



HAL
open science

Large-time asymptotics of the wave fronts length I The Euclidean disk

Yves Colin de Verdière, David Vicente

► **To cite this version:**

Yves Colin de Verdière, David Vicente. Large-time asymptotics of the wave fronts length I
The Euclidean disk. 2020. hal-03009034v1

HAL Id: hal-03009034

<https://hal.science/hal-03009034v1>

Preprint submitted on 18 Nov 2020 (v1), last revised 27 Nov 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Large-time asymptotics of the wave fronts length I The Euclidean disk

Yves Colin de Verdière*

David Vicente, †

November 18, 2020

In the paper [Vi-20], the second author proves that the length $|S_t|$ of the wave front S_t at time t of a wave propagating in an Euclidean disk \mathbb{D} of radius 1, starting from a source q , admits a linear asymptotics as $t \rightarrow +\infty$: $|S_t| = \lambda(q)t + o(t)$ with $\lambda(q) = 2\arcsin a$ and $a = d(0, q)$. We will give a more direct proof and compute the oscillating corrections to this linear asymptotics. The proof is based on the “stationary phase” approximation.

1 Wave fronts

Let us consider a 2D-Riemannian compact manifold (X, g) possibly with a smooth convex boundary. We denote by $g^* : T^*X \rightarrow \mathbb{R}$ the half of the dual metric which is the Hamiltonian of the geodesic flow.

We denote by π_X the canonical projection of T^*X onto X and $\phi_t : T^*X \rightarrow T^*X$, $t \in \mathbb{R}$ the Hamiltonian flow of g^* which is the geodesic flow. If X has a non empty boundary, we define ϕ_t using the law of reflection. Let $q \in X$ be given. For any $t > 0$, we define the wave front S_t at time t as the set of

*Université Grenoble-Alpes, Institut Fourier, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402-Saint Martin d’Hères Cedex (France); yves.colin-de-verdiere@univ-grenoble-alpes.fr

†Université d’Orléans, Institut Denis Poisson, Unité mixte de recherche CNRS 7013, Rue de Chartres, B.P. 6759, 45067 Orléans Cedex 2 (France) david.vicente154@gmail.com

points of X of the form $\pi_X(\phi_t(\Sigma^q))$ where $\Sigma^q := \{(q, \xi) \in T^*X | g^*(q, \xi) = 1\}$. The set S_t could also be defined as the image by the exponential map at q of the circle Σ_t of radius t in the tangent space T_qX .

Let us define the length of S_t and denote it by $|S_t|$. The wave front S_t is a curve parametrized by a circle: $S_t := \exp_q(\Sigma_t)$. This allows to define its length using the Riemannian metric. Note that S_t can admit some singular points. The length of the corresponding part vanishes and the remaining part is an immersed co-oriented curve with only transversal self-intersections.

In this article, we focus on the case where X is the unit disk in \mathbb{R}^2 and g is the Euclidean metric. In this context, we will prove that the following expansion holds:

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + \mathcal{O}(1) \quad (1)$$

as $t \rightarrow +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

where a is the distance from the point q to the center of the disk.

The case of closed surfaces with integrable geodesic flows will be the subject of [CV-20].

2 Numerics

In this section, we will compare the expansion given by (1) with the numerical calculations. We introduce a (small) time step $\delta_t > 0$, a (large) number of points n which compose the wave front, two vectors M and V in $(\mathbb{R}^2)^n$ such that, for any $k \in \llbracket 1, n \rrbracket$, $X_k \in \mathbb{R}^2$ represents the position and $V_k \in \mathbb{R}^2$ the speed of the k th point of the wave front at a given time. We fix $a \in]0, 1[$ such that $(a, 0)$ are the coordinates of the source q . Thus, we introduce the

