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# Large-time asymptotics of the wave fronts length I The Euclidean disk

Yves Colin de Verdière\*

David Vicente, †

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In the paper [Vi-20], the second author proves that the length  $|S_t|$  of the wave front  $S_t$  at time  $t$  of a wave propagating in an Euclidean disk  $\mathbb{D}$  of radius 1, starting from a source  $q$ , admits a linear asymptotics as  $t \rightarrow +\infty$ :  $|S_t| = \lambda(q)t + o(t)$  with  $\lambda(q) = 2 \arcsin a$  and  $a = d(0, q)$ . We will give a more direct proof and compute the oscillating corrections to this linear asymptotics. The proof is based on the “stationary phase” approximation.

## 1 Wave fronts

Let us consider a 2D-Riemannian compact manifold  $(X, g)$  possibly with a smooth convex boundary. We denote by  $g^* : T^*X \rightarrow \mathbb{R}$  the half of the dual metric which is the Hamiltonian of the geodesic flow.

We denote by  $\pi_X$  the canonical projection of  $T^*X$  onto  $X$  and  $\phi_t : T^*X \rightarrow T^*X$ ,  $t \in \mathbb{R}$  the Hamiltonian flow of  $g^*$  which is the geodesic flow. If  $X$  has a non empty boundary, we define  $\phi_t$  using the law of reflection. Let  $q \in X$  be given. For any  $t > 0$ , we define the wave front  $S_t$  at time  $t$  as the set of

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\*Université Grenoble-Alpes, Institut Fourier, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402-Saint Martin d’Hères Cedex (France); [yves.colin-de-verdiere@univ-grenoble-alpes.fr](mailto:yves.colin-de-verdiere@univ-grenoble-alpes.fr)

†Université d’Orléans, Institut Denis Poisson, Unité mixte de recherche CNRS 7013, Rue de Chartres, B.P. 6759, 45067 Orléans Cedex 2 (France) [david.vicente154@gmail.com](mailto:david.vicente154@gmail.com)

points of  $X$  of the form  $\pi_X(\phi_t(\Sigma^q))$  where  $\Sigma^q := \{(q, \xi) \in T^*X | g^*(q, \xi) = 1\}$ . The set  $S_t$  could also be defined as the image by the exponential map at  $q$  of the circle  $\Sigma_t$  of radius  $t$  in the tangent space  $T_qX$ .

Let us define the length of  $S_t$  and denote it by  $|S_t|$ . The wave front  $S_t$  is a curve parametrized by a circle:  $S_t := \exp_q(\Sigma_t)$ . This allows to define its length using the Riemannian metric. Note that  $S_t$  can admit some singular points. The length of the corresponding part vanishes and the remaining part is an immersed co-oriented curve with only transversal self-intersections.

In this article, we focus on the case where  $X$  is the unit disk in  $\mathbb{R}^2$  and  $g$  is the Euclidean metric. In this context, we will prove that the following expansion holds:

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + \mathcal{O}(1) \quad (1)$$

as  $t \rightarrow +\infty$ , with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

where  $a$  is the distance from the point  $q$  to the center of the disk.

The case of closed surfaces with integrable geodesic flows will be the subject of [CV-20].

## 2 Numerics

In this section, we will compare the expansion given by (1) with the numerical calculations. We introduce a (small) time step  $\delta_t > 0$ , a (large) number of points  $n$  which compose the wave front, two vectors  $M$  and  $V$  in  $(\mathbb{R}^2)^n$  such that, for any  $k \in \llbracket 1, n \rrbracket$ ,  $X_k \in \mathbb{R}^2$  represents the position and  $V_k \in \mathbb{R}^2$  the speed of the  $k$ th point of the wave front at a given time. We fix  $a \in ]0, 1[$  such that  $(a, 0)$  are the coordinates of the source  $q$ . Thus, we introduce the

