u^{j-1}] + \overline{\delta}\kappa^n \delta u^{j-1} \\ &= h^2[\frac{h}{2}\kappa'(x_n)+\kappa^n]u''(x_{j-1}) + h^2\kappa'(x_n)u'(x_{j-1}) + O(h^3) \\ &= \frac{h^3}{2}\kappa'(x_n)u''(x_{j-1}) + h^2\kappa^n[u''(q-h)+(u''(x_{j-1})-u''(q-h))] \\ &\quad + h^2\kappa'(x_n)[u'(q-h)+(u'(x_{j-1})-u'(q-h))] + O(h^3) \\ &= h^2\kappa^n u''(q-h) + h^2\kappa'(x_n)u'(q) + O(h^3),\end{aligned}$$

for any $q \in (x_{j-1}, x_j)$. One the other hand, using equation (22) we have

$$h\, G_h^n = \sum_{j=n+1}^{P} \int_{I_j} \omega_{1-\alpha}(q-x_n)[\kappa^n u'(q) - \kappa^{n-1}u'(q-h)]\, dq - \kappa^{n-1}\int_{I_P} \omega_{1-\alpha}(s-x_{n-1})u'(s)\, ds,$$

where, by Taylor series expansion,

$$\begin{aligned}\kappa^n u'(q) - \kappa^{n-1}u'(q-h) &= \kappa^n[u'(q)-u'(q-h)] + \delta\kappa^n\, u'(q-h) \\ &= h\kappa^n u''(q-h) + h\kappa'(x_n)u'(q-h) + O(h^2).\end{aligned}$$

Inserting the above estimates in (21), we obtain for $1 \le n \le P-1$

$$T_h^n = E^n + O(h), \quad E^n := -h^{-2}\int_{I_P} \omega_{1-\alpha}(s-x_{n-1})[h\kappa^{n-1}u'(s) + \kappa^{n-1/2}u^{P-1}]ds.$$

Since $\kappa^{n-1/2}u^{P-1} = [\kappa^{n-1}+O(h)][-hu'(x_{P-1})+O(h^2)] = -h\kappa^{n-1}u'(x_{P-1}) + O(h^2)$,

$$E^n = O(1)\int_{I_P} \omega_{1-\alpha}(s-x_{n-1})ds = O(h)\omega_{1-\alpha}(\xi - x_{n-1}), \quad \text{for some } \xi \in I_P.$$

This completes the proof of the RS truncation error. □

4 Two-sided fractional derivative

In this section, we return to the two-sided fractional differential equation (1). To construct our finite difference approximation we simply combine the finite difference schemes introduced in the two previous sections for the LS and RS fractional derivatives. Specifically, using (7) and (18), the finite difference solution $U^n \approx u^n$ of the fractional model problem (1) is given by the equations

$$\theta\mathcal{A}_h^{\alpha,1}U(x_n) + (1-\theta)\mathcal{A}_h^{\alpha,0}U(x_n) = f^n, \quad \text{for } n = 1, \cdots, P-1 \text{ with } U^0 = U^P = 0.$$

The finite difference solution is obtained by solving the linear system $\mathbf{BU} = \mathbf{F}$, where $\mathbf{B} = \theta\mathbf{B}_L + (1-\theta)\mathbf{B}_R$, with the definitions of the matrices $\mathbf{B}_L$ and $\mathbf{B}_R$ given in the previous sections.