following iterative scheme

$$\left\{ \begin{array}{l} \mathbf{initialization:} \\ M \leftarrow ((a, 0), \dots, (a, 0)) \in (\mathbb{R}^2)^n, \\ \text{for any } k \in \llbracket 1, n \rrbracket, V_k \leftarrow \left(\cos\left(\frac{2k\pi}{n}\right), \sin\left(\frac{2k\pi}{n}\right) \right), \\ \mathbf{iterative step:} \\ \widetilde{M} \leftarrow M + \delta_t V, \\ \text{for any } k \in \llbracket 1, n \rrbracket, \left\{ \begin{array}{l} \text{if } \widetilde{M}_k \in \mathbb{D} \text{ then } M_k \leftarrow \widetilde{M}_k, \\ \text{else} \\ \text{compute } \delta_t^k \text{ s.t. } \|M_k + \delta_t^k V_k\| = 1 \text{ and } \delta_t^k \geq 0, \\ M_k \leftarrow M_k + \delta_t^k V_k, \\ V_k \leftarrow V_k - 2 \left\langle V_k \mid \frac{M_k}{\|M_k\|} \right\rangle \frac{M_k}{\|M_k\|}, \\ M_k \leftarrow M_k + (\delta_t - \delta_t^k) V_k. \end{array} \right. \end{array} \right.$$

The iterative loop consists in the computation of a linear motion outside the boundary and at the boundary one applies the familiar law *the angle of incidence equals the angle of reflection*. After p iterations, M represents the points of the wave front (see Figure 1 and *Videos*¹).

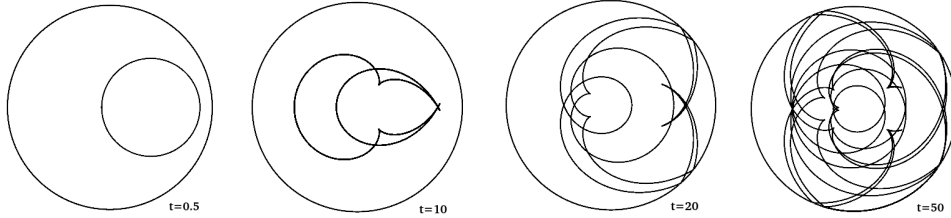


Figure 1: Wave Front for $a = 0.5$ and $t \in \{0.5, 10, 20, 50\}$

First, we can observe that $|S_t|$ admits a linear asymptotic as t grows to $+\infty$. Then, the oscillations are of period 2 with a phase independent of a (see Figure 2). One may remark the following points.

1. For $a = 0$, the family of curves $(S_t)_t$ are concentric circles and $|S_t|$ is of period 2.
2. For $a = 0.5$, the terms $J_k^{\text{approx}}(t)$ vanish for any t and, in this case, this expansion is not able to capture the oscillating part of $t \mapsto |S_t|$.

¹<https://www.youtube.com/channel/UCMTvpxuhYwbYBYDERs1UOEA/>

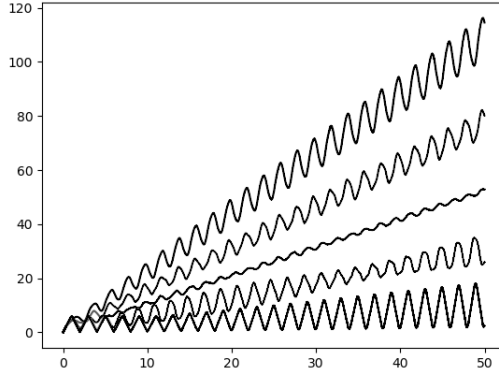


Figure 2: Length of the wave front $|S_t|$ for $t \in [0, 50]$ and for different values of a , respectively (from bottom to top), for $a = 0.1$, $a = 0.3$, $a = 0.5$, $a = 0.7$ and $a = 0.9$

3. The terms $|J_k^{\text{approx}}(t)|$ are bounded by $Ck^{-5/2}t^{-1/2}$, where C is a constant. For t fixed, this ensures the (fast) convergence of the series $\sum_k J_k^{\text{approx}}(t)$ and then the amplitude of $t \mapsto t \sum_{k \in \mathbb{N}} J_k^{\text{approx}}(t)$ is of order $t^{1/2}$ (see Figure 2).

