



**HAL**  
open science

# A Nonsmooth Newton Method for the Design of State Feedback Stabilizers under Structure Constraints

Mohamed Yagoubi, Redouane Chaibi

► **To cite this version:**

Mohamed Yagoubi, Redouane Chaibi. A Nonsmooth Newton Method for the Design of State Feedback Stabilizers under Structure Constraints. *Journal of The Franklin Institute*, 2020, 358 (1), pp.800-813. 10.1016/j.jfranklin.2020.11.010 . hal-03006105

**HAL Id: hal-03006105**

**<https://hal.science/hal-03006105>**

Submitted on 2 Jan 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# A Nonsmooth Newton Method for the Design of State Feedback Stabilizers under Structure Constraints

Mohamed Yagoubi<sup>a,\*</sup>, Redouane Chaibi<sup>b</sup>

<sup>a</sup>*IMT Atlantique, LS2N, CNRS UMR-6004, Nantes, France*

<sup>b</sup>*Sidi-Mohamed Ben-Abdellah University, Fès, Morocco*

---

## Abstract

The paper proposes a method for structured state-feedback controllers design for linear time-invariant systems. A necessary and sufficient condition for structured state-feedback stabilizability of linear systems, making an appeal to the linear-quadratic (LQ) regulator theory, is first proposed. The latter is presented in the form of a nonlinear matrix equation. Then, it is recast as a nonsmooth unconstrained equation using projection onto the positive semi-definite matrices cone. Thereby, a nonsmooth Newton's iterative algorithm, based on the Clarke generalized Jacobian of said projection, is proposed. This method has a guaranteed local convergence. Finally, numerical examples illustrate the effectiveness of the proposed method.

*Keywords:* Structured control, LQ theory, Clarke generalized Jacobian, Nonsmooth Newton.

---

## 1. Introduction

State feedback control subject to structural constraints achieving asymptotic stability is a topic that, although it has been studied extensively ([1], [2], [3], [4], [5], [6], [7], [8], [9] and references therein) still remains a challenge to the control community. It is still a problem of major importance from a practical point of view [10]. This prominent control problem has many applications, including

---

\*Corresponding author

*Email addresses:* [mohamed.yagoubi@imt-atlantique.fr](mailto:mohamed.yagoubi@imt-atlantique.fr) (Mohamed Yagoubi),  
[c.redouane.chaibi@gmail.com](mailto:c.redouane.chaibi@gmail.com) (Redouane Chaibi)

sparse feedback design, decentralized control, distributed and networked control systems, overlapping control, etc.

It is customary to obtain stabilizing controllers for a wide class of systems  
10 by solving convex optimization problems under (standard or extended/dilated)  
Linear Matrix Inequality (LMI) constraints [11], [12]. However, when a struc-  
tured set of interconnected subsystems is considered the design requires the  
introduction of information constraints leading to non-convex and even NP-  
hard problems [13]. The key drawback in the context of the LMI approach lies  
15 in the fact that the feedback gain matrix cannot be explicitly isolated (see [2],  
[14] and references therein). LMI relaxations can then be considered in this  
instance. From most to least conservative, these methods hinge on imposing  
a specific structure either directly to the Lyapunov matrix or indirectly to a  
slack matrix variable [10]. These computationally efficient methods, however,  
20 are often conservative and may even not lead to feasible designs.

Another approach consists in enforcing the controller gain to belong to a par-  
ticular set, such as quadratically invariant sets with respect to the considered  
system, ensuring thus the convexity of the problem [15], [16]. This approach  
concerns only some particular classes of system-structure pairs. Furthermore,  
25 the separation of the Lyapunov matrix and the controller gain can also be en-  
sured by alternating projections or reflections like optimization algorithms (see  
for instance [17]). Such splitting methods, frequently employed for finding ze-  
ros of sums of maximally monotone operators have an outstanding success in  
the context of closed convex non-empty sets. However, when one of the sets is  
30 non-convex their convergence is only observed in practice [18].

There is, however, a different line of research which makes an appeal to  
the LQ control theory [19], [20], [21]. This approach stems from the key idea  
that every stabilizing static state feedback minimizes some quadratic cost func-  
tional. The state feedback can be related to some specific algebraic Riccati  
35 equations [22], [23] linked to LQ cost functionals with a cross weighting term.  
Taking advantage of these well-known results and the Finsler's lemma [24], the  
present paper proposes a new necessary and sufficient condition for structured

feedback stabilizability of linear systems. The latter, presented in the form of a nonlinear matrix equation, is solved directly and efficiently through a non-smooth Newton's iterative method. In more detail, the nonlinear equation is a matrix-valued function in a positive definite matrix. Hence, a projection function onto the cone of positive definite matrices is included. This projection is a non-smooth matrix-valued function. Hence, a particular Newton's method, proposed for instance in [25] and [26] and making use of an explicit form of the Clarke generalized Jacobian of said projection, is adapted to solve this problem.

Finally, authors draw attention to some notable differences with the well-known non-smooth optimization based methods such those in [27] or [28]. These methods generally deal with a min-max optimization of the closed-loop matrix spectral abscissa (which is a non-smooth function). Hence, they optimize directly the gain matrix without searching for any Lyapunov matrix. In contrast, the proposed non-smooth Newton's method search for the matrix gain indirectly through the Lyapunov matrix. It is, however, easily implementable, modified at will and suitable for Riccati-like equations which represents a potential advantage in the case of large scale systems. Local convergence is established in both cases.

