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Large-time asymptotics of the wave fronts length I The Euclidean disk

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In the paper [Vi-20], the second author proves that the length $|S_t|$ of the wave front S_t at time t of a wave propagating in an Euclidean disk \mathbb{D} of radius 1, starting from a source q , admits a linear asymptotics as $t \rightarrow +\infty$: $|S_t| = \lambda(q)t + o(t)$ with $\lambda(q) = 2 \arcsin a$ and $a = d(0, q)$. We will give a more direct proof and compute the oscillating corrections to this linear asymptotics. The proof is based on the “stationary phase” approximation.

1 Wave fronts

Let us consider a 2D-Riemannian compact manifold (X, g) possibly with a smooth convex boundary. We denote by $g^* : T^*X \rightarrow \mathbb{R}$ the half of the dual metric which is the Hamiltonian of the geodesic flow.

We denote by π_X the canonical projection of T^*X onto X and $\phi_t : T^*X \rightarrow T^*X$, $t \in \mathbb{R}$ the Hamiltonian flow of g^* which is the geodesic flow. If X has a non empty boundary, we define ϕ_t using the law of reflection. Let $q \in X$ be given. For any $t > 0$, we define the wave front S_t at time t as the set of

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points of X of the form $\pi_X(\phi_t(\Sigma^q))$ where $\Sigma^q := \{(q, \xi) \in T^*X | g^*(q, \xi) = 1\}$. The set S_t could also be defined as the image by the exponential map at q of the circle Σ_t of radius t in the tangent space T_qX .

Let us define the length of S_t and denote it by $|S_t|$. The wave front S_t is a curve parametrized by a circle: $S_t := \exp_q(\Sigma_t)$. This allows to define its length using the Riemannian metric. Note that S_t can admit some singular points. The length of the corresponding part vanishes and the remaining part is an immersed co-oriented curve with only transversal self-intersections.

In this article, we focus on the case where X is the unit disk in \mathbb{R}^2 and g is the Euclidean metric. In this context, we will prove that the following expansion holds:

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + \mathcal{O}(1) \quad (1)$$

as $t \rightarrow +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

where a is the distance from the point q to the center of the disk.

The case of closed surfaces with integrable geodesic flows will be the subject of [CV-20].

2 Numerics

In this section, we will compare the expansion given by (1) with the numerical calculations. We introduce a (small) time step $\delta_t > 0$, a (large) number of points n which compose the wave front, two vectors M and V in $(\mathbb{R}^2)^n$ such that, for any $k \in \llbracket 1, n \rrbracket$, $X_k \in \mathbb{R}^2$ represents the position and $V_k \in \mathbb{R}^2$ the speed of the k th point of the wave front at a given time. We fix $a \in]0, 1[$ such that $(a, 0)$ are the coordinates of the source q . Thus, we introduce the

following iterative scheme

$$\left\{ \begin{array}{l} \textbf{initialization:} \\ M \leftarrow ((a, 0), \dots, (a, 0)) \in (\mathbb{R}^2)^n, \\ \text{for any } k \in \llbracket 1, n \rrbracket, V_k \leftarrow \left(\cos \left(\frac{2k\pi}{n} \right), \sin \left(\frac{2k\pi}{n} \right) \right), \\ \textbf{iterative step:} \\ \widetilde{M} \leftarrow M + \delta_t V, \\ \text{for any } k \in \llbracket 1, n \rrbracket, \left\{ \begin{array}{l} \text{if } \widetilde{M}_k \in \mathbb{D} \text{ then } M_k \leftarrow \widetilde{M}_k, \\ \text{else} \\ \text{compute } \delta_t^k \text{ s.t. } \|M_k + \delta_t^k V_k\| = 1 \text{ and } \delta_t^k \geq 0, \\ M_k \leftarrow M_k + \delta_t^k V_k, \\ V_k \leftarrow V_k - 2 \left\langle V_k \mid \frac{M_k}{\|M_k\|} \right\rangle \frac{M_k}{\|M_k\|}, \\ M_k \leftarrow M_k + (\delta_t - \delta_t^k) V_k. \end{array} \right. \end{array} \right.$$

The iterative loop consists in the computation of a linear motion outside the boundary and at the boundary one applies the familiar law *the angle of incidence equals the angle of reflection*. After p iterations, M represents the points of the wave front (see Figure 1 and *Videos*¹).

