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# On the Cheeger inequality for convex sets

Ilias Ftouhi

November 23, 2020

## Abstract

In this paper, we prove new sharp bounds for the Cheeger constant of planar convex sets that we use to study the relations between the Cheeger constant and the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. This problem is closely related to the study of the so-called Cheeger inequality for which we provide an improvement in the class of planar convex sets. We then provide an existence theorem that highlights the tight relation between improving the Cheeger inequality and proving the existence of a minimizer of the functional  $J_n := \lambda_1/h^2$  in any dimension  $n$ . We finally, provide some new sharp bounds for the first Dirichlet eigenvalue of planar convex sets and a new sharp upper bound for triangles which is better than the conjecture stated in [32] in the case of thin triangles.

## Contents

<b>1</b>	<b>Introduction and main results</b>	<b>1</b>
<b>2</b>	<b>Sharp estimates for the Cheeger constant: Proof of Theorem 1.2</b>	<b>4</b>
<b>3</b>	<b>Improving the Cheeger inequality for planar convex sets</b>	<b>5</b>
3.1	Proof of Theorem 1.1 . . . . .	5
3.2	A slight improvement of the result of Theorem 1.1 . . . . .	6
3.3	Improvements for special classes of shapes . . . . .	7
<b>4</b>	<b>On the existence of a minimizer in higher dimensions</b>	<b>8</b>
4.1	Proof of Theorem 1.3 . . . . .	8
4.2	Discussion of the hypothesis $\beta_n < \beta_{n-1}$ . . . . .	11
<b>5</b>	<b>Appendix: Some applications</b>	<b>13</b>
5.1	Some sharp upper bounds for the first Dirichlet eigenvalue . . . . .	13
5.1.1	General planar convex sets . . . . .	13
5.1.2	Sets that are homothetic to their form bodies: in particular "triangles" . . . . .	14
5.2	A sharp Cheeger-type inequality . . . . .	15

## 1 Introduction and main results

A celebrated inequality due to Jeff Cheeger states that for every open bounded set  $\Omega \in \mathbb{R}^n$  (where  $n \geq 2$ ) one has:

$$\lambda_1(\Omega) \geq \frac{1}{4}h(\Omega)^2,$$

where  $\lambda_1(\Omega)$  is the first Dirichlet eigenvalue and  $h(\Omega)$  is the Cheeger constant of  $\Omega$ , which is defined as follows:

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \mid E \text{ measurable and } E \subset \Omega \right\}, \quad (1)$$

where  $P(E)$  is the perimeter of De-Giorgi of  $E$  measured with respect to  $\mathbb{R}^n$  (see for example [26] for definitions) and  $|E|$  is the  $n$ -dimensional Lebesgue measure of  $E$ . Any set  $C_\Omega \subset \Omega$  for which the infimum is attained is called (when it exists) a *Cheeger set* of  $\Omega$ . We refer to [26] for an introduction to the Cheeger problem.

In the present paper,  $d$  and  $r$  respectively correspond to the diameter and the inradius functionals.

Recently, E. Parini [27] remarked that the constant  $\frac{1}{4}$  can be improved for the class  $\mathcal{K}^2$  (for every  $n \in \mathbb{N}^*$ , we denote  $\mathcal{K}^n$  the class of bounded convex subsets of  $\mathbb{R}^n$ ). He proved the following inequality:

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{16} h(\Omega)^2, \quad (2)$$

and noted that the constant  $\frac{\pi^2}{16}$  is also not optimal. He then took a shape optimization point of view by introducing the functional  $J_2 : \Omega \in \mathcal{K}^2 \mapsto J_2(\Omega) := \frac{\lambda_1(\Omega)}{h(\Omega)^2}$  for which he proves the existence of a minimizer in  $\mathcal{K}^2$  and conjectures that it is the square; in which case the optimal lower bound would be given by:

$$\min_{\Omega \in \mathcal{K}^2} J_2(\Omega) = J_2((0, 1)) = \frac{2\pi^2}{(2 + \sqrt{\pi})^2} \approx 1.387\dots$$

Nevertheless, as far as we know, as mentioned in [27, Section 6] the existence of an optimal shape in higher dimensions ( $n \geq 3$ ) remains open.

Moreover, in the same paper [27], the author proved the following reverse Cheeger's inequality:

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) < \frac{\pi^2}{4} h(\Omega)^2, \quad (3)$$

which is sharp as it is asymptotically attained by any sequence  $(\Omega_k)_{k \in \mathbb{N}}$  of planar convex sets such that  $|\Omega_k| = V$  (where  $V > 0$ ) and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ , see [27, Proposition 4.1]. It was then remarked by L. Brasco [7, Remark 1]

that the main argument used by Parini, which is Polya's inequality  $\lambda_1(\Omega) < \frac{\pi^2}{4} \left( \frac{P(\Omega)}{|\Omega|} \right)^2$  holds in higher dimensions (see [21]). Thus, the reverse Cheeger inequality also holds for higher dimensions and is sharp as any sequence  $(\omega \times (0, 1/k))_k$ , where  $\omega \in \mathcal{K}^{n-1}$ , provides asymptotic equality when  $k$  tends to  $+\infty$ .

For the lower bound, one can combine the inequality  $h(\Omega) \leq \frac{n}{r(\Omega)}$  (which is obtained by taking the inscribed ball  $B_{r(\Omega)}$  as a test set in the definition of the Cheeger constant  $h(\Omega)$ ) and Protter's inequality [29]:

$$\forall \Omega \in \mathcal{K}^n, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{4} \left( \frac{1}{r(\Omega)^2} + \frac{n-1}{d(\Omega)^2} \right), \quad (4)$$

which generalises Hersch's inequality [19] (used by Parini for the planar case) to higher dimensions. We then obtain the following lower bound:

$$\forall n \geq 2, \forall \Omega \in \mathcal{K}^n, \quad J_n(\Omega) := \frac{\lambda_1(\Omega)}{h(\Omega)^2} > \frac{\pi^2}{4n^2},$$

which improves the original constant  $\frac{1}{4}$  given by J. Cheeger only for  $n \in \{2, 3\}$ . In which cases, we have:

$$\forall \Omega \in \mathcal{K}^2, \quad J_2(\Omega) > \frac{\pi^2}{16} \approx 0.616\dots \quad \text{and} \quad \forall \Omega \in \mathcal{K}^3, \quad J_3(\Omega) > \frac{\pi^2}{36} \approx 0.274\dots$$

In the present paper, we improve the Cheeger-Parini's inequality (2). Our result in this direction is stated as follows:

**Theorem 1.1.** *We have:*

$$\forall \Omega \in \mathcal{K}^2, \quad J_2(\Omega) = \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq \left( \frac{\pi j_{01}}{2j_{01} + \pi} \right)^2 \approx 0.902\dots$$

where  $j_{01}$  denotes the first zero of the first Bessel function.

This result relies on the combination of Protter's inequality (4) and a diametrical Faber-Krahn type inequality (see [4, Theorem 2.1] or [13, Theorem 3.3]) in order to bound  $\lambda_1(\Omega)$  from below.

The study of complete systems of inequalities relating some given functionals is an interesting subject for its own. It is closely related to the so called *Blaschke-Santaló* diagrams, we refer to the original works of Blaschke [2] and Santaló [30] and to the more recent works [6, 11, 16, 17, 18] for some interesting examples involving geometric functionals.

In the present paper we provide a complete system of inequalities relating the Cheeger constant  $h$ , the inradius  $r$  and the area  $|\cdot|$  of planar convex sets, which corresponds to a complete description of the related Blaschke-Santaló diagram introduced in the following Theorem:

**Theorem 1.2.** *We have:*

$$\forall \Omega \in \mathcal{K}^2, \quad \frac{1}{r(\Omega)} + \frac{\pi r(\Omega)}{|\Omega|} \leq h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}, \quad (5)$$

*These inequalities are sharp as equalities are obtained for stadiums in the lower estimate and for domains that are homothetic to their form bodies<sup>1</sup> in the upper one.*

*Moreover, we have the following explicit description of the Blaschke-Santaló diagram:*

$$\left\{ \left( \frac{1}{r(\Omega)}, h(\Omega) \right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\} = \left\{ (x, y) \mid x \geq \frac{1}{r(B)} = \sqrt{\pi} \text{ and } x + \frac{\pi}{x} \leq y \leq x + \sqrt{\pi} \right\},$$

where  $B \subset \mathbb{R}^2$  is a ball of unit area.