*Notations:* The following notations are used in this paper.  $Z_{ij}$  are all  $(i, j)^{th}$  entries,  $Z^T$  the transpose and  $Z^\dagger$  an outer inverse (such as Moore-Penrose inverse) of a given real matrix  $Z \in \mathbb{R}^{n \times m}$ .  $Z > 0$  ( $Z \geq$ ) means that  $Z$  is a positive-definite (positive semi-definite) matrix.  $vec(Z)$  denotes the vec-operator that stacks the columns of  $Z$  into a vector.  $\|Z\| = (\sum_{i,j} Z_{ij}^2)^{\frac{1}{2}}$  designates the Frobenius norm.  $\mathbb{S}_n$  the space of real symmetric matrices of dimensions  $n \times n$  and  $\mathbb{S}_n^+$  ( $\mathbb{S}_n^{+*}$ ) is the cone of real symmetric positive semi-definite (respectively definite) matrices of dimensions  $n \times n$ .  $\odot$  denotes the Hadamard product (or direct product) of matrices. The matrix  $diag(Z_1, Z_2, \dots, Z_q)$  represents the block-diagonal matrix with matrices  $Z_1, Z_2, \dots, Z_q$  as diagonal blocks. For a symmetric matrix represented block-wise, off-diagonal blocks are abbreviated with  $\bullet$ . Finally,  $I_n$  ( $0_n$ ) designates, henceforth, the identity (zero) matrix of dimensions  $n \times n$ .

## 2. Structured stabilizability

Let us consider a linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (1)$$

and a static state feedback

$$u = Kx \quad (2)$$

One of the most fundamental problems in control theory is the so-called infinite horizon LQ problem of finding a control law (2) such that the closed-loop system  $A_c := A + BK$  is asymptotically stable and minimizes a cost functional

$$J := \int_0^\infty (x^\top Qx + 2u^\top Lx + u^\top Ru) dt \quad (3)$$

with  $R \in \mathbb{S}_n^{+*}$ ,  $(Q - LR^{-1}L^\top) \in \mathbb{S}_n^+$ ,  $x(0) = x_0$ . In such a case, we cannot dispense with the standing assumptions that the pair  $(A, B)$  is stabilizable and the pair  $(A, Q)$  is detectable. The solution is then given by

$$K = -R^{-1}(B^\top P + L) \quad (4)$$

where the  $P \in \mathbb{S}_n^{+*}$  is the solution of the algebraic Riccati equation

$$A^\top P + PA - (B^\top P + L)^\top R^{-1}(B^\top P + L) + Q = 0 \quad (5)$$

Our aim is to propose a parametrization of all structured stabilizing state  
70 feedback gains. Since the standard Riccati equation (5) involves products between the Lyapunov matrix  $P$  and the cross weighting matrix  $L$ , the emphasis is placed on a different LQ cost function. Admittedly, this new cost function is with a more complex form than (3) (we will see that it includes bilinear products of  $K$  and the cross weighting matrix  $L$ ), but it leads to a matrix equation with  
75 no product between  $P$  and  $L$ .

As a first step, Theorem 1 presents a necessary and sufficient condition for the parametrization of stabilizing state feedback gains.

**Theorem 1** *Suppose that  $(A, B)$  is stabilizable.*

$$K = -B^\top P + L \quad (6)$$

is a state feedback that stabilizes system (1) if and only if there exist a parameter matrix  $L \in \mathbb{R}^{m \times n}$  and a matrix  $P \in \mathbb{S}_n^+$  such that for a given matrix  $Q \in \mathbb{S}_n^+$  the following equation holds

$$A^\top P + PA - PBB^\top P + Q + L^\top L = 0 \quad (7)$$

**Proof (Necessity)** Suppose that, for a given state feedback  $K$ ,  $A_c = A + BK$  is stable. Since  $I_n + K^\top K$  is a positive semi-definite matrix, then, according to [29] Lemma 12.1, there exists a unique matrix  $P \in \mathbb{S}^+$  such that

$$(A + BK)^\top P + P(A + BK) + I_n + K^\top K = 0 \quad (8)$$

rearranging the last equation, we obtain

$$A^\top P + PA - PBB^\top P + I_n + (K + B^\top P)^\top (K + B^\top P) = 0 \quad (9)$$

Hence, setting  $L = K + B^\top P$  and  $Q = I_n$  implies that (7) holds.

(Sufficiency) Suppose that (6) and (7) hold. It follows that

$$A^\top P + PA - PBB^\top P + Q + (K + B^\top P)^\top (K + B^\top P) = 0 \quad (10)$$

which means that

$$(A + BK)^\top P + P(A + BK) + Q + K^\top K = 0. \quad (11)$$

for some matrix  $P \in \mathbb{S}^+$ . (11) leads to the stability of  $A_c$ .

80 **Remark 1** At first glance the result in Theorem 1 may seem to be more or less similar to the results in [23] and [30]. Both cases, however, do not share the same weighting matrices as in (7) that is  $Q$  is free and  $R$  is fixed to identity.

Note that the condition (7) implies that the state-feedback gain (6) minimizes the non-standard cost functional

$$\bar{J} = \int_0^\infty (x^\top (Q + K^\top L + L^\top K)x - 2u^\top Lx + u^\top u) dt \quad (12)$$

This cost functional has a notable difference with (3) since it leads to equation (7) with a separation between the parameter  $L$  and the Lyapunov matrix  $P$  85 which is not the case in (5).