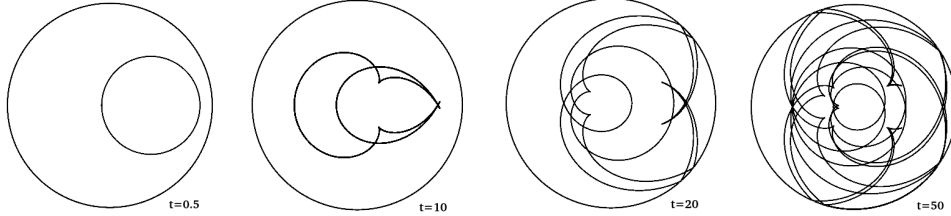


Figure 1: Wave Front for $a = 0.5$ and $t \in \{0.5, 10, 20, 50\}$

First, we can observe that $|S_t|$ admits a linear asymptotic as t grows to $+\infty$. Then, the oscillations are of period 2 with a phase independent of a (see Figure 2). One may remark the following points.

1. For $a = 0$, the family of curves $(S_t)_t$ are concentric circles and $|S_t|$ is of period 2.
2. For $a = 0.5$, the terms $J_k^{\text{approx}}(t)$ vanish for any t and, in this case, this expansion is not able to capture the oscillating part of $t \mapsto |S_t|$.

¹<https://www.youtube.com/channel/UCMTvpxuhYwbYBYDERs1UOEA/>

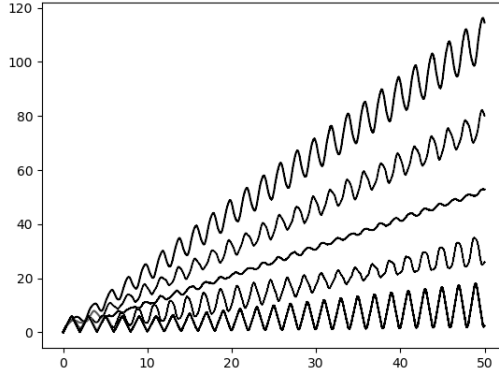


Figure 2: Length of the wave front $|S_t|$ for $t \in [0, 50]$ and for different values of a , respectively (from bottom to top), for $a = 0.1$, $a = 0.3$, $a = 0.5$, $a = 0.7$ and $a = 0.9$

3. The terms $|J_k^{\text{approx}}(t)|$ are bounded by $Ck^{-5/2}t^{-1/2}$, where C is a constant. For t fixed, this ensures the (fast) convergence of the series $\sum_k J_k^{\text{approx}}(t)$ and then the amplitude of $t \mapsto t \sum_{k \in \mathbb{N}} J_k^{\text{approx}}(t)$ is of order $t^{1/2}$ (see Figure 2).

3 A short proof of the Arcsinus formula

In the paper [Vi-20], the author was able to prove by elementary calculations the

Theorem 3.1 *If X is the unit disk, $|S_t| = \lambda(q)t + o(t)$ as $t \rightarrow +\infty$ with radius 1 with $\lambda(q) = 2 \arcsin a$ where a is the distance from q to the center of the disk.*

We will reprove it using tools which will be extended to integrable geodesic flows in a forthcoming paper. For this, we will prove an integral formula:

Theorem 3.2 *Let ψ be the function periodic of period 1 whose restriction to $[0, 1]$ is given by $\psi(\theta) = |2\theta - 1|$. We have*

$$|S_t| = t \sum_{\pm} \int_{I_{\alpha_0}} \psi \left(\theta_2^{\pm, q}(\xi) - \frac{t}{2 \sin \xi} \right) d\xi + O(1) \quad (2)$$