This shows that the numerical scheme amounts to inverting a system of $(P-1)$ linear equations in the $P-1$ unknowns, so the existence of the finite difference solution follows from its uniqueness. Following a similar path as for the proof of uniqueness for the cases of the LS and RS fractional derivative schemes, we get

$$[\theta\mathbf{W}^\alpha + (1-\theta)\mathbf{W}^{\alpha T}]\Phi = \psi\mathbf{K}, \quad \text{with } \psi = \theta\delta U^1 + (1-\theta)\sum_{j=1}^{P} w_{j,1}\delta U^j. \tag{24}$$

By [21, Lemma A.2], the matrix $\mathbf{W}^{\alpha}$ in (12) is positive definite and so is $\mathbf{W}^{\alpha T}$. Thus, the Toeplitz matrix $\theta\mathbf{W}^{\alpha} + (1-\theta)\mathbf{W}^{\alpha T}$ is also positive definite and hence, has an inverse, denoted by $\mathbf{E}_{\alpha,\theta}$, with entries $e_{i,j}$. From (24), $\Phi = \mathbf{E}_{\alpha,\theta}\mathbf{K}\psi$ and thus, $\sum_{i=1}^{P} \delta U^i = \psi \sum_{i=1}^{P}\sum_{j=1}^{P} e_{i,j}\, k_j$. Since $\sum_{j=1}^{P} \delta U^j = 0$,

$$\psi \sum_{j=1}^{P} k_j \sum_{i=1}^{P} e_{i,j} = 0, \quad \text{where} \quad k_j > 0. \tag{25}$$

Recall that, the sequence $\{b_j\}_{j\geq 0}$ is positive, slowly decaying and is also strictly *log-convex*, then by following the arguments for the case of LS fractional derivative, we conclude that the matrix $\mathbf{E}_{\alpha,\theta}$ and its transpose $\mathbf{E}_{\alpha,\theta}^{T}$ are strictly diagonally dominant [18], with $e_{i,i} > 0$ and $e_{i,j} \leq 0$ for $i \neq j$. Hence, $\sum_{i=1}^{P} e_{i,j} > 0$ and thus, $\psi = 0$ from (25). Substitute this in (24) yields $\Phi = \mathbf{0}$ and it follows that $U^n = 0$ for $1 \leq n \leq P-1$ because $U^0 = U^P = 0$. This completes the proof of the existence and uniqueness of U.

Furthermore, by combining the results of Theorems 2 and 4, it is trivial to show that the truncation error is of order $O(h)$ (not near the boundaries at $x = a, b$).

5 Numerical results

In this section we present several numerical experiments to support the theoretical analyses of the previous sections. Specifically, we consider the model problem in (1) over $\Omega = (0,1)$, subject to homogeneous Dirichlet (absorbing) boundary conditions, and we set $\kappa = 1 + \exp(x)$. The finite difference discretization uses uniform spatial meshes with $P = 2^l$ subintervals, for $l > 1$, such that $h = 1/P$. The solution error E_h is measured using the discrete L^{∞}-norm $\|v\|_h = \max_{0\leq i\leq P} |v(x_i)|$. Based on this error definition, the numerical estimate of convergence rates σ_h of the finite difference solutions is obtained from the relation $\sigma_h = \log_2(E_{2h}/E_h)$.

Example 1. We first consider the source term f leading to the exact solution

$$u_{\text{ex}}(x) = x^{4-\theta(1-\alpha)}(1-x)^{4-(1-\theta)(1-\alpha)}. \tag{26}$$

We first fix $\theta = 1/2$, $P = 1024$ and report in Fig. 1 the estimates σ_h as a function of α. The plot shows that $\sigma_h \sim 1$, denoting an error in $O(h)$, for almost all values of α except in the immediate neighborhood of $\alpha = 1$. When $\alpha \to 1$, σ_h exhibits a rapidly varying behavior to reach the expected $O(h^2)$ rate at $\alpha = 1$.

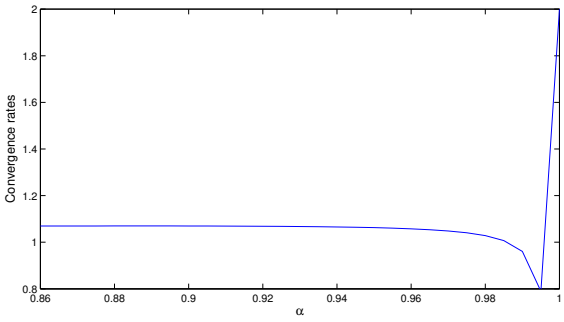

Figure 1: Graphical plot of σ_h against the diffusion exponent α. Computations use $\theta = 1/2$ and $P = 1024$.

