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# Robustness of sampled-data homogeneous systems

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## Abstract

The main goal of this article is to study the robustness of continuous homogeneous systems with sampled-data inputs subject to external disturbances. We characterize how the perturbations affect the system and we show that global asymptotic set stability is achieved.

*Key words:* Lyapunov stability, sampled-data systems, homogeneous systems, robustness.

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## Introduction

Over the last decades, many research articles have been focused on robustness of dynamical systems [1–3]. Recently, the need of robustness results for sampled-data systems has been increased by new fields of research, like Networked Control Systems, where the effects of sampling cannot be neglected. It is well known that linear systems become unstable when a certain sampling time threshold, called the Schur limit, is exceeded [4]. Hence, since sampling can destabilize the system, it is important to study its effects when the sampling time increases, especially when the system is subject to other external disturbances as well.

The effects of sampling on linear and nonlinear systems have been widely studied [5–11]. However, in the literature, smoothness assumptions are almost always made, like  $C^1$  or Lipschitz continuity. When dealing with mere continuous systems, these smoothness assumptions do not hold. Moreover, the effects of sampling are often considered alone, without taking into account any other source of perturbations. When both sampling and external perturbations are applied to a nonlinear system, their conjugated effects are not necessarily the sum of their individual effects. Therefore, when dealing with

mere continuous systems subject to both sampling and external disturbances, no generic results are available.

Homogeneous systems generalize linear systems and some polynomial systems. They have scaling properties that allow local behaviors to be extended globally. Many works study the stability of homogeneous systems [12, 13]. Especially, the authors of [14] shows the links between homogeneity and finite-time global asymptotic stability.

In [15], it was proven that if a feedback control is built such that the closed-loop continuous-time system is homogeneous, then sampling its control law preserves stability, though under a weaker form; indeed, depending on the degree of homogeneity, global asymptotic stability is replaced by practical or local stability. However, this paper only considers the effects of the sampling on the dynamical system without taking into account external disturbances. Let us mention that other recent works have been devoted to the effects of sampling on homogeneous systems (like [16, 17],) but, again, no external disturbances are considered.

The robustness properties of homogeneous systems have also been widely studied: under reasonable assumptions, homogeneous systems are Input-to-State Stable when subject to perturbations due to external causes (like measurement noise or unmodelled forces); see [18–21] for weighted homogeneity and [22] for the more general case of geometric homogeneity. Input-to-state stability is a popular theory that emerged three decades ago [23]

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and which guarantees Lyapunov stability provided that a Lyapunov function verifies sufficient conditions. In [24], the links between finite-time stability and ISS properties are established. However, none of these works take into account the effects of sampling.

The contribution of this paper consists in studying the combined effects of two kinds of perturbations, endogenous (coming from sampling) and exogenous (coming from external disturbances) on a continuous homogeneous system. It complements the aforementioned papers, especially [15] and [22]. We consider a negative degree homogeneous system simultaneously subject to sampled-data inputs and an exogenous disturbance. Under the assumptions that the continuous-time closed-loop system without disturbance is globally asymptotically stable and that the disturbed system with sampled-data inputs verifies a suitable homogeneity property, we show that practical stability is achieved. Furthermore, we show that the system stays asymptotically bounded in a region which size is an explicit function of both the maximum sampling time and the exogenous disturbance intensity.

The article is outlined as follows: Section 1 introduces notation and definitions used throughout the paper; Section 2 presents the problem statement; Section 3 is devoted to the main result, its proof and a discussion on the results; Section 4 shows computer simulations illustrating the main result; and finally a conclusion sums up the paper.

## 1 Notations and definitions

Let us introduce the following notations:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers.
- $\mathbb{N}^*$  denotes the set of non-zero natural numbers.
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$  and  $\|\cdot\|$  the Euclidean norm on  $\mathbb{R}^n$ .
- For  $x \in \mathbb{R}$  and  $\alpha > 0$ , we denote  $|x|^\alpha = \text{sign}(x)|x|^\alpha$ .
- For  $r_1, r_2, \dots, r_n$ ,  $\text{Diag}(r_1, \dots, r_n)$  denotes the diagonal matrix of dimension  $n \times n$  with  $k^{\text{th}}$  diagonal entry  $r_k$ .
- A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is unbounded.
- A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}_\infty$  for each fixed  $t \in \mathbb{R}_+$  and if for each fixed  $s \in \mathbb{R}_+$  the function  $t \mapsto \beta(s, t)$  is decreasing to 0.
- The notation  $d_x V$  (resp.  $d_x \Phi$ ) stands for the differential of the function  $V$  (resp. the diffeomorphism  $\Phi$ ) at point  $x$ .

**Definition 1** A vector field  $\nu$  on  $\mathbb{R}^n$  is called an Euler

vector field if  $\nu$  is of class  $C^1$ , complete (i.e. the maximal solutions of  $\dot{x} = \nu(x)$  are defined on  $\mathbb{R}$ ) and if the origin is a globally asymptotically stable equilibrium of  $-\nu$ .

**Definition 2** Let  $\nu$  be an Euler vector field on  $\mathbb{R}^n$ ,  $\Phi^s(x)$  denotes the value of the flow of  $\nu$  at time  $s$  with initial condition  $x$ . A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\nu$ -homogeneous of degree  $\kappa \in \mathbb{R}$  if for all  $x \in \mathbb{R}^n$  and all  $s \in \mathbb{R}$  we have  $V(\Phi^s(x)) = e^{\kappa s} V(x)$ . A vector field  $f$  on  $\mathbb{R}^n$  is  $\nu$ -homogeneous of degree  $\kappa \in \mathbb{R}$  if for all  $x \in \mathbb{R}^n$  and all  $s \in \mathbb{R}$  we have  $f(\Phi^s(x)) = e^{\kappa s} d_x \Phi^s f(x)$ .

**Remark 3** • If we consider a matrix  $A \in \mathbb{R}^{n \times n}$  such that  $-A$  is Hurwitz, the vector field defined by  $\forall x \in \mathbb{R}^n, \nu(x) = Ax$ , is an Euler vector field and the flow of  $\nu$  verifies,  $\forall s \in \mathbb{R}_+, \Phi^s(x) = \exp(As)x$ .

- If  $A = \text{Diag}(r_1, \dots, r_n)$  with  $r_1, \dots, r_n > 0$ , the vector field defined by  $\forall x \in \mathbb{R}^n, \nu(x) = Ax$ , is Euler and we find  $\forall s \in \mathbb{R}_+, \Phi^s(x) = \text{Diag}(e^{r_1 s}, \dots, e^{r_n s})x$ . This particular case of homogeneity is usually referred to as weighted homogeneity, the coefficients  $r_1, \dots, r_n$  are called the weights and  $\mathbf{r} = [r_1, \dots, r_n]$  is called the generalized weight. Homogeneity with respect to an Euler vector field defined by a generalized weight  $\mathbf{r}$  is usually simply referred to as  $\mathbf{r}$ -homogeneity.