**Remark 2** *Another perceived advantage of the cost functional (12), when compared to (3), is that only the weighting matrix  $Q$  has to be chosen. For this purpose, clearly the best alternative is to reduce this choice to a single tuning positive scalar. In [31], the following pertinent choice  $Q := (\tau G_c(\tau))^{-1}$  where*  
 90  *$G_c(\tau)$  is the partial controllability Gramian associated to the pair  $(A, B)$ , is proposed.*

On another note, a zero-nonzero structure constraint is introduced by a matrix  $S \in \mathbb{R}^{m \times n}$ ,  $S_{ij} \in \{0, 1\}$  and the Hadamard product of matrices such that:

$$K \odot S = 0 \quad (13)$$

For easier reading, a specific notation for structured matrices will be used. For a given matrix  $Z \in \mathbb{R}^{m \times n}$ ,

$$(Z)_S = Z \odot S, \quad (Z)_{\bar{S}} = Z \odot \bar{S} := Z \odot (\Omega - S) \quad (14)$$

where  $\Omega_{ij} = 1, i = 1, \dots, m, j = 1, \dots, n$ .

Let us recall an excerpt of the well-known Finsler's lemma which is originally attributed to (Finsler, 1937) [32].

95 **Lemma 1** *Let  $\Phi = \Phi^\top \in \mathbb{R}^{n \times n}$  and  $\Gamma \in \mathbb{R}^{n \times n}$  such that  $\text{rank}(\Gamma) < n$ . The following statements are equivalent*

- $\exists \mu \in \mathbb{R}^{+*} : \Phi - \mu \Gamma^\top \Gamma < 0$
- $\exists \Psi \in \mathbb{R}^{n \times m} : \Phi + \Psi \Gamma + \Gamma^\top \Psi^\top < 0$

Lemma 1 has many available proofs in the literature as, for instance, in [24].

100 Now we are ready to introduce a new necessary and sufficient condition for the parametrization of structured stabilizing state-feedback gains.

**Theorem 2** *Assume that the pair  $(A, B)$  is stabilizable. System (1) admits a stabilizing structured state-feedback, with respect to a structure matrix  $S$ , given by*

$$K = -B^\top P + (B^\top P)_S \quad (15)$$

if and only if there exist matrices  $Q \in \mathbb{S}_n^+$ ,  $P \in \mathbb{S}_n^{+*}$  such that

$$A^\top P + PA - PBB^\top P + Q + (B^\top P)_S^\top (B^\top P)_S = 0 \quad (16)$$

**Proof** (Sufficiency) Suppose that (16) holds. According to Theorem 1,  $K = -B^\top P + L$  with  $L := (B^\top P)_S$  is a state feedback that stabilizes system (1). (Necessity) Suppose that  $K = -B^\top P + (B^\top P)_S$  is stabilizing for system (16). Then according to Theorem 1, since  $K$  is stabilizing there exist matrices  $L \in \mathbb{R}^{m \times n}$  and  $\bar{P} \in \mathbb{S}_n^{+*}$  such that for a given matrix  $\bar{Q} \in \mathbb{S}_n^+$  the following equation holds,

$$A^\top \bar{P} + \bar{P}A - \bar{P}BB^\top \bar{P} + \bar{Q} + L^\top L = 0 \quad (17)$$

with  $K = (-B^\top \bar{P} + L)$ . Moreover,  $K$  is supposed to be structured with respect to  $S$ . Thus,  $K \odot S = 0$  which means that  $(L)_S = (B^\top \bar{P})_S$ . Since  $L = (L)_S + (L)_{\bar{S}}$ , (17) writes

$$\begin{aligned} A^\top \bar{P} + \bar{P}A - \bar{P}BB^\top \bar{P} + \bar{Q} + (L)_S^\top (L)_S + (L)_{\bar{S}}^\top (L)_{\bar{S}} \\ + (L)_S^\top (L)_{\bar{S}} + (L)_{\bar{S}}^\top (L)_S = 0 \end{aligned} \quad (18)$$

Hence, for a sufficiently small  $\epsilon > 0$

$$\begin{aligned} A^\top \bar{P} + \bar{P}A - \bar{P}BB^\top \bar{P} + \bar{Q} + (L)_S^\top (L)_S + (L)_{\bar{S}}^\top (L)_{\bar{S}} \\ + (L)_S^\top (L)_{\bar{S}} + (L)_{\bar{S}}^\top (L)_S - \epsilon I_n < 0 \end{aligned} \quad (19)$$

which is equivalent, in virtue of Finsler's lemma 1, to

$$\begin{aligned} A^\top \bar{P} + \bar{P}A - \bar{P}BB^\top \bar{P} + \bar{Q} + (L)_S^\top (L)_S + (L)_{\bar{S}}^\top (L)_{\bar{S}} \\ - \mu (L)_{\bar{S}}^\top (L)_{\bar{S}} - \epsilon I_n < 0 \end{aligned} \quad (20)$$

for some  $\mu \in \mathbb{R}^{+*}$ . Since the terms  $(L)_{\bar{S}}^\top (L)_{\bar{S}}$ ,  $\mu (L)_{\bar{S}}^\top (L)_{\bar{S}}$  and  $\epsilon I_n$  are all semi-definite positive, it is always possible to choose  $\bar{Q}$  verifying the following inequality:

$$\bar{Q} + (L)_{\bar{S}}^\top (L)_{\bar{S}} \geq \mu (L)_{\bar{S}}^\top (L)_{\bar{S}} + \epsilon I_n. \quad (21)$$