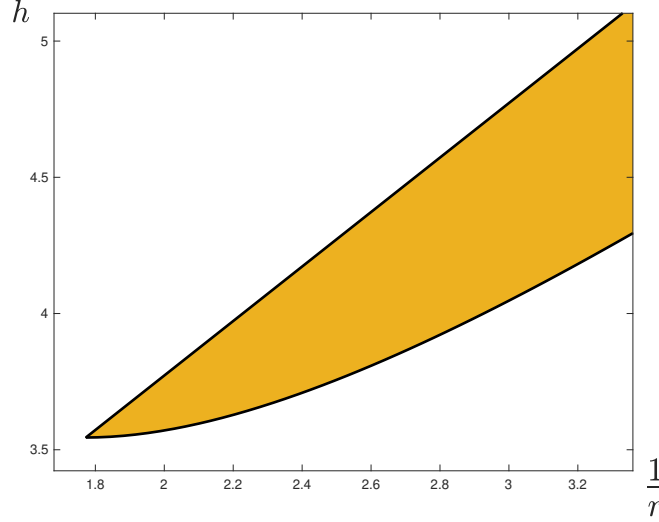


Figure 1: The diagram of the triplet  $(r, h, |\cdot|)$ .

At last, we are interested by the question of the existence of a minimizer of  $J_n$  for higher dimensions  $n \geq 3$ . We prove the following Theorem:

**Theorem 1.3.** *Let us define the real sequence  $(\beta_n)_n$  as follows:*

$$\forall n \in \mathbb{N}^*, \quad \beta_n := \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega).$$

*We have:*

1.  $(\beta_n)_n$  is a decreasing sequence.
2.  $\lim_{n \rightarrow +\infty} \beta_n = \frac{1}{4}$ .
3. For  $n \geq 2$ , if the strict inequality  $\beta_n < \beta_{n-1}$  holds, we have the following existence result:

$$\exists \Omega_n^* \in \mathcal{K}^n, \quad J_n(\Omega_n^*) = \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega).$$

Let us give a few interesting comments on Theorem 1.3:

- The convergence result  $\lim_{n \rightarrow +\infty} \beta_n = \frac{1}{4}$  shows that the constant  $\frac{1}{4}$  given in the original Cheeger inequality [10] is optimal in the sense that there exists no constant  $C > \frac{1}{4}$  such that:

$$\forall n \geq 1, \forall \Omega \in \mathcal{K}^n, \quad \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq C.$$

<sup>1</sup>We refer to [24, Section 1.1] for the definition of form bodies and to [31] for more details.

- We believe that the assertion  $\beta_n < \beta_{n-1}$  is true for any  $n \geq 2$ . This conjecture is motivated by the discussion of Section 4.2. In particular, when  $n = 2$ , we have:

$$\inf_{\Omega \in \mathcal{K}^2} J_2(\Omega) < \frac{\pi^2}{4} = \inf_{\omega \in \mathcal{K}^1} J_1(\omega).$$

Thus, we retrieve Parini's result of existence in the class of planar sets without using the explicit formulae of Cheeger constants of planar convex sets.

The present paper is organized as follows: in Section 2, we provide the proof of the sharp estimates of the Cheeger constant given in Theorem 1.2. Section 3 is devoted to the improvement of the Cheeger-Parini's inequality for planar convex sets (2), we also give improved results for some special shapes (triangles, rhombii and stadiums), see Proposition 3.1. We then prove the existence result of Theorem 1.3 in Section 4. We finally discuss some new sharp inequalities involving the first Dirichlet eigenvalue, the Cheeger constant, the inradius and the area of planar convex sets in Appendix 5.

## 2 Sharp estimates for the Cheeger constant: Proof of Theorem 1.2

The proof of Theorem 1.2 is presented in 3 parts:

### The upper bound:

Let  $\Omega \in \mathcal{K}^2$ . We have by [25, Theorem 2]:

$$\forall t \in (0, r(\Omega)), \quad |\Omega_{-t}| \geq |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2, \quad (6)$$

with equality if and only if  $\Omega$  is homothetic to its form body.

If  $\Omega$  is homothetic to its form body, we have by solving the equation  $|\Omega_{-t}| = |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2 = \pi t^2$ :

$$h(\Omega) = \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}.$$

From now on, we assume that  $\Omega$  is not homothetic to its form body. Let us introduce the functions:

- $f : t \in (0, r(\Omega)) \mapsto |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2 - \pi t^2 = |\Omega| - \frac{2|\Omega|}{r(\Omega)}t + \left(\frac{|\Omega|}{r(\Omega)^2} - \pi\right)t^2$ ,
- $g : t \in (0, r(\Omega)) \mapsto |\Omega_{-t}| - \pi t^2$ .

By (6), we have:

$$\begin{cases} g(0) = f(0), \\ \forall t \in (0, r(\Omega)), \quad g(t) > f(t), \end{cases}$$

This implies that  $1/h(\Omega)$ , the first zero of  $g$  on  $[0, r(\Omega)]$ , is strictly larger than the first zero of  $f$  given by  $\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^{-1}$  (see Figure 2), which proves the inequality.

### The lower bound:

Let  $\Omega \in \mathcal{K}^2$ , we denote  $C_\Omega \in \mathcal{K}^2$  its (unique) Cheeger set. Let us show that:

$$r(\Omega) = r(C_\Omega).$$

By the characterization of the Cheeger set of planar convex sets of [22], we have  $C_\Omega = \Omega_{-\frac{1}{h(\Omega)}} + \frac{1}{h(\Omega)}B_1$ , where  $B_1$  is the ball of unit radius centred at the origin. We then have:

$$r(C_\Omega) = r\left(\Omega_{-\frac{1}{h(\Omega)}} + \frac{1}{h(\Omega)}B_1\right) = r\left(\Omega_{-\frac{1}{h(\Omega)}}\right) + r\left(\frac{1}{h(\Omega)}B_1\right) = r(\Omega) - \frac{1}{h(\Omega)} + \frac{1}{h(\Omega)}r(B_1) = r(\Omega).$$

Since, the Cheeger set  $C_\Omega$  is convex, we can use the following Bonnesen's inequality [5]:

$$P(C_\Omega) \geq \pi r(C_\Omega) + \frac{|C_\Omega|}{r(C_\Omega)},$$

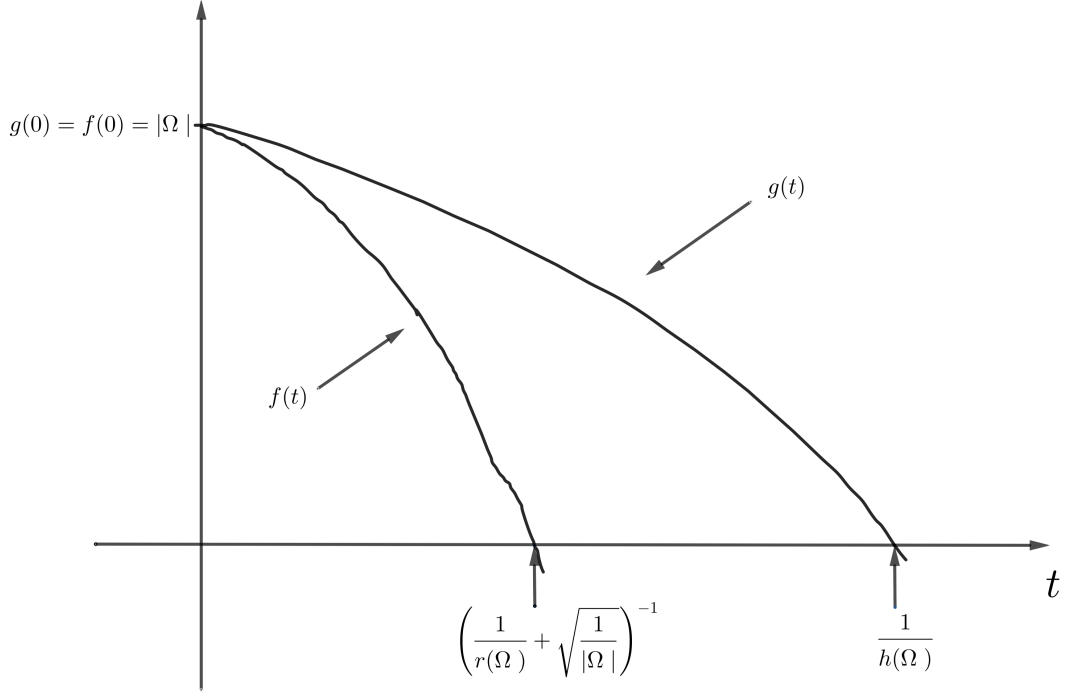


Figure 2: Idea of proof of the upper bound.