**Definition 4** Let  $\nu$  be an Euler vector field on  $\mathbb{R}^n$ . A  $\nu$ -homogeneous norm is a function  $N : \mathbb{R}^n \mapsto \mathbb{R}$  such that:

- (1)  $N$  is positive definite;
- (2)  $N$  is  $\nu$ -homogeneous of degree 1;
- (3)  $N$  is continuous.

**Remark 5** Let  $\mathbf{r} = [r_1, \dots, r_n]$  be a generalized weight. For any  $\rho > 0$ , these following functions define  $\mathbf{r}$ -homogeneous norms on  $\mathbb{R}^n$ :

$$N_\rho(x) = \left( \sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}} \right)^{\frac{1}{\rho}} ; \quad N_\infty(x) = \sup_i |x_i|^{\frac{1}{r_i}}.$$

**Remark 6** A homogeneous norm  $N$  is always radially unbounded and proper [14](lemma 4.1). Consequently, the set  $\mathcal{S} = \{x \in \mathbb{R}^n : N(x) = 1\}$  is always compact.

The following proposition is a direct consequence of [14](lemma 4.2).

**Proposition 7** Let  $N_1$  and  $N_2$  be two  $\nu$ -homogeneous norms. Then there exist  $a, b > 0$  such that, for all  $x \in \mathbb{R}^n$ :

$$aN_1(x) \leq N_2(x) \leq bN_1(x)$$

**Theorem 8** (Rosier[25]) Let a homogeneous system of degree  $\kappa \in \mathbb{R}$ :

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n, n \geq 1$$

with  $f$  continuous, which satisfies the fact that the zero solution of this system is locally asymptotically stable, then there exists a smooth Lyapunov function belonging to  $C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \cap C^p(\mathbb{R})$ , with  $p \in \mathbb{N}^*$  as large as wanted, homogeneous of degree  $\mu > 0$ , such that  $\kappa + \mu > 0$ .

Let us consider the following nonlinear system

$$\dot{x} = F(x, \Delta) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state,  $\Delta \in \mathcal{L}_{loc}^\infty$  is the external input and  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is continuous.

Let us recall the definition of input-to-state stability (ISS).

**Definition 9 ([26])** *The system (1) is (globally) input-to-state stable (ISS) if there exist a  $\mathcal{KL}$  function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  and a class  $\mathcal{K}$  function  $\gamma$  such that, for each input  $u \in \mathcal{L}_{loc}^\infty$  and each  $x_0 \in \mathbb{R}^n$ , the following inequality holds*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \text{ess sup}_{s \in [0, t]} \gamma(\|\Delta(s)\|), \quad \forall t \geq 0.$$

where  $x(t)$  is the solution of the system (1) satisfying  $x(0) = x_0$ . The function  $\gamma$  is called a nonlinear asymptotic gain.

**Definition 10** *Consider the system  $\dot{x} = f(t, x)$ . A compact set  $K \subset \mathbb{R}^n$  is:*

- *stable if for any neighborhood  $U$  of  $K$ , there exists a neighborhood  $V$  of  $K$  such that for any  $x_0 \in V$ , any maximal solution  $x(t)$  with  $x(0) = x_0$  is defined for all  $t \geq 0$  and verifies  $x(t) \in U$  for all  $t \geq 0$ ;*
- *locally attractive if there exists a neighborhood  $U$  of  $K$  such that for any maximal solution  $x(t)$  such that  $x(0) \in U$ , then  $x(t)$  is defined for all  $t \geq 0$  and  $x(t) \rightarrow K$  when  $t \rightarrow +\infty$ ; the domain of attraction of a locally attractive set is the biggest set  $U$  for which the preceding point hold;*
- *globally attractive if it is locally attractive and if its domain of attraction is  $\mathbb{R}^n$ ;*
- *locally (resp. globally) asymptotically stable if it is stable and locally (resp. globally) attractive w.r.t. the system (1) ;*
- *unstable if it is not stable.*

## 2 Problem statement

Let us consider the following nonlinear system:

$$\dot{x} = f(x, u, d) \quad (2)$$

where  $x \in \mathbb{R}^n$  is the state,  $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}^n$  a continuous function,  $u : \mathbb{R}^n \mapsto \mathbb{R}^m$  a continuous static feedback such that the origin is a globally asymptotically stable equilibrium of the closed-loop system  $\dot{x} =$

$f(x, u(x), 0)$ , and  $d : t \mapsto d(t) \in \mathbb{R}^p$  indicates the perturbation, which is supposed essentially bounded. Since in networked communication, the state information is only updated at discrete time instants, we consider a sequence of sampling times  $(t_k)_{k \in \mathbb{N}}$  such that  $t_0 = 0$ , and a maximum sampling period  $h > 0$  and

$$0 < t_{k+1} - t_k \leq h. \quad (3)$$

Due to the sampling, the control is now  $u_{SD}(t) = u(x(t_k))$  for all  $t \in [t_k, t_{k+1})$  (sample and hold). The system can therefore be rewritten under the following form

$$\dot{x}(t) = f(x(t), u(x(t_k)), d(t)) \quad t \in [t_k, t_{k+1}). \quad (4)$$

Our aim in this paper is to study the influence of both the sampling and the external perturbation, acting simultaneously, on the stability of our nominal system. Given the aforementioned robustness properties of homogeneous systems, we will focus ourselves on systems which exhibit a suitable homogeneity property.

**Assumption 11** *There exist a matrix  $A \in \mathbb{R}^{n \times n}$  and a matrix  $\bar{A} \in \mathbb{R}^{p \times p}$  such that  $-A$  and  $-\bar{A}$  are Hurwitz and a degree  $\kappa < 0$  such that*

$$f(e^{As}x, u(e^{\bar{A}s}z), e^{\bar{A}s}d) = e^{\kappa s} e^{As} f(x, u(z), d)$$

for all  $x, z \in \mathbb{R}^n, d \in \mathbb{R}^p$  and all  $s \in \mathbb{R}$ .

**Remark 12** *It is worth noting that, in practice, this assumption is always verified as long as the nominal system is homogeneous. Indeed, the effects of a disturbance  $d$  on the nominal system are often written under the form  $f(x, u) + d$ . For such a disturbance, with a homogeneous  $f$ , Assumption 11 always holds with  $\bar{A} = A + \kappa I$ . Moreover, in practice, the control  $u$  itself shows homogeneity properties like  $u(e^{As}x) = e^{\bar{A}s}u(x)$  for a given  $\bar{A}$  matrix. In this situation, Assumption 11 would also hold for a disturbed system written as  $f(x, u + d)$ .*

In the next section, we will show that, under Assumption 11, practical stability is achieved.

## 3 Main result

We will first prove a technical lemma.