θ	$-\log_2 h$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
		E_h	σ_h	E_h	σ_h	E_h	σ_h
0.0	7	1.040e-04	0.9929	8.028e-05	0.9659	5.164e-05	0.9030
	8	5.214e-05	0.9960	4.080e-05	0.9765	2.723e-05	0.9234
	9	2.611e-05	0.9976	2.064e-05	0.9834	1.421e-05	0.9382
	10	1.307e-05	0.9986	1.040e-05	0.9882	7.357e-06	0.9496
0.25	7	1.784e-04	0.9838	9.622e-05	0.9636	4.275e-05	0.9255
	8	8.875e-05	1.0071	4.843e-05	0.9905	2.200e-05	0.9588
	9	4.325e-05	1.0369	2.393e-05	1.0173	1.108e-05	0.9887
	10	2.033e-05	1.0894	1.150e-05	1.0569	5.432e-06	1.0290
0.5	7	2.865e-04	0.9280	1.045e-04	0.9530	3.705e-05	9.4381
	8	1.461e-04	0.9713	5.269e-05	0.9883	1.868e-05	9.8776
	9	7.289e-05	1.0036	2.599e-05	1.0198	9.157e-06	1.0287
	10	3.545e-05	1.0398	1.243e-05	1.0643	4.304e-06	1.0893
0.75	7	1.714e-04	0.9672	9.392e-05	0.9527	4.190e-05	0.9147
	8	8.632e-05	0.9898	4.757e-05	0.9812	2.167e-05	0.9513
	9	4.289e-05	1.0092	2.370e-05	1.0054	1.097e-05	0.9820
	10	2.094e-05	1.0341	1.156e-05	1.0359	5.408e-06	1.0205
1.0	7	1.034e-04	0.9855	7.929e-05	0.9546	5.048e-05	0.8893
	8	5.197e-05	0.9922	4.048e-05	0.9700	2.677e-05	0.9149
	9	2.607e-05	0.9956	2.053e-05	0.9794	1.403e-05	0.9326
	10	1.306e-05	0.9975	1.037e-05	0.9857	7.283e-06	0.9457

Table 1: Discrete L^∞-norm errors E_h and estimated numerical convergence rates σ_h for different values of α and θ.

Next, we fix $P = 512$ and plot E_h against α for different values of θ. Results are reported in Fig. 2. We observe that the errors are almost the same for $\theta = 0.25$ and $\theta = 0.75$, and for $\theta = 0$ and $\theta = 1$. This is due to the similar singularity behavior near the boundaries of the exact solution in (26) for any choice of $\theta = c$ and $\theta = 1 - c$. Note that the errors are decreasing as $\alpha \to 1$ for all θ. Interestingly enough, Fig. 2 also shows that for $\alpha < 0.6$, the error is lower for extreme values of θ, that is close to 0 or 1, and on the contrary E_h is lower for intermediate values ($\approx 1/2$) when $\alpha > 0.6$.

Table 1 reports the L^∞-norm of E_h and the corresponding estimates of convergence rate for different values of α, θ and the discretization step size h. The table confirms the $O(h)$ errors.

Example 2. (nonsmooth solutions) In practice, due to the presence of the two-sided fractional derivative, the solution u of (1) admits end-point singularities even if the source term f is smooth. It was proved in [28] that, for $\theta = 1/2$, the leading singularity term takes the form $x^{\frac{1+\alpha}{2}}(1-x)^{\frac{1+\alpha}{2}}$ when the diffusivity coefficient κ is constant. Similarly, one can show that leading singularity term takes the form $(x-a)^\alpha$, with $a = 0$ presently, in the case of LS fractional derivative ($\theta = 1$), and the form $(b-x)^\alpha$, with $b = 1$ presently, in the case of RS fractional derivatives ($\theta = 0$). For smooth κ, we conjecture the same singular behavior. Furthermore, we suggest that for $\theta \in [0, 1]$, the leading singularity term has the generic form $(x-a)^{1-\theta(1-\alpha)}(b-x)^{1-(1-\theta)(1-\alpha)}$ ($a = 0$ and $b = 1$). However, demonstrating this point remains an open problem and it will be a subject of future work. Noting