Let  $\tilde{Q} := \bar{Q} + (1 - \mu)(L)_{\bar{S}}^\top (L)_{\bar{S}} - \epsilon I_n \geq 0$ . The latter along with inequality (20) ensure the existence of a positive definite matrix  $\hat{Q} \in \mathbb{S}_n^{+*}$  such that

$$A^\top \bar{P} + \bar{P}A - \bar{P}BB^\top \bar{P} + (L)_S^\top (L)_S + \tilde{Q} = -\hat{Q}$$

Hence, taking  $Q = \tilde{Q} + \hat{Q} \geq 0$  leads to (16). According to Theorem 1, this means also that  $K^* = -B^\top \bar{P} + (B^\top \bar{P})_S$  is stabilizing and verifies  $K^* \odot S = 0$ . This ends the proof.

### 105 3. A nonsmooth Newton's method

Let us first consider the following notation

$$\begin{aligned} F(P) &:= A^\top P + PA - PBB^\top P + Q + (B^\top P)_S^\top (B^\top P)_S \\ F(P) &= 0, \quad P \in \mathbb{S}_n^+ \end{aligned} \quad (22)$$

where  $F : \mathbb{S}_n \rightarrow \mathbb{S}_n$  is a continuously differentiable function and  $\mathcal{P}_+ : \mathbb{S}_n \rightarrow \mathbb{S}_n^+$  denotes the orthogonal projection of symmetric real matrices on the cone  $\mathbb{S}_n^+$ . Equation (22) can be addressed through the following equations in  $P \in \mathbb{S}_n$ :

$$\begin{cases} F(P) = 0 & (i) \\ G(P) := \mathcal{P}_+(P) - P = 0 & (ii) \end{cases} \quad (23)$$

Therefore, some properties of the function  $\mathcal{P}_+(\cdot)$  should be recalled.

#### 3.1. Differentiability of the projection

Let  $X = T\Lambda T^\top$  be a spectral decomposition of  $X \in \mathbb{S}_n$  where  $\Lambda$  is the diagonal matrix of eigenvalues of  $X$  and  $T$  is a corresponding orthogonal matrix of orthonormal eigenvectors. The said projection is given by

$$\mathcal{P}_+(X) = T\Lambda_+T^\top \quad (24)$$

where  $\Lambda_+$  is a diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of  $\Lambda$  that is

$$(\Lambda_+)_{ij} := \max(\Lambda_{ij}, 0).$$

Hence, the function  $\mathcal{P}_+(\cdot)$  is non-smooth. Leveraging on these facts, a particular Newton's like method based on an explicit form of the Clarke generalized  
 110 Jacobian of the projection  $\mathcal{P}_+(\cdot)$ , proposed for instance in [33], [25] or [26], is used hereafter to solve equation (23).

**Proposition 1** [34], [35]  $\mathcal{P}_+(\cdot)$  is Fréchet differentiable at  $X$  in  $\mathbb{S}_n$  if and only if  $X$  is non-singular. Moreover,  $\mathcal{P}_+(\cdot)$  is directionally differentiable everywhere in  $\mathbb{S}_n$ .

115 Furthermore, [33], [36] (for instance) gave *inter alia* an explicit formula for the Clarke's generalized Jacobian  $\partial\mathcal{P}_+$  of  $\mathcal{P}_+(\cdot)$ . Based on this a directional derivative of  $\mathcal{P}_+(\cdot)$  at  $X$  for a given direction  $D$ , denoted henceforth  $\mathcal{P}'_+(X, D)$ , can be obtained. The computation of said derivative is summarized briefly in what follows.

Consider the spectral decomposition (24) of a given matrix  $X \in \mathbb{S}_n$ . Let  $\lambda_k$  be the eigenvalues of  $X$ . Subsequently, define three sets of positive, zero and negative eigenvalues of  $X$  as follows:

$$\lambda^p := \{k : \lambda_k > 0\}, \lambda^0 := \{k : \lambda_k = 0\}, \lambda^n := \{k : \lambda_k < 0\}. \quad (25)$$

On the basis of this decomposition one can write

$$\Lambda = \text{diag}(\Lambda_{\lambda^p}, 0, \Lambda_{\lambda^n}), \quad T = [T_{\lambda^p}, T_{\lambda^0}, T_{\lambda^n}] \quad (26)$$

Let us also define the matrix  $U \in \mathbb{S}_n$  with

$$U_{ij} := \frac{\max(\lambda_i, 0) + \max(\lambda_j, 0)}{|\lambda_i| + |\lambda_j|}, \quad i, j = 1, \dots, n. \quad (27)$$

Hence, a directional derivative is given by

$$\mathcal{P}'_+(X, D) := TD_+T^\top$$

$$D_+ := \begin{bmatrix} \tilde{D}_{\lambda^p\lambda^p} & \tilde{D}_{\lambda^p\lambda^0} & U_{\lambda^p\lambda^n} \odot \tilde{D}_{\lambda^p\lambda^n} \\ \bullet & \mathcal{P}_+(\tilde{D}_{\lambda^p\lambda^0}) & 0 \\ \bullet & \bullet & 0 \end{bmatrix} \quad (28)$$

120 where  $\tilde{D} := T^\top DT$ .