**Lemma 13** *Let  $\nu$  an Euler field on  $\mathbb{R}^n$  and  $N$  a  $\nu$ -homogeneous norm and  $\mathcal{S} = \{x \in \mathbb{R}^n : N(x) = 1\}$ . We consider the functions:*

$$\begin{aligned} \underline{h} : c \in \mathbb{R}_+ &\mapsto \inf\{N(z) : \exists y \in \mathcal{S}, N(y - z) \leq c\}, \\ \bar{h} : c \in \mathbb{R}_+ &\mapsto \sup\{N(z) : \exists y \in \mathcal{S}, N(y - z) \leq c\}. \end{aligned}$$

The functions  $\underline{h}$  and  $\bar{h}$  are continuous on  $\mathbb{R}_+$ ,  $\underline{h}$  is decreasing on  $[0, 1]$  and  $\bar{h}$  is increasing on  $\mathbb{R}_+$ . Furthermore,  $\underline{h}(0) = \bar{h}(0) = 1$ ,  $\underline{h}(c) = 0$  if  $c \geq 1$  and  $\lim_{+\infty} \bar{h} = +\infty$ .

**Proof.** Let us consider  $c \in \mathbb{R}_+$  and  $(c_n)_{n \geq 0}$  a sequence of reals in  $\mathbb{R}_+$  converging to  $c$ .

1) We denote  $E_c = \{z \in \mathbb{R}^n \mid \exists y \in \mathcal{S}, N(y - z) \leq c\}$ . We have  $E_c = \mathcal{S} + \{x \in \mathbb{R}^n, N(x) \leq c\}$ . As  $N$  is a homogeneous norm,  $\mathcal{S}$  and  $\{x \in \mathbb{R}^n, N(x) \leq c\}$  are compact, so  $E_c$  is compact.

2) By compactness, there exists a  $z \in E_c$  such that  $N(z) = \underline{h}(c)$ . By definition of  $E_c$ , there exists  $y \in \mathcal{S}$  such that  $N(y - z) \leq c$ . Suppose that there exists a  $y \in \mathcal{S}$  such that  $N(y - z) < c$ . By continuity of  $N$ , there exists  $\varepsilon > 0$  such that  $B(z, \varepsilon)$  is included in  $E_c$ , where  $B(z, \varepsilon)$  refers the euclidian ball of centre  $z$  and radius  $\varepsilon$ . Let us consider, for  $t \in \mathbb{R}$ , the curve  $t \mapsto \Phi^t(z)$ , with  $\Phi$  the flow of  $\nu$ . So for small values of  $t < 0$ ,  $\Phi^t(z) \in B(z, \varepsilon) \subset E_c$ . But  $N(\Phi^t(z)) = e^t N(z) < N(z) = \underline{h}(c)$  which is contradictory as  $\Phi^t(z)$  belongs to  $E_c$ . We deduce that for all  $z \in E_c$  such that  $N(z) = \underline{h}(c)$  and for all  $y \in \mathcal{S}$ , we have  $N(y - z) \geq c$ .

3) Since  $E_c$  is compact, there exists  $z \in E_c$  such that  $\underline{h}(c) = N(z)$  and then  $y \in \mathcal{S}$  such that  $N(y - z) = c$ , according to point 2). Let us note  $z_n = y - \Phi^{\ln(c_n/c)}(y - z)$ . We have

$$N(y - z_n) = N(\Phi^{\ln(c_n/c)}(y - z)) = \frac{c_n}{c} N(y - z) = c_n$$

so  $z_n \in E_{c_n}$  so  $\underline{h}(c_n) \leq N(z_n)$ . Furthermore, the sequence  $(z_n)_{n \geq 0}$  converges to  $z$  so by continuity  $(N(z_n))_{n \geq 0}$  converges to  $N(z) = \underline{h}(c)$ . We deduce that:

$$\limsup_{n \rightarrow +\infty} \underline{h}(c_n) \leq \limsup_{n \rightarrow +\infty} N(z_n) = \underline{h}(c).$$

4) Let  $(z_n)_{n \geq 0}$  a sequence of vectors such that  $z_n \in E_{c_n}$  for all  $n$  and  $N(z_n) = \underline{h}(c_n)$ . As  $\underline{h}$  is bounded on  $\mathbb{R}_+$  and non increasing, we have  $\underline{h}(1) = 0 \leq \underline{h}(c) \leq \underline{h}(0) = 1$  for all  $c \in \mathbb{R}_+$ , so  $(\underline{h}(c_n))_{n \geq 0}$  is bounded. So we can extract from  $(c_n)_{n \geq 0}$  a sub-sequence, denoted  $(c_{n_k})_{k \geq 0}$ , in such a way that  $(\underline{h}(c_{n_k}))_{k \geq 0}$  converges to a limit  $\ell \in [0, 1]$ . As the sequence  $(c_n)_{n \geq 0}$  converges, it is bounded by a positive constant  $c_{\max}$ . Moreover, for all  $n$ ,  $z_n$  belongs to  $E_{c_n} \subset E_{c_{\max}}$  which is compact, we can extract a convergent sub-sequence  $(z_n)_{n \geq 0}$ , noted  $(z_{n_p})_{p \geq 0}$ , with limit  $z^*$ . For each  $z_n$ , there exists  $y_n \in \mathcal{S}$  such that  $N(y_n - z_n) = c_n$ . As  $\mathcal{S}$  is compact, we extract from  $(y_{n_p})_{p \geq 0}$  a convergent sub-sequence, noted  $(y_{n_q})_{q \geq 0}$ , which converges to  $y^* \in \mathcal{S}$ . By going to the limit, we have  $N(y^* - z^*) = c$ , so  $z^* \in E_c$ . Thus, it yields

$$\underline{h}(c) \leq N(z^*) = \lim_{k \rightarrow +\infty} N(z_{n_k}) = \lim_{k \rightarrow +\infty} \underline{h}(c_{n_k}) = \ell.$$

Since this inequality is true for any accumulation point  $\ell$  of the sequence  $(\underline{h}(c_n))_{n \geq 0}$ , we conclude that  $\underline{h}(c) \leq \liminf_{n \rightarrow +\infty} \underline{h}(c_n)$ , which with point 3) proves the continuity.

5)  $\underline{h}$  is clearly non increasing. Let us show that it is decreasing. Suppose that it exists  $c_1$  and  $c_2$  verifying  $0 \leq c_1 < c_2 \leq 1$  such that  $\underline{h}(c_1) = \underline{h}(c_2)$ . This means that it exists  $y_1$  and  $y_2 \in \mathcal{S}$  and  $z_1 \in E_{c_1}$ ,  $z_2 \in E_{c_2}$  such that  $N(y_1 - z_1) = c_1$  and  $N(y_2 - z_2) = c_2$  from points 1) and 2). Since  $E_{c_1} \subset E_{c_2}$  then  $z_1 \in E_{c_2}$ . From point 2), since  $y_1 \in \mathcal{S}$ , then  $N(y_1 - z_1) \geq c_2$ , which implies  $c_1 \geq c_2$ , that is a contradiction.