3 A short proof of the Arcsinus formula

In the paper [Vi-20], the author was able to prove by elementary calculations the

Theorem 3.1 *If X is the unit disk, $|S_t| = \lambda(q)t + o(t)$ as $t \rightarrow +\infty$ with radius 1 with $\lambda(q) = 2 \arcsin a$ where a is the distance from q to the center of the disk.*

We will reprove it using tools which will be extended to integrable geodesic flows in a forthcoming paper. For this, we will prove an integral formula:

Theorem 3.2 *Let ψ be the function periodic of period 1 whose restriction to $[0, 1]$ is given by $\psi(\theta) = |2\theta - 1|$. We have*

$$|S_t| = t \sum_{\pm} \int_{I_{\alpha_0}} \psi \left(\theta_2^{\pm, q}(\xi) - \frac{t}{2 \sin \xi} \right) d\xi + O(1) \quad (2)$$

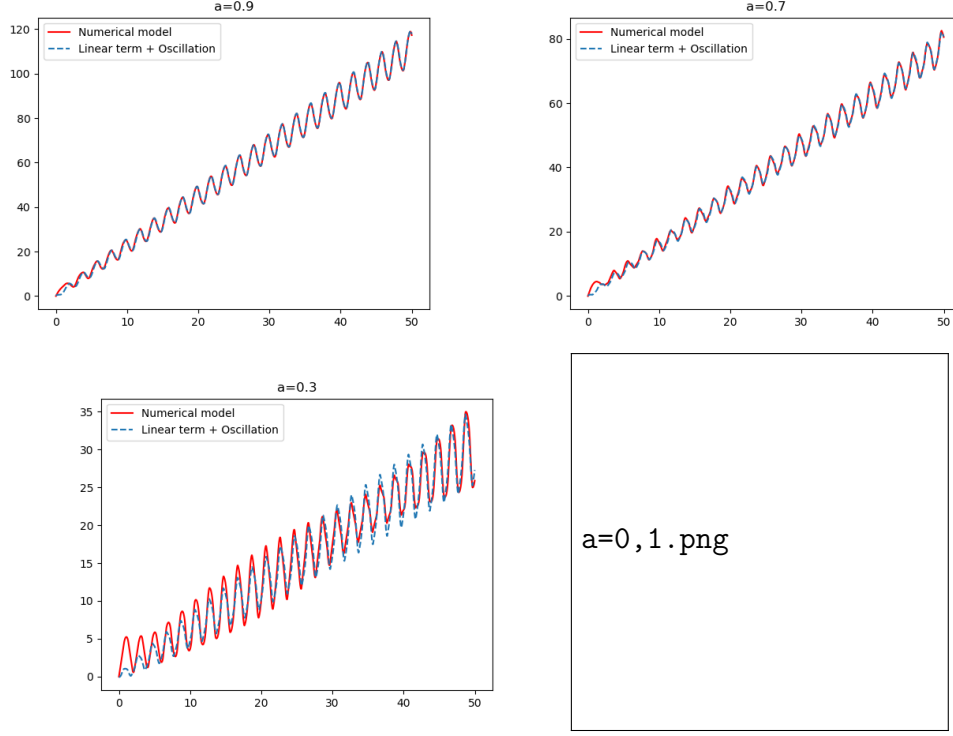


Figure 3: Comparison between $|S_t|$ and $2 \arcsin(a)t + t \sum_{k=0}^N J_k^{\text{approx}}(t)$ for $N = 10$, $t \in [0, 50]$ and $a \in \{0.1, 0.3, 0.7, 0.9\}$.

as $t \rightarrow +\infty$, where $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$, $\alpha_0 = \arcsin a$ and

$$\theta_2^{\pm, a}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

This integral can also be written as an integral over \mathbb{T} :

$$|S_t| = t \int_{\mathbb{T}} \psi \left(\frac{1}{2} - \frac{a \cos \alpha + t}{2\sqrt{1 - a^2 \sin^2 \alpha}} \right) \frac{a \cos \alpha}{\sqrt{1 - a^2 \sin^2 \alpha}} d\alpha + O(1) \quad (3)$$

as $t \rightarrow +\infty$.

Let us show how Theorem 3.1 follows from Theorem 3.2. We consider an integral

$$I(t) = \int_{I_{\alpha_0}} \psi \left(\theta(\xi) - \frac{t}{2 \sin \xi} \right) d\xi$$

with θ smooth. We first approximate uniformly ψ by a sequence of trigonometric polynomials $\psi_N(u) = \sum_{|n| \leq N} a_n \exp(2i\pi n u)$ with $a_0 = \int_0^1 \psi(\theta) d\theta = \frac{1}{2}$. This way we get

$$I_N(t) = 2\alpha_0 + \sum_{|n| \leq N, n \neq 0} a_n \int_{I_{\alpha_0}} e^{2i\pi n \theta(\xi)} e^{-2i\pi n t / \sin \xi} d\xi$$

It follows from the stationary phase approximations that all these integrals tend to 0 as $t \rightarrow \infty$, Theorem 3.1 follows.