following iterative scheme

$$\left\{ \begin{array}{l} \mathbf{initialization:} \\ M \leftarrow ((a, 0), \dots, (a, 0)) \in (\mathbb{R}^2)^n, \\ \text{for any } k \in \llbracket 1, n \rrbracket, V_k \leftarrow \left( \cos\left(\frac{2k\pi}{n}\right), \sin\left(\frac{2k\pi}{n}\right) \right), \\ \mathbf{iterative step:} \\ \widetilde{M} \leftarrow M + \delta_t V, \\ \text{for any } k \in \llbracket 1, n \rrbracket, \left\{ \begin{array}{l} \text{if } \widetilde{M}_k \in \mathbb{D} \text{ then } M_k \leftarrow \widetilde{M}_k, \\ \text{else} \\ \text{compute } \delta_t^k \text{ s.t. } \|M_k + \delta_t^k V_k\| = 1 \text{ and } \delta_t^k \geq 0, \\ M_k \leftarrow M_k + \delta_t^k V_k, \\ V_k \leftarrow V_k - 2 \left\langle V_k \mid \frac{M_k}{\|M_k\|} \right\rangle \frac{M_k}{\|M_k\|}, \\ M_k \leftarrow M_k + (\delta_t - \delta_t^k) V_k. \end{array} \right. \end{array} \right.$$

The iterative loop consists in the computation of a linear motion outside the boundary and at the boundary one applies the familiar law *the angle of incidence equals the angle of reflection*. After  $p$  iterations,  $M$  represents the points of the wave front (see Figure 1 and *Videos*<sup>1</sup>).

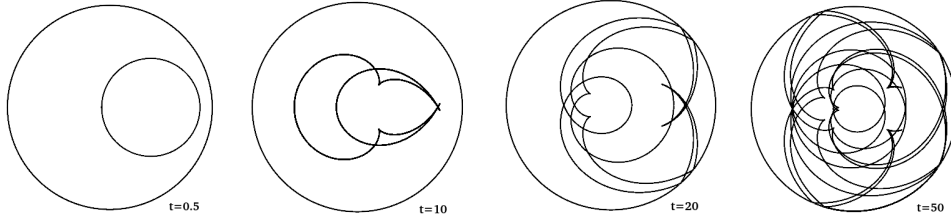


Figure 1: Wave Front for  $a = 0.5$  and  $t \in \{0.5, 10, 20, 50\}$

First, we can observe that  $|S_t|$  admits a linear asymptotic as  $t$  grows to  $+\infty$ . Then, the oscillations are of period 2 with a phase independent of  $a$  (see Figure 2). One may remark the following points.

1. For  $a = 0$ , the family of curves  $(S_t)_t$  are concentric circles and  $|S_t|$  is of period 2.
2. For  $a = 0.5$ , the terms  $J_k^{\text{approx}}(t)$  vanish for any  $t$  and, in this case, this expansion is not able to capture the oscillating part of  $t \mapsto |S_t|$ .

<sup>1</sup><https://www.youtube.com/channel/UCMTvpxuhYwbYBYDERs1UOEA/>

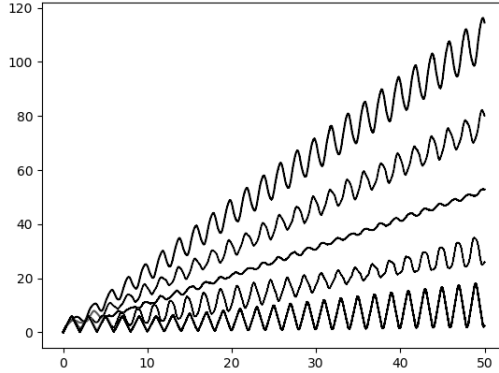


Figure 2: Length of the wave front  $|S_t|$  for  $t \in [0, 50]$  and for different values of  $a$ , respectively (from bottom to top), for  $a = 0.1$ ,  $a = 0.3$ ,  $a = 0.5$ ,  $a = 0.7$  and  $a = 0.9$

3. The terms  $|J_k^{\text{approx}}(t)|$  are bounded by  $Ck^{-5/2}t^{-1/2}$ , where  $C$  is a constant. For  $t$  fixed, this ensures the (fast) convergence of the series  $\sum_k J_k^{\text{approx}}(t)$  and then the amplitude of  $t \mapsto t \sum_{k \in \mathbb{N}} J_k^{\text{approx}}(t)$  is of order  $t^{1/2}$  (see Figure 2).

