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On the Cheeger inequality for convex sets

Ilias Ftouhi*

June 16, 2021

Abstract

In this paper, we prove new sharp bounds for the Cheeger constant of planar convex sets that we use to study the relations between the Cheeger constant and the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. This problem is closely related to the study of the so-called Cheeger inequality for which we provide an improvement in the class of planar convex sets. We then provide an existence theorem that highlights the tight relation between improving the Cheeger inequality and proving the existence of a minimizer of a the functional $J := \lambda_1/h^2$ in any dimension n. We finally, provide some new sharp bounds for the first Dirichlet eigenvalue of planar convex sets and a new sharp upper bound for triangles which is better than the conjecture stated in [32] in the case of thin triangles.

Keywords: Cheeger constant, complete systems of inequalities, Blaschke-Santaló diagrams, convex sets, sharp spectral inequalities.

AMS classification:: 52A10, 52A40, 49R99, 65K15

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1 Introduction and main results

A celebrated inequality due to Jeff Cheeger states that for every open bounded set $\Omega \in \mathbb{R}^n$ (where $n \geq 2$) one has:

$$\lambda_1(\Omega) \geq \frac{1}{4} h(\Omega)^2,$$

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where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue and $h(\Omega)$ is the Cheeger constant of Ω, which is defined as follows:

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \;\middle|\; E \text{ measurable and } E \subset \Omega \right\}, \tag{1}$$

where $P(E)$ is the perimeter of De-Giorgi of E measured with respect to $\mathbb{R}^n$ (see for example [26] for definitions) and $|E|$ is the n-dimensional Lebesgue measure of E. Any set $C_\Omega \subset \Omega$ for which the infimum is attained is called (when it exists) a *Cheeger set of* Ω. We refer to [26] for an introduction to the Cheeger problem.

In the present paper, d and r respectively correspond to the diameter and the inradius functionals.

Recently, E. Parini [27] remarked that the constant $\frac{1}{4}$ can be improved for the class $\mathcal{K}^2$ (for every $n \in \mathbb{N}^*$, we denote $\mathcal{K}^n$ the class of bounded convex bodies of $\mathbb{R}^n$ of non-empty interior). He proved the following inequality:

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{16} h(\Omega)^2, \tag{2}$$

and noted that the constant $\frac{\pi^2}{16}$ is also not optimal. He then took a shape optimization point of view by introducing the functional $J : \Omega \in \mathcal{K}^2 \longmapsto J(\Omega) := \frac{\lambda_1(\Omega)}{h(\Omega)^2}$ for which he proves the existence of a minimizer in $\mathcal{K}^2$ and conjectures that it is the square; in which case the optimal lower bound would be given by:

$$\min_{\Omega \in \mathcal{K}^2} J(\Omega) = J\big((0,1)^2\big) = \frac{2\pi^2}{(2+\sqrt{\pi})^2} \approx 1.387...$$

Nevertheless, as far as we know, as mentioned in [27, Section 6], the existence of an optimal shape in higher dimensions ($n \geq 3$) remains open.

In order to obtain a lower bound of J on the class $\mathcal{K}^n$, one can combine the inequality $h(\Omega) \leq \frac{n}{r(\Omega)}$ (which is obtained by taking the inscribed ball $B_{r(\Omega)}$ as a test set in the definition of the Cheeger constant $h(\Omega)$) and Protter's inequality [29]:

$$\forall \Omega \in \mathcal{K}^n, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{4} \left(\frac{1}{r(\Omega)^2} + \frac{n-1}{d(\Omega)^2} \right), \tag{3}$$

which generalises Hersch's inequality [20] (used by Parini for the planar case) to higher dimensions. We then obtain the following lower bound:

$$\forall n \geq 2, \forall \Omega \in \mathcal{K}^n, \quad J(\Omega) := \frac{\lambda_1(\Omega)}{h(\Omega)^2} > \frac{\pi^2}{4n^2},$$

which improves the original constant $\frac{1}{4}$ given by J. Cheeger only for $n \in \{2,3\}$. In which cases, we have:

$$\forall \Omega \in \mathcal{K}^2, \quad J(\Omega) > \frac{\pi^2}{16} \approx 0.616... \quad \text{and} \quad \forall \Omega \in \mathcal{K}^3, \quad J(\Omega) > \frac{\pi^2}{36} \approx 0.274...$$

In the present paper, we improve the Cheeger-Parini's inequality (2). Our result in this direction is stated as follows:

Theorem 1.1. *We have:*

$$\forall \Omega \in \mathcal{K}^2, \quad J(\Omega) = \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq \left(\frac{\pi j_{01}}{2j_{01} + \pi} \right)^2 \approx 0.902...$$

where j_{01} denotes the first zero of the first Bessel function.

At last, we are interested by the question of the existence of an minimizer of J for higher dimensions $n \geq 3$. We prove the following Theorem:

Theorem 1.2. *Let us define the real sequence $(\beta_n)_n$ as follows:*

$$\forall n \in \mathbb{N}^*, \quad \beta_n := \inf_{\Omega \in \mathcal{K}^n} J(\Omega).$$

We have:

1. *$(\beta_n)_n$ is a decreasing sequence.*
2. $\lim_{n \to +\infty} \beta_n = \frac{1}{4}$.
3. *For $n \geq 2$, if the <u>strict</u> inequality $\beta_n < \beta_{n-1}$ holds, we have the following existence result:*

$$\exists \Omega_n^* \in \mathcal{K}^n, \quad J(\Omega_n^*) = \inf_{\Omega \in \mathcal{K}^n} J(\Omega).$$

Let us give a few interesting comments on Theorem 1.2:

- The convergence result $\lim_{n\to+\infty} \beta_n = \frac{1}{4}$ of shows that the constant $\frac{1}{4}$ given in the original Cheeger inequality [10] is optimal in the sense that there exists no constant $C > \frac{1}{4}$ such that:

$$\forall n \geq 1, \forall \Omega \in \mathcal{K}^n, \quad \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq C.$$

- We believe that the assertion $\beta_n < \beta_{n-1}$ is true for any $n \geq 2$. This conjecture is motivated by the discussion of Section 4.2. In particular, when $n = 2$, we have:

$$\inf_{\Omega\in\mathcal{K}^2} J(\Omega) < \frac{\pi^2}{4} = \inf_{\omega\in\mathcal{K}^1} J_1(\omega).$$

  Thus, we retrieve Parini's result of existence in the class of planar sets without using the explicit formulae of Cheeger constants of planar convex sets.

At last, we note that the result of Theorem 1.1 relies on the combination of Protter's inequality (3) and the Faber-Krahn inequality [12, 23] to bound $\lambda_1(\Omega)$ from below and an upper sharp estimate of the Cheeger constant in terms of the inradius and the area.

The study of complete systems of inequalities relating some given functionals is an interesting subject for its own. It is closely related to the so called *Blaschke-Santaló* diagrams, we refer to the original works of Blaschke [2] and Santaló [30] and to the more recent works [5, 11, 17, 18, 19] for some interesting examples involving geometrical functionals and to [14, 15, 24, 34, 9, 35] for recent examples dealing with diagrams involving spectral and geometrical quantities.

In the present paper we provide a complete system of inequalities relating the Cheeger constant h, the inradius r and the area $|\cdot|$ of planar convex sets, which corresponds to a complete description of the related Blaschke-Santaló diagram introduced in Theorem 1.3.

Before stating the result, let us provide some notations and define various important notions: we denote d^H the Hausdorff distance (for more details we refer for example to [16, Chapter 2]), $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ the unit sphere and $B \subset \mathbb{R}^n$ a ball of unit volume. If $\Omega \in \mathcal{K}^n$, we denote by $h_\Omega : u \in \mathbb{S}^{n-1} \longmapsto \sup_{x\in\Omega}\langle x, u\rangle$ the *support function* of the convex body Ω, it is the function that describes the distance from the origin to the supporting hyperplane of Ω with normal u. In what follows we denote such a supporting hyperplane by $H(\Omega, u)$. We then have the following characterization of the supporting hyperplanes of Ω:

$$H(\Omega, u) = \{x \in \mathbb{R}^n, \ \langle x, u\rangle = h_\Omega(u)\}.$$

A point $x \in \partial\Omega$ is called regular if the supporting hyperplane at x is uniquely defined, that is if there is a unique $u \in \mathbb{S}^{n-1}$ such that $x \in H(\Omega, u) \cap \Omega$. The set of all regular points of $\partial\Omega$ is denoted by reg(Ω). We also introduce $\mathcal{U}(\Omega)$ the set of all outward pointing unit normals to $\partial\Omega$ at points of reg(Ω). We are now in position to define the *form body* Ω_* of Ω as in [31]:

$$\Omega_* := \bigcap_{u\in\mathcal{U}(\Omega)} \{x \in \mathbb{R}^n, \ \langle x, u\rangle = 1\}.$$

We are now ready to state the following result:

Theorem 1.3. *We have:*

$$\forall \Omega \in \mathcal{K}^2, \qquad \frac{1}{r(\Omega)} + \frac{\pi r(\Omega)}{|\Omega|} \leq h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}, \tag{4}$$

These inequalities are sharp as equalities are obtained for stadiums in the lower estimate and for domains that are homothetic to their form bodies in the upper one.