Similarly, we can show that the function  $\bar{h}$  defined by

$$\bar{h} : c \in \mathbb{R}_+ \mapsto \sup\{N(z) : \exists y \in \mathcal{S}, N(y - z) \leq c\},$$

is continuous and increasing on  $[0, 1]$ . ■

We can now state the main result of this paper.

**Theorem 14** *Assume that the sampled system (4) is such that the sampling times satisfy (3) and Assumption 11 holds. Consider  $N$  any  $\nu$ -homogeneous norm and  $\bar{N}$  any  $\bar{\nu}$ -homogeneous norm. Then there exist constants  $c_1 > 0$ ,  $c_2 > 0$  such that the set  $\mathcal{B} = \{x \in \mathbb{R}^n \mid N(x) \leq c_1 h^{-\frac{1}{\kappa}} + c_2 d_{\max}\}$  is globally asymptotically stable w.r.t. the system (2), where  $d_{\max} = \operatorname{ess\,sup}_{t \in \mathbb{R}_+} \bar{N}(d(t))$ .*

**Proof.** The proof is organized in different parts. First, three preliminaries allow us to construct several constants that will be instrumental in the sequel of the proof. Then, we proceed to the main part of the proof, itself divided in six steps. In the first five steps, we show that the state of the system eventually reaches the set  $\{x \in \mathbb{R}^n : N(x) \leq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{\max}\}$ , where  $C_1$  and  $C_2$  are positive constants. Finally, the last step concludes the proof by showing set stability.

Let us denote  $V$  a  $\nu$ -homogeneous smooth Lyapunov function of degree  $\mu > 0$  with  $\kappa + \mu > 0$  of the asymptotically stable closed-loop system  $\dot{x} = f(x, u(x), 0)$ , whose existence is provided by Theorem 8. Thereafter, we fix a  $\nu$ -homogeneous norm  $N$  defined by  $N = V^{\frac{1}{\mu}}$  and we denote  $\mathcal{S} = \{x \in \mathbb{R}^n : N(x) = 1\}$  and  $\Delta(t) = x(t_k) - x(t) = x_k - x(t)$ . We will prove the theorem for this particular homogeneous norm; the result for any homogeneous norm is then straightforward from Proposition 7. Let us define  $a = -\sup_{z \in \mathcal{S}} d_z V f(z, u(z), 0)$ , which is positive since  $V$  is a Lyapunov function for the system  $\dot{x} = f(x, u(x), 0)$ , and  $b = \sup_{z \in \mathcal{S}} \|d_z V\|$ .

**Preliminary 1:** There exists  $\varepsilon > 0$  such that if  $t \in [t_k, t_{k+1})$  verifies  $N(\Delta(t)) < \varepsilon N(x(t))$ , then

$$d_{x(t)} V f(x(t), u(x_k), 0) < -\frac{a}{2} V(x(t))^{\frac{\kappa+\mu}{\mu}}.$$

Indeed, let us fix  $t \in [t_k, t_{k+1})$  such that  $x(t) \neq 0$ . For the sake of clarity, we will write  $x$  instead of  $x(t)$ . Consider  $s \in \mathbb{R}$  such that  $N(x) = e^s$  and denote  $\tilde{x} = e^{-As}x$  and  $\tilde{\Delta} = e^{-As}\Delta(t)$ . Note that  $N(\tilde{x}) = 1$ . By homogeneity of  $V$  and  $f$ , we have

$$\begin{aligned} d_x V f(x, u(x_k), 0) &= d_x V f(x, u(x + \Delta(t)), 0) \\ &= e^{(\kappa+\mu)s} d_{\tilde{x}} V f(\tilde{x}, u(\tilde{x} + \tilde{\Delta}), 0). \end{aligned}$$

The function  $\delta \in \mathbb{R}^n \mapsto \sup_{z \in \mathcal{S}} d_z V f(z, u(z + \delta), 0)$  is continuous, since  $V$  is  $\mathcal{C}^1$ ,  $f$  is continuous and  $\mathcal{S}$  is compact. If  $\delta = 0$ , we have  $\sup_{z \in \mathcal{S}} d_z V f(z, u(z), 0) = -a < 0$ . So by

continuity, there exists  $\varepsilon > 0$  such that for all  $\tilde{\Delta}$  verifying  $N(\tilde{\Delta}) < \varepsilon$ :

$$d_{\tilde{x}} V f(\tilde{x}, u(\tilde{x} + \tilde{\Delta}), 0) < -\frac{a}{2}.$$

Since  $e^s = N(x)$ ,  $N(\tilde{\Delta}) < \varepsilon$  is equivalent to  $N(\Delta) < \varepsilon N(x)$ . Therefore, if this condition is satisfied, we have:

$$e^{(\kappa+\mu)s} d_{\tilde{x}} V f(\tilde{x}, u(\tilde{x} + \tilde{\Delta}), 0) < -\frac{a}{2} e^{(\kappa+\mu)s}$$

which leads to the announced inequality.

**Preliminary 2:** Consider the functions  $\underline{h}$  and  $\bar{h}$  defined in Lemma 13 and define  $c^*(\alpha) = \min\{\underline{h}^{-1}(1 - \alpha), \bar{h}^{-1}(1 + \alpha)\}$ . Then there exists  $\alpha^* \in (0, 1)$  such that  $c^*(\alpha^*) < (1 - \alpha^*)\varepsilon$ , where  $\varepsilon$  is defined in Preliminary 1.

Let us first show that the function  $c^*$  is well-defined on  $(0, 1)$ . Fix  $\alpha \in (0, 1)$ . The function  $\underline{h}$  is continuous and decreasing from  $[0, 1]$  to  $[0, 1]$ , so is bijective and  $\underline{h}^{-1}$  is continuous and decreasing from  $[0, 1]$  onto  $[0, 1]$ . Similarly,  $\bar{h}$  is an increasing function from  $[0, +\infty[$  to  $[1, +\infty[$ , so  $\bar{h}^{-1}$  is well-defined, continuous and increasing from  $[1, +\infty[$  onto  $[0, +\infty[$ . Therefore,  $c^*(\alpha) = \min\{\underline{h}^{-1}(1 - \alpha), \bar{h}^{-1}(1 + \alpha)\}$  exists, belongs to  $[0, 1]$  and verifies:

$$1 - \alpha \leq \underline{h}(c^*(\alpha)) \leq \bar{h}(c^*(\alpha)) \leq 1 + \alpha.$$

Since  $\lim_{x \rightarrow 0^+} \underline{h}(x) = 1^+$  we have  $\lim_{y \rightarrow 1^+} \underline{h}^{-1}(y) = 0^+$ , so

$$\lim_{\alpha \rightarrow 0^+} \underline{h}^{-1}(1 - \alpha) = 0^+. \text{ Similarly, } \lim_{\alpha \rightarrow 0^+} \bar{h}^{-1}(1 + \alpha) = 0^+.$$

Hence  $\lim_{\alpha \rightarrow 0^+} \frac{c^*(\alpha)}{1 - \alpha} = 0$  and we can select an  $\alpha^* \in (0, 1)$  such that  $c^*(\alpha^*) < (1 - \alpha^*)\varepsilon$ .