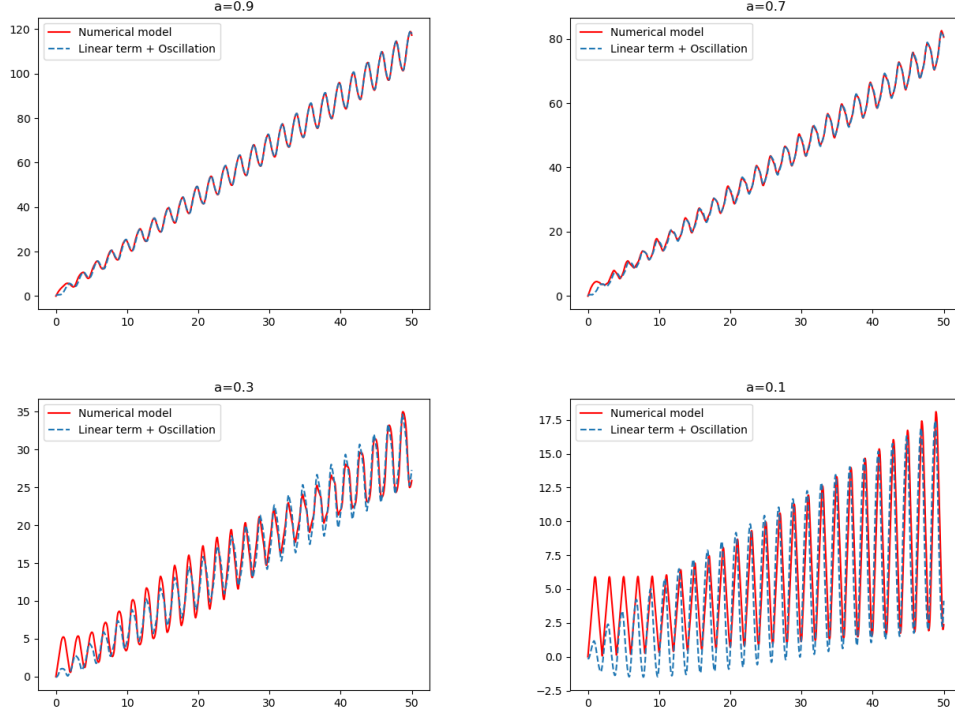


Figure 3: Comparison between $|S_t|$ and $2 \arcsin(a)t + t \sum_{k=0}^N J_k^{\text{approx}}(t)$ for $N = 10$, $t \in [0, 50]$ and $a \in \{0.1, 0.3, 0.7, 0.9\}$.

as $t \rightarrow +\infty$, where $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$, $\alpha_0 = \arcsin a$ and

$$\theta_2^{\pm, a}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

This integral can also be written as an integral over \mathbb{T} :

$$|S_t| = t \int_{\mathbb{T}} \psi \left(\frac{1}{2} - \frac{a \cos \alpha + t}{2\sqrt{1 - a^2 \sin^2 \alpha}} \right) \frac{a \cos \alpha}{\sqrt{1 - a^2 \sin^2 \alpha}} d\alpha + O(1) \quad (3)$$

as $t \rightarrow +\infty$.

Let us show how Theorem 3.1 follows from Theorem 3.2. We consider an integral

$$I(t) = \int_{I_{\alpha_0}} \psi \left(\theta(\xi) - \frac{t}{2 \sin \xi} \right) d\xi$$

with θ smooth. We first approximate uniformly ψ by a sequence of trigonometric polynomials $\psi_N(u) = \sum_{|n| \leq N} a_n \exp(2i\pi n u)$ with $a_0 = \int_0^1 \psi(\theta) d\theta = \frac{1}{2}$. This way we get

$$I_N(t) = 2\alpha_0 + \sum_{|n| \leq N, n \neq 0} a_n \int_{I_{\alpha_0}} e^{2i\pi n \theta(\xi)} e^{-2i\pi n t / \sin \xi} d\xi$$

It follows from the stationary phase approximations that all these integrals tend to 0 as $t \rightarrow \infty$, Theorem 3.1 follows.