**Remark 3** The projection  $\mathcal{P}_+(\tilde{D}_{\lambda^p\lambda^0})$  in (28) can be efficiently computed using a Schur decomposition or a matrix sign based method.

**Remark 4** Note that in [33], for instance, it is stated that either the identity or the zero mappings can replace the projection  $\mathcal{P}_+(\tilde{D}_{\lambda^p\lambda^0})$  in (28). This 125 simplifies the computation of the directional derivative (28).

As noted in Remark 4, the derivative can be expressed as a linear mapping in  $\tilde{D}$  using Hadamard product as follows

$$T^\top(\mathcal{P}'_+(X, D))T = \tilde{D} \odot \Gamma_1 + U_{\lambda^p \lambda^n} \odot \tilde{D} \odot \Gamma_2 \quad (29)$$

where

$$\Gamma_1 := \text{diag}(\Lambda_{\lambda^p \lambda^0}, 0_{\lambda^n}), \quad \Gamma_2 := \begin{bmatrix} 0_{\lambda^p} & 0 & \Lambda_{\lambda^n} \\ 0 & 0_{\lambda^0} & 0 \\ 0 & 0 & 0_{\lambda^n} \end{bmatrix}.$$

### 3.2. A modified Newton's method

The goal henceforth is to find a symmetric matrix  $P$  verifying (23). For this purpose, the use of a Newton's method is intended. Hence, one has first to compute the Fréchet derivative at  $P$  of  $F(\cdot)$ .

**Definition 1** *The Fréchet derivative of function  $F : \mathbb{S}_n \rightarrow \mathbb{S}_n$  at a matrix  $X \in \mathbb{S}_n$  is a linear mapping:*

$$\begin{aligned} \mathbb{S}_n &\rightarrow \mathbb{S}_n \\ \Delta &\rightarrow L_F(X, \Delta) \end{aligned}$$

such that for all  $\Delta \in \mathbb{S}_n$

$$F(X + \Delta) - F(X) - L_F(X, \Delta) = \mathcal{O}(\|\Delta\|) \quad (30)$$

The Fréchet derivative therefore describes the first order effect on  $F$  of perturbations in  $X$ . For the equation (i) in (23), this derivative is given by:

$$\begin{aligned} F'(P)\Delta &= L_F(P, \Delta) \\ F'(P)\Delta &= (A - BB^\top P)^\top \Delta + \Delta(A - BB^\top P) \\ &\quad + (B^\top P)_S^\top (B^\top \Delta)_S + (B^\top \Delta)_S^\top (B^\top P)_S. \end{aligned} \quad (31)$$

In Newton's method, the next iterate can be obtained by solving the equation

$$F'(P_i)\Delta_i = -F(P_i) \quad (32)$$

and updating by

$$P_{i+1} = P_i + \Delta_i \quad (33)$$

Following the same lines, for the equation (ii) in (23), a directional derivative is given by:

$$(T^\top(G(P_i))T)'\Delta = T^\top(\mathcal{P}'_+(P_i, \Delta) - I)T \quad (34)$$

130 On the strength of these results a generalized non-smooth Newton type algorithm for solving (23) is proposed.

**Remark 5** Note that for some  $\delta > 0$ , if  $F(P) + \delta G(P) = 0$  for  $P \in \mathbb{S}_+^n$  then (16) holds for some matrix  $Q' = Q + \delta G(P) \in \mathbb{S}_+^n$  and  $P \in \mathbb{S}_+^n$ . Based on this observation, Algorithm 1 solves the equation  $F(P) + \delta G(P) = 0$  instead of the  
135 matrix equation system (23).

---

**Algorithm 1** Non-smooth Newton Algorithm

---

1: Initialization : For a predefined  $\delta > 0$ , choose  $P_1 \in \mathbb{S}_n$  and  $\varepsilon$  a sufficiently small positive real.

2: Determine solutions  $\Delta_i^1 \in \mathbb{S}_n$ ,  $\Delta_i^2 \in \mathbb{S}_n$  of the following linear matrix equations

$$\begin{aligned} F'(P_i)\Delta_i^1 &= -F(P_i) \\ T^\top(\mathcal{P}'_+(P_i, \Delta_i^2) - I)T\Delta_i^2 &= -T^\top(G(P_i))T. \end{aligned} \quad (35)$$

3: Update

$$P_{i+1} = P_i + (\Delta_i^1 + \delta\Delta_i^2) \quad (36)$$

4: If  $\|P_{i+1} - P_i\| > \varepsilon$ , set  $i = i + 1$  and return to Step 2 else Stop.▷ return  $P^*$

---

**Theorem 3** Algorithm 1 is quadratically convergent in a neighborhood of a local minimizer  $P^*$ .

**Proof** The local convergence properties of Algorithm 1 are formulated by analyzing smoothness properties of the functions  $F$  and  $G$ . On one side, the map  
140  $F(\cdot)$  is differentiable and has a Lipschitz continuous derivative. On the other side, it is stated in [36] that  $G(\cdot)$  is a strongly semi-smooth matrix function (interested readers can find the definition of this property in [25]). Hence,  $F(\cdot) + \delta G(\cdot)$ ,  $\delta > 0$  is a strongly semi-smooth function. The latter property

ensures the quadratic convergence, in a neighborhood of a local minimizer, according to [25].