Moreover, we have the following explicit description of the Blaschke-Santaló diagram:

$$\mathcal{D} := \left\{ \left(\frac{1}{r(\Omega)}, h(\Omega)\right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\} = \left\{ (x, y) \;\middle|\; x \geq \frac{1}{r(B)} = \sqrt{\pi} \text{ and } x + \frac{\pi}{x} \leq y \leq x + \sqrt{\pi} \right\},$$

where $B \subset \mathbb{R}^2$ is a ball of unit area.

The present paper is organized as follows: in Section 2, we provide the proof of the sharp estimates of the Cheeger constant given in Theorem 1.3. Section 3 is devoted to the improvement of the Cheeger-Parini's inequality for planar convex sets (2), we also give improved results for some special shapes (triangles, rhombii and stadiums), see Proposition 3.1. We then prove the existence result of Theorem 1.2 in Section 4. We finally discuss some new sharp inequalities involving the first Dirichlet eigenvalue, the Cheeger constant, the inradius and the area of planar convex sets in Appendix 5.

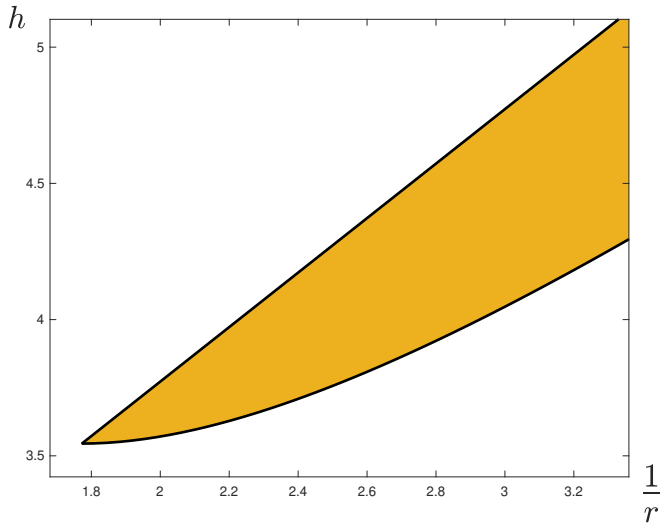

Figure 1: The diagram of the triplet $(r, h, |\cdot|)$.

2 Sharp estimates for the Cheeger constant: Proof of Theorem 1.3

The proof of Theorem 1.3 is presented in 3 parts:

2.1 The lower bound:

Let $\Omega \in \mathcal{K}^2$, we denote $C_\Omega \in \mathcal{K}^2$ its (unique) Cheeger set. Let us show that:

$$r(\Omega) = r(C_\Omega).$$

By the characterization of the Cheeger set of planar convex sets of [22], we have $C_\Omega = \Omega_{-\frac{1}{h(\Omega)}} + \frac{1}{h(\Omega)} B_1$, where B_1 is the ball of unit radius centred at the origin. We then have:

$$r(C_\Omega) = r\left(\Omega_{-\frac{1}{h(\Omega)}} + \frac{1}{h(\Omega)} B_1\right) = r\left(\Omega_{-\frac{1}{h(\Omega)}}\right) + r\left(\frac{1}{h(\Omega)} B_1\right) = r(\Omega) - \frac{1}{h(\Omega)} + \frac{1}{h(\Omega)} r(B_1) = r(\Omega).$$

Since, the Cheeger set C_Ω is convex, we can use the following Bonnesen's inequality [4]:

$$P(C_\Omega) \geq \pi r(C_\Omega) + \frac{|C_\Omega|}{r(C_\Omega)},$$

with equality if and only if C_Ω is a stadium (note that does not mean that Ω is a stadium). Thus:

$$h(\Omega) = \frac{P(C_\Omega)}{|C_\Omega|} \geq \frac{\pi r(C_\Omega) + \frac{|C_\Omega|}{r(C_\Omega)}}{|C_\Omega|} = \frac{\pi r(\Omega)}{|C_\Omega|} + \frac{1}{r(\Omega)} \geq \frac{\pi r(\Omega)}{|\Omega|} + \frac{1}{r(\Omega)},$$

where the last inequality is a consequence of the inclusion $C_\Omega \subset \Omega$ and thus is an equality if and only if $\Omega = C_\Omega$. Finally, we proved the lower bound and the equality holds if and only if Ω is a stadium.

2.2 The upper bound:

Let $\Omega \in \mathcal{K}^2$. We have by [25, Theorem 2]:

$$\forall t \in \big(0, r(\Omega)\big), \qquad |\Omega_{-t}| \geq |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2, \tag{5}$$

with equality if and only if Ω is homothetic to its form body.

If Ω is homothetic to its form body, we have by solving the equation $|\Omega_{-t}| = |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2 = \pi t^2$:

$$h(\Omega) = \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}.$$

From now on, we assume that Ω is not homothetic to its form body. Let us introduce the functions:

- $f : t \in \big(0, r(\Omega)\big) \longmapsto |\Omega| \left(1 - \frac{t}{r(\Omega)}\right)^2 - \pi t^2 = |\Omega| - \frac{2|\Omega|}{r(\Omega)} t + \left(\frac{|\Omega|}{r(\Omega)^2} - \pi\right) t^2,$
- $g : t \in \big(0, r(\Omega)\big) \longmapsto |\Omega_{-t}| - \pi t^2.$

By (5), we have:

$$\begin{cases} g(0) = f(0), \\ \forall t \in \big(0, r(\Omega)\big), \qquad g(t) > f(t). \end{cases}$$

This implies that $1/h(\Omega)$, the first zero of g on $[0, r(\Omega)]$, is strictly larger than the first zero of f given by $\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^{-1}$ (see Figure 2) , which proves the inequality.

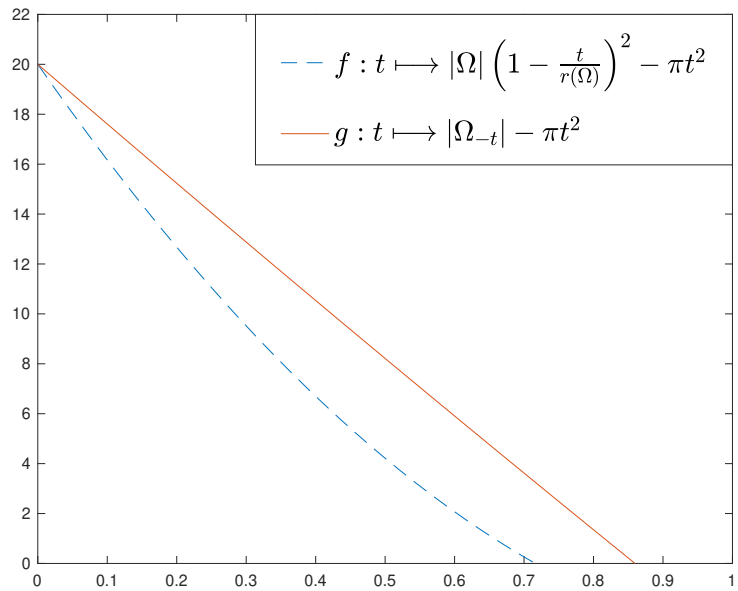

Figure 2: Idea of proof of the upper bound, with $\Omega = [-1, 1] \times [0, 10]$.

2.3 The diagram:

The inequalities (4) imply that

$$\mathcal{D} := \left\{ \left(\frac{\sqrt{|\Omega|}}{r(\Omega)}, \sqrt{|\Omega|} h(\Omega) \right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\} \subset \left\{ (x, y) \;\middle|\; x \geq \frac{1}{r(B)} = \sqrt{\pi} \text{ and } x + \frac{\pi}{x} \leq y \leq x + \sqrt{\pi} \right\}.$$

It remains to prove the reverse inclusion. The proof follows the following steps:

1. We explicit a continuous family $(S_r)_{r \leq r(B)}$ of convex bodies which fill the lower boundary of the diagram.
2. We explicit a continuous family $(U_r)_{r \leq r(B)}$ of convex bodies which fill the upper boundary of the diagram.
3. We use the latter domains to construct (via Minkowski sums) a family of continuous paths $(\Gamma_r)_{r \leq r(B)}$ which connect the upper domains to the lower ones and prove that we are able to cover all the area between the upper and lower boundaries.