**Preliminary 3:** There exists  $\eta > 0$  such that, if  $\bar{N}(\tilde{d}) \leq \eta$  then  $\sup_{(\tilde{x}, \tilde{x}_k) \in \mathcal{C}} \|f(\tilde{x}, u(\tilde{x}_k), \tilde{d}) - f(\tilde{x}, u(\tilde{x}_k), 0)\| < \frac{a}{4b}$ , where

$$\mathcal{C} = \mathcal{S} \times N^{-1} \left( \left[ \frac{1}{1 + \alpha^*}, \frac{1}{1 - \alpha^*} \right] \right). \quad (5)$$

Since  $\mathcal{C}$  is compact, the function:

$$\tilde{d} \mapsto \sup_{(\tilde{x}, \tilde{x}_k) \in \mathcal{C}} \|f(\tilde{x}, u(\tilde{x}_k), \tilde{d}) - f(\tilde{x}, u(\tilde{x}_k), 0)\|$$

is continuous. Moreover, this function vanishes for  $\tilde{d} = 0$ . The claim follows.

We shall now proceed to the main part of the proof. Let us define, for  $r \geq 0$ :

$$\begin{aligned} \mathcal{K}_r &= \mathcal{S} \times N^{-1}([0, c^*(\alpha^*)]) \times \bar{N}^{-1}([0, r]), \\ g(r) &= \sup_{(\tilde{x}_k, \tilde{\Delta}, \tilde{d}) \in \mathcal{K}_r} |d_{\tilde{\Delta}} V f(\tilde{x}_k - \tilde{\Delta}, u(\tilde{x}_k), \tilde{d})|, \\ C_1 &= \left( \frac{c^*(\alpha^*)^\mu}{g((1 - \alpha^*)\eta)} \right)^{\frac{1}{\kappa}} \text{ and } C_2 = \frac{1}{(1 - \alpha^*)\eta}. \end{aligned}$$

**Step 1:** If  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$  then

$$N(\Delta(t)) \leq c^*(\alpha^*) N(x_k) \quad \forall t \in [t_k, t_{k+1}).$$

Indeed,  $\mathcal{K}_r$  is compact for any  $r \geq 0$  and  $g$  is a non decreasing function. A direct rewriting yields  $\tilde{\Delta} = -f(x_k - \Delta, u(x_k), d)$ . Let us denote  $t^* = \inf\{t \geq t_k : N(\Delta(t)) \geq c^*(\alpha^*) N(x_k)\}$ . For all  $t$  in  $[t_k, t^*]$ , we have  $N(\Delta(t)) \leq c^*(\alpha^*) N(x_k)$ . Moreover,  $N(\Delta(t_k)) = 0 < c^*(\alpha^*) N(x_k)$ , thus  $t^* > t_k$ . Consider  $s \in \mathbb{R}$  such that  $e^s = N(x_k)$ , and denote, for  $t \in [t_k, t^*]$ ,  $\tilde{x}_k = e^{-As}x_k$ ,  $\tilde{\Delta}(t) = e^{-As}\Delta(t)$  and  $\tilde{d}(t) = e^{-As}d(t)$ . Note that  $N(\tilde{x}_k) = 1$ . By homogeneity, we have  $N(\tilde{\Delta}(t)) \leq c^*(\alpha^*)$  and  $\bar{N}(\tilde{d}) \leq \frac{d_{max}}{N(x_k)}$ .

Now a direct computation gives:

$$\begin{aligned}
V(\Delta(t^*)) &= \int_{t_k}^{t^*} -d_{\Delta(\tau)} V f(x_k - \Delta(\tau), u(x_k), d(\tau)) d\tau \\
&= \int_{t_k}^{t^*} -e^{(\mu+\kappa)s} d_{\tilde{\Delta}(\tau)} V f(\tilde{x}_k - \tilde{\Delta}(\tau), u(\tilde{x}_k), \tilde{d}(\tau)) d\tau \\
&\leq (t^* - t_k) e^{(\mu+\kappa)s} \\
&\quad \times \sup_{t \in [t_k, t^*]} |d_{\tilde{\Delta}(t)} V f(\tilde{x}_k - \tilde{\Delta}(t), u(\tilde{x}_k), \tilde{d}(t))| \\
&\leq (t^* - t_k) e^{(\mu+\kappa)s} \\
&\quad \times \sup_{(\tilde{x}_k, \tilde{\Delta}, \tilde{d}) \in \mathcal{K} \frac{d_{max}}{N(x_k)}} |d_{\tilde{\Delta}} V f(\tilde{x}_k - \tilde{\Delta}, u(\tilde{x}_k), \tilde{d})| \\
&\leq (t^* - t_k) N(x_k)^{\kappa+\mu} g \left( \frac{d_{max}}{N(x_k)} \right).
\end{aligned}$$

Since  $N(x_k) \geq C_2 d_{max} = \frac{d_{max}}{(1-\alpha^*)\eta}$ , then we have  $\frac{d_{max}}{N(x_k)} \leq (1-\alpha^*)\eta$  and therefore, since  $g$  is non decreasing :

$$V(\Delta(t^*)) \leq (t^* - t_k) N(x_k)^{\mu+\kappa} g((1-\alpha^*)\eta).$$

By continuity of  $V$ , we have  $V(\Delta(t^*)) = c^*(\alpha^*)^\mu N(x_k)^\mu$ . Hence  $c^*(\alpha^*)^\mu N(x_k)^\mu \leq (t^* - t_k) N(x_k)^{\mu+\kappa} g((1-\alpha^*)\eta)$ , that is

$$t^* - t_k \geq \frac{c^*(\alpha^*)^\mu}{N(x_k)^\kappa g((1-\alpha^*)\eta)}.$$

Now, the condition  $N(x_k) \geq C_1 h^{-1/\kappa}$  rewrites  $\frac{c^*(\alpha^*)^\mu}{N(x_k)^\kappa g((1-\alpha^*)\eta)} \geq h$  and we get  $t^* \geq t_{k+1}$ . Hence, for all  $t \in [t_k, t_{k+1}]$ , we have  $N(\Delta(t)) \leq c^*(\alpha^*) N(x_k)$  which concludes Step 1.

**Step 2:** If  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$ , then for all  $t \in [t_k, t_{k+1}]$ , we have  $(1-\alpha^*)N(x_k) \leq N(x) \leq (1+\alpha^*)N(x_k)$ .