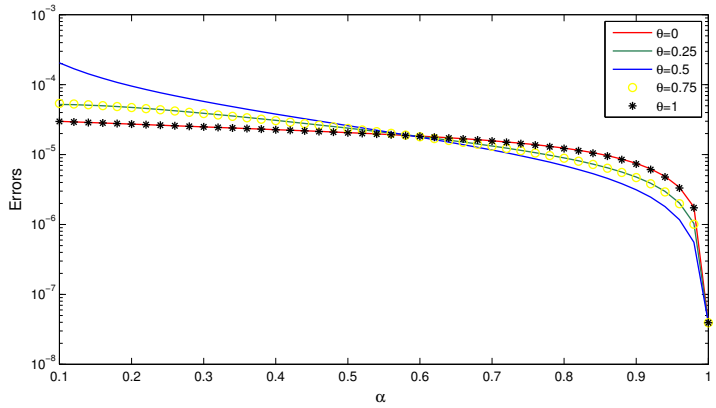

Figure 2: The error E_h against the diffusion exponent α, for $P = 512$ and different values of θ as indicated.

that, for $\kappa = 1$ and for $0 \leq \theta \leq 1$, the authors in [11] studied the regularity properties of u where the fractional derivative operator is not of the Riemann-Liouville type, see [11, Equations (1.3) and (3.11)].

To support our claim, we choose now the source term f such that $u_{\text{ex}}(x) = x^{1-\theta(1-\alpha)}(1-x)^{1-(1-\theta)(1-\alpha)}$ is the exact solution of the problem with other settings as before. One can easily check that the truncation errors analyses provided above are not valid in this situation. We then apply to this problem our finite difference scheme for the LS ($\theta = 1$) and RS ($\theta = 0$) fractional derivatives cases for different values of α and h. Table 2 reports the discrete L^∞-norm of E_h and estimates of the convergence rates σ_h. The results clearly indicate a convergence rate of the error in $O(h^\alpha)$.

		$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
θ	$-\log_2 h$	E_h	σ_h	E_h	σ_h	E_h	σ_h
0.0	9	3.527e-02	0.2567	9.510e-03	0.5037	1.618e-03	0.7556
	10	2.959e-02	0.2534	6.716e-03	0.5019	9.601e-04	0.7533
	11	2.485e-02	0.2517	4.745e-03	0.5010	5.702e-04	0.7518
	12	2.088e-02	0.2509	3.354e-03	0.5005	3.388e-04	0.7510
1.0	9	3.498e-02	0.2449	9.465e-03	0.4970	1.606e-03	0.7454
	10	2.946e-02	0.2475	6.700e-03	0.4985	9.562e-04	0.7478
	11	2.480e-02	0.2487	4.740e-03	0.4993	5.690e-04	0.7490
	12	2.086e-02	0.2494	3.352e-03	0.4996	3.384e-04	0.7495

Table 2: Discrete L^∞-norm errors E_h and estimated numerical convergence rates σ_h for different values of α and θ.