Step 1: The lower boundary of the diagram:

As proved above, the lower boundary corresponds to stadiums. Let us consider the family of stadiums $(Q_t)_{t \geq 0}$ given by convex hulls of the balls of unit radius centred in $O(0, 0)$ and $O_t(t, 0)$. The functions

- $t \in [0, +\infty) \longmapsto \sqrt{|Q_t|}/r(Q_t) = \pi + 2t$

- $t \in [0,+\infty) \longmapsto \sqrt{|Q_t|}h(Q_t) = \frac{2(\pi+t)}{\sqrt{\pi+2t}}$.

are continuous and strictly increasing to infinity (when $t \to +\infty$). Thus, we have by the intermediate values Theorem:

$$\left\{ \left(\frac{\sqrt{|Q_t|}}{r(Q_t)}, \sqrt{|Q_t|}h(Q_t) \right) \;\middle|\; t \geq 0 \right\} = \left\{ (x,y) \;\middle|\; x \geq \frac{1}{r(B)} = \sqrt{\pi} \text{ and } y = x + \frac{\pi}{x} \right\}.$$

Step 2: The upper boundary of the diagram:

Since equality $h(\Omega) = \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}$ holds for sets that are homothetical to their form bodies, we use such domains to fill the upper boundary.

Let us consider the family $(C_d)_{d\geq 2}$ of the so-called symmetrical cup-bodies, which are given by convex hulls of the unit ball (centred in $O(0,0)$ of radius 1) and the points of coordinates $(-d/2,0)$ and $(d/2,0)$. By using formula (7) of [19], we have for every $d \geq 2$:

$$|C_d| = \sqrt{d^2-4} + \pi - 2\arccos\left(\frac{2}{d}\right), \quad r(C_d) = 1 \quad \text{and} \quad h(C_d) = \frac{1}{r(C_d)} + \sqrt{\frac{\pi}{|C_d|}}.$$

The functions

- $d \in [2,+\infty) \longmapsto \sqrt{|C_d|}/r(C_d) = \sqrt{\sqrt{d^2-4} + \pi - 2\arccos\left(\frac{2}{d}\right)}$,
- $d \in [2,+\infty) \longmapsto \sqrt{|C_d|}h(C_d) = \sqrt{\sqrt{d^2-4} + \pi - 2\arccos\left(\frac{2}{d}\right)} + \sqrt{\pi}$.

are continuous and strictly increasing to infinity (when $d \to +\infty$), this shows by the intermediate values Theorem that:

$$\left\{ \left(\frac{\sqrt{|C_d|}}{r(C_d)}, \sqrt{|C_d|}h(C_d) \right) \;\middle|\; d \geq 2 \right\} = \left\{ (x,y) \;\middle|\; x \geq \frac{1}{r(B)} = \sqrt{\pi} \text{ and } y = x + \sqrt{\pi} \right\}.$$

Step 3: Continuous paths:

For every $d \geq 2$ and $t \geq 0$, we denote $C'_d := \frac{C_d}{\sqrt{|C_d|}}$ and $Q'_t := \frac{Q_t}{\sqrt{|Q_t|}}$ the rescaled versions of C_d and Q_t (such that $|C'_d| = 1$ and $|Q'_t| = 1$). Since the functions

- $t \in [0,+\infty) \longmapsto \frac{1}{r(Q'_t)} = \frac{\sqrt{|Q_t|}}{r(Q_t)} = \frac{2(\pi+t)}{\sqrt{\pi+2t}}$
- $d \in [2,+\infty) \longmapsto \frac{1}{r(C'_d)} = \frac{\sqrt{|C_d|}}{r(C_d)} = 2\sqrt{\sqrt{d^2-1} + 2\arcsin\frac{2}{d}}$

are continuous and strictly increasing, we have that for every $r \leq r(B)$, there exists a unique $(t_r, d_r) \in [0,+\infty) \times [2,+\infty)$ such that $r(Q_{t_r}) = r(C_{d_r}) = r$. From now on we denote $S_r := Q_{t_p}$ and $L_r := C_{d_p}$.

For every $r \leq r(B)$, we introduce the closed and continuous path Γ_r (see Figure 3) :

$$\begin{array}{rcccl} \Gamma_r & : & [0,3] & \longrightarrow & \mathbb{R}^2 \\ & & t & \longmapsto & \begin{cases} \left(1/r(K_r^t), h(K_r^t)\right) & \text{if } t \in [0,1], \\ \left(x_{r,1}(t), f_1(x_{r,1}(t))\right) & \text{if } t \in [1,2], \\ \left(x_{r,2}(t), f_2(x_{r,1}(t))\right) & \text{if } t \in [2,3], \end{cases} \end{array}$$

where

- $K_r^t := \frac{tS_r + (1-t)L_r}{\sqrt{|tS_r + (1-t)L_r|}} \in \mathcal{K}_1^2 := \{\Omega \in \mathcal{K}^2,\ |\Omega| = 1\}$,
- $x_{r,1} : t \longmapsto (t-1)\frac{1}{r(B)} + (2-t)\frac{1}{r}$,
- $x_{r,2} : t \longmapsto (3-t)\frac{1}{r(B)} + (t-2)\frac{1}{r}$,
- $f_1 : x \in [1/r(B),+\infty) \longmapsto x + \sqrt{\pi}$,
- $f_2 : x \in [1/r(B),+\infty) \longmapsto x + \frac{\pi}{x}$.

The application $t \in [0,1] \longmapsto tS_r + (1-t)L_r \in (\mathcal{K}^2, d^H)$ is continuous and since the measure is continuous for the Hausdorff distance, we deduce that $t \in [0,1] \longmapsto K_r^t \in (\mathcal{K}_1^2, d^H)$ is continuous, thus by continuity of the inradius and the Cheeger constant for the Hausdorff distance (see the proof of [11, Proposition 1] and [27, Proposition 3.1]), the path $t \in [0,3] \longmapsto \Gamma_r(t) \in \mathbb{R}^2$ is a continuous curve.

We also notice that thanks to the linearity of the inradius for the Minkowski sum, as well as the Brunn-Minkowski inequality (see for example [31, Theorem 7.1.1]), one has:

$$r(tS_r + (1-t)U_r) = t \times r(S_r) + (1-t) \times r(U_r) = t \times r + (1-t) \times r = r \text{ and } |tS_r + (1-t)U_r| \geq 1.$$

We then have:

$$\forall t \in [0,1], \qquad \frac{1}{r(K_r^t)} = \frac{\sqrt{|t \times r(S_r) + (1-t) \times r(U_r)|}}{t \times r(S_r) + (1-t) \times r(U_r)} \geq \frac{1}{r}. \tag{6}$$

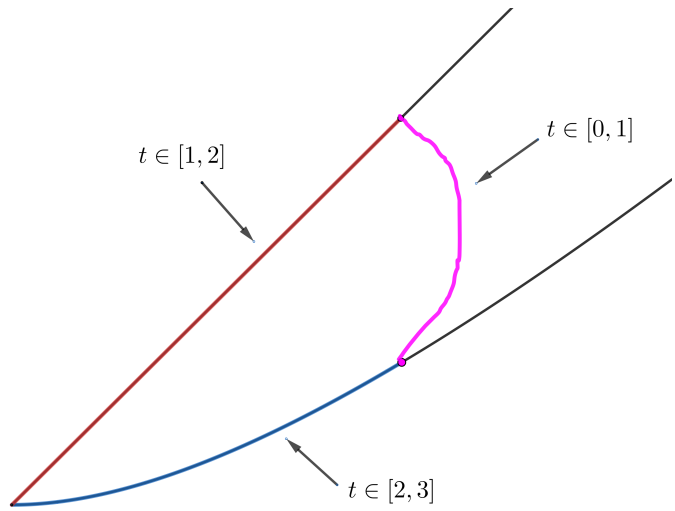

Figure 3: The continuous path Γ_r.

Step 4: Stability of the paths:

For $X = (x_1, x_2) \in \mathbb{R}^2$, we denote $\|X\| = \max(|x_1|, |x_2|)$.