### 3 A short proof of the Arcsinus formula

In the paper [Vi-20], the author was able to prove by elementary calculations the

**Theorem 3.1** *If  $X$  is the unit disk,  $|S_t| = \lambda(q)t + o(t)$  as  $t \rightarrow +\infty$  with radius 1 with  $\lambda(q) = 2 \arcsin a$  where  $a$  is the distance from  $q$  to the center of the disk.*

We will reprove it using tools which will be extended to integrable geodesic flows in a forthcoming paper. For this, we will prove an integral formula:

**Theorem 3.2** *Let  $\psi$  be the function periodic of period 1 whose restriction to  $[0, 1]$  is given by  $\psi(\theta) = |2\theta - 1|$ . We have*

$$|S_t| = t \Sigma_{\pm} \int_{I_{\alpha_0}} \psi \left( \theta_2^{\pm, q}(\xi) - \frac{t}{2 \sin \xi} \right) d\xi + O(1) \quad (2)$$

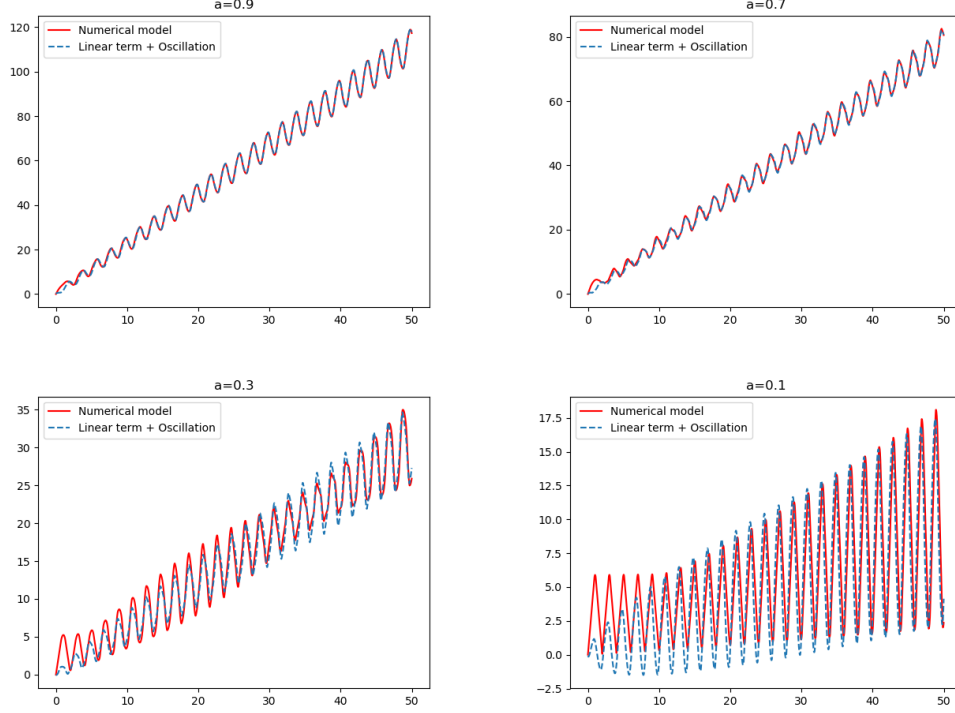


Figure 3: Comparison between  $|S_t|$  and  $2 \arcsin(a)t + t \sum_{k=0}^N J_k^{\text{approx}}(t)$  for  $N = 10$ ,  $t \in [0, 50]$  and  $a \in \{0.1, 0.3, 0.7, 0.9\}$ .

as  $t \rightarrow +\infty$ , where  $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$ ,  $\alpha_0 = \arcsin a$  and

$$\theta_2^{\pm, q}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

This integral can also be written as an integral over  $\mathbb{T}$ :

$$|S_t| = t \int_{\mathbb{T}} \psi \left( \frac{1}{2} - \frac{a \cos \alpha + t}{2\sqrt{1 - a^2 \sin^2 \alpha}} \right) \frac{a \cos \alpha}{\sqrt{1 - a^2 \sin^2 \alpha}} d\alpha + O(1) \quad (3)$$

as  $t \rightarrow +\infty$ .

Let us show how Theorem 3.1 follows from Theorem 3.2. We consider an integral

$$I(t) = \int_{I_{\alpha_0}} \psi \left( \theta(\xi) - \frac{t}{2 \sin \xi} \right) d\xi$$

with  $\theta$  smooth. We first approximate uniformly  $\psi$  by a sequence of trigonometric polynomials  $\psi_N(u) = \sum_{|n| \leq N} a_n \exp(2i\pi n u)$  with  $a_0 = \int_0^1 \psi(\theta) d\theta = \frac{1}{2}$ . This way we get

$$I_N(t) = 2\alpha_0 + \sum_{|n| \leq N, n \neq 0} a_n \int_{I_{\alpha_0}} e^{2i\pi n \theta(\xi)} e^{-2i\pi n t / \sin \xi} d\xi$$

It follows from the stationary phase approximations that all these integrals tend to 0 as  $t \rightarrow \infty$ , Theorem 3.1 follows.