This degradation of the convergence rate was expected because the low regularity of the solution: $u_{\text{ex}} \in C^\alpha[0,1]$. In the context of time-stepping schemes for fractional diffusion of fractional wave equations, adapted meshes with refinement (clustering of elements) around the singularity successfully improve the errors and consequently, the convergence rates, see [30, 32]. To check if such refinement

approach could be useful in our problem of (steady) spatial fractional diffusion problem, we set $\theta = 1$ (LS singularity) and consider a family of graded spatial meshes of $\Omega = (0, 1)$ based on a sequence of points given by $x_i = (i/P)^\gamma$, $i = 0, \ldots, P$ and $\gamma \geq 1$ is a refinement parameter. The objective is to refine the mesh at the boundary $x = 0$ where the solution has a singularity. Table 5 reports the evolution with $\log_2(P)$ of the L^∞-norm of the error and estimated convergence rate σ_h and using $\gamma = 2, 3$ and 4. The results show that one can obtain an $O(h^{\alpha\gamma})$ convergence rate. Finally, Fig. 3 compares the pointwise errors obtained for uniform and nonuniform meshes with $\gamma = 3$ when using the same number of discretization points $P = 256$, 512, 1024 and 2048. The reduction of the error due to the mesh refinement is clearly visible. Note that similar results can be obtained for $\theta = 0$ using discretization points defined by $x_i = 1 - ((P - i)/P)^\gamma$ to refine the mesh at the endpoint $x = 1$.

	$\gamma = 2$		$\gamma = 3$		$\gamma = 4$	
$\log_2 P$	E_h	σ_h	E_h	σ_h	E_h	σ_h
9	8.140e-03	0.4998	1.711e-03	0.7498	3.597e-04	0.9997
10	5.756e-03	0.4999	1.018e-03	0.7499	1.800e-04	0.9999
11	4.070e-03	0.4999	6.051e-04	0.7499	8.994e-05	0.9999
12	2.878e-03	0.5002	3.600e-04	0.7500	4.497e-05	0.9998

Table 3: Discrete L^∞-norm errors E_h and estimated numerical convergence rates σ_h for $\alpha = 0.25$, $\theta = 1$ (LS fractional derivatives), different number of discretization points (P) and refinement parameters γ.

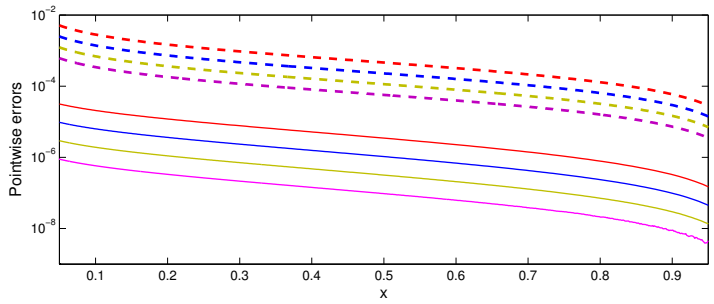

Figure 3: Pointwise errors using uniform (dashed lines) and nonuniform meshes with $\gamma = 3$ (solid lines), for $\alpha = 0.25$ with $P = 256, 512, 1024, 2048$ (in order from top to bottom).

6 Concluding remarks

The objective of this work was to develop and analyze a finite-difference scheme for the solution of fractional elliptic problems with a variable diffusion coefficient. We proved the existence and uniqueness of the proposed numerical solution and established the order of convergence for the truncation error. Some numerical results were also presented for problems admitting both smooth and nonsmooth solutions.

This paper will form a stepping stone for the researchers who are interested in computational solutions of variable coefficient two-sided fractional derivative problems. The results obtained in this work lead to several questions that will have to be addressed in the future. First, it will crucial to address the reason(s) for the dramatic deterioration in the order of convergence of the finite difference scheme

when the fractional order α immediately departs from 1 (classical case)? Second, it will be interesting to explore the possibility of incorporating the fractional exponent α directly in the finite difference discretization, that is, fractionalizing the numerical scheme. A possible route along this direction could be inspired by the recent research papers on the fractionalization of the Crank-Nicolson time-scheme for solving time-fractional diffusion equation, see [8]. Another extension that would be worthwhile to explore concerns generalization of the present methodology to multiple space dimensions. An attractive avenue consists in combining the presently considered staggered grid machinery with suitable representations of the fractional diffusion flux [26] evaluated at cell faces. Finally, mechanisms for determining the order of singularity near the boundaries in the case of variable diffusivity remains to be developed. These and other related open questions will be the subject of future research.