In this step, we prove a continuity result on the paths $\left(\Gamma_r\right)_{r \leq r(B)}$: let us take $r_0 \in (0, r(B)]$ and $\varepsilon > 0$, we show that:

$$\exists\, \alpha_\varepsilon > 0, \forall r \in (r_0 - \alpha_\varepsilon, r_0 + \alpha_\varepsilon) \cap (0, r(B)], \qquad \sup_{t \in [0,3]} \| \Gamma_r(t) - \Gamma_{r_0}(t) \| \leq \varepsilon. \tag{7}$$

Let us take $r \in [r_0/2, r(B)]$, we have for every $t \in [1,2]$:

$$\begin{aligned}
\|\Gamma_r(t) - \Gamma_{r_0}(t)\| &= \max\left(|x_{r,1}(t) - x_{r_0,1}(t)|, |f_1(x_{r,1}(t)) - f_1(x_{r_0,1}(t))|\right) \\
&\leq \max\left(|x_{r,1}(t) - x_{r_0,1}(t)|, \sup_{s \in [\sqrt{\pi}, +\infty)} |f_1'(s)| \times |x_{r,1}(t) - x_{r_0,1}(t)|\right) \\
&= |x_{r,1}(t) - x_{r_0,1}(t)| = (2-t)\left|\frac{1}{r} - \frac{1}{r_0}\right| \leq \frac{1}{r \times r_0}|r - r_0| \leq \frac{2}{r_0^2}|r - r_0|.
\end{aligned}$$

With similar computations we obtain that for every $t \in [2,3]$, we have $\|\Gamma_r(t) - \Gamma_{r_0}(t)\| \leq \frac{2}{r_0}|r - r_0|$.

We then write:

$$\sup_{t \in [1,3]} \| \Gamma_r(t) - \Gamma_{r_0}(t) \| \leq \frac{2}{r_0^2}|r - r_0|. \tag{8}$$

The remaining case ($t \in [0,1]$) requires more computations. For every $t \in [0,1]$, we have

$$\|\Gamma_r(t) - \Gamma_{r_0}(t)\| \leq \underbrace{\max\left(\frac{2}{r_0^2}, \frac{128}{r_0^6}\right)}_{C_{r_0} > 0} d^H(K_r^t, K_{r_0}^t).$$

Indeed, we used:

- for the term with inradii

$$\left|\frac{1}{r(K_r^t)}-\frac{1}{r(K_{r_0}^t)}\right|=\left|\frac{1}{r}-\frac{1}{r_0}\right|=\frac{1}{r\times r_0}|r-r_0|\leq\frac{2}{r_0^2}|r-r_0|,$$

- the first assertion of [14, Lemma 2.7] for the term with the Cheeger constants, with the sets K_p^t and $K_{p_0}^t$ that we assume to contain the origin O and whose radial functions are denoted $f_{p,t}, f_{p_0,t}$ (if Ω is a convex set that contains the origin we denote $f_\Omega:\theta\in[0,2\pi]\longmapsto\sup\{\lambda\geq 0,\lambda\binom{\cos\theta}{\sin\theta}\in\Omega\}$ the radial function of Ω).

$$\begin{aligned}
|h(K_r^t)-h(K_{r_0}^t)| &\leq \frac{2}{\min(r,r_0)^2}\times\|f_{r,t}-f_{r_0,t}\|_\infty \quad \text{(by the first assertion of [14, Lemma 2.7])}\\
&\leq \frac{8}{r_0^2}\times\frac{\|f_{p,t}\|_\infty\|f_{p_0,t}\|_\infty}{\min\left(r(K_r^t),r(K_{r_0}^t)\right)^2}\times d^H(K_r^t,K_{r_0}^t) \quad \text{(by [6, Proposition 2])}\\
&\leq \frac{8}{r_0^2}\times\frac{d(K_r^t)d(K_{r_0}^t)}{\min(r,r_0)^2}\times d^H(K_r^t,K_{r_0}^t) \quad \text{(because } \|f_{r,t}\|_\infty,\|f_{r_0,t}\|_\infty\leq)\\
&\leq \frac{32}{r_0^4}\times\frac{4}{r_0^2}\times d^H(K_r^t,K_{r_0}^t) \quad \text{(because } \|f_{r_0,t}\|_\infty\leq\|f_{r_0,t}\|_\infty\leq d(K_{r_0}^t) \text{ and } d(K_{r_0}^t)\leq\frac{2}{r_0}\text{, see (9) below)}\\
&= \frac{128}{r_0^6}\times d^H(K_r^t,K_{r_0}^t),
\end{aligned}$$

where we used that for every $\Omega\in\mathcal{K}^2$ with unit area which contains a ball of radius $r_0/2$, we have:

$$d(\Omega)\leq\frac{1}{2}P(\Omega)\leq\frac{|\Omega|}{r(\Omega)}\leq\frac{2|\Omega|}{r_0}=\frac{2}{r_0}. \tag{9}$$

Moreover, we recall that $h_{K_{r_0}^t}$ and $h_{K_r^t}$ respectively correspond to the support functions of the convex sets $K_{r_0}^t$ and K_r^t, we have:

$$\begin{aligned}
d^H(K_r^t,K_{r_0}^t) &= \left\|h_{K_r^t}-h_{K_{r_0}^t}\right\|_\infty\\
&= \left\|\frac{(1-t)h_{L_{r_0}}+th_{S_{r_0}}}{\sqrt{|(1-t)L_{r_0}+tS_{r_0}|}}-\frac{(1-t)h_{L_r}+th_{S_r}}{\sqrt{|(1-t)L_r+tS_r|}}\right\|_\infty\\
&\leq (1-t)\left\|\frac{h_{L_{r_0}}}{\sqrt{|(1-t)L_{r_0}+tS_{r_0}|}}-\frac{h_{L_r}}{\sqrt{|(1-t)L_r+tS_r|}}\right\|_\infty\\
&\quad +t\left\|\frac{h_{S_{r_0}}}{\sqrt{|(1-t)L_{r_0}+tS_{r_0}|}}-\frac{h_{S_r}}{\sqrt{|(1-t)L_p+tS_p|}}\right\|_\infty\\
&\leq \frac{1}{\sqrt{|(1-t)L_r+tS_r|}}\left(\left\|h_{S_{r_0}}-h_{S_r}\right\|_\infty+\left\|h_{L_{r_0}}-h_{L_r}\right\|_\infty\right)\\
&\quad +\left(\left\|h_{S_{r_0}}\right\|_\infty+\left\|h_{L_{r_0}}\right\|_\infty\right)\left|\frac{1}{\sqrt{|(1-t)L_r+tS_r|}}-\frac{1}{\sqrt{|(1-t)L_{r_0}+tS_{r_0}|}}\right|\\
&\leq \left(d^H(S_{r_0},S_r)+d^H(L_{r_0},L_r)\right)\\
&\quad +\left(\left\|h_{S_{r_0}}\right\|_\infty+\left\|h_{L_{r_0}}\right\|_\infty\right)\times\Big|\,|(1-t)L_r+tS_r|-|(1-t)L_{r_0}+tS_{r_0}|\,\Big|\\
&\leq \left(d^H(S_{r_0},S_r)+d^H(L_{r_0},L_r)\right)\\
&\quad \underbrace{+\left(\left\|h_{S_{r_0}}\right\|_\infty+\left\|h_{L_{r_0}}\right\|_\infty\right)\times\sum_{k=0}^{2}|W_k(L_r,S_r)-W_k(L_{r_0},S_{r_0})|}_{G_{r_0}(r)},
\end{aligned}$$

where W_1, W_2 and W_3 are the classical Minkowski mixed volumes (we refer to [31] for definitions and more properties).

Thus, we obtain:

$$\sup_{t\in[0,1]}\|\,\Gamma_r(t)-\Gamma_{r_0}(t)\,\|\leq\max\left(\frac{2}{r_0^2},\frac{128}{r_0^6}\right)\times G_{r_0}(r). \tag{10}$$

By combining (8) and (10), we write:

$$\sup_{t\in[0,3]} \| \Gamma_r(t) - \Gamma_{r_0}(t) \| \leq \max\left(\max\left(\frac{2}{r_0^2}, \frac{128}{r_0^6}\right) \times G_{r_0}(r), \frac{2}{r_0^2}|r-r_0|\right) \underset{r\to r_0}{\longrightarrow} 0,$$

where the convergence $G_{r_0}(r) \underset{r\to r_0}{\longrightarrow} 0$ follows from the fact that $d^H(S_r, S_{r_0}) \underset{r\to r_0}{\longrightarrow} 0$, $d^H(L_r, L_{r_0}) \underset{r\to r_0}{\longrightarrow} 0$ and the continuity of the Minkowski mixed volumes W_1, W_2 and W_3 for the Hausdorff distance (see [31]).

Finally, we deduce that $\lim_{r\to r_0} \sup_{t\in[0,3]} \| \Gamma_r(t) - \Gamma_{r_0}(t)\| = 0$, which proves (7).