Proof of Theorem 3.2.–

We will first parametrize the dynamics using angle coordinates on tori. Let us denote by $m(s) = (\cos s, \sin s)$ on the circle and by \vec{u}_s the vector $\overrightarrow{0m(s)}$. Let us introduce a set of coordinates. In what follows, we parametrize the 2D-submanifold of the phase space consisting of oriented chords joining a point $m(s)$ to $m(s + 2\xi)$ with speed 1 by $\xi \in]0, \pi[$. Changing the orientation of the chords moves ξ into $\pi - \xi$. For $\xi \in]0, \pi[$ and $r \in [0, 2 \sin \xi]$, we define $F_\xi(s, r) = m(s) + r\vec{u}_{s+\xi+\pi/2}$. This describes the chord C_ξ between $m(s)$ and $m(s + 2\xi)$. The function F_ξ is extended as a function on \mathbb{R}^2 periodic with respect to the lattice L_ξ spanned by the vectors $(2\pi, 0)$ and $(2\xi, -2 \sin \xi)$. The function F_ξ is continuous, but only piecewise smooth. The pull-back under F_ξ on \mathbb{R}^2 of the billiard dynamics is generated by the vector ∂_r .

The coordinates (s, r) range over a torus \mathbb{R}^2/L_ξ . In order to continue the computation, we need to fix the lattice \mathbb{Z}^2 . For that we introduce the linear map $M_\xi : \mathbb{R}_{\theta_1, \theta_2}^2 \rightarrow \mathbb{R}_{s, r}^2$ sending the canonical basis of \mathbb{Z}^2 onto the previous basis of L_ξ . The dynamics on the torus \mathbb{R}^2/L_ξ is the image of ∂_r under M_ξ^{-1} ; let us denote it by V . We get

$$V = \frac{1}{2\pi \sin \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2})$$

Then, we need to compute the Euclidean norm of $F'_\xi(M_\xi(\partial_\xi V))$. We have

$$\partial_\xi V = \frac{-\cos \xi}{2\pi \sin^2 \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2}) + \frac{1}{2\pi \sin \xi} \partial_{\theta_1}$$

Hence

$$M_\xi(\partial_\xi V) = \frac{1}{\sin \xi} (-\cos \xi \partial_r + \partial_s)$$

Then

$$F'_\xi(\partial_r) = \vec{u}_{s+\xi+\pi/2}, \quad F'_\xi(\partial_s) = \vec{u}_{s+\pi/2} - r\vec{u}_{s+\xi}$$

This gives

$$\|\partial_\xi V\| = \frac{|r - \sin \xi|}{\sin \xi}$$

As could have been anticipated, this length vanishes on the caustic! We now take the pull back of $\|\partial_\xi V\|$ under M_ξ and get $|2\theta_2 - 1|$.

Let us parametrize the chords starting from q by the angle $\alpha \in \mathbb{T}$ defined by $\alpha := \langle q, C_\xi \rangle$. We get $\cos \xi = a \sin \alpha$. Hence ξ is the smooth function $\xi(\alpha) = \arccos(a \sin \alpha)$. The length $|S_t|$ is given by

$$|S_t| = \int_{\mathbb{T}} \left\| \frac{d}{d\alpha} (\phi_t(\vec{u}_\alpha)) \right\| d\alpha$$

where ϕ_t is the geodesic flow. Let us denote by $\theta(\alpha)$ the coordinates of q in \mathbb{T}_θ^2 . We get, using the parametrization of the flow on the tori \mathbb{T}_θ ,

$$\begin{aligned} |S_t| &= \int_{\mathbb{T}} \left\| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)} (\theta'(\alpha) + tV'(\alpha)) \right\| d\alpha, \\ &= t \int_{\mathbb{T}} \left\| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)} (V'(\alpha)) \right\| d\alpha + O(1) \end{aligned}$$

as $t \rightarrow +\infty$. We rewrite the integral in terms of ξ , using $\cos \xi = a \sin \alpha$ and $\theta_2(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$ with $+$ if $\alpha \in [\pi/2, 3\pi/2]$ and $-$ otherwise. From this follows the result.