Proof of Theorem 3.2.–

We will first parametrize the dynamics using angle coordinates on tori. Let us denote by $m(s) = (\cos s, \sin s)$ on the circle and by \vec{u}_s the vector $\overrightarrow{0m(s)}$. Let us introduce a set of coordinates. In what follows, we parametrize the 2D-submanifold of the phase space consisting of oriented chords joining a point $m(s)$ to $m(s + 2\xi)$ with speed 1 by $\xi \in]0, \pi[$. Changing the orientation of the chords moves ξ into $\pi - \xi$. For $\xi \in]0, \pi[$ and $r \in [0, 2 \sin \xi]$, we define $F_\xi(s, r) = m(s) + r\vec{u}_{s+\xi+\pi/2}$. This describes the chord C_ξ between $m(s)$ and $m(s + 2\xi)$. The function F_ξ is extended as a function on \mathbb{R}^2 periodic with respect to the lattice L_ξ spanned by the vectors $(2\pi, 0)$ and $(2\xi, -2 \sin \xi)$. The function F_ξ is continuous, but only piecewise smooth. The pull-back under F_ξ on \mathbb{R}^2 of the billiard dynamics is generated by the vector ∂_r .

The coordinates (s, r) range over a torus \mathbb{R}^2/L_ξ . In order to continue the computation, we need to fix the lattice \mathbb{Z}^2 . For that we introduce the linear map $M_\xi : \mathbb{R}_{\theta_1, \theta_2}^2 \rightarrow \mathbb{R}_{s, r}^2$ sending the canonical basis of \mathbb{Z}^2 onto the previous basis of L_ξ . The dynamics on the torus \mathbb{R}^2/L_ξ is the image of ∂_r under M_ξ^{-1} ; let us denote it by V . We get

$$V = \frac{1}{2\pi \sin \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2})$$

Then, we need to compute the Euclidean norm of $F'_\xi(M_\xi(\partial_\xi V))$. We have

$$\partial_\xi V = \frac{-\cos \xi}{2\pi \sin^2 \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2}) + \frac{1}{2\pi \sin \xi} \partial_{\theta_1}$$

Hence

$$M_\xi(\partial_\xi V) = \frac{1}{\sin \xi} (-\cos \xi \partial_r + \partial_s)$$

Then

$$F'_\xi(\partial_r) = \vec{u}_{s+\xi+\pi/2}, \quad F'_\xi(\partial_s) = \vec{u}_{s+\pi/2} - r\vec{u}_{s+\xi}$$

This gives

$$\|\partial_\xi V\| = \frac{|r - \sin \xi|}{\sin \xi}$$

As could have been anticipated, this length vanishes on the caustic! We now take the pull back of $\|\partial_\xi V\|$ under M_ξ and get $|2\theta_2 - 1|$.

Let us parametrize the chords starting from q by the angle $\alpha \in \mathbb{T}$ defined by $\alpha := \langle q, C_\xi \rangle$. We get $\cos \xi = a \sin \alpha$. Hence ξ is the smooth function $\xi(\alpha) = \arccos(a \sin \alpha)$. The length $|S_t|$ is given by

$$|S_t| = \int_{\mathbb{T}} \left\| \frac{d}{d\alpha} (\phi_t(\vec{u}_\alpha)) \right\| d\alpha$$

where ϕ_t is the geodesic flow. Let us denote by $\theta(\alpha)$ the coordinates of q in \mathbb{T}_θ^2 . We get, using the parametrization of the flow on the tori \mathbb{T}_θ ,

$$\begin{aligned} |S_t| &= \int_{\mathbb{T}} \|(F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)}(\theta'(\alpha) + tV'(\alpha))\| d\alpha, \\ &= t \int_{\mathbb{T}} \|(F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)}(V'(\alpha))\| d\alpha + O(1) \end{aligned}$$

as $t \rightarrow +\infty$. We rewrite the integral in terms of ξ , using $\cos \xi = a \sin \alpha$ and $\theta_2(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$ with $+$ if $\alpha \in [\pi/2, 3\pi/2]$ and $-$ otherwise. From this follows the result.