Step 5: Conclusion:

Now that we proved that the boundaries $\{(x, x+\sqrt{\pi}) \mid x \geq 1/r(B)\}$ and $\{(x, x+\pi/x) \mid x \geq 1/r(B)\}$ are included in the diagram $\mathcal{D}$, it remains to show that it is also the case for the set of points contained between them. We argue by contradiction, assuming that there exists $A(x_A, y_A) \in \{(x,y) \mid x > 1/r(B) \text{ and } x+\pi/x < y < x+\sqrt{\pi}\}$, such that $A \notin \mathcal{D}$.

We consider the function $\phi_A : x \in [1/r(B), +\infty) \longmapsto \text{ind}(\Gamma_{1/x}, A)$, where $\text{ind}(\Gamma_{1/x}, A)$ is the index of A with respect to $\Gamma_{1/x}$ (also called the winding number of the closed curve $\Gamma_{1/x}$ around the point A).

- By Step 4 and the continuity of the index, the function ϕ_A is constant on $[1/r(B), +\infty)$.
- By inequality (6), for every $x > x_A$ the point A is in the interior of $\Gamma_{1/x}$, thus $\phi_A(x) \neq 0$.
- On the other hand, the point A is in the exterior of $\Gamma_{1/r(B)} = \{(1/r(B), 1/r(B)+\sqrt{\pi})\}$, thus $\phi_A(1/r(B)) = 0$.

By the last three points we get a contradiction, thus $A \in \mathcal{D}$. Finally, we get the equality

$$\mathcal{D} = \left\{(x,y) \mid x \geq x_0 \text{ and } x + \frac{\pi}{x} \leq y \leq x + \sqrt{\pi}\right\}.$$

3 Improving the Cheeger inequality for planar convex sets

In this section, we provide the proof of Theorem 1.1 and prove some improved bounds for J in some special subclasses of $\mathcal{K}^2$, namely: triangles, rhombii and stadiums.

3.1 Proof of Theorem 1.1

Let $\Omega \in \mathcal{K}^2$. We have by Hersch inequality [20] and Faber-krahn inequaliy [12, 23]:

$$|\Omega|\lambda_1(\Omega) \geq \max\left(\pi j_{01}^2, \frac{\pi^2|\Omega|}{4r(\Omega)^2}\right).$$

On the other hand, we recall the upper estimate of Theorem 1.3:

$$\sqrt{|\Omega|}h(\Omega) \leq \frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi}.$$

Thus, we have:

$$J(\Omega) = \frac{\lambda_1(\Omega)}{h(\Omega)^2} \geq \frac{\max\left(\pi j_{01}^2, \frac{\pi^2|\Omega|}{4r(\Omega)^2}\right)}{\left(\frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi}\right)^2} \geq \min_{x\geq\sqrt{\pi}} \frac{\max\left(\pi j_{01}^2, \frac{\pi^2 x^2}{4}\right)}{(x+\sqrt{\pi})^2} = \left(\frac{\pi j_{01}}{2j_{01}+\pi}\right)^2 \approx 0.902...$$

The minimum is taken over $[\sqrt{\pi}, +\infty)$ because $\frac{\sqrt{|\Omega|}}{r(\Omega)} \geq \sqrt{\pi}$ for every $\Omega \in \mathcal{K}^2$. Moreover, it is attained for $x = \frac{\pi j_{01}}{\sqrt{\pi}}$, see Figure 4.

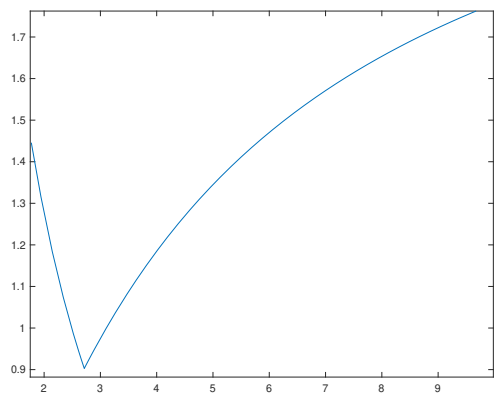

Figure 4: Curve of the function $x \longmapsto \frac{\max\left(\pi j_{01}^2, \frac{\pi^2 x^2}{4}\right)}{(x+\sqrt{\pi})^2}$.

3.2 A slight improvement of the result of Theorem 1.1

We note that one can combine the following Protter's inequality [29]:

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) \geq \frac{\pi^2}{4}\left(\frac{1}{r(\Omega)^2} + \frac{1}{d(\Omega)^2}\right),$$

which is an improvement of Hersch's inequality [20, Section 8] with the optimal inequality (7) of [19]:

$$\forall \Omega \in \mathcal{K}^2, \quad \frac{|\Omega|}{r(\Omega)^2} \geq \sqrt{1 + \left(\frac{d(\Omega)}{r(\Omega)}\right)^2} + 2\arcsin\left(\frac{2r(\Omega)}{d(\Omega)}\right) := \varphi\left(\frac{d(\Omega)}{r(\Omega)}\right), \tag{11}$$

to provide a slight improvement of the lower bound of Theorem 1.1.

Indeed, the function φ is continuous and strictly increasing on $[2, +\infty)$ (we note that $d(\Omega)/r(\Omega) \in [2, +\infty)$, thus by considering the inverse function denoted φ^{-1}, inequality 11 becomes:

$$\forall \Omega \in \mathcal{K}^2, \quad \frac{d(\Omega)}{r(\Omega)} \leq \varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)$$

. We then write:

$$\begin{aligned}
\frac{\lambda_1(\Omega)}{h(\Omega)^2} &\geq \frac{\max\left(\frac{\pi^2}{4r(\Omega)^2} \times \left(1 + \left(\frac{1}{\varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)}\right)\right)^2, \frac{\pi j_{01}^2}{|\Omega|}\right)}{\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^2} \\
&= \frac{\max\left(\frac{\pi^2}{4} \times \frac{|\Omega|}{r(\Omega)^2} \times \left(1 + \left(\frac{1}{\varphi^{-1}\left(\frac{|\Omega|}{r(\Omega)^2}\right)}\right)^2\right), \pi j_{01}^2\right)}{\left(\left(\frac{|\Omega|}{r(\Omega)^2}\right)^{1/2} + \sqrt{\pi}\right)^2} \\
&\geq \min_{x \in [\pi, +\infty)} \frac{\max\left(\frac{\pi^2}{4} x \left(1 + \frac{1}{\varphi^{-1}(x)^2}, \pi j_{01}^2\right)\right)}{(\sqrt{x} + \sqrt{\pi})} \quad \text{(because } \frac{|\Omega|}{r(\Omega)^2} \geq \frac{\pi r(\Omega)^2}{r(\Omega)^2} = \pi\text{)}.
\end{aligned}$$

Numerical computations show that the latter minimum is approximately equal to $0.914...$, which slightly improves the lower bound of Theorem 1.1.

3.3 Improvements for special classes of shapes

We provide some improved lower bounds of J for some special classes of planar convex sets. We note that the numerical bounds which appear in the following proposition are not exact.

Proposition 3.1. *Let $\Omega \in \mathcal{K}^2$.*

1. *If Ω is a triangle, then $J(\Omega) > 1.2076$.*
2. *If Ω is a rhombus, then $J(\Omega) \geq 1.3819$.*
3. *If Ω is a stadium (i.e. the convex hull of two identical balls), then $J(\Omega) \geq 1.3673$.*

Proof. Let $\Omega \in \mathcal{K}^2$, since J is invariant by homothety (due to scaling properties of λ_1 and h), we may assume without loss of generality that $|\Omega| = 1$.