**Remark 6** A part from the LMI based method in [10] which proves to be rather powerful and has the ability to eliminate the rigid block-diagonal structural constraints on the Lyapunov matrix, the proposed non-smooth Newton's approach has many advantages when compared to LMI based methods. One can quote, without claiming to be exhaustive, the fact that it relies on an original parametrization of all structured stabilizing state-feedbacks using a Riccati-like equation (which is potentially suitable for large scale systems) and it does not need neither initialization point nor relaxations (that allow to deal with a convex problem instead of the original non-convex one).

#### 4. Numerical examples

For all the following examples, the weighting matrix  $Q$  is fixed, according to Remark 2, such as  $Q := (\tau G_c(\tau))^{-1}$  with  $\tau = 1$ .

##### 4.1. Example 1

This numerical example is borrowed from [4]. Let us consider a system of the form (1) where

$$A = \begin{bmatrix} -1.75 & 1.5 & 0 & 0 & 0 & 0 \\ 0.25 & -0.25 & 0.5 & 0.5 & -0.25 & 0.5 \\ 1 & 1.5 & -2.75 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & -2 & 0 \\ 0.5 & 0.5 & -0.25 & -0.25 & -2.75 & -1 \\ 0 & 0 & 0 & 3 & -3 & -3.75 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (37)$$

Matrix  $A$  has a pair of unstable eigenvalues  $\{0.268 \pm 0.577i\}$ . Our objective is to design a stabilizing state-feedback (2) with a matrix gain subject to the specific nonzero pattern

$$K = \begin{bmatrix} 0 & * & 0 & 0 & 0 & * \\ 0 & * & * & * & * & 0 \end{bmatrix} \quad (38)$$

corresponding to a structure matrix  $S$  of the form :

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (39)$$

Algorithm 1 is applied with  $\delta = 1$ ,  $\varepsilon = 10^{-10}$ . Moreover,  $P_1$  is fixed with all entries chosen randomly in  $] -10^3, 10^3[$ .  $P_1$  has eigenvalues with positive and negative real parts  $10^3\{-1.4217, -0.7813, 0.0475, 1.5205, 2.9206, 3.9483\}$ . 26 iterations are needed to obtain a positive definite solution  $P^*$  with eigenvalues  $10^3\{0.0059, 0.0062, 0.2173, 0.4250, 1.5610, 4.8641\}$  and a stabilizing structured feedback gain :

$$K^* = -B^\top P^* + (B^\top P^*)_S = \begin{bmatrix} 0 & 36.3395 & 0 & 0 & 0 & 0.6041 \\ 0 & 11.4518 & 0.6041 & 12.4612 & -11.6747 & 0 \end{bmatrix} \quad (40)$$

The closed-loop matrix  $A_c = A - BK^*$  has stable eigenvalues  $\{-2.0005 \pm 4.8587i, -1.7356 \pm 1.7773i, -1.6656, -2.8622\}$ .

#### 4.2. Example 2

The following example is borrowed from [2] and also considered in [10]. It deals with the control with overlapping information structure constraints for a system of the form (1) with the following data

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top. \quad (41)$$

The objective is to design a stabilizing state feedback gain subject to the nonzero pattern

$$K = \begin{bmatrix} * & * & 0 \\ 0 & * & * \end{bmatrix} \quad (42)$$

corresponding to a structure matrix  $S$  of the form :

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (43)$$

For this aim, Algorithm 1 is applied with  $\delta = 1$ ,  $\varepsilon = 10^{-10}$  and initialization matrix  $P_1$  with all entries chosen randomly in  $]-10^3, 10^3[$

$$P_1 = 10^3 \begin{bmatrix} -0.4116 & -0.3805 & -1.7158 \\ \bullet & -1.6339 & 0.3758 \\ \bullet & \bullet & 1.1589 \end{bmatrix}.$$

$P_1$  has eigenvalues with positive and negative real parts

$$\sigma(P_1) = 10^3 \{-1.7578, -1.4582, 2.3294\}.$$

After 28 iterations, one gets a positive definite solution  $P^*$

$$P^* = \begin{bmatrix} 6.0749 & 11.8611 & 2.8537 \\ \bullet & 49.1367 & 0.1159 \\ \bullet & \bullet & 10.1896 \end{bmatrix}$$

with eigenvalues  $\{3.0195, 7.5410, 54.8407\}$  from which a stabilizing structured state feedback gain is obtained:

$$K^* = \begin{bmatrix} 6.0749 & 11.8611 & 0 \\ 0 & 10.5042 & 10.1896 \end{bmatrix}. \quad (44)$$

The closed-loop matrix  $A_c = A - BK^*$  has stable eigenvalues  $\{-5.8242, -2.2202 \pm 3.7466i\}$ .