4 Local asymptotics of the length

In this section, we describe the asymptotics of the length of the intersection of the wave front with a smooth domain K included in the disk \mathbb{D} . We have

Theorem 4.1 *We have*

$$l(S_t \cap K) \sim \frac{2t}{\pi} \int_K \Psi(\sqrt{x^2 + y^2}) |dxdy|$$

as $t \rightarrow +\infty$, where

$$\Psi(r) = \frac{\min(r, a)}{\sqrt{1 - \min(r, a)^2}}$$

Note that the function Ψ is continuous, vanishes at $r = 0$ and is constant for $a \leq r \leq 1$. This implies that the density of the wave front is smaller near the center of the disk.

Proof.— Let $\phi \in C(\mathbb{D}, \mathbb{R}^+)$, we want to calculate the asymptotics of the length $|S_{t,\phi}|$ of S_t computed in the metric $\phi^2 \text{Eucl}$. Following the proof of Theorem 3.1, we get $|S_{t,\phi}|/t \rightarrow \lambda(q, \phi)$ as $t \rightarrow +\infty$, with

$$\lambda(q, \phi) = 2 \int_{I_{\alpha_0}} \int_{\mathbb{T}^2} |2\theta_2 - 1| \phi \circ G(\theta, \xi) |d\xi d\theta|$$

with $G(\theta, \xi) = F_\xi \circ M_\xi(\theta)$. We will first make the change of variable $(\theta, \xi) \rightarrow (s, r, \xi)$ whose Jacobian is $4\pi \sin \xi$. This gives

$$\lambda(q, \phi) = \frac{1}{2\pi} \int_{I_{\alpha_0}} \int_{\mathbb{R}^2/L_\xi} \left| \frac{r - \sin \xi}{\sin^2 \xi} \right| \phi \circ F_\xi(s, r) |d\xi ds dr|$$

Finally, we pass from (s, r) to (x, y) . We have $|dxdy| = |r - \sin \xi| |dsdr|$. The domain of integration is $\rho = \sqrt{x^2 + y^2} \geq \cos \xi$ which is covered twice by the torus \mathbb{R}^2/L_ξ , we get hence

$$\lambda(q, \phi) = \frac{1}{\pi} \int_{I_{\alpha_0}} \int_{\cos \xi \leq \rho} \left| \frac{1}{\sin^2 \xi} \right| \phi(x, y) |d\xi dxdy|$$

An elementary calculus gives then

$$\lambda(q, \phi) = \frac{2}{\pi} \int_{\mathbb{D}} \Psi(\rho) \phi(x, y) |dxdy|$$

The result follows then by approximating the characteristic function of K by continuous functions. \square

5 Oscillations of the length

The numerical computations of the second author in [Vi-20] show clearly some regular oscillations of the length $|S_t|$ around the linear asymptotics. These oscillations are given in the

Theorem 5.1 *The following expansion holds:*

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + O(1)$$

as $t \rightarrow +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The oscillations have an amplitude of the order of \sqrt{t} , are periodic of period 2. If $a = \frac{1}{2}$, we get $|S_t| = \pi t/3 + O(1)$ as $t \rightarrow +\infty$.

We start from the formula given by Equation (2):

$$|S_t| = t \Sigma_{\pm} \int_{I_{\alpha_0}} \psi\left(\theta_2^{\pm,q}(\xi) - \frac{t}{2 \sin \xi}\right) d\xi + O(1)$$

as $t \rightarrow +\infty$, where $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$ and ψ restricted to $[0, 1]$ is given by $\psi(\theta) = |2\theta - 1|$ and ψ is periodic of period 1. We have

$$\theta_2^{\pm,q}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

The idea is to start with the Fourier expansion of ψ and then to apply the stationary phase asymptotics.

We have

$$\psi(\theta) = \frac{1}{2} + \sum_{n \in \mathbb{Z}} \frac{2}{\pi^2(2n+1)^2} e^{2(2n+1)i\pi\theta}$$