*Proof of Theorem 3.2.*–

We will first parametrize the dynamics using angle coordinates on tori. Let us denote by  $m(s) = (\cos s, \sin s)$  on the circle and by  $\vec{u}_s$  the vector  $\overrightarrow{0m(s)}$ . Let us introduce a set of coordinates. In what follows, we parametrize the 2D-submanifold of the phase space consisting of oriented chords joining a point  $m(s)$  to  $m(s + 2\xi)$  with speed 1 by  $\xi \in ]0, \pi[$ . Changing the orientation of the chords moves  $\xi$  into  $\pi - \xi$ . For  $\xi \in ]0, \pi[$  and  $r \in [0, 2 \sin \xi]$ , we define  $F_\xi(s, r) = m(s) + r\vec{u}_{s+\xi+\pi/2}$ . This describes the chord  $C_\xi$  between  $m(s)$  and  $m(s + 2\xi)$ . The function  $F_\xi$  is extended as a function on  $\mathbb{R}^2$  periodic with respect to the lattice  $L_\xi$  spanned by the vectors  $(2\pi, 0)$  and  $(2\xi, -2 \sin \xi)$ . The function  $F_\xi$  is continuous, but only piecewise smooth. The pull-back under  $F_\xi$  on  $\mathbb{R}^2$  of the billiard dynamics is generated by the vector  $\partial_r$ .

The coordinates  $(s, r)$  range over a torus  $\mathbb{R}^2/L_\xi$ . In order to continue the computation, we need to fix the lattice  $\mathbb{Z}^2$ . For that we introduce the linear map  $M_\xi : \mathbb{R}_{\theta_1, \theta_2}^2 \rightarrow \mathbb{R}_{s, r}^2$  sending the canonical basis of  $\mathbb{Z}^2$  onto the previous basis of  $L_\xi$ . The dynamics on the torus  $\mathbb{R}^2/L_\xi$  is the image of  $\partial_r$  under  $M_\xi^{-1}$ ; let us denote it by  $V$ . We get

$$V = \frac{1}{2\pi \sin \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2})$$

Then, we need to compute the Euclidean norm of  $F'_\xi(M_\xi(\partial_\xi V))$ . We have

$$\partial_\xi V = \frac{-\cos \xi}{2\pi \sin^2 \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2}) + \frac{1}{2\pi \sin \xi} \partial_{\theta_1}$$

Hence

$$M_\xi(\partial_\xi V) = \frac{1}{\sin \xi} (-\cos \xi \partial_r + \partial_s)$$

Then

$$F'_\xi(\partial_r) = \vec{u}_{s+\xi+\pi/2}, \quad F'_\xi(\partial_s) = \vec{u}_{s+\pi/2} - r\vec{u}_{s+\xi}$$

This gives

$$\|\partial_\xi V\| = \frac{|r - \sin \xi|}{\sin \xi}$$

As could have been anticipated, this length vanishes on the caustic! We now take the pull back of  $\|\partial_\xi V\|$  under  $M_\xi$  and get  $|2\theta_2 - 1|$ .

Let us parametrize the chords starting from  $q$  by the angle  $\alpha \in \mathbb{T}$  defined by  $\alpha := \langle q, C_\xi \rangle$ . We get  $\cos \xi = a \sin \alpha$ . Hence  $\xi$  is the smooth function  $\xi(\alpha) = \arccos(a \sin \alpha)$ . The length  $|S_t|$  is given by

$$|S_t| = \int_{\mathbb{T}} \left\| \frac{d}{d\alpha} (\phi_t(\vec{u}_\alpha)) \right\| d\alpha$$

where  $\phi_t$  is the geodesic flow. Let us denote by  $\theta(\alpha)$  the coordinates of  $q$  in  $\mathbb{T}_\theta^2$ . We get, using the parametrization of the flow on the tori  $\mathbb{T}_\theta$ ,

$$\begin{aligned} |S_t| &= \int_{\mathbb{T}} \left\| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)} (\theta'(\alpha) + tV'(\alpha)) \right\| d\alpha, \\ &= t \int_{\mathbb{T}} \left\| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)} (V'(\alpha)) \right\| d\alpha + O(1) \end{aligned}$$

as  $t \rightarrow +\infty$ . We rewrite the integral in terms of  $\xi$ , using  $\cos \xi = a \sin \alpha$  and  $\theta_2(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$  with  $+$  if  $\alpha \in [\pi/2, 3\pi/2]$  and  $-$  otherwise. From this follows the result.