References

[1] M. K. Almoaeet, M. Shamsi H. Khosravian-Arab, and D. F. M. Torres, A collocation method of lines for two-sided space-fractional advection-diffusion equations with variable coefficients, Math. Meth. Appl. Sci., 42, 3465–3480, 2019.

[2] D. Benson, S. W. Wheatcraft and M. M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resource Res., 36, 1413–1423, 2000.

[3] C. Celik and M. Duman, Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, J. Comput. Phys., 231, 1743–1750, 2012.

[4] A. Chaves, Fractional diffusion equation to describe Lévy fights, Phys. Lett. A, 239, 12–16, 1998.

[5] A. V. Chechkin, J. Klafter and I. M. Sokolov, Fractional Fokker–Planck equation for ultraslow kinetics, Europhys. Lett., 63, 326–332, 2003.

[6] D. del-Castillo-Negrete, Chaotic transport in zonal flows in analogous geophysical and plasma systems, Phys. Plasmas, 7, 1702–1711, 2000.

[7] W. Deng and J.S. Hesthaven, Local discontinuous Galerkin methods for fractional diffusion equations, ESAIM: Mathematical Modelling and Numerical Analysis, 47, 1845–1864, 2013.

[8] Y. Dimitrov, Numerical approximations for fractional differential equations. J. Fract. Calc. Appl., 5(suppl. 3S):Paper no. 22, 1–45, 2014.

[9] H. Ding, C. Li and Y. Chen, High—order algorithms for Riesz derivative and their applications I, Abstract and Applied Analysis, 2014, 1–17, 2014.

[10] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods. Partial Differential Equations, 22, 559–576, 2006.

[11] V. J. Ervin, N. Heuer and J. P. Roop, Regularity of the solution to $1-$D fractional order diffusion equations, Math. Comp., 87, 2273–2294, 2018.

[12] L. Feng, P. Zhuang, F. Liu, I. Turner, V. Anh, J. Li, A fast second-order accurate method for a two-sided space-fractional diffusion equation with variable coefficients, Comput. Math. Appl., 73, 1155–1171, 2017.

[13] L. Feng, P. Zhuang, F. Liu, I. Turner and Q. Yang, Second–order approximation for the space fractional diffusion equation with variable coefficient, Prog. Frac. Diff. Appl., 1, 23–35, 2017.

[14] N. J. Ford, D. V. Savostyanov and N. L. Zamarashkin, On the Decay of the elements of inverse triangular Toeplitz matrices, SIAM J. Matrix Anal. Appl., 35, 1288–1302, 2014.

[15] X. Guo, Y. Li, and H. Wang, A fourth-order scheme for space fractional diffusion equations, J. Comput. Phy., 373, 410–424, 2018.

[16] Z. Hao, M. Park, G. Lin, and Z. Cai, Finite element method for two-sided fractional differential equations with variable coefficients: Galerkin approach, J. Sci. Comput., 79, 700–717, 2019.

[17] G. H. Hardy, Divergent Series, Clarendon Press, Oxford, 1949.

[18] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, New York, 2013.

[19] Y. Huang and A. Oberman, Numerical methods for the fractional Laplacian: a finite difference-quadrature approach, SIAM J. Numer. Anal., 52, 3056–3084, 2014.

[20] B. Jin, R. Lazarov, J. Pasciak and W. Rundell, Variational formulation of problems involving fractional order differential operators, Math. Comp., 84, 2665–2700, 2015.

[21] K. N. Le, W. McLean and K. Mustapha, Numerical solution of the time-fractional Fokker–Planck equation with general forcing, SIAM J. Numer. Anal., 54, 1763–1784, 2016.

[22] E. Kharazmi, M. Zayernouri and G. Em Karniadakis, A Petrov-Galerkin spectral element method for fractional elliptic problems, Comput. Methods Appl. Mech. Engrg., 324, 512–536, 2017.