We saw in Step 1 that if  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$ , then it leads to

$$N(x_k - x(t)) \leq c^*(\alpha^*) N(x_k) \quad \forall t \in [t_k, t_{k+1}]. \quad (6)$$

Denoting  $e^s = N(x_k)$ ,  $\tilde{x}_k = e^{-As} x_k$  and  $\tilde{x}(t) = e^{-As} x(t)$ , this is equivalent to

$$N(\tilde{x}_k - \tilde{x}) \leq c^*(\alpha^*) \quad \forall t \in [t_k, t_{k+1}].$$

So according to the definition of  $\alpha^*$ ,  $c^*$ ,  $\bar{h}$  and  $\underline{h}$ , we get

$$1 - \alpha^* \leq \underline{h}(c^*(\alpha^*)) \leq N(\tilde{x}) \leq \bar{h}(c^*(\alpha^*)) \leq 1 + \alpha^*$$

and then

$$(1 - \alpha^*) N(x_k) \leq N(x) \leq (1 + \alpha^*) N(x_k). \quad (7)$$

**Step 3:** If  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$  then for all  $t \in [t_k, t_{k+1}]$  we have

$$|d_x V f(x, u(x_k), d) - d_x V f(x, u(x_k), 0)| \leq \frac{a}{4} V(x)^{\frac{\kappa+\mu}{\mu}}.$$

Assume that  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$ . Denoting  $e^s = N(x)$ ,  $\tilde{x}_k = e^{-As} x_k$ ,  $\tilde{x} = e^{-As} x$  and  $\tilde{d} = e^{-As} d$ , we have

$$\begin{aligned}
&|d_x V f(x, u(x_k), d) - d_x V f(x, u(x_k), 0)| \\
&= e^{(\kappa+\mu)s} |d_x V [f(\tilde{x}, u(\tilde{x}_k), \tilde{d}) - f(\tilde{x}, u(\tilde{x}_k), 0)]| \\
&\leq e^{(\kappa+\mu)s} b \sup_{(\tilde{x}, \tilde{x}_k) \in \mathcal{C}} \|f(\tilde{x}, u(\tilde{x}_k), \tilde{d}) - f(\tilde{x}, u(\tilde{x}_k), 0)\|
\end{aligned}$$

From Step 2, we obtain

$$\frac{1}{1+\alpha^*} \leq N(\tilde{x}_k) \leq \frac{1}{1-\alpha^*} \quad \forall t \in [t_k, t_{k+1}]$$

and then, given that  $C_2 = \frac{1}{\eta(1-\alpha^*)}$ ,  $\bar{N}(d) \leq d_{max} \leq \eta(1-\alpha^*)N(x_k) \leq \eta N(x)$ , or equivalently,  $\bar{N}(\tilde{d}) \leq \eta$ . But, according to Preliminary 3, if  $\bar{N}(\tilde{d}) \leq \eta$  then we have  $\sup_{(\tilde{x}, \tilde{x}_k) \in \mathcal{C}} \|f(\tilde{x}, u(\tilde{x}_k), \tilde{d}) - f(\tilde{x}, u(\tilde{x}_k), 0)\| < \frac{a}{4b}$ . Not-

ing that  $e^s = N(x) = V(x)^{\frac{1}{\mu}}$ , this yields

$$|d_x V f(x, u(x_k), d) - d_x V f(x, u(x_k), 0)| \leq \frac{a}{4} V(x)^{\frac{\kappa+\mu}{\mu}}.$$

**Step 4:** If  $N(x_k) \geq C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$  then it leads to

$$d_x V f(x, u(x_k), d) < -\frac{a}{4} V(x)^{\frac{\kappa+\mu}{\mu}} \quad \forall t \in [t_k, t_{k+1}]. \quad (8)$$

Using consecutively the results from Step 1, Step 2 and Preliminary 2, we get  $N(\Delta) \leq \varepsilon N(x)$ . Then from Preliminary 1 we get

$$d_x V f(x, u(x_k), 0) < -\frac{a}{2} V(x)^{\frac{\kappa+\mu}{\mu}} \quad \forall t \in [t_k, t_{k+1}]. \quad (9)$$

Writing

$$\begin{aligned}
d_x V f(x, u(x_k), d) &= d_x V f(x, u(x_k), 0) \\
&\quad + d_x V f(x, u(x_k), d) - d_x V f(x, u(x_k), 0)
\end{aligned}$$

we get (8) from Step 3 and (9).

**Step 5:** If  $N(x_k) < R = C_1 h^{-\frac{1}{\kappa}} + C_2 d_{max}$ , there exists  $\omega_0 > 0$  such that  $N(x(t)) \leq (1 + \omega_0)^{\frac{1}{\mu}} R$ , for all  $t \in [t_k, t_{k+1}]$ .

Denote, for  $\omega \geq 0$ :

$$\theta(\omega) = (1 + \omega)^{\frac{\kappa + \mu}{\mu}} \sup_{(\tilde{x}, \tilde{x}_k, \tilde{d}) \in \mathcal{M}_\omega} |d_x \tilde{V} f(\tilde{x}, u(\tilde{x}_k), \tilde{d})|,$$

where  $\mathcal{M}_\omega = \left\{ (\tilde{x}, \tilde{x}_k, \tilde{d}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \mid V(\tilde{x}) \leq 1, V(\tilde{x}_k) \leq \frac{1}{1+\omega}, \bar{N}(\tilde{d}) \leq \frac{d_{max}}{R(1+\omega)^{\frac{1}{\mu}}} \right\}$ , and  $\varphi(\omega) = \omega/\theta(\omega)$ .

Given that  $\theta(\omega) > 0$  for any  $\omega \geq 0$ ,  $\varphi$  is well-defined. Moreover,  $\varphi(0) = 0$  and  $\varphi(\omega) \underset{+\infty}{\sim} \frac{1}{\xi} \omega^{-\frac{\kappa}{\mu}}$ , where  $\xi =$

$\sup_{V(\tilde{x}) \leq 1} |d_x \tilde{V} f(\tilde{x}, u(0), 0)| > 0$ . But  $-\kappa/\mu > 0$  and we conclude that  $\varphi(\omega) \rightarrow +\infty$  when  $\omega \rightarrow +\infty$ . Since  $\varphi$  is continuous, this proves that there exists  $\omega_0 > 0$  such that  $\varphi(\omega_0) = hR^\kappa$ , i.e.  $\frac{\omega_0}{\theta(\omega_0)} = hR^\kappa$ .