with equality if and only if  $C_\Omega$  is a stadium (note that does not mean that  $\Omega$  is a stadium). Thus:

$$h(\Omega) = \frac{P(C_\Omega)}{|C_\Omega|} \geq \frac{\pi r(C_\Omega) + \frac{|C_\Omega|}{r(C_\Omega)}}{|C_\Omega|} = \frac{\pi r(\Omega)}{|C_\Omega|} + \frac{1}{r(\Omega)} \geq \frac{\pi r(\Omega)}{|\Omega|} + \frac{1}{r(\Omega)},$$

where the last inequality is a consequence of the inclusion  $C_\Omega \subset \Omega$  and thus is an equality if and only if  $\Omega = C_\Omega$ . Finally, we proved the lower bound and the equality holds if and only if  $\Omega$  is a stadium.

### The diagram:

The demonstration is exactly similar to the case of the diagram relating the Cheeger constant, the area and the perimeter studied in detail in [15], one just has to replace the perimeter by the inradius and reproduce the same steps of the proof of [15, Theorem 1].

## 3 Improving the Cheeger inequality for planar convex sets

In this section, we provide the proof of Theorem 1.1 and prove some improved bounds for  $J_2$  in some special subclasses of  $\mathcal{K}^2$ , namely: triangles, rhombii and stadiums.

### 3.1 Proof of Theorem 1.1

Let  $\Omega \in \mathcal{K}^2$ . We have by Hersch inequality [19] and Faber-krahn inequality [12, 23]:

$$|\Omega| \lambda_1(\Omega) \geq \max \left( \pi j_{01}^2, \frac{\pi^2 |\Omega|}{4r(\Omega)^2} \right).$$

On the other hand, we recall the upper estimate of Theorem 1.2:

$$\sqrt{|\Omega|} h(\Omega) \leq \frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi}.$$

Thus, we have:

$$J(\Omega) = \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq \frac{\max \left( \pi j_{01}^2, \frac{\pi^2 |\Omega|}{4r(\Omega)^2} \right)}{\left( \frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi} \right)^2} \geq \min_{x \geq \sqrt{\pi}} \frac{\max \left( \pi j_{01}^2, \frac{\pi^2 x^2}{4} \right)}{(x + \sqrt{\pi})^2} = \left( \frac{\pi j_{01}}{2j_{01} + \pi} \right)^2 \approx 0.902\dots$$

The minimum is taken over  $[\sqrt{\pi}, +\infty)$  because  $\frac{\sqrt{|\Omega|}}{r(\Omega)} \geq \sqrt{\pi}$  for every  $\Omega \in \mathcal{K}^2$ . Moreover, it is attained for  $x = \frac{\pi j_{01}^2}{\sqrt{\pi}}$ , see Figure 3.

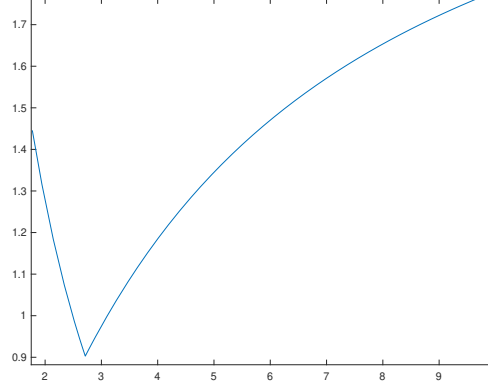


Figure 3: Curve of the function  $x \mapsto \frac{\max\left(\pi j_{01}^2, \frac{\pi^2 x^2}{4}\right)}{(x+\sqrt{\pi})^2}$ .

### 3.2 A slight improvement of the result of Theorem 1.1

We note that one can combine the following Protter's inequality [29]:

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{4} \left( \frac{1}{r(\Omega)^2} + \frac{1}{d(\Omega)^2} \right),$$

which is an improvement of Hersch's inequality [19, Section 8] with the optimal inequality (7) of [18]:

$$\forall \Omega \in \mathcal{K}^2, \quad \frac{|\Omega|}{r(\Omega)^2} \geq \sqrt{1 + \left(\frac{d(\Omega)}{r(\Omega)}\right)^2} + 2 \arcsin\left(\frac{2r(\Omega)}{d(\Omega)}\right) := \varphi\left(\frac{d(\Omega)}{r(\Omega)}\right), \quad (7)$$

to provide a slight improvement of the lower bound of Theorem 1.1.

Indeed, the function  $\varphi$  is continuous and strictly increasing on  $[2, +\infty)$  (we note that  $d(\Omega)/r(\Omega) \in [2, +\infty)$ , thus by considering the inverse function denoted  $\varphi^{-1}$ , inequality 7 becomes:

$$\forall \Omega \in \mathcal{K}^2, \quad \frac{d(\Omega)}{r(\Omega)} \leq \varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)$$

. We then write:

$$\begin{aligned} \frac{\lambda_1(\Omega)}{h(\Omega)^2} &\geq \frac{\max\left(\frac{\pi^2}{4r(\Omega)^2} \times \left(1 + \left(\frac{1}{\varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)}\right)^2\right), \frac{\pi j_{01}^2}{|\Omega|}\right)}{\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^2} \\ &= \frac{\max\left(\frac{\pi^2}{4} \times \frac{|\Omega|}{r(\Omega)^2} \times \left(1 + \left(\frac{1}{\varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)}\right)^2\right), \pi j_{01}^2\right)}{\left(\left(\frac{|\Omega|}{r(\Omega)^2}\right)^{1/2} + \sqrt{\pi}\right)^2} \\ &\geq \min_{x \in [\pi, +\infty)} \frac{\max\left(\frac{\pi^2}{4} x \left(1 + \frac{1}{\varphi^{-1}(x)^2}, \pi j_{01}^2\right)\right)}{(\sqrt{x} + \sqrt{\pi})} \quad (\text{because } \frac{|\Omega|}{r(\Omega)^2} \geq \frac{\pi r(\Omega)^2}{r(\Omega)^2} = \pi). \end{aligned}$$

Numerical computations show that the latter minimum is approximately equal to 0.914..., which slightly improves the lower bound of Theorem 1.1.

### 3.3 Improvements for special classes of shapes

We state and prove the following Proposition:

**Proposition 3.1.** *Let  $\Omega \in \mathcal{K}^2$ .*

1. *If  $\Omega$  is a triangle, then  $J_2(\Omega) > 1.2076$ .*
2. *If  $\Omega$  is a rhombus, then  $J_2(\Omega) \geq 1.3819$ .*
3. *If  $\Omega$  is a stadium (i.e. the convex hull of two identical balls), then  $J_2(\Omega) \geq 1.3673$ .*

*Proof.* Let  $\Omega \in \mathcal{K}^2$ , since  $J_2$  is invariant by homothety (due to scaling properties of  $\lambda_1$  and  $h$ ), we may assume without loss of generality that  $|\Omega| = 1$ .

1. Let us assume  $\Omega$  to be a triangle and denote  $d$  its diameter and  $L$  its perimeter. To bound  $\lambda_1(\Omega)$  from below, we make use of two inequalities:

- The first one is the polygonal Faber-Krahn inequality for triangles, which states that between triangles of the same area, the regular one minimizes the first Dirichlet eigenvalue:

$$\lambda_1(\Omega) \geq \lambda_1(T_{\text{eq}}) = \frac{4\pi^2}{\sqrt{3}},$$

where  $T_{\text{eq}}$  is the equilateral triangle of unit area (whose diameter is  $d_{\text{eq}} = \frac{2}{3^{1/4}}$ ).