[23] X. Li and C. Xu, The existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation, Commun. Comput. Phys., 8, 1016–1051, 2010.

[24] X-l. Lin, M. K. Ng, and H-W. Sun, Stability and convergence analysis of finite difference schemes for time-dependent space-fractional diffusion equations with variable diffusion coefficients, J. Sci. Comput., 75, 1102–1127, 2018.

[25] Y. Liu, Y. Yan, and M. Khan, Discontinuous Galerkin time stepping method for solving linear space fractional partial differential equations, Appl. Numer. Math., 115, 200–213, 2017.

[26] M. Lucchesi, S. Allouch, O. P. Le Maître, K. A. Mustapha, and O. M. Knio, Particle simulation of space-fractional diffusion equations, Comp. Part. Mech., 2019, https://doi.org/10.1007/s40571-019-00275-8.

[27] V. Lynch, B. Carreras, D. del Castillo-Negrete, K. Ferreira-Mejias, and H. Hicks, Numerical methods for the solution of partial differential equations of fractional order, J. Comput. Phys., 192, 406–421, 2003.

[28] Z. Mao, S. Chen, and J. Shen, Efficient and accurate spectral method using generalized Jacobi functions for solving Riesz fractional differential equations, Appl. Numer. Math., 106, 165–181, 2016.

[29] Z. Mao and J. Shen, Spectral element method with geometric mesh for two-sided fractional differential equations, Adv. Comput. Math., 44, 745–771, 2018.

[30] W. McLean and K. Mustapha, A second-order accurate numerical method for a fractional wave equation, Numer. Math., 105, 481–510, 2007.

[31] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Physics Reports*, 339, 1–77, 2000.

[32] K. Mustapha, An implicit finite difference time-stepping method for a sub-diffusion equation, with spatial discretization by finite elements, IMA J. Numer. Anal., 31. 719–739, 2011.

[33] M. D. Ortigueira, Riesz potential operators and inverses via fractional centered derivatives, Int. J. Math. Mathematical Sci., 1–12, 2006.

[34] T. H. Solomon, E. R. Weeks, and H. L. Swinney, Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow, Phys. Rev. Lett., 71, 3975–3978, 1993.

[35] E. Sousa, Finite difference approximations for a fractional advection diffusion problem, J. Comput. Phys., 228, 4038–4054, 2009.

[36] M. Stynes and J. L. Gracia, A finite difference method for a two-point boundary value problem with a Caputo fractional derivative, IMA J. Numer. Anal., 35, 2015.

[37] C. Tadjeran, M. M. Meerschaert, and H.-P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation, J. Comput. Phys., 205–213, 2006.

[38] W. Tian, H. Zhou, and W. Deng, A class of second order difference approximations for solving space fractional diffusion equations, Math. Comp., 84, 1703–1727, 2015.

[39] S. Vong and P. Lyu, On a second order scheme for space fractional diffusion equations with variable coefficients, Appl. Numer. Math., 137, 34–48, 2019.

[40] H. Wang and D. Yang, Wellposedness of variable-coefficient conservative fractional elliptic differential equations, SIAM J. Numer. Anal., 51, 1088–1107, 2013.

[41] H. Wang, D. Yang, and S. Zhu, Inhomogeneous Dirichlet boundary-value problems of space–fractional diffusion equations and their finite element approximations, SIAM J. Numer. Anal., 52, 1292–1310, 2014.

[42] H. Wang, D. Yang, and S. Zhu, A Petrov-Galerkin finite element method for variable-coefficient fractional diffusion equations, Comput. Methods Appl. Mech. Engrg., 290, 45–56, 2015.

[43] H. Zhang, F. Liu, and V. Anh, Garlerkin finite element approximations of symmetric space-fractional partial differential equations, Appl. Math. Comput., 217, 2534–2545, 2010.

[44] X. Zheng, V. J. Ervin, and H. Wang, Spectral approximation of a variable coefficient fractional diffusion equation in one space dimension, Appl. Math. Comput., 15, 98–111, 2019.