1. Let us assume Ω to be a triangle and denote d its diameter and L its perimeter. To bound $\lambda_1(\Omega)$ from below, we make use of two inequalities:

   - The first one is the polygonal Faber-Krahn inequality for triangles, which states that between triangles of the same area, the regular one minimizes the first Dirichlet eigenvalue:
   $$\lambda_1(\Omega) \geq \lambda_1(T_{\text{eq}}) = \frac{4\pi^2}{\sqrt{3}},$$
   where T_{eq} is the equilateral triangle of unit area (whose diameter is $d_{eq} = \frac{2}{3^{1/4}}$).
   - The second (more recent) is due to P. Freitas and B. Siudeja [13, Corollary 4.1]:
   $$\lambda_1(\Omega) \geq \frac{\pi^2}{4|\Omega|^2}\left(d(\Omega) + \frac{2|\Omega|}{d(\Omega)}\right)^2.$$

   We then have on the one hand:
   $$\lambda_1(\Omega) \geq \max\left(\frac{4\pi^2}{\sqrt{3}}, \frac{\pi^2}{4}\left(d + \frac{2}{d}\right)^2\right),$$
   and on the other hand, the Cheeger constant of the triangle Ω is given by:
   $$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = \frac{L + \sqrt{4\pi}}{2} \leq \frac{L_{iso} + \sqrt{4\pi}}{2},$$
   where L_{iso} is the perimeter of the isoceles triangle whose diameter is equal to d and area equal to 1. By using Pythagoras' theorem, we have:
   $$L_{iso} = 2d + \sqrt{\left(d - \sqrt{d^2 - \frac{4}{d^2}}\right)^2 + \frac{4}{d^2}}.$$
   Finally, we obtain the following inequality:
   $$J(\Omega) \geq \phi_1(d) := \frac{\max\left(\frac{4\pi^2}{\sqrt{3}}, \frac{\pi^2}{4}\left(d + \frac{2}{d}\right)^2\right)}{\left(\frac{2d + \sqrt{\left(d - \sqrt{d^2 - \frac{4}{d^2}}\right)^2 + \frac{4}{d^2}} + \sqrt{4\pi}}{2}\right)^2}.$$
   We note that $d \geq d_{eq}$. Indeed, by the isoperimetric inequality of triangles:
   $$3d_{eq} = L_{eq} \leq L \leq 3d.$$
   Numerically, we obtain $\min\limits_{d \geq d_{eq}} \phi_1(d) \approx 1.2076...$

2. Let us assume Ω to be the rhombus of unit area whose vertices are given by $(-d/2, 0)$, $(0, -1/d)$, $(d/2, 0)$ and $(0, 1/d)$.

   We bound $\lambda_1(\Omega)$ from below by using the following Hooker and Protter's estimate for for rhombi [21]:
   $$\lambda_1(\Omega) \geq \pi^2\left(\frac{d}{2} + \frac{1}{d}\right)^2.$$

As for the Cheeger constant, since Ω is a circumscribed polygon, we have:

$$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = 2\sqrt{\frac{1}{d^2} + \frac{d^2}{4}} + \sqrt{\pi}$$

we use its explicit value in term of d.

$$J(\Omega) \geq \phi_2(d) = \frac{\pi^2 \left(\frac{d}{2} + \frac{1}{d}\right)^2}{\left(2\sqrt{\frac{1}{d^2} + \frac{d^2}{4}} + \sqrt{\pi}\right)^2}.$$

Numerically, we obtain $\min\limits_{d \geq \sqrt{2}} \phi_2(d) \approx 1.3819...$

3. Let us assume Ω to be a stadium of unit area whose diameter is given by $a + 2r$, where $r > 0$ is the radius of the ball of its extremity and $a > 0$. The condition $|\Omega| = 1$ implies that $\pi r^2 + 2ar = 1$, which is equivalent to $a = \frac{1-\pi r^2}{2r}$. We use the monotonicity of λ_1 for inclusion (for $\Omega \subset (-r, r) \times (0, a + 2r)$) and Faber-Krahn inequality to write:

$$\lambda_1(\Omega) \geq \max\left(\lambda_1(B), \pi^2\left(\frac{1}{4r^2} + \frac{1}{(a+2r)^2}\right)\right) = \max\left(\lambda_1(B), \pi^2\left(\frac{4r^2}{(1+(4-\pi)r^2)^2} + \frac{1}{4r^2}\right)\right).$$

It is classical that the stadiums are Cheeger of themselves, see [22], we then have:

$$h(\Omega) = \frac{P(\Omega)}{|\Omega|} = 2a + 2\pi r = \frac{1 + \pi r^2}{r}.$$

Then:

$$J(\Omega) \geq \phi_3(r) := \frac{\max\left(\lambda_1(B), \pi^2\left(\frac{4r^2}{(1+(4-\pi)r^2)^2} + \frac{1}{4r^2}\right)\right)}{\left(\frac{1+\pi r^2}{r}\right)^2}$$

Numerically, we obtain $\min\limits_{r \in \left(0, \frac{1}{\sqrt{\pi}}\right]} \phi_3(r) \approx 1.3673...$

□

4 On the existence of a minimizer in higher dimensions

4.1 Proof of Theorem 1.2

1. Let $n \geq 2$, let us first prove that:

$$\beta_n := \inf_{\Omega \in \mathcal{K}^n} J(\Omega) \leq \inf_{\omega \in \mathcal{K}^{n-1}} J(\omega) =: \beta_{n-1}.$$

The idea is to prove that for any $\omega \in \mathcal{K}^{n-1}$, there exists a family $(\Omega_d)_{d>0}$ of elements of $\mathcal{K}^n$ such that:

$$J(\omega) = \lim_{d \to +\infty} J(\Omega_d).$$

The proof is decomposed in 3 steps.

Step 1: Lower estimates for λ_1 and h

Let us take $\Omega \in \mathcal{K}^n$. We can assume without loss of generality that $\inf\{t \in \mathbb{R} \mid \Omega \cap \{x_1 = t\} \neq \varnothing\} = 0$ and denote $M_\Omega := \sup\{t \in \mathbb{R} \mid \Omega \cap \{x_1 = t\} \neq \varnothing\}$. We introduce the functions

$$\phi_\lambda : t \in [0, M_\Omega] \longmapsto \begin{cases} \lambda_1(\Omega \cap \{x_1 = t\}) \in \mathbb{R}_+^* & \text{if } t \in (0, M_\Omega), \\ \lambda_1(\Omega \cap \{x_1 = t\}) \in \mathbb{R}_+^* & \text{if } t \in \{0, M_\Omega\} \text{ and } |\Omega \cap \{x_1 = t\}| > 0, \\ +\infty & \text{if } t \in \{0, M_\Omega\} \text{ and } |\Omega \cap \{x_1 = t\}| = 0, \end{cases}$$

and

$$\phi_h : t \in [0, M_\Omega] \longmapsto \begin{cases} h(\Omega \cap \{x_1 = t\}) \in \mathbb{R}_+^* & \text{if } t \in (0, M_\Omega), \\ h(\Omega \cap \{x_1 = t\}) \in \mathbb{R}_+^* & \text{if } t \in \{0, M_\Omega\} \text{ and } |\Omega \cap \{x_1 = t\}| > 0, \\ +\infty \quad \text{if } t \in \{0, M_\Omega\} \text{ and } |\Omega \cap \{x_1 = t\}| = 0, \end{cases}$$

Since the function $t \in (0, M_\Omega) \longmapsto \Omega \cap \{x_1 = t\} \in (\mathcal{K}^{n-1}, d^H)$ is continuous and λ_1 and h are continuous for the Hausdorff distance d^H on $\mathcal{K}^{n-1}$, we have that the functions ϕ_λ and ϕ_h are continuous on $(0, M_\Omega)$. We then distinguish the cases "$\phi_\lambda(0) = +\infty$ or $\phi_\lambda(M_\Omega) = +\infty$" (resp. "$\phi_h(0) = +\infty$ or $\phi_h(M_\Omega) = +\infty$") and "$\phi_\lambda(0) \in \mathbb{R}$ and $\phi_\lambda(M_\Omega) \in \mathbb{R}$" (resp. "$\phi_h(0) \in \mathbb{R}$ and $\phi_h(M_\Omega) \in \mathbb{R}$") and use the intermediate values theorem to show that there exist $t_\lambda, t_h \in [0, M_\Omega]$ such that $\min\limits_{t \in (0, M_\Omega)} \phi_\lambda(t) = \phi_\lambda(t_\lambda)$ and $\min\limits_{t \in (0, M_\Omega)} \phi_h(t) = \phi_h(t_\lambda)$.

In the proof of [7, Lemma 7.63], the authors prove that:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega \cap \{x_1 = t_\lambda\}).$$

Let us prove a similar result for Cheeger's constant (i.e. $h(\Omega) \geq h(\Omega \cap \{x_1 = t_h\})$.). We have

$$\begin{aligned} h(\Omega) = \frac{P(C_\Omega)}{|C_\Omega|} = \frac{\int_0^d P(C_\Omega \cap \{x_1 = t\})dt}{\int_0^d |C_\Omega \cap \{x_1 = t\}|dt} &\geq \frac{\int_0^d h(\Omega \cap \{x_1 = t\})|C_\Omega \cap \{x_1 = t\}|dt}{\int_0^d |C_\Omega \cap \{x_1 = t\}|dt} \\ &\geq \frac{h(\Omega \cap \{x_1 = t_h\}) \int_0^d |C_\Omega \cap \{x_1 = t\}|dt}{\int_0^d |C_\Omega \cap \{x_1 = t\}|dt} \\ &= h(\Omega \cap \{x_1 = t_h\}). \end{aligned}$$

Step 2: Study of sets with increasing diameters and fixed volume

Let (Ω_k) a sequence of elements $\mathcal{K}^n$ of the same volumes 1, such that $d_k := d(\Omega_k) \to +\infty$. Let us prove that:

$$\liminf_{k \to +\infty} J(\Omega_k) \geq \inf_{\omega \in \mathcal{K}^{n-1}} J(\omega).$$

For every $k \in \mathbb{N}$, we consider A_k and A'_k two diametrical points of Ω_k (ie. such as $|A_k A'_k| = d_k$). Since J is invariant by rigid motions we can assume without loss of generality that $A_k = (0, ..., 0)$ and $A'_k = (d_k, 0, ..., 0)$.