#### 4.3. Example 3

The following feedback control under structure information case is borrowed from [37]. Consider the system (1) with the following data

$$A = \begin{bmatrix} -4 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 3 & 0 & -2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The objective is to design a stabilizing state feedback gain subject to the nonzero pattern

$$K = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \quad (45)$$

corresponding to a structure matrix  $S$  of the form :

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad (46)$$

This structured case corresponds to a decentralized output feedback case where  $y = Cx$  with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By applying Algorithm 1 with  $\delta = 1$ ,  $\varepsilon = 10^{-10}$  and an initialization matrix

$$P_1 = 10^3 \cdot \begin{bmatrix} 0.1002 & -0.9869 & 0.9581 & -0.3222 & 0.3648 \\ \bullet & -0.9785 & -1.6319 & 0.3626 & 0.6498 \\ \bullet & \bullet & 1.6844 & 1.0932 & 0.5852 \\ \bullet & \bullet & \bullet & -1.0853 & -0.5875 \\ \bullet & \bullet & \bullet & \bullet & -0.3238 \end{bmatrix}.$$

The eigenvalues of  $P_1$  has positive and negative real parts

$$\sigma(P_1) = 10^3 \{-2.7906, -1.3115, -0.0142, 0.3667, 3.1467\}.$$

After 40 iterations, one gets a positive-definite solution

$$P^* = \begin{bmatrix} 4.7023 & -3.4023 & 0.2574 & 0.2627 & 0.8838 \\ \bullet & 3.8559 & -1.5033 & -0.2481 & -0.1571 \\ \bullet & \bullet & 22.9771 & 0.3788 & -1.0026 \\ \bullet & \bullet & \bullet & 1.9112 & 0.1139 \\ \bullet & \bullet & \bullet & \bullet & 2.3128 \end{bmatrix}$$

with eigenvalues  $\{0.6905, 1.8683, 2.2708, 7.7679, 23.1616\}$  from which a stabilizing structured state-feedback  $u = K^*x$  is obtained such that:

$$K^* = \begin{bmatrix} 1.3 & 0 & 0 & 0 & 0 \\ 0 & -0.2481 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.3128 \end{bmatrix}. \quad (47)$$

Note that this structured state feedback verifies :  $K^* = K_D C$  with  $K_D$  a decentralized output feedback  $u = K_D y$  with:

$$K_D = \begin{bmatrix} 1.3 & 0 & 0 \\ 0 & -0.2481 & 0 \\ 0 & 0 & 2.3128 \end{bmatrix}. \quad (48)$$

<sup>165</sup> The closed-loop matrix  $A_c = A - BK_D C$  has stable eigenvalues  $\{-4.9454 \pm 1.0885i, -0.7221, -1.5000 \pm 1.8039i\}$ .

#### 4.4. Example 4

A formation of  $q = 10$  vehicles in a plane is considered. The aim is to keep constant distances between each two successive vehicles. A well-known model of these independently actuated vehicles using double-integrators, in both horizontal and vertical directions is given by the following state-space representation

$$A = \text{diag}(A_1, \dots, A_q), \quad B = \text{diag}(B_1, \dots, B_q) \quad (49)$$

$$A_i = \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}, \quad i = 1, \dots, q$$

A decentralized state feedback controller subject to the nonzero pattern  $K \in \mathbb{R}^{20 \times 40}$ :

$$K = \begin{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} & 0 & 0 \\ & \ddots & 0 \\ & & 0 & \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} \end{bmatrix} \quad (50)$$

has to be found. The method presented in Section III is applied with  $\delta = 1$ ,  $\varepsilon = 10^{-10}$  and  $P_1 \in \mathbb{S}_{40}$  with all entries chosen randomly in  $]-10^3, 10^3[$ . 19 iterations yield to a decentralized stabilizing state feedback of the form (50)  $K_D = \text{diag}(K^*, \dots, K^*)$  with

$$K^* = \begin{bmatrix} 1 & 0 & 1.7321 & 0 \\ 0 & 1 & 0 & 1.7321 \end{bmatrix}.$$

The closed-loop is stable with 20 complex conjugate pairs  $-0.866 \pm 0.5i$ .

## 5. CONCLUSION

170 In this paper, the design of structured stabilizing state feedback gains for  
LTI systems is considered. An appeal to the LQ regulator theory is made. A  
necessary and sufficient condition for stabilizability subject to a fixed pattern  
is given by means of a nonlinear matrix equation. The latter is recast as a  
175 non-smooth unconstrained equation using orthogonal projection onto the pos-  
itive semi-definite matrices cone. Motivated by the property of strongly semi-  
smoothness of this matrix-valued function, a non-smooth Newton's algorithm to  
find a feasible solution is developed. The proposed approach does not require a  
stabilizing controller or a positive definite Riccati equation solution to initialize  
the iterative procedure and its utility is illustrated by several examples.

180 Although the focus is put on stability, in our ongoing efforts we shall en-  
deavor to extended the method to the  $H_\infty$  case using the parametrization of all  
stabilizing  $H_\infty$  state feedbacks.

## References

### References

- 185 [1] D. D. Šiljak, Decentralized Control of Complex Systems, Dover Books on  
Electrical Engineering, 2013.
- [2] A. I. Zečević, D. D. Šiljak, Control of complex systems: structural con-  
straints and uncertainty, Springer, 2010.