- The second (more recent) is due to P. Freitas and B. Siudeja [14, Corollary 4.1]:

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4|\Omega|^2} \left( d(\Omega) + \frac{2|\Omega|}{d(\Omega)} \right)^2.$$

We then have on the one hand:

$$\lambda_1(\Omega) \geq \max \left( \frac{4\pi^2}{\sqrt{3}}, \frac{\pi^2}{4} \left( d + \frac{2}{d} \right)^2 \right),$$

and on the other hand, the Cheeger constant of the triangle  $\Omega$  is given by:

$$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = \frac{L + \sqrt{4\pi}}{2} \leq \frac{L_{\text{iso}} + \sqrt{4\pi}}{2},$$

where  $L_{\text{iso}}$  is the perimeter of the isocles triangle whose diameter is equal to  $d$  and area equal to 1. By using Pythagoras' theorem, we have:

$$L_{\text{iso}} = 2d + \sqrt{\left( d - \sqrt{d^2 - \frac{4}{d^2}} \right)^2 + \frac{4}{d^2}}.$$

Finally, we obtain the following inequality:

$$J_2(\Omega) \geq \phi_1(d) := \frac{\max \left( \frac{4\pi^2}{\sqrt{3}}, \frac{\pi^2}{4} \left( d + \frac{2}{d} \right)^2 \right)}{\left( \frac{2d + \sqrt{\left( d - \sqrt{d^2 - \frac{4}{d^2}} \right)^2 + \frac{4}{d^2}} + \sqrt{4\pi}}{2} \right)^2}.$$

We note that  $d \geq d_{\text{eq}}$ . Indeed, by the isoperimetric inequality of triangles:

$$3d_{\text{eq}} = L_{\text{eq}} \leq L \leq 3d.$$

Numerically, we obtain  $\min_{d \geq d_{\text{eq}}} \phi_1(d) \approx 1.2076\dots$



2. Let us assume  $\Omega$  to be the rhombus of unit area whose vertices are given by  $(-d/2, 0)$ ,  $(0, -1/d)$ ,  $(d/2, 0)$  and  $(0, 1/d)$ .

We bound  $\lambda_1(\Omega)$  from below by using the following Hooker and Protter's estimate for rhombi [20]:

$$\lambda_1(\Omega) \geq \pi^2 \left( \frac{d}{2} + \frac{1}{d} \right)^2.$$

As for the Cheeger constant, since  $\Omega$  is a circumscribed polygon, we have:

$$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = 2\sqrt{\frac{1}{d^2} + \frac{d^2}{4}} + \sqrt{\pi}$$

we use its explicit value in term of  $d$ .

$$J_2(\Omega) \geq \phi_2(d) = \frac{\pi^2 \left( \frac{d}{2} + \frac{1}{d} \right)^2}{\left( 2\sqrt{\frac{1}{d^2} + \frac{d^2}{4}} + \sqrt{\pi} \right)^2}.$$

Numerically, we obtain  $\min_{d \geq \sqrt{2}} \phi_2(d) \approx 1.3819\dots$

3. Let us assume  $\Omega$  to be a stadium of unit area whose diameter is given by  $a + 2r$ , where  $r > 0$  is the radius of the ball of its extremity and  $a > 0$ . The condition  $|\Omega| = 1$  implies that  $\pi r^2 + 2ar = 1$ , which is equivalent to  $a = \frac{1 - \pi r^2}{2r}$ . We use the monotonicity of  $\lambda_1$  for inclusion (for  $\Omega \subset (-r, r) \times (0, a + 2r)$ ) and Faber-Krahn inequality to write:

$$\lambda_1(\Omega) \geq \max \left( \lambda_1(B), \pi^2 \left( \frac{1}{4r^2} + \frac{1}{(a + 2r)^2} \right) \right) = \max \left( \lambda_1(B), \pi^2 \left( \frac{4r^2}{(1 + (4 - \pi)r^2)^2} + \frac{1}{4r^2} \right) \right).$$

It is classical that the stadiums are Cheeger of themselves, see [22], we then have:

$$h(\Omega) = \frac{P(\Omega)}{|\Omega|} = 2a + 2\pi r = \frac{1 + \pi r^2}{r}.$$

Then:

$$J_2(\Omega) \geq \phi_3(r) := \frac{\max \left( \lambda_1(B), \pi^2 \left( \frac{4r^2}{(1 + (4 - \pi)r^2)^2} + \frac{1}{4r^2} \right) \right)}{\left( \frac{1 + \pi r^2}{r} \right)^2}$$

Numerically, we obtain  $\min_{r \in (0, \frac{1}{\sqrt{\pi}}]} \phi_3(r) \approx 1.3673\dots$

□

## 4 On the existence of a minimizer in higher dimensions

### 4.1 Proof of Theorem 1.3

1. Let  $n \geq 2$ , let us first prove that:

$$\beta_n := \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega) \leq \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega) =: \beta_{n-1}.$$

The idea is to prove that for any  $\omega \in \mathcal{K}^{n-1}$ , there exists a family  $(\Omega_d)_{d>0}$  of elements of  $\mathcal{K}^n$  such that:

$$J_{n-1}(\omega) = \lim_{d \rightarrow +\infty} J_n(\Omega_d).$$

As the proof is quite involved, we decompose it in 3 steps.

### **Step 1: Lower estimates for $\lambda_1$ and $h$**

Let us take  $\Omega \in \mathcal{K}^n$  and assume without loss of generality that  $\inf\{t \in \mathbb{R} \mid \Omega \cap \{x = t\} \neq \emptyset\} = 0$  and denote  $d := \sup\{t \in \mathbb{R} \mid \Omega \cap \{x = t\} \neq \emptyset\}$ . In the proof of [8, Lemma 6.11], the authors prove that:

$$\exists t_\lambda \in [0, d], \quad \lambda_1(\Omega) \geq \lambda_1(\Omega \cap \{x = t_\lambda\}).$$

We adapt there method to Cheeger's constant and state that:

$$\exists t_h \in [0, d], \quad h(\Omega) \geq h(\Omega \cap \{x = t_h\}).$$

For sake of completeness, we present the proofs of both cases.

Let  $t_\lambda, t_h \in (0, d)$  such that:

$$\lambda_1(\Omega \cap \{x = t_\lambda\}) = \inf_{t \in \mathbb{R}} \lambda_1(\Omega \cap \{x = t\}) \quad \text{and} \quad h(\Omega \cap \{x = t_h\}) = \inf_{t \in \mathbb{R}} h(\Omega \cap \{x = t\}),$$

and we denote:  $\omega_\lambda = \Omega \cap \{x = t_\lambda\}$  and  $\omega_h = \Omega \cap \{x = t_h\}$ .

Let  $u \in H_0^1(\Omega)$  a positive eigenfunction corresponding to  $\lambda_1(\Omega)$  such that  $\|u\|_2 = 1$ . We have:

$$\begin{aligned} \lambda_1(\Omega) &= \int_{\Omega} |\nabla u|^2 dx \\ &\geq \int_0^d \left( \int_{\Omega \cap \{x=t\}} |\nabla' u|^2 dx' \right) dt \\ &\geq \int_0^d \left( \lambda_1(\Omega \cap \{x = t\}) \int_{\Omega \cap \{x=t\}} u^2 dx' \right) dt \\ &\geq \lambda_1(\omega_\lambda) \int_0^d \left( \int_{\Omega \cap \{x=t\}} u^2 dx' \right) dt \\ &= \lambda_1(\omega_\lambda) \end{aligned}$$

As for Cheeger's constant, we have  $\chi_{C_\Omega} \in BV(\Omega)$  (where  $BV(\Omega)$  stands for the set of functions of bounded variations on  $\Omega$ , we refer to [1] for definitions and more details), thus there exists a sequence  $(f_p)$  of functions of  $BV(\Omega) \cap C^\infty(\Omega)$  such as:  $f_k \rightarrow \chi_{C_\Omega}$  in  $L^1(\Omega)$  and  $\lim_{k \rightarrow +\infty} |Df_k|(\mathbb{R}^n) = |D\chi_{C_\Omega}|(\mathbb{R}^n) = P(C_\Omega)$ .