4 Local asymptotics of the length

In this section, we describe the asymptotics of the length of the intersection of the wave front with a smooth domain K included in the disk \mathbb{D} . We have

Theorem 4.1 *We have*

$$l(S_t \cap K) \sim \frac{2t}{\pi} \int_K \Psi(\sqrt{x^2 + y^2}) |dxdy|$$

as $t \rightarrow +\infty$, where

$$\Psi(r) = \frac{\min(r, a)}{\sqrt{1 - \min(r, a)^2}}$$

Note that the function Ψ is continuous, vanishes at $r = 0$ and is constant for $a \leq r \leq 1$. This implies that the density of the wave front is smaller near the center of the disk.

Proof.— Let $\phi \in C(\mathbb{D}, \mathbb{R}^+)$, we want to calculate the asymptotics of the length $|S_{t,\phi}|$ of S_t computed in the metric $\phi^2 \text{Eucl}$. Following the proof of Theorem 3.1, we get $|S_{t,\phi}|/t \rightarrow \lambda(q, \phi)$ as $t \rightarrow +\infty$, with

$$\lambda(q, \phi) = 2 \int_{I_{\alpha_0}} \int_{\mathbb{T}^2} |2\theta_2 - 1| \phi \circ G(\theta, \xi) |d\xi d\theta|$$

with $G(\theta, \xi) = F_\xi \circ M_\xi(\theta)$. We will first make the change of variable $(\theta, \xi) \rightarrow (s, r, \xi)$ whose Jacobian is $4\pi \sin \xi$. This gives

$$\lambda(q, \phi) = \frac{1}{2\pi} \int_{I_{\alpha_0}} \int_{\mathbb{R}^2/L_\xi} \left| \frac{r - \sin \xi}{\sin^2 \xi} \right| \phi \circ F_\xi(s, r) |d\xi ds dr|$$

Finally, we pass from (s, r) to (x, y) . We have $|dxdy| = |r - \sin \xi| |dsdr|$. The domain of integration is $\rho = \sqrt{x^2 + y^2} \geq \cos \xi$ which is covered twice by the torus \mathbb{R}^2/L_ξ , we get hence

$$\lambda(q, \phi) = \frac{1}{\pi} \int_{I_{\alpha_0}} \int_{\cos \xi \leq \rho} \left| \frac{1}{\sin^2 \xi} \right| \phi(x, y) |d\xi dxdy|$$

An elementary calculus gives then

$$\lambda(q, \phi) = \frac{2}{\pi} \int_{\mathbb{D}} \Psi(\rho) \phi(x, y) |dxdy|$$

The result follows then by approximating the characteristic function of K by continuous functions. \square

5 Oscillations of the length

The numerical computations of the second author in [Vi-20] show clearly some regular oscillations of the length $|S_t|$ around the linear asymptotics. These oscillations are given in the

Theorem 5.1 *The following expansion holds:*

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + O(1)$$

as $t \rightarrow +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The oscillations have an amplitude of the order of \sqrt{t} , are periodic of period 2. If $a = \frac{1}{2}$, we get $|S_t| = \pi t/3 + O(1)$ as $t \rightarrow +\infty$.

We start from the formula given by Equation (2):

$$|S_t| = t \Sigma_{\pm} \int_{I_{\alpha_0}} \psi\left(\theta_2^{\pm,q}(\xi) - \frac{t}{2 \sin \xi}\right) d\xi + O(1)$$

as $t \rightarrow +\infty$, where $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$ and ψ restricted to $[0, 1]$ is given by $\psi(\theta) = |2\theta - 1|$ and ψ is periodic of period 1. We have

$$\theta_2^{\pm,q}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

The idea is to start with the Fourier expansion of ψ and then to apply the stationary phase asymptotics.