## 4 Local asymptotics of the length

In this section, we describe the asymptotics of the length of the intersection of the wave front with a smooth domain  $K$  included in the disk  $\mathbb{D}$ . We have

**Theorem 4.1** *We have*

$$l(S_t \cap K) \sim \frac{2t}{\pi} \int_K \Psi(\sqrt{x^2 + y^2}) |dxdy|$$

as  $t \rightarrow +\infty$ , where

$$\Psi(r) = \frac{\min(r, a)}{\sqrt{1 - \min(r, a)^2}}$$



Note that the function  $\Psi$  is continuous, vanishes at  $r = 0$  and is constant for  $a \leq r \leq 1$ . This implies that the density of the wave front is smaller near the center of the disk.

*Proof.*— Let  $\phi \in C(\mathbb{D}, \mathbb{R}^+)$ , we want to calculate the asymptotics of the length  $|S_{t,\phi}|$  of  $S_t$  computed in the metric  $\phi^2 \text{Eucl}$ . Following the proof of Theorem 3.1, we get  $|S_{t,\phi}|/t \rightarrow \lambda(q, \phi)$  as  $t \rightarrow +\infty$ , with

$$\lambda(q, \phi) = 2 \int_{I_{\alpha_0}} \int_{\mathbb{T}^2} |2\theta_2 - 1| \phi \circ G(\theta, \xi) |d\xi d\theta|$$

with  $G(\theta, \xi) = F_\xi \circ M_\xi(\theta)$ . We will first make the change of variable  $(\theta, \xi) \rightarrow (s, r, \xi)$  whose Jacobian is  $4\pi \sin \xi$ . This gives

$$\lambda(q, \phi) = \frac{1}{2\pi} \int_{I_{\alpha_0}} \int_{\mathbb{R}^2/L_\xi} \left| \frac{r - \sin \xi}{\sin^2 \xi} \right| \phi \circ F_\xi(s, r) |d\xi ds dr|$$

Finally, we pass from  $(s, r)$  to  $(x, y)$ . We have  $|dxdy| = |r - \sin \xi| |dsdr|$ . The domain of integration is  $\rho = \sqrt{x^2 + y^2} \geq \cos \xi$  which is covered twice by the torus  $\mathbb{R}^2/L_\xi$ , we get hence

$$\lambda(q, \phi) = \frac{1}{\pi} \int_{I_{\alpha_0}} \int_{\cos \xi \leq \rho} \left| \frac{1}{\sin^2 \xi} \right| \phi(x, y) |d\xi dxdy|$$

An elementary calculus gives then

$$\lambda(q, \phi) = \frac{2}{\pi} \int_{\mathbb{D}} \Psi(\rho) \phi(x, y) |dxdy|$$

The result follows then by approximating the characteristic function of  $K$  by continuous functions.  $\square$

## 5 Oscillations of the length

The numerical computations of the second author in [Vi-20] show clearly some regular oscillations of the length  $|S_t|$  around the linear asymptotics. These oscillations are given in the

**Theorem 5.1** *The following expansion holds:*

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + O(1)$$

as  $t \rightarrow +\infty$ , with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The oscillations have an amplitude of the order of  $\sqrt{t}$ , are periodic of period 2. If  $a = \frac{1}{2}$ , we get  $|S_t| = \pi t/3 + O(1)$  as  $t \rightarrow +\infty$ .

We start from the formula given by Equation (2):

$$|S_t| = t\Sigma_{\pm} \int_{I_{\alpha_0}} \psi\left(\theta_2^{\pm,q}(\xi) - \frac{t}{2\sin\xi}\right) d\xi + O(1)$$

as  $t \rightarrow +\infty$ , where  $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$  and  $\psi$  restricted to  $[0, 1]$  is given by  $\psi(\theta) = |2\theta - 1|$  and  $\psi$  is periodic of period 1. We have

$$\theta_2^{\pm,q}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2\xi}}{2\sin\xi}$$

The idea is to start with the Fourier expansion of  $\psi$  and then to apply the stationary phase asymptotics.