We have:

$$\begin{aligned} h(\Omega) &= \frac{P(C_\Omega)}{|C_\Omega|} = \frac{|D\chi_{C_\Omega}|(\mathbb{R}^n)}{\int_{\Omega} |\chi_{C_\Omega}|} = \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} |\nabla f_k|}{\int_{\Omega} |f_k|} \\ &\geq \lim_{k \rightarrow +\infty} \frac{\int_0^d \int_{\Omega \cap \{x=t\}} |\nabla' f_k| dx' dt}{\int_{\Omega} |f_k|} \\ &\geq \lim_{k \rightarrow +\infty} \frac{\int_0^d \left( h(\Omega \cap \{x = t\}) \int_{\Omega \cap \{x=t\}} |f_k| dx' \right) dt}{\int_{\Omega} |f_k|} \\ &\geq \lim_{k \rightarrow +\infty} h(\Omega \cap \{x = t_h\}) \frac{\int_0^d \left( \int_{\Omega \cap \{x=t\}} |f_k| dx' \right) dt}{\int_{\Omega} |f_k|} \\ &= h(\Omega \cap \{x = t_h\}) \end{aligned}$$

### **Step 2: Study of sets with increasing diameters and fixed volume**

Let  $(\Omega_k)$  a sequence of elements  $\mathcal{K}^n$  of the same volumes 1, such that  $d_k := d(\Omega_k) \rightarrow +\infty$ . Let us prove that:

$$\liminf_{k \rightarrow +\infty} J_n(\Omega_k) \geq \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega).$$

For every  $k \in \mathbb{N}$ , we consider  $A_k$  and  $A'_k$  two diametrical points of  $\Omega_k$  (ie. such as  $|A_k A'_k| = d_k$ ). Since  $J_n$  is invariant by rigid motions we can assume without loss of generality that  $A_k = (0, \dots, 0)$  and  $A'_k = (d_k, 0, \dots, 0)$ .

By Step 1, we have for all  $k \in \mathbb{N}$ :

$$\lambda_1(\Omega_k) \geq \lambda_1(\omega_k)$$

where  $\omega_k := \Omega_k \cap \{x = t_k\}$ .

We can assume without loss of generality that  $t_k \geq d_k/2$ . Let  $\mathcal{T}_k$  be the cone obtained by taking the convex hull of  $\{A_k\} \cup C_k$ , where  $C_k$  is the Cheeger set of the convex section  $\omega_k$ .

Let  $\alpha \in ]0, 1[$ , we introduce the tube  $U_k^\alpha := \alpha C_k \times (0, (1 - \alpha)t_k)$ . By convexity, we have the following inclusions:

$$U_n^\alpha \subset \mathcal{T}_k \subset \Omega_k$$

By definition of the Cheeger constant, we have:

$$h(\Omega_k) \leq \frac{P(U_k^\alpha)}{|U_k^\alpha|} = \frac{2\alpha^{n-1}|C_k| + \alpha^{n-2}(1 - \alpha)P(C_k)t_k}{\alpha^{n-1}(1 - \alpha)|C_k|t_k} = \frac{2}{(1 - \alpha)t_k} + \frac{h(\omega_k)}{\alpha} \underset{k \rightarrow +\infty}{\sim} \frac{h(\omega_k)}{\alpha}.$$

Indeed:  $\frac{1}{t_k} = o_{k \rightarrow \infty} \left( \frac{P(C_k)}{|C_k|} \right)$ , because:

$$\frac{|C_k|}{P(C_k)} = \frac{|C_k|}{P\left(|C_k|^{\frac{1}{n}} \times \frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} = \frac{|C_k|}{|C_k|^{\frac{n-1}{n}} P\left(\frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} = \frac{|C_k|^{\frac{1}{n}}}{P\left(\frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} \leq \frac{|C_k|^{\frac{1}{n}}}{P(B_{n-1})} \leq \frac{|\Omega_k|^{\frac{1}{n}} n^{\frac{1}{n}}}{P(B_{n-1})} \times \frac{1}{t_k^{1/n}} = o_{k \rightarrow \infty}(t_k),$$

where  $B_{n-1} \subset \mathbb{R}^{n-1}$  is a ball of volume 1.

We deduce that:

$$\forall \alpha \in (0, 1), \quad J_n(\Omega_k) \geq \frac{\lambda_1(\omega_k)}{\left(\frac{2}{(1-\alpha)t_k} + \frac{h(\omega_k)}{\alpha}\right)^2} \underset{k \rightarrow +\infty}{\sim} \alpha^2 J_{n-1}(\omega_k)$$

Thus:

$$\forall \alpha \in (0, 1), \quad \liminf_{k \rightarrow +\infty} J_n(\Omega_k) \geq \alpha^2 \liminf_{k \rightarrow +\infty} J_{n-1}(\omega_k) \geq \alpha^2 \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega)$$

By letting  $\alpha \rightarrow 1$ , we obtain:

$$\liminf_{k \rightarrow +\infty} J_n(\Omega_k) \geq \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega)$$

### Step 3: Study of long tubes

In this step, we show that when the height of a tube goes to infinity, the value of  $J_n$  of this tube converges to the value corresponding to the  $(n - 1)$ -dimensional section given by its basis. More precisely, if we take  $\omega \in \mathcal{K}^{n-1}$ , we prove that

$$\lim_{d \rightarrow +\infty} J_n((0, d) \times \omega) = J_{n-1}(\omega).$$

We have by Step 2:

$$\liminf_{k \rightarrow +\infty} J_n(\Omega_k) \geq J_{n-1}(\omega)$$

We recall that  $\lambda_1((0, d) \times \omega) = \left(\frac{\pi}{d}\right)^2 + \lambda_1(\omega)$  and use the second assertion of Step 1 to bound  $J_n((0, d) \times \omega)$  from above:

$$J_n((0, d) \times \omega) = \frac{\lambda_1((0, d) \times \omega)}{h((0, d) \times \omega)^2} \leq \frac{\left(\frac{\pi}{d}\right)^2 + \lambda_1(\omega)}{h(\omega)^2}.$$

By passing to superior limit:

$$\limsup_{d \rightarrow +\infty} J_n((0, d) \times \omega) \leq J_{n-1}(\omega).$$

Then:

$$\lim_{d \rightarrow +\infty} J_n((0, d) \times \omega) = J_{n-1}(\omega).$$

At last, we write:

$$\beta_{n-1} = \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega) = \inf_{\omega \in \mathcal{K}^{n-1}} \left( \lim_{d \rightarrow +\infty} \underbrace{J_n((0, d) \times \omega)}_{\geq \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega)} \right) \geq \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega) = \beta_n.$$

2. For every  $n \geq 2$ , we take a ball  $B_n \subset \mathbb{R}^n$  of unit radius, we have:

$$\frac{1}{4} \leq \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega) \leq J_n(B_n) = \frac{\lambda_1(B_n)}{h(B_n)^2} = \frac{j_{\frac{n}{2}-1,1}^2}{n^2} \underset{n \rightarrow +\infty}{\sim} \frac{\left(\frac{n}{2}\right)^2}{n^2} = \frac{1}{4},$$

where  $j_{\frac{n}{2}-1,1}$  is the first root of the  $n^{\text{th}}$  Bessel function of first kind. We refer to [33] for the equivalence  $j_{\frac{n}{2}-1,1} \underset{n \rightarrow +\infty}{\sim} \frac{n}{2}$ .

### 3. **The existence result:**

Now, we assume that:  $\inf_{\Omega \in \mathcal{K}^n} J_n(\Omega) < \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega)$ . Let us prove the existence of a minimizer of  $J_n$  on  $\mathcal{K}^n$ .

Let  $(\Omega_k)$  be a minimizing sequence of  $\mathcal{K}^n$  (ie. such as  $\lim_{k \rightarrow +\infty} J_n(\Omega_k) = \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega)$ ). Since  $J_n$  is scaling invariant we can assume without loss of generality that  $|\Omega_k| = 1$  for all  $n \in \mathbb{N}$ .

If  $(d(\Omega_k))$  is not bounded, we can extract a subsequence  $(\Omega_{\varphi(k)})$  such as

$$\lim_{k \rightarrow +\infty} d(\Omega_{\varphi(k)}) = +\infty.$$

Thus, by Step 2:

$$\inf_{\Omega \in \mathcal{K}^n} J_n(\Omega) = \lim_{k \rightarrow +\infty} J_n(\Omega_k) \geq \inf_{\omega \in \mathcal{K}^{n-1}} J_{n-1}(\omega).$$

which contradicts hypothesis  $\beta_{n-1} > \beta_n$ .