- [3] L. Bakule, Decentralized control: An overview, *Annual Reviews in Control* 32 (1) (2008) 87–98.
- [4] A. I. Zečević, D. D. Šiljak, Control design with arbitrary information structure constraints, *Automatica* 44 (10) (2008) 2642 – 2647.
- [5] J. C. Geromel, J. Bernussou, P. L. D. Peres, Decentralized control through parameter space optimization, *Automatica* 30 (10) (1994) 1565–1578.
- [6] P. Grosdidier, M. Morari, Interaction measures for systems under decentralized control, *Automatica* 22 (1986) 309–312.
- [7] M. Hovd, S. Skogestad, Sequential design of decentralized controllers, *Automatica* 30 (10) (1994) 1601–1607.
- [8] D. D. Šiljak, Decentralized control and computations : Status and prospects, *Automatica Reviews in Control* 20 (1996) 131–141.
- [9] A. Iftar, Overlapping decentralized dynamic optimal control, *International Journal of Control* 58 (1993) 187–209.
- [10] F. Ferrante, F. Dabbene, C. Ravazzi, On the design of structured stabilizers for LTI systems, *IEEE Control Systems Letters* 4 (2) (2020) 289–294.
- [11] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Studies in Applied Mathematics, Philadelphia, Pennsylvania, 1994.
- [12] Y. Ebihara, D. Peaucelle, D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*, Communications and Control Engineering, Springer-Verlag London, 2015.
- [13] V. Blondel, J. N. Tsitsiklis, NP-hardness of some linear control design problems, *SIAM Journal of Control and Optimization* (1997) 2118–2127.
- [14] J. Rubio, J. Rossell, H. Karimi, F. Palacios, Static output-feedback control under information structure constraints, *Automatica* 49 (1) (2013) 313–316.

- 215 [15] M. Rotkowitz, S. Lall, A characterization of convex problems in decentralized control, *IEEE Transactions on Automatic Control* 51 (2) (2006) 274–286.
- [16] M. Rotkowitz, S. Lall, On computation of optimal controllers subject to quadratically invariant sparsity constraints, 2004.
- 220 [17] M. Yagoubi, Projection/reflection-based techniques for multi-objective control synthesis under information structure constraints, 2019, pp. 5811–5818.
- [18] H. H. Bauschke, W. Loursi, On the Douglas-Rachford algorithm, *Mathematical Programming* 164 (1) (2017) 263–284.
- [19] R. E. Kalman, Contributions to the theory of optimal control, *Bol. Soc. Matem. Mex.* (1960) 102–119.
- 225 [20] R. E. Kalman, When is a linear control system optimal?, *Trans. ASME, Ser.D: J. Basic Eng.*, 86 (1964) 1–10.
- [21] E. Kreindler, A. Jameson, Decentralized control: An overview, *IEEE Transactions on Automatic Control* 17 (3) (1972) 349–351.
- 230 [22] A. Trofino-Neto, V. Kučera, Stabilization via static output feedback, *IEEE Trans. on Automatic Control* 38 (5) (1993) 764–765.
- [23] V. Kučera, C. E. D. Souza, A necessary and sufficient condition for output feedback stabilizability, *Automatica* 31 (9) (1995) 1357–1359.
- [24] M. C. de Oliveira, R. E. Skelton, Stability tests for constrained linear systems., *Perspectives in robust control*. London: Springer-Verlag 268 (2001) 241–257.
- 235 [25] L. Qi, J. Sun, A nonsmooth version of Newtons method, *Mathematical Programming* 58 (1993) 353–367.
- [26] J. S. Pang, D. Sun, J. Sun, Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems, *Mathematics of Operations Research* 28 (2003) 39–63.
- 240

- [27] J. V. Burke, A. S. Lewis., M. L. Overton, Optimization and pseudospectra, with applications to robust stability, *SIAM Journal on Matrix Analysis and Applications* 25 (1) (2003) 80–104. doi:10.1137/S0895479802402818.
- 245 [28] S. Gumussoy, D. Henrion, M. Millstone, M. L. Overton, Multiobjective robust control with hifoo 2.0, *IFAC Proceedings Volumes* 42 (6) (2009) 144 – 149, 6th IFAC Symposium on Robust Control Design.
- [29] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. Springer-Verlag, New York, 2015.
- 250 [30] J. Gadewadikar, F. L. Lewis, L. Xie, V. Kucera, M. Abu-Khalaf, Parameterization of all stabilizing  $H_\infty$  static state-feedback gains: Application to output-feedback design, *Automatica* 43 (9) (2007) 1597 – 1604.
- [31] P. D. Larminat, *Analysis and control of linear systems*, ISTE, London, 2007.
- 255 [32] P. Finsler, Über das vorkommen definiten und semidefiniten formen in scharen quadratischer formen, *Math. Helvetici* 9 (1997) 188–192.
- [33] H. S. S. J. Malick, Clarke generalized Jacobian of the projection onto the cone of positive semidefinite matrices, *Set-Valued Analysis*, Springer 14 (2006) 273–293.
- 260 [34] J. F. Bonnans, R. Cominetti, A. Shapiro, Sensitivity analysis of optimization problems under second order regularity constraints, *Mathematics of Operations Research* 23 (1998) 803–832.
- [35] J. F. Bonnans, R. Cominetti, A. Shapiro, Second order optimality conditions based on parabolic second order tangent sets, *SIAM Journal on Optimization* 9 (1999) 466–493.
- 265 [36] D. Sun, The strong second-order sufficient condition and constraint non-degeneracy in nonlinear semidefinite programming and their implications, *Mathematics of Operations Research* 31 (4) (2006) 761–776.

- [37] J. Rubió-Massegú, J. Rossell, H. Karimi, F. Palacios-Quinonero, Static  
270 output-feedback control under information structure constraints, *Automatica* 49 (1) (2013) 313 – 316.