We have

$$\psi(\theta) = \frac{1}{2} + \sum_{n \in \mathbb{Z}} \frac{2}{\pi^2(2n+1)^2} e^{2(2n+1)i\pi\theta}$$

We need to evaluate the integrals

$$I_n(t) = \frac{-4}{((2n+1)\pi)^2} \int_{I_{\alpha_0}} \cos\left((2n+1)\pi \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}\right) e^{-i\pi(2n+1)\frac{t}{\sin\xi}} d\xi$$

and then we have

$$|S_t| = 2\alpha_0 t + t \sum_{n \in \mathbb{Z}} I_n(t) + O(1)$$

as  $t \rightarrow +\infty$ . Note first that the function  $\cos\left((2n+1)\frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}\right)$  is smooth on  $I_{\alpha_0}$  with a non vanishing derivative at the boundaries. The non vanishing contributions come from the critical point  $\xi = \pi/2$  and the boundaries of  $I_{\alpha_0}$ . The boundary contributions are  $O(1/t)$ . They contribute to the  $O(1)$  remainder. The contribution of the critical point can be calculated using the

formula (4). We get an asymptotic for  $J_n = I_n + I_{-n-1}$ ,  $n = 0, 1, \dots$  given by

$$J_n(t) \sim J_n^{\text{approx}} = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The previous calculation is only formal. We need to control the remainder terms in a uniform way with respect to  $n$ . Let us rewrite the integral  $I_n$  as combination of integrals of the form

$$\int_{I_{\alpha_0}} e^{-i\pi(2n+1)t\left(\frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}\right)} d\xi$$

and apply the stationary phase with the phase functions depending on  $t$ :  $\Phi_t(\xi) = \frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}$ . This phase function is non degenerate and converges in  $C^\infty$  topology to  $\frac{1}{\sin\xi}$  as  $t \rightarrow \infty$ . Hence the remainder is  $O((nt)^{-3/2})$  as  $t \rightarrow +\infty$ , uniformly with respect to  $n$ .

## A Stationary phase

For this section, we refer the reader to [GS-77], chap. 1.

We want to evaluate the asymptotics as  $t \rightarrow +\infty$  of integrals of the form

$$I(t) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

where  $S$  is a real valued smooth function. We assume that the critical points of  $S$ , ie the zeroes of  $S'$ , are non degenerate, ie  $S''(x) \neq 0$ . We will first assume that  $a \in C_o^\infty(\mathbb{R})$  with only one critical point  $x = 0$  in the support of  $a$ . Then  $I(t)$  admits a full asymptotic expansion given by

$$I(t) = \frac{\sqrt{2\pi} e^{i\varepsilon\pi/4}}{|tS''(0)|^{\frac{1}{2}}} e^{itS(0)} (a(0) + O(t)) \quad (4)$$

as  $t \rightarrow +\infty$ , with  $\varepsilon = \pm 1$  depending on the sign of  $S''(0)$ . We will need some uniform estimates in the remainder term. This is provided by the following

**Proposition A.1** *Let us consider the integrals*

$$I(t; S, a) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

Let  $S_0$  be a smooth real valued Morse function and  $a_0$  be a smooth function. Let  $S_\lambda$  and  $a_\lambda$  be smoothly dependent of a real parameter  $\lambda$ . Then, for  $\lambda$  small enough,

$$I(t; S_\lambda, a_\lambda) := I_{\text{asymp}}(t, \lambda) + O(t^{-3/2})$$

as  $t \rightarrow +\infty$ , where the  $O$  is uniform and  $I_{\text{asymp}}(t, \lambda)$  is the sum of terms given by the formula (4) for all critical points of  $S_\lambda$ .

If  $\lambda$  is small enough,  $S_\lambda$  is still a Morse function. We localize the integrals near the critical points and apply the Morse Lemma with parameters. We are then reduced locally to the case where  $S_\lambda(x) = \pm x^2$ . We apply then any proof of the stationary phase approximation.

It will also be useful to consider the case of an integral on a closed interval  $[c, d]$  with  $c < d$ .

$$I(t) := \int_c^d e^{itS(x)} a(x) dx$$

Assuming that  $S'$  does not vanish on the support of  $a$  and that  $a$  is  $C^1$ , we have

$$I(t) = \frac{1}{it} \left( \frac{a(d)e^{itS(d)}}{S'(d)} - \frac{a(c)e^{itS(c)}}{S'(c)} + O(t) \right) \quad (5)$$

as  $t \rightarrow +\infty$ .

Note that in both asymptotic formulae, the remainders “ $O(t)$ ” are uniform if  $S'$  (resp.  $a'$ ) is close to  $S$  (resp. close to  $a$ ) in  $C^2$  topology.

The previous asymptotics extend to higher dimensional integrals.

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