We deduce that the sequence of diameters  $(d(\Omega_k))$  is bounded, then by compactness, there exists  $\Omega^* \in \mathcal{K}^n$  and a strictly increasing map  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\Omega_{\sigma(k)} \xrightarrow[k \rightarrow \infty]{} \Omega^*$  for Hausdorff distance. We then have by continuity of  $J_n$  for the same metric (see [27, Proposition 3.2]):

$$J_n(\Omega^*) = \lim_{k \rightarrow +\infty} J_n(\Omega_k) = \inf_{\Omega \in \mathcal{K}^n} J_n(\Omega).$$

## 4.2 Discussion of the hypothesis $\beta_n < \beta_{n-1}$

We believe that hypothesis  $\beta_n < \beta_{n-1}$  is true for any dimension  $n$  and that one can use convex cylinders (i.e. those of the form  $\omega \times (0, d)$ , where  $\omega \in \mathcal{K}^{n-1}$  and  $d > 0$ ) to show it.

Let us analyse what happens when  $n = 2$ . In this case convex cylinders are rectangles. We consider the family of cylinders  $\Omega_d = (0, 1) \times (0, d)$  (where  $d > 0$ ) and denote

$$\Psi_{(0,1)} : d > 0 \mapsto J_2(\Omega_d) = \frac{\lambda_1(\Omega_d)}{h(\Omega_d)^2} = \frac{\pi^2 \left(1 + \frac{1}{d^2}\right)}{\left(\frac{4-\pi}{1+d-\sqrt{(d-1)^2+\pi d}}\right)^2}.$$

We plot the curve of  $\Psi_{(0,1)}$  in Figure 4.

We remark that when  $d$  goes to infinity,  $\Psi_{(0,1)} = J_2(\Omega_d)$  tends to  $J_1((0, 1)) = \frac{\pi^2}{4}$  from below, in the sense that there exists an order from which the function  $\Psi_{(0,1)}$  should be strictly increasing and thus cannot have values above the limit  $J_1((0, 1))$ . We believe that the same property should hold in higher dimensions: let  $n \geq 2$ ,  $\Omega_d := \omega \times (0, d)$  where  $d > 0$  and  $\omega \in \mathcal{K}^{n-1}$ , as before we denote  $\Psi_\omega : d > 0 \mapsto J_n(\Omega_d)$ . We have already proved above that:

$$\lim_{d \rightarrow +\infty} \Psi_\omega(d) = \lim_{d \rightarrow +\infty} J_n(\Omega_d) = J_{n-1}(\omega).$$

It remains to prove that for large values of  $d$  one has:

$$\Psi_\omega(d) = J_n(\Omega_d) < J_{n-1}(\omega).$$

To do so, we propose to show that function  $\Psi_\omega$  is strictly increasing for large values of  $d$  by studying the derivative  $\Psi'_\omega(d)$ .

Let us take  $d > 0$ , we have for  $t > 0$  sufficiently small:

$$\lambda_1(\Omega_{d+t}) = \lambda_1(\omega) + \frac{\pi^2}{(d+t)^2} = \lambda_1(\omega) + \frac{\pi^2}{d^2} - \frac{2\pi^2}{d^3}t + o_{t \rightarrow 0}(t),$$

and

$$h(\Omega_{d+t}) = h(\Omega_d) + h'(\Omega_d, V_d) \times t + o_{t \rightarrow 0}(t),$$

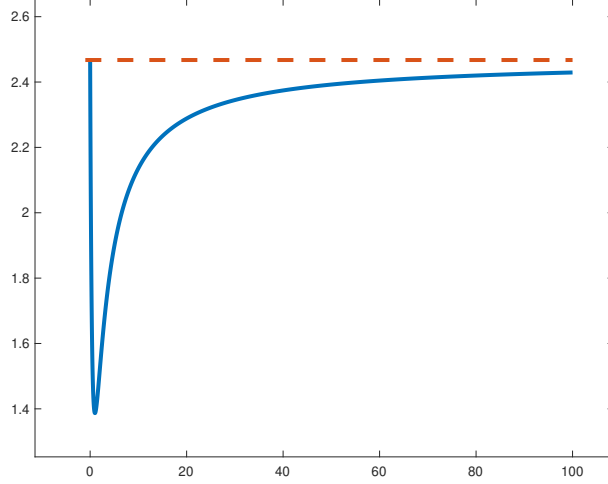


Figure 4: The curve of  $\Psi_{(0,1)}$ .

where  $V_d : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the smooth dilatation field such that  $V_d(x_1, \dots, x_n) = (0, \dots, 0, \frac{x_n}{d})$ . As proved in [28], we have:

$$h'(\Omega_d, V_d) = \frac{1}{|C_{\Omega_d}|} \int_{\partial\Omega_d \cap \partial C_{\Omega_d}} (\kappa - h(\Omega_d)) \langle V_d, n \rangle d\sigma,$$

where  $\kappa$  is the mean curvature and  $C_{\Omega_d}$  is the Cheeger set of  $\Omega_d$ . Since  $\langle V_d, n \rangle = 0$  on all  $\partial\Omega_d \cap \partial C_{\Omega_d}$  except on the upper basis  $\partial\Omega_d \cap \partial C_{\Omega_d} \cap \{x_n = d\}$  where  $\kappa$  is null, we have the following formula for the shape derivative:

$$h'(\Omega_d, V_d) = - \frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|} h(\Omega_d).$$

By straightforward computations we obtain:

$$\begin{aligned} \Psi'_\omega(d) &= \frac{1}{h(\Omega_d)^2} \left( -\frac{2\pi^2}{d^3} + 2 \left( \lambda_1(\omega) + \frac{\pi^2}{d^2} \right) \frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|} \right) \\ &> \frac{2\pi^2}{h(\Omega_d)^2} \left( -\frac{1}{d^3} + \lambda_1(\omega) \frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|} \right) \\ &\geq \frac{2\pi^2}{d \times h(\Omega_d)^2} \left( \frac{\lambda_1(\omega)}{|\omega|} |\partial C_{\Omega_d} \cap \{x_n = d\}| - \frac{1}{d^2} \right) \quad (\text{because } C_{\Omega_d} \subset \Omega_d, \text{ thus } |C_{\Omega_d}| \leq |\Omega_d| = |\omega| \times d). \end{aligned}$$

Finally, it remains to prove that for sufficiently large values of  $d$  one can prove estimate of the type:

$$|\partial C_{\Omega_d} \cap \{x_n = d\}| > \frac{1}{d^2}.$$

One can check that this assertion is correct when  $n = 2$ . Indeed, if we consider the cylinder  $\Omega_d = (0, 1) \times (0, d)$ , we can easily check that

$$|\partial C_{\Omega_d} \cap \{x_n = d\}| \underset{d \rightarrow +\infty}{\sim} M_2 \times \frac{1}{d} > \frac{1}{d^2},$$

where  $M_2$  is some dimensional constant.

At last, let us mention the very recent work of E. Parini and V. Bobkov [3] where they manage to explicitly describe the Cheeger sets of rationally invariant sets in any dimension and thus compute their Cheeger values. By applying these results to cylinders of the form  $B_{n-1} \times (0, d)$ , where  $B_{n-1} \subset \mathbb{R}^{n-1}$  is a ball, we remark as expected that  $\Psi_{B_{n-1}}$  is strictly increasing for higher values of  $d$  and thus converges to  $J_{n-1}(B_{n-1})$  from below, which supports our strategy.

## 5 Appendix: Some applications

In this Appendix, we apply the sharp estimates given in (5) to obtain some new bounds for the first Dirichlet eigenvalue in the case of planar convex sets.

### 5.1 Some sharp upper bounds for the first Dirichlet eigenvalue

#### 5.1.1 General planar convex sets

**Proposition 5.1.** *We have the following sharp inequality:*

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) < \frac{\pi^2}{4} \left( \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}} \right)^2, \quad (8)$$

where equality is asymptotically attained by any family of convex sets  $(\Omega_k)_{k \in \mathbb{N}}$  such as  $|\Omega_k| = V_0$  for any  $k \in \mathbb{N}$  (where  $V_0$  is a positive constant) and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ .

*Proof.* We have for every  $\Omega \in \mathcal{K}^2$ :

$$\lambda_1(\Omega) < \frac{\pi^2}{4} h(\Omega)^2 \leq \frac{\pi^2}{4} \left( \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}} \right)^2,$$

where the first inequality is the reverse Cheeger inequality proved by E. Parini in [27, Proposition 4.1] and the second inequality corresponds to the upper bound given in (5).

Let us now prove the sharpness inequality (8). Let  $V_0 > 0$  and  $(\Omega_k)_{k \in \mathbb{N}}$  a family of convex sets such as  $|\Omega_k| = V_0$  for any  $k \in \mathbb{N}$  and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ . We have on the one hand:

$$\forall k \in \mathbb{N}^*, \quad \frac{\pi^2}{4r(\Omega_k)^2} < \lambda_1(\Omega_k) < \frac{\pi^2}{4} \left( \frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{|\Omega_k|}} \right)^2 = \frac{\pi^2}{4} \left( \frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{V_0}} \right)^2,$$

on the other hand, we have:

$$\frac{1}{r(\Omega_k)} \geq \frac{P(\Omega_k)}{2|\Omega_k|} \geq \frac{d(\Omega_k)}{V_0} \xrightarrow[k \rightarrow +\infty]{} +\infty,$$

thus:

$$\lambda_1(\Omega_k) \underset{k \rightarrow +\infty}{\sim} \frac{\pi^2}{4r(\Omega_k)^2}, \quad (9)$$

which proves the sharpness of inequality (8).  $\square$

**Remark 5.2.** *We note that one can use inequalities (5), to provide a similar equivalence as (9) for the Cheeger constant. Indeed, let us consider  $V_0 > 0$  and  $(\Omega_k)_{k \in \mathbb{N}}$  a family of convex sets such as  $|\Omega_k| = V_0$  for any  $k \in \mathbb{N}$  and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ . We have by (5):*

$$\forall k \in \mathbb{N}, \quad \frac{1}{r(\Omega_k)} + \frac{\pi r(\Omega_k)}{V_0} \leq h(\Omega_k) \leq \frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{|V_0|}},$$

and since  $\frac{1}{r(\Omega_k)} \xrightarrow[k \rightarrow +\infty]{} +\infty$ , we have the following equivalence:

$$h(\Omega_k) \underset{k \rightarrow +\infty}{\sim} \frac{1}{r(\Omega_k)}. \quad (10)$$

By combining (9) and (10), we retrieve (with an alternative method) the asymptotic result of [27, Proposition 4.1]:

$$\lim_{k \rightarrow +\infty} J_2(\Omega_k) = \lim_{k \rightarrow +\infty} \frac{\lambda_1(\Omega_k)}{h(\Omega_k)^2} = \frac{\pi^2}{4}.$$

It is interesting to compare inequality (8) with other inequalities involving the inradius and the area. One immediate estimate can be obtained by considering the inclusion  $B_{r(\Omega)} \subset \Omega$  (where  $B_{r(\Omega)}$  is an inscribed ball of  $\Omega$  (with radius  $r(\Omega)$ )). We have by the monotonicity of  $\lambda_1$ :

$$\lambda_1(\Omega) \leq \lambda_1(B_{r(\Omega)}) = \frac{j_{01}^2}{r(\Omega)^2}, \quad (11)$$

where  $j_{01}$  denotes the first zero of the first Bessel function. This inequality was already stated in [27, inequality (3)] and in [9, inequality (1.5)] in higher dimensions and for a more general setting. In Figure 5, we plot the curves corresponding to the latter inequalities and an approximation of the Blaschke-Santaló diagram corresponding to the functionals  $\lambda_1$ , the inradius  $r$  and the area  $|\cdot|$ , that is the set of points:

$$\mathcal{D} := \left\{ \left( \frac{1}{r(\Omega)}, \lambda_1(\Omega) \right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\}.$$

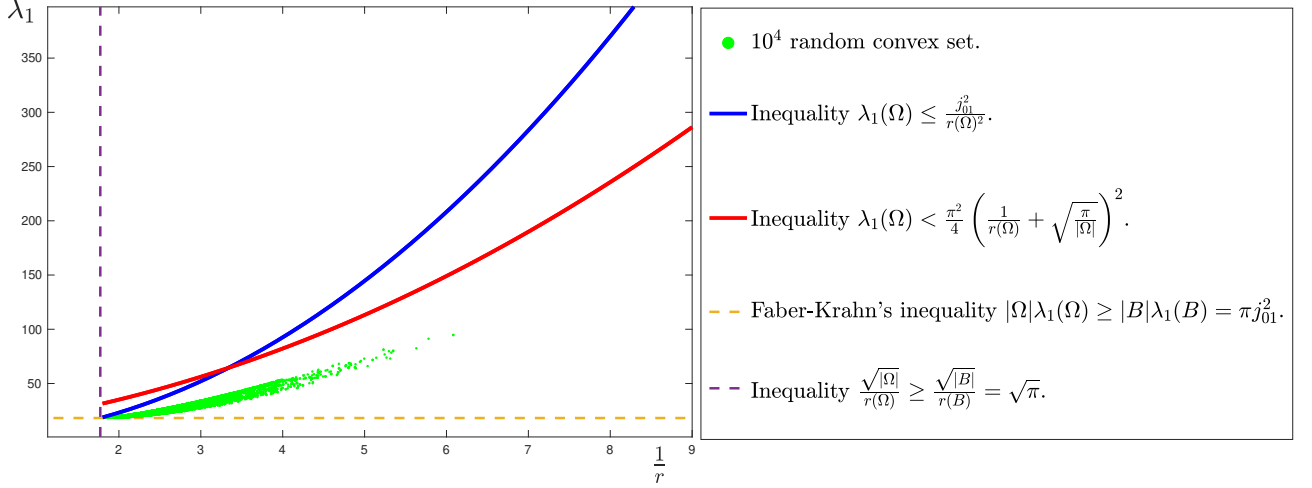


Figure 5: Inequality (8) improves (11) for convex sets with small inradius (ie. large  $\frac{1}{r}$ ).

### 5.1.2 Sets that are homothetic to their form bodies: in particular "triangles"

We recall that in the case of sets that are homothetic to their form bodies, one has  $\frac{1}{2}P(\Omega)r(\Omega) = |\Omega|$  and:

$$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}.$$

Thus one can write the following result, which is an immediate Corollary of the reverse Cheeger's inequality of [27, Proposition 4.1]:

**Corollary 1.** *For every set  $\Omega \in \mathcal{K}^2$ , that is homothetic to its form body (in particular triangles), we have the following inequality:*

$$\lambda_1(\Omega) < \frac{\pi^2}{4} \times \left( \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}} \right)^2 = \frac{\pi^2}{16} \times \left( \frac{P(\Omega)}{|\Omega|} + 2\sqrt{\frac{\pi}{|\Omega|}} \right)^2. \quad (12)$$

The inequality is sharp as it is asymptotically attained by any sequence of convex sets  $(\Omega_k)$  of unit area that are homothetic to their form bodies such that  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ .

The most important thing about this upper bound is that in the case of triangles, inequality (12) is better than the following bound obtained by B. Siudeja in [32, Theorem 1.1] for "thin" triangles:

$$\lambda_1(T) \leq \frac{\pi^2}{9} \times \left( \frac{P(T)}{|T|} \right)^2. \quad (13)$$

It is also interesting to note that inequality (12) is even better (also for thin triangles) than the following upper bound stated in [32, Conjecture 1.2]:

**Conjecture 1.** *For every triangle  $T$ , one has:*

$$\lambda_1(T) \leq \frac{\pi^2}{12} \times \left( \frac{P(T)}{|T|} \right)^2 + \frac{\sqrt{3}\pi^2}{3|T|}. \quad (14)$$

Here also, let us compare the different estimates in a Blaschke-Santaló diagram: we consider the one involving the perimeter, the area and  $\lambda_1$  in the class of triangles, that is the set of points:

$$\mathcal{T} := \{ (P(T), \lambda_1(T)) \mid T \text{ is a triangle such that } |T| = 1 \}.$$