Denote  $t^* = \inf\{t \geq t_k, V(x(t)) \geq (1 + \omega_0)R^\mu\} > t_k$ . Therefore, for all  $t \in [t_k, t^*]$ , we have  $V(x(t)) \leq (1 + \omega_0)R^\mu$  and

$$\begin{aligned} V(x(t^*)) &= V(x_k) + \int_{t_k}^{t^*} d_x V f(x, u(x_k), d) dt \\ &\leq R^\mu + (t^* - t_k) \sup_{t \in [t_k, t^*]} |d_x V f(x, u(x_k), d)|. \end{aligned}$$

Denote  $s = \frac{\ln[(1+\omega)R^\mu]}{\mu}$ ,  $\tilde{x}_k = e^{-As}x_k$ ,  $\tilde{x} = e^{-As}x$  and  $\tilde{d} = e^{-\bar{A}s}d$ . Note that we have  $V(\tilde{x}) \leq 1$ ,  $V(\tilde{x}_k) \leq \frac{1}{(1+\omega)}$  and  $\bar{N}(\tilde{d}) \leq \frac{d_{max}}{R(1+\omega)^{\frac{1}{\mu}}}$ . We get

$$\begin{aligned} V(x(t^*)) &\leq R^\mu + (t^* - t_k)e^{(\kappa + \mu)s} \\ &\quad \times \sup_{(\tilde{x}, \tilde{x}_k, \tilde{d}) \in \mathcal{M}_{\omega_0}} |d_x \tilde{V} f(\tilde{x}, u(\tilde{x}_k), \tilde{d})|. \end{aligned}$$

Since  $e^{(\kappa + \mu)s} = R^{\kappa + \mu}(1 + \omega_0)^{\frac{\kappa + \mu}{\mu}}$ , we have

$$\begin{aligned} V(x(t^*)) &\leq R^\mu + (t^* - t_k)R^{\kappa + \mu}\theta(\omega_0) \\ &\leq R^\mu + (t^* - t_k)R^\mu \frac{\omega_0}{h} \\ &\leq R^\mu \left( 1 + \frac{t^* - t_k}{h} \omega_0 \right). \end{aligned}$$

By continuity of  $V$ , we get  $V(x(t^*)) = (1 + \omega_0)R^\mu$  and thus  $h \leq t^* - t_k$ , i.e.  $t^* \geq t_{k+1}$ . Finally, if  $N(x_k) < R$  then we obtain for all  $t \in [t_k, t_{k+1}]$ ,  $V(x(t)) \leq (1 + \omega_0)R^\mu$ , or equivalently  $N(x(t)) \leq (1 + \omega_0)^{\frac{1}{\mu}} R$ .

### Step 6: Conclusion

Let us consider  $\mathcal{B} = \{x \in \mathbb{R}^n \mid N(x) \leq (1 + \omega_0)^{\frac{1}{\mu}} R\}$  and select  $t \geq 0$ . There exists  $k \in \mathbb{N}$  such that  $t \in [t_k, t_{k+1})$ . Denote  $x_k = x(t_k)$ . If  $N(x_k) \geq R$ , then from Step 4 we

have  $d_x V f(x, u(x_k), d) < -\frac{a}{4} V(x)^{\frac{\kappa + \mu}{\mu}}$ . If  $N(x_k) < R$ , then from Step 5 we obtain  $N(x(t)) \leq (1 + \omega_0)^{\frac{1}{\mu}} R$ . This shows that  $\mathcal{B}$  is positively invariant and then that it is globally asymptotically stable. This concludes the proof by taking  $c_1 = (1 + \omega_0)^{\frac{1}{\mu}} C_1$  and  $c_2 = (1 + \omega_0)^{\frac{1}{\mu}} C_2$ . ■

**Remark 15** *Theorem 14 is a result about practical stability. Indeed, the result implies:*

$$\limsup_{t \rightarrow +\infty} N(x(t)) \leq c_1 h^{-\frac{1}{\kappa}} + c_2 d_{max},$$

*which says that, ultimately, the state of the system reaches a (homogeneous) ball around the origin which size is determined by  $c_1 h^{-\frac{1}{\kappa}} + c_2 d_{max}$ . The smaller the sampling step and the maximum disturbance, the closer the system will get to the origin.*

**Remark 16** *Only mere continuity is assumed on the system in Theorem 14, meaning it can be applied when the system is not Lipschitz continuous.*

*In practice, the constants  $c_1$  and  $c_2$  given by Theorem 11 are very hard to compute; however, they can be numerically estimated (see Section 4).*

We considered a system perturbed by two phenomena: an endogenous disturbance, namely the effects of the sampling, alongside an exogenous disturbance called  $d$ . We notice that the effects of each disturbance can be analyzed separately and their conjugated effect is not bigger than the sum of both. This is surprising given that the vector field  $f$  is only supposed continuous and homogeneous and may present high non-linearities.

Theorem 14 is another illustration of the robustness properties of homogeneous systems. It was already known [22],[27] that homogeneous systems were robust with respect to both external disturbances and the effects of sampling when these perturbations were applied separately; Theorem 14 shows that homogeneous systems are robust with respect to both applied together.

Another consequence of the main result is that, when sampling a disturbed homogeneous system, increasing the sampling frequency has diminishing returns. Indeed, when the maximum sampling interval  $h$  is small, the size of the practical convergence zone is already basically  $c_2 d_{max}$ .

## 4 Simulations

In this section, we would like to illustrate the theoretical result of the paper with a simple academic example and show how it is possible to numerically estimate the



constants  $c_1$  and  $c_2$  given by theorem 14. We consider the following controlled system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + d(t) \end{cases} \quad (10)$$

with  $(x_1, x_2)$  being the state of the system,  $u$  its control and  $d(t)$  an external disturbance, verifying  $|d(t)| \leq d_{\max}$  for a given  $d_{\max} > 0$ . The selected control is :

$$u(x_1, x_2) = -k_1|x_1|^{\frac{1}{3}} - k_2|x_2|^{\frac{1}{2}}, \quad (11)$$

where  $k_1, k_2 > 0$  are positive gains. Control laws of this type have been widely used in the literature (for instance [14,27]). Remark that, with this control law, the system is *not* Lipschitz continuous. The closed loop system (10)–(11) with  $d(t) = 0$  is well-known to be homogeneous of degree  $-0.5$  with respect to  $A = \text{diag}(1.5, 1)$  as well as being finite-time stable, see [14, Proposition 8.1]. Let us check Assumption 11. Indeed, taking  $\bar{A} = (0.5)$  and denoting  $f(x, u, d)$  the right-hand side of (10), we get

$$\begin{aligned} & f(e^{As}x, u(e^{As}z), e^{\bar{A}s}d) \\ &= \begin{pmatrix} e^s x_2 \\ -k_1[e^{1.5s}z_1]^{\frac{1}{3}} - k_2[e^s z_2]^{\frac{1}{2}} + e^{s/2}d \end{pmatrix} \\ &= \begin{pmatrix} e^s x_2 \\ e^{s/2}(-k_1|z_1|^{\frac{1}{3}} - k_2|z_2|^{\frac{1}{2}} + d) \end{pmatrix} \\ &= e^{-s/2}e^{As}f(x, u(z), d). \end{aligned}$$

Hence, a direct application of Theorem 14 shows that, under a periodic sampling given by  $t_k = kh$ , with  $h > 0$ , the disturbed sampled system is practically stable.