We need to evaluate the integrals

$$I_n(t) = \frac{-4}{((2n+1)\pi)^2} \int_{I_{\alpha_0}} \cos\left((2n+1)\pi \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi}\right) e^{-i\pi(2n+1)\frac{t}{\sin \xi}} d\xi$$

and then we have

$$|S_t| = 2\alpha_0 t + t \sum_{n \in \mathbb{Z}} I_n(t) + O(1)$$

as $t \rightarrow +\infty$. Note first that the function $\cos\left((2n+1)\frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi}\right)$ is smooth on I_{α_0} with a non vanishing derivative at the boundaries. The non vanishing contributions come from the critical point $\xi = \pi/2$ and the boundaries of I_{α_0} . The boundary contributions are $O(1/t)$. They contribute to the $O(1)$ remainder. The contribution of the critical point can be calculated using the

formula (4). We get an asymptotic for $J_n = I_n + I_{-n-1}$, $n = 0, 1, \dots$ given by

$$J_n(t) \sim J_n^{\text{approx}} = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi\left((2n+1)t + \frac{1}{4}\right)\right)$$

The previous calculation is only formal. We need to control the remainder terms in a uniform way with respect to n . Let us rewrite the integral I_n as combination of integrals of the form

$$\int_{I_{\alpha_0}} e^{-i\pi(2n+1)t\left(\frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}\right)} d\xi$$

and apply the stationary phase with the phase functions depending on t : $\Phi_t(\xi) = \frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2\xi}}{\sin\xi}$. This phase function is non degenerate and converges in C^∞ topology to $\frac{1}{\sin\xi}$ as $t \rightarrow \infty$. Hence the remainder is $O((nt)^{-3/2})$ as $t \rightarrow +\infty$, uniformly with respect to n .

A Stationary phase

For this section, we refer the reader to [GS-77], chap. 1.

We want to evaluate the asymptotics as $t \rightarrow +\infty$ of integrals of the form

$$I(t) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

where S is a real valued smooth function. We assume that the critical points of S , ie the zeroes of S' , are non degenerate, ie $S''(x) \neq 0$. We will first assume that $a \in C_o^\infty(\mathbb{R})$ with only one critical point $x = 0$ in the support of a . Then $I(t)$ admits a full asymptotic expansion given by

$$I(t) = \frac{\sqrt{2\pi} e^{i\varepsilon\pi/4}}{|tS''(0)|^{\frac{1}{2}}} e^{itS(0)} (a(0) + O(t)) \quad (4)$$

as $t \rightarrow +\infty$, with $\varepsilon = \pm 1$ depending on the sign of $S''(0)$. We will need some uniform estimates in the remainder term. This is provided by the following

Proposition A.1 *Let us consider the integrals*

$$I(t; S, a) := \int_{\mathbb{T}} e^{itS(x)} a(x) dx$$

Let S_0 be a smooth real valued Morse function and a_0 be a smooth function. Let S_λ and a_λ be smoothly dependent of a real parameter λ . Then, for λ small enough,

$$I(t; S_\lambda, a_\lambda) := I_{\text{asymp}}(t, \lambda) + O(t^{-3/2})$$

as $t \rightarrow +\infty$, where the O is uniform and $I_{\text{asymp}}(t, \lambda)$ is the sum of terms given by the formula (4) for all critical points of S_λ .

If λ is small enough, S_λ is still a Morse function. We localize the integrals near the critical points and apply the Morse Lemma with parameters. We are then reduced locally to the case where $S_\lambda(x) = \pm x^2$. We apply then any proof of the stationary phase approximation.

It will also be useful to consider the case of an integral on a closed interval $[c, d]$ with $c < d$.

$$I(t) := \int_c^d e^{itS(x)} a(x) dx$$

Assuming that S' does not vanish on the support of a and that a is C^1 , we have

$$I(t) = \frac{1}{it} \left(\frac{a(d)e^{itS(d)}}{S'(d)} - \frac{a(c)e^{itS(c)}}{S'(c)} + O(t) \right) \quad (5)$$

as $t \rightarrow +\infty$.

Note that in both asymptotic formulae, the remainders “ $O(t)$ ” are uniform if S' (resp. a') is close to S (resp. close to a) in C^2 topology.

The previous asymptotics extend to higher dimensional integrals.

References

- [CV-20] Yves Colin de Verdière. *Large time asymptotics of the wave fronts length II: closed surfaces with integrable geodesic flow*. In preparation (2020?).
- [GS-77] Victor Guillemin & Shlomo Sternberg. *Geometric asymptotics*. AMS (1977).
- [Ta-05] Serge Tabachnikov. *Geometry and Billiards*. AMS (2005).
- [Vi-20] David Vicente. *Une goutte d'eau dans un bol*. *Quadrature* **117**:13–22, 45 (2020).