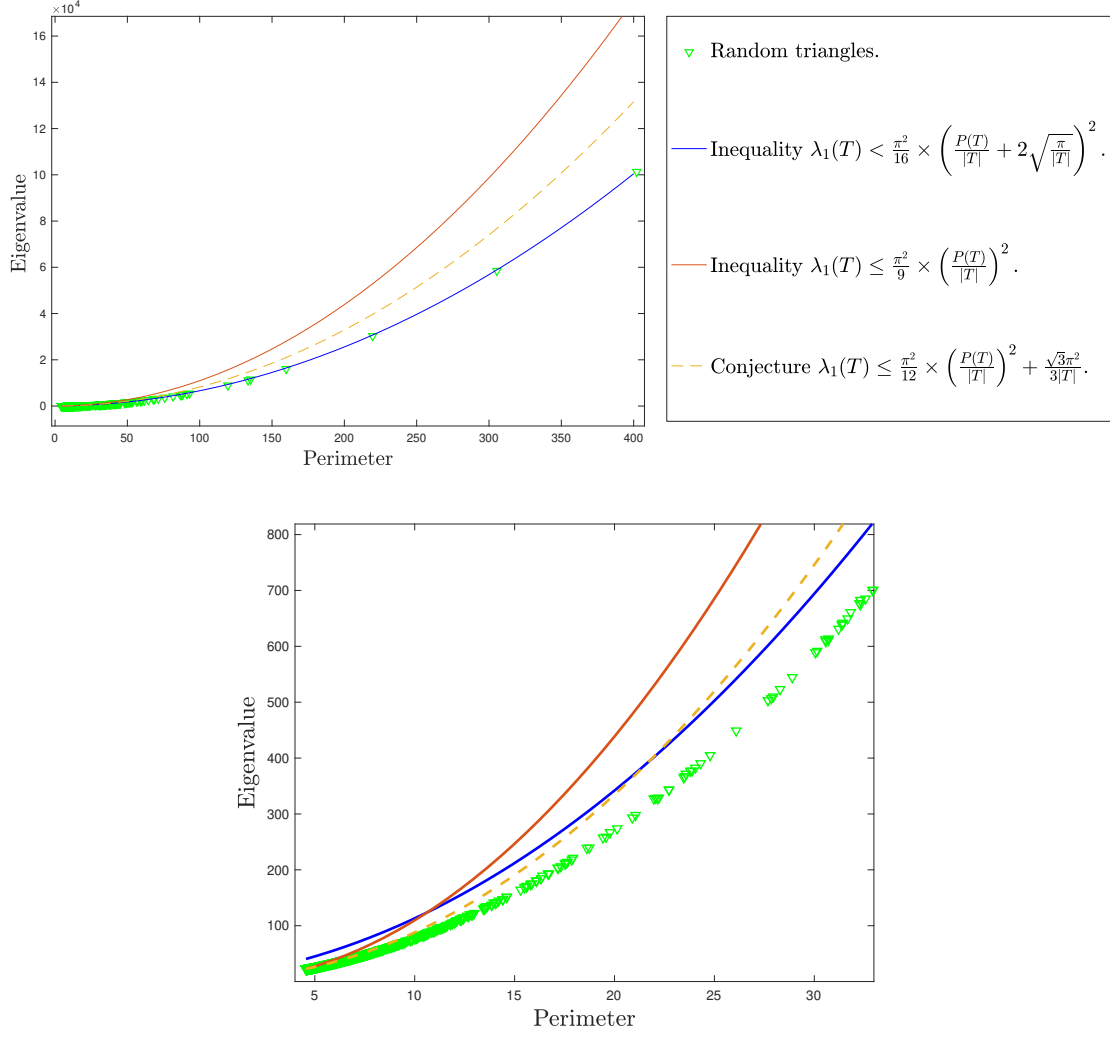


Figure 6: Comparison between inequalities (12) and (13) and Conjecture (14) with a zoom on smaller values of the perimeter.

## 5.2 A sharp Cheeger-type inequality

**Proposition 5.3.** *We have the following sharp Cheeger-type inequality:*

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) > \frac{\pi^2}{4} \left( h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}} \right)^2, \quad (15)$$

where equality is asymptotically attained by any family of convex sets  $(\Omega_k)_{k \in \mathbb{N}}$  such as  $|\Omega_k| = V_0$  for any  $k \in \mathbb{N}$  (where  $V_0$  is a positive constant) and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ .

*Proof.* Let  $\Omega \in \mathcal{K}^2$ , we have:

$$\lambda_1(\Omega) > \frac{\pi^2}{4} \times \frac{1}{r(\Omega)^2} \geq \frac{\pi^2}{4} \left( h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}} \right)^2,$$

where the first inequality is the classical Hersch's inequality [19] and the second follows from is the upper estimate of (5).

As for the equality case, let  $(\Omega_k)$  a family of convex sets  $(\Omega_k)_{k \in \mathbb{N}}$  such as  $|\Omega_k| = V_0$  for any  $k \in \mathbb{N}$  and  $d(\Omega_k) \xrightarrow[k \rightarrow +\infty]{} +\infty$ . By [27, Proposition 4.1], we have:  $\lambda_1(\Omega_k) \underset{k \rightarrow +\infty}{\sim} \frac{\pi^2}{4} h(\Omega_k)^2$  and by the equivalence (10) and  $\lim_{k \rightarrow +\infty} \frac{1}{r(\Omega_k)} = +\infty$  (see the proof of Proposition 5.1), we have  $\lim_{k \rightarrow +\infty} h(\Omega_k) = +\infty$  which implies the

$$\text{equivalence: } \lambda_1(\Omega) \underset{k \rightarrow +\infty}{\sim} \frac{\pi^2}{4} \left( h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}} \right)^2. \quad \square$$



We note that inequality (15) is better than the improved Cheeger inequality of Theorem 1.1 (and even the conjecture  $J_2(\Omega) \geq J_2((0, 1)^2)$ ) for thin planar convex domains, see Figure 7, where we provide an approximation of the following Blaschke-Santaló diagram relating  $\lambda_1$ , the Cheeger constant and the area:

$$\mathcal{C} := \{ (h(\Omega), \lambda_1(\Omega)) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \}.$$

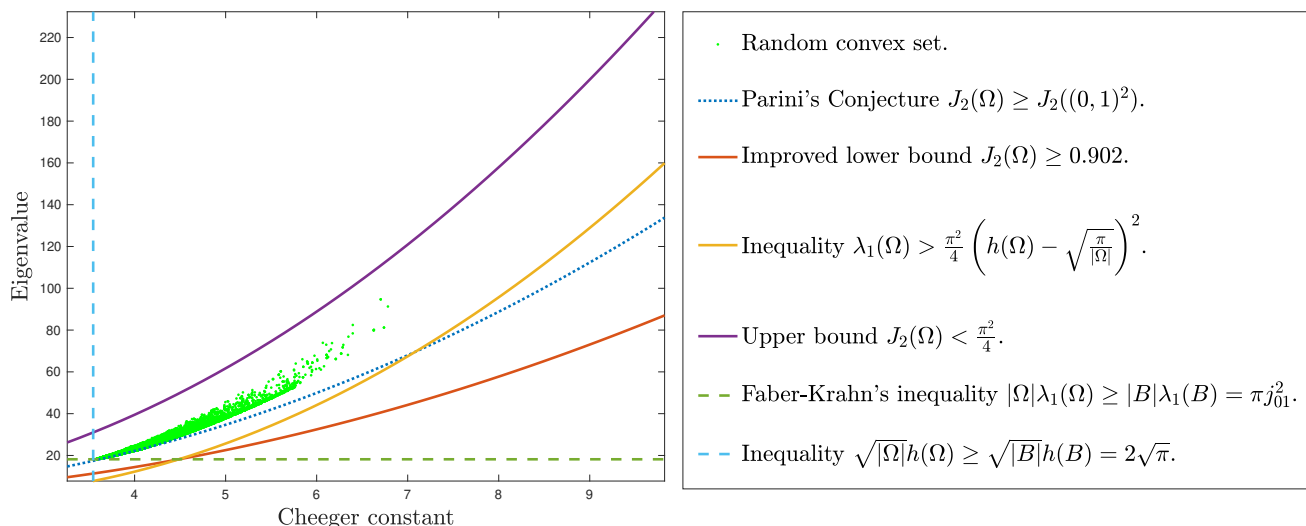


Figure 7: Approximation of the Blaschke-Santaló diagram  $\mathcal{C}$  and relevant inequalities.

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