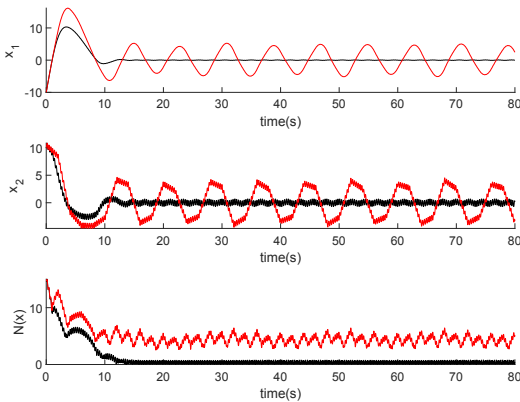


Fig. 1. Time evolution of the state  $x$  of (10) and  $N(x)$  for  $h = 0.1s$  and  $h = 2s$  for a constant external disturbance  $d_{\max} = 10$ .

Figure 1 shows the time evolution of the state of the system (10) and the time evolution of the homogeneous norm  $N(x) = |x_1|^{\frac{2}{3}} + |x_2|$  in two different cases by varying the sampling rate. For the black curves, the sampling rate is  $h = 0.1s$ , and for the red curves  $h = 2s$ . In both cases the external disturbance is  $d_{\max} = 10$ . Firstly, Figure 1 shows that the practical stability is reached as expected. Then, we observe that the size of the practical convergence set increases as  $h$  becomes larger.

In Figure 2, we focus on the influence of the external disturbance. The black curves shows the evolution of the state of the system and the homogeneous norm in the case of  $d_{\max} = 1$  and the red curves in the case of  $d_{\max} = 81$ , with a constant sampling rate  $h = 2s$ . As expected, the practical convergence set increases as  $d_{\max}$  becomes larger. However, we notice that a small increase of the sampling rate has a great influence on the size of the practical convergence set. On the other side, a large increase of the disturbance has a tiny influence on the size of the practical convergence set.

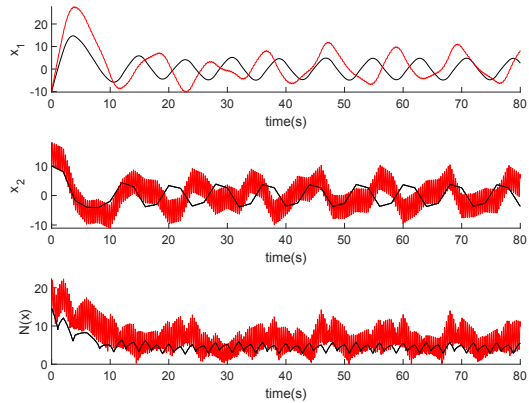


Fig. 2. Time evolution of the state  $x$  of (10) and  $N(x)$  for  $d_{\max} = 1$  and  $d_{\max} = 81$  for a constant sampling rate  $h = 2s$

Figures 3 and 4 respectively show the influence of the sampling rate  $h$  and of the maximum allowed disturbance  $d_{\max}$  on the asymptotic behavior of the system then the other parameter is fixed. Considering the homogeneous norm  $N(x_1, x_2) = |x_1|^{\frac{2}{3}} + |x_2|$ , we estimate the maximum value of  $N(x_1, x_2)$  when the state has reached the practical stability set. To do this, the system is simulated for a duration of 200 time periods and the upper norm is estimated by taking the maximum value of  $N(x_1, x_2)$  over the last 50 time periods. The operation is repeated for 100 different initial values, and the final upper norm is given by the maximum of them. We see that

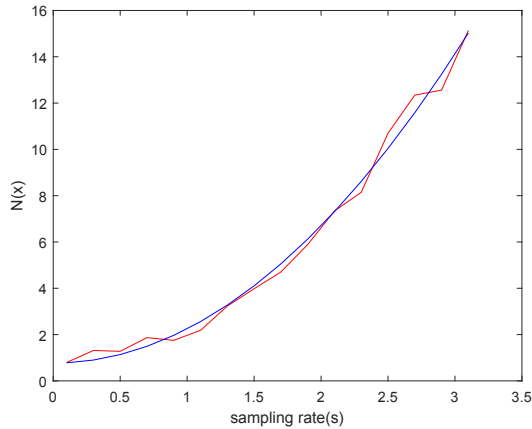


Fig. 3. Evolution of the upper norm  $N(x)$  for system (10) depending on the sampling rate  $h$  for a fixed  $d_{\max} = 10$

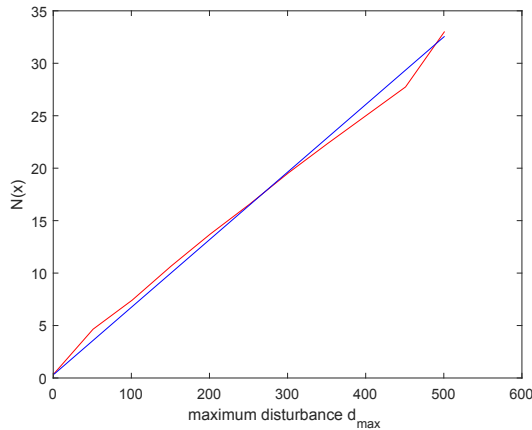


Fig. 4. Evolution of the upper norm  $N(x)$  for system (10) depending on the external disturbance  $d_{\max}$  for a fixed  $h = 0.5s$

the curves have the expected shape: a parabola in Fig. 3, a straight line in Fig. 4. The blue curves are computed with a least squares method to separately estimate the coefficients  $c_1$  and  $c_2$ .

To get a more precise estimation of the coefficients  $c_1$  and  $c_2$ , the whole process is performed for a set of values for  $h$  (from 0.1 to 3.1 by step 0.2) and for  $d_{\max}$  (from 1 to 501 by step 50) varying independently. All the computed values are then used to numerically estimate the coefficients  $c_1$  and  $c_2$  of Theorem 14 by a least squares method. The obtained numerical values are  $c_1 = 2.3667$  and  $c_2 = 0.079$ . They confirm the results obtained in Figures 1 and 2: the influence of the sampling rate on the size of the practical convergence set is much greater than the external disturbance one.

## Conclusion

In this paper, we proved that a negative degree homogeneous controlled system achieves practical stability when it is simultaneously subject to an external disturbance and to a sampling of its control law. Similarly to the case of a sampled-data system, as in [15], where the convergence area is only function of the sampling step, here it is function of both the sampling step and the maximum allowed disturbance. Moreover, the two disturbances overlap but do not interfere with each other and act separately.

In future works, we plan to relax the assumption of continuity of the system in order to apply it to more general frameworks such as Sliding Mode Control.

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