By Step 1, we have for all $k \in \mathbb{N}$:

$$\lambda_1(\Omega_k) \geq \lambda_1(\omega_k)$$

where $\omega_k := \Omega_k \cap \{x = t_k\}$.

We can assume without loss of generality that $t_k \geq d_k/2$. Let $\mathcal{T}_k$ be the cone obtained by taking the convex hull of $\{A_k\} \cup C_k$, where C_k is the Cheeger set of the convex section ω_k.

Let $\alpha \in]0, 1[$, we introduce the tube $U_k^\alpha := \alpha C_k \times (0, (1-\alpha)t_k)$. By convexity, we have the following inclusions:

$$U_k^\alpha \subset \mathcal{T}_k \subset \Omega_k$$

By definition of the Cheeger constant, we have:

$$h(\Omega_k) \leq \frac{P(U_k^\alpha)}{|U_k^\alpha|} = \frac{2\alpha^{n-1}|C_k| + \alpha^{n-2}(1-\alpha)P(C_k)t_k}{\alpha^{n-1}(1-\alpha)|C_k|t_k} = \frac{2}{(1-\alpha)t_k} + \frac{h(\omega_k)}{\alpha} \underset{k \to +\infty}{\sim} \frac{h(\omega_k)}{\alpha}.$$

Indeed: $\frac{1}{t_k} = \underset{k \to \infty}{o}\left(\frac{P(C_k)}{|C_k|}\right)$, because:

$$\frac{|C_k|}{P(C_k)} = \frac{|C_k|}{P\left(|C_k|^{\frac{1}{n}} \times \frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} = \frac{|C_k|}{|C_k|^{\frac{n-1}{n}} P\left(\frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} = \frac{|C_k|^{\frac{1}{n}}}{P\left(\frac{C_k}{|C_k|^{\frac{1}{n}}}\right)} \leq \frac{|C_k|^{\frac{1}{n}}}{P(B_{n-1})} \leq \frac{|\Omega_k|^{\frac{1}{n}} n^{\frac{1}{n}}}{P(B_{n-1})} \times \frac{1}{t_k^{1/n}} = \underset{k \to \infty}{o}(t_k),$$

where $B_{n-1} \subset \mathbb{R}^{n-1}$ is a ball of volume 1.

We deduce that:

$$\forall \alpha \in (0, 1), \quad J(\Omega_k) \geq \frac{\lambda_1(\omega_k)}{\left(\frac{2}{(1-\alpha)t_k} + \frac{h(\omega_k)}{\alpha}\right)^2} \underset{k \to +\infty}{\sim} \alpha^2 J(\omega_k)$$

Thus:

$$\forall \alpha \in (0,1), \quad \liminf_{k\to+\infty} J(\Omega_k) \geq \alpha^2 \liminf_{k\to+\infty} J(\omega_k) \geq \alpha^2 \inf_{\omega\in\mathcal{K}^{n-1}} J(\omega)$$

By letting $\alpha \to 1$, we obtain:

$$\liminf_{k\to+\infty} J(\Omega_k) \geq \inf_{\omega\in\mathcal{K}^{n-1}} J(\omega)$$

<u>Step 3: Study of long tubes</u>

In this step, we show that when the height of a tube goes to infinity, the value of J of this tube converges to the value corresponding to the $(n-1)$-dimensional section given by its basis. More precisely, if we take $\omega \in \mathcal{K}^{n-1}$, we prove that

$$\lim_{d\to+\infty} J\big([0,d]\times\omega\big) = J(\omega).$$

We have by Step 2:

$$\liminf_{k\to+\infty} J(\Omega_k) \geq J(\omega)$$

It is classical that

$$\lambda_1\big([0,d]\times\omega\big) = \left(\frac{\pi}{d}\right)^2 + \lambda_1(\omega),$$

which follows from the use the separation of variables and the orthogonality of Laplacian eigenfunctions.

As for the Cheeger constant, we use the result of Step 1:

$$h\big([0,d]\times\omega\big) \geq h(\omega).$$

Thus, we have:

$$J\big([0,d]\times\omega\big) = \frac{\lambda_1\big([0,d]\times\omega\big)}{h\big([0,d]\times\omega\big)^2} \leq \frac{\left(\frac{\pi}{d}\right)^2 + \lambda_1(\omega)}{h(\omega)^2}.$$

By passing to superior limit:

$$\limsup_{d\to+\infty} J\big([0,d]\times\omega\big) \leq J(\omega).$$

Then:

$$\lim_{d\to+\infty} J\big([0,d]\times\omega\big) = J(\omega).$$

At last, we write:

$$\beta_{n-1} = \inf_{\omega\in\mathcal{K}^{n-1}} J(\omega) = \inf_{\omega\in\mathcal{K}^{n-1}} \Big(\lim_{d\to+\infty} \underbrace{J\big([0,d]\times\omega\big)}_{\geq \inf_{\Omega\in\mathcal{K}^n} J(\Omega)}\Big) \geq \inf_{\Omega\in\mathcal{K}^n} J(\Omega) = \beta_n.$$

2. For every $n \geq 2$, we take a ball $B_n \subset \mathbb{R}^n$ of unit radius, we have:

$$\frac{1}{4} \leq \inf_{\Omega\in\mathcal{K}^n} J(\Omega) \leq J(B_n) = \frac{\lambda_1(B_n)}{h(B_n)^2} = \frac{j^2_{\frac{n}{2}-1,1}}{n^2} \underset{n\to+\infty}{\sim} \frac{\left(\frac{n}{2}\right)^2}{n^2} = \frac{1}{4},$$

where $j_{\frac{n}{2}-1,1}$ is the first root of the nth Bessel function of first kind. We refer to [33] for the equivalence $j_{\frac{n}{2}-1,1} \underset{n\to+\infty}{\sim} \frac{n}{2}$.

3. **<u>The existence result</u>**:

Now, we assume that: $\inf_{\Omega\in\mathcal{K}^n} J(\Omega) < \inf_{\omega\in\mathcal{K}^{n-1}} J(\omega)$. Let us prove the existence of a minimizer of J on $\mathcal{K}^n$.

Let (Ω_k) be a minimizing sequence of $\mathcal{K}^n$ (ie. such as $\lim_{k\to+\infty} J(\Omega_k) = \inf_{\Omega\in\mathcal{K}^n} J(\Omega)$). Since J is scaling invariant we can assume without loss of generality that $|\Omega_k| = 1$ for all $n \in \mathbb{N}$.

If $\big(d(\Omega_k)\big)$ is not bounded, we can extract a subsequence $(\Omega_{\varphi(k)})$ such as

$$\lim_{k\to+\infty} d(\Omega_{\varphi(k)}) = +\infty.$$

Thus, by Step 2:

$$\inf_{\Omega\in\mathcal{K}^n} J(\Omega) = \lim_{k\to+\infty} J(\Omega_k) \geq \inf_{\omega\in\mathcal{K}^{n-1}} J(\omega).$$

which contradicts hypothesis $\beta_{n-1} > \beta_n$.

We deduce that the sequence of diameters $\big(d(\Omega_k)\big)$ is bounded, then by compactness, there exists $\Omega^* \in \mathcal{K}^n$ and a strictly increasing map $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\Omega_{\sigma(k)} \underset{k\to\infty}{\longrightarrow} \Omega^*$ for Hausdorff distance. We then have by continuity of J for the same metric (see [27, Proposition 3.2]):

$$J(\Omega^*) = \lim_{k\to+\infty} J(\Omega_k) = \inf_{\Omega\in\mathcal{K}^n} J(\Omega).$$

4.2 Discussion of the hypothesis $\beta_n < \beta_{n-1}$

We believe that hypothesis $\beta_n < \beta_{n-1}$ is true for any dimension n and that one can use convex cylinders (i.e. those of the form $\omega \times [0, d]$, where $\omega \in \mathcal{K}^{n-1}$ and $d > 0$) to show it.