We have

$$\psi(\theta) = \frac{1}{2} + \sum_{n \in \mathbb{Z}} \frac{2}{\pi^2(2n+1)^2} e^{2(2n+1)i\pi\theta}$$

We need to evaluate the integrals

$$I_n(t) = \frac{-4}{((2n+1)\pi)^2} \int_{I_{\alpha_0}} \cos\left((2n+1)\pi \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi}\right) e^{-i\pi(2n+1)\frac{t}{\sin \xi}} d\xi$$

and then we have

$$|S_t| = 2\alpha_0 t + t \sum_{n \in \mathbb{Z}} I_n(t) + O(1)$$

as $t \rightarrow +\infty$. Note first that the function $\cos\left((2n+1)\frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi}\right)$ is smooth on I_{α_0} with a non vanishing derivative at the boundaries. The non vanishing contributions come from the critical point $\xi = \pi/2$ and the boundaries of I_{α_0} . The boundary contributions are $O(1/t)$. They contribute to the $O(1)$ remainder. The contribution of the critical point can be calculated using the

formula (4). We get an asymptotic for $J_n = I_n + I_{-n-1}$, $n = 0, 1, \dots$ given by

$$J_n(t) \sim J_n^{\text{approx}} = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The previous calculation is only formal. We need to control the remainder terms in a uniform way with respect to n . Let us rewrite the integral I_n as combination of integrals of the form

$$\int_{I_{\alpha_0}} e^{-i\pi(2n+1)t\left(\frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}\right)} d\xi$$

and apply the stationary phase with the phase functions depending on t : $\Phi_t(\xi) = \frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}$. This phase function is non degenerate and converges in C^∞ topology to $\frac{1}{\sin\xi}$ as $t \rightarrow \infty$. Hence the remainder is $O((nt)^{-3/2})$ as $t \rightarrow +\infty$, uniformly with respect to n .

A Stationary phase

For this section, we refer the reader to [GS-77], chap. 1.

We want to evaluate the asymptotics as $t \rightarrow +\infty$ of integrals of the form

$$I(t) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

where S is a real valued smooth function. We assume that the critical points of S , ie the zeroes of S' , are non degenerate, ie $S''(x) \neq 0$. We will first assume that $a \in C_o^\infty(\mathbb{R})$ with only one critical point $x = 0$ in the support of a . Then $I(t)$ admits a full asymptotic expansion given by

$$I(t) = \frac{\sqrt{2\pi} e^{i\varepsilon\pi/4}}{|tS''(0)|^{\frac{1}{2}}} e^{itS(0)} (a(0) + O(t)) \quad (4)$$

as $t \rightarrow +\infty$, with $\varepsilon = \pm 1$ depending on the sign of $S''(0)$. We will need some uniform estimates in the remainder term. This is provided by the following

Proposition A.1 *Let us consider the integrals*

$$I(t; S, a) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

Let S_0 be a smooth real valued Morse function and a_0 be a smooth function. Let S_λ and a_λ be smoothly dependent of a real parameter λ . Then, for λ small enough,

$$I(t; S_\lambda, a_\lambda) := I_{\text{asymp}}(t, \lambda) + O(t^{-3/2})$$

as $t \rightarrow +\infty$, where the O is uniform and $I_{\text{asymp}}(t, \lambda)$ is the sum of terms given by the formula (4) for all critical points of S_λ .

If λ is small enough, S_λ is still a Morse function. We localize the integrals near the critical points and apply the Morse Lemma with parameters. We are then reduced locally to the case where $S_\lambda(x) = \pm x^2$. We apply then any proof of the stationary phase approximation.

It will also be useful to consider the case of an integral on a closed interval $[c, d]$ with $c < d$.

$$I(t) := \int_c^d e^{itS(x)} a(x) dx$$

Assuming that S' does not vanish on the support of a and that a is C^1 , we have

$$I(t) = \frac{1}{it} \left(\frac{a(d)e^{itS(d)}}{S'(d)} - \frac{a(c)e^{itS(c)}}{S'(c)} + O(t) \right) \quad (5)$$

as $t \rightarrow +\infty$.

Note that in both asymptotic formulae, the remainders “ $O(t)$ ” are uniform if S' (resp. a') is close to S (resp. close to a) in C^2 topology.

The previous asymptotics extend to higher dimensional integrals.

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