Let us analyse what happens when $n = 2$. In this case convex cylinders are rectangles. We consider the family of cylinders $\Omega_d = [0, 1] \times [0, d]$ (where $d > 0$) and denote

$$\Psi_{[0,1]} : d > 0 \longmapsto J(\Omega_d) = \frac{\lambda_1(\Omega_d)}{h(\Omega_d)^2} = \frac{\pi^2\left(1+\frac{1}{d^2}\right)}{\left(\frac{4-\pi}{1+d-\sqrt{(d-1)^2+\pi d}}\right)^2}.$$

We plot the curve of $\Psi_{[0,1]}$ in Figure 5. We note that for sufficiently high values of d the function $\Psi_{[0,1]} : d > 0 \longmapsto J(\Omega_d)$ is strictly increasing and converges to $J\big((0, 1)\big) = \frac{\pi^2}{4}$.

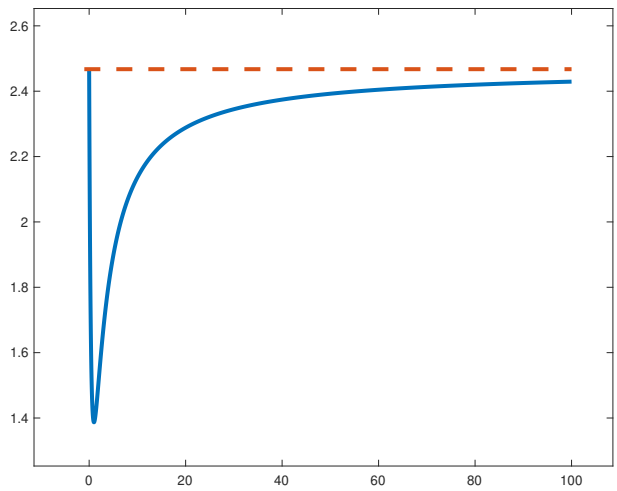

Figure 5: Curve of the function $\Psi_{[0,1]} : d > 0 \longmapsto J\big([0, 1] \times [0, d]\big)$.

We believe that the monotonicity property should hold in higher dimensions: let $n \geq 2$, $\Omega_d := \omega \times [0, d]$ where $d > 0$ and $\omega \in \mathcal{K}^{n-1}$, as before we denote $\Psi_\omega : d > 0 \longmapsto J(\Omega_d)$. We have already proved above that:

$$\lim_{d\to+\infty} \Psi_\omega(d) = \lim_{d\to+\infty} J(\Omega_d) = J(\omega).$$

It remains to prove that for large values of d one has:

$$\Psi_\omega(d) = J(\Omega_d) < J(\omega).$$

To do so, we propose to show that function Ψ_ω is strictly increasing for large values of d by studying the derivative $\Psi'_\omega(d)$.

Let us take $d > 0$, we have for $t > 0$ sufficiently small:

$$\lambda_1(\Omega_{d+t}) = \lambda_1(\omega) + \frac{\pi^2}{(d+t)^2} = \lambda_1(\omega) + \frac{\pi^2}{d^2} - \frac{2\pi^2}{d^3}t + \underset{t\to 0}{o}(t),$$

and

$$h(\Omega_{d+t}) = h(\Omega_d) + h'(\Omega_d, V_d) \times t + \underset{t\to 0}{o}(t),$$

where $V_d : \mathbb{R}^n \to \mathbb{R}^n$ is the smooth dilatation field such that $V_d(x_1, \cdots, x_n) = \left(0, \cdots, 0, \frac{x_n}{d}\right)$. As proved in [28], we have:

$$h'(\Omega_d, V_d) = \frac{1}{|C_{\Omega_d}|} \int_{\partial\Omega_d \cap \partial C_{\Omega_d}} \big(\kappa - h(\Omega_d)\big)\langle V_d, n\rangle d\sigma,$$

where κ is the mean curvature and C_{Ω_d} is the Cheeger set of Ω_d. Since $\langle V_d, n\rangle = 0$ on all $\partial\Omega_d \cap \partial C_{\Omega_d}$ except on the upper basis $\partial\Omega_d \cap \partial C_{\Omega_d} \cap \{x_n = d\}$ where κ is null, we have the following formula for the shape derivative:

$$h'(\Omega_d, V_d) = -\frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|} h(\Omega_d).$$

By straightforward computations we obtain:

$$\begin{aligned}
\Psi'_\omega(d) &= \frac{1}{h(\Omega_d)^2}\left(-\frac{2\pi^2}{d^3} + 2\left(\lambda_1(\omega) + \frac{\pi^2}{d^2}\right)\frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|}\right) \\
&> \frac{2\pi^2}{h(\Omega_d)^2}\left(-\frac{1}{d^3} + \lambda_1(\omega)\frac{|\partial C_{\Omega_d} \cap \{x_n = d\}|}{|C_d|}\right) \\
&\geq \frac{2\pi^2}{d \times h(\Omega_d)^2}\left(\frac{\lambda_1(\omega)}{|\omega|}|\partial C_{\Omega_d} \cap \{x_n = d\}| - \frac{1}{d^2}\right) \quad \text{(because } C_{\Omega_d} \subset \Omega_d\text{, thus } |C_{\Omega_d}| \leq |\Omega_d| = |\omega| \times d\text{)}.
\end{aligned}$$

Finally, it remains to prove that for sufficiently large values of d one can prove estimate of the type:

$$|\partial C_{\Omega_d} \cap \{x_n = d\}| > \frac{1}{d^2}.$$

One can check that this assertion is correct when $n = 2$. Indeed, if we consider the cylinder (rectangle) $\Omega_d = [0, 1] \times [0, d]$, we use the explicit expression of the Cheeger constant of rectangles that can be found in [22]:

$$h(\Omega_d) = \frac{4 - \pi}{d + 1 - \sqrt{(d-1)^2 + \pi d}} = 1 + \frac{1}{d} + \sqrt{1 + \frac{\pi - 2}{d}} = 2 + \frac{\pi - 1}{2d} + \underset{d\to+\infty}{o}\left(\frac{1}{d}\right).$$

Thus, as shown in Figure 6, we have:

$$|\partial C_{\Omega_d} \cap \{x_2 = d\}| = 1 - \frac{2}{h(\Omega_d)} = \frac{\pi - 1}{4d} + \underset{d\to+\infty}{o}\left(\frac{1}{d}\right),$$

which proves that

$$|\partial C_{\Omega_d} \cap \{x_2 = d\}| \underset{d\to+\infty}{\sim} \frac{\pi - 1}{4d} > \frac{1}{d^2}.$$

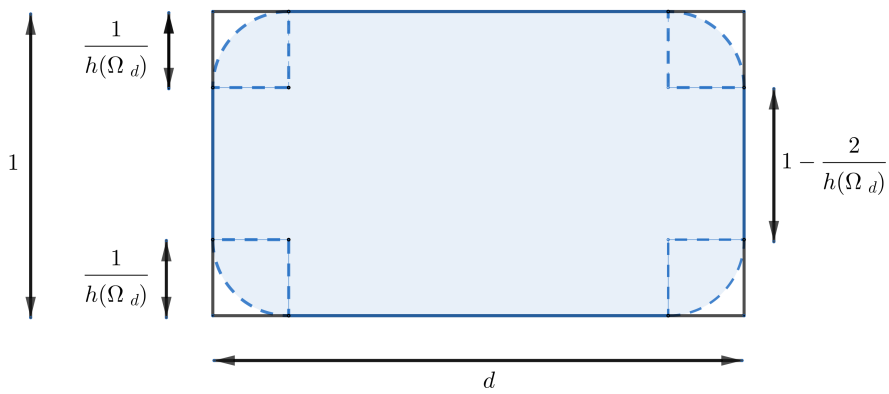

Figure 6: Cheeger set of the rectangle Ω_d.

At last, let us mention the very recent work of E. Parini and V. Bobkov [3] where they manage to explicitly describe the Cheeger sets of rationally invariant sets in any dimension and thus compute their Cheeger values. By applying these results to cylinders of the form $B_{n-1} \times [0, d]$, where $B_{n-1} \subset \mathbb{R}^{n-1}$ is a ball, we remark as expected that $\Psi_{B_{n-1}}$ is strictly increasing for higher values of d and thus converges to $J(B_{n-1})$ from below, which supports our strategy.

5 Appendix: Some applications

In this Appendix, we apply the sharp estimates given in (4) to obtain some new bounds for the first Dirichlet eigenvalue in the case of planar convex sets.

5.1 Some sharp upper bounds for the first Dirichlet eigenvalue

5.1.1 General planar convex sets

Proposition 5.1. *We have the following sharp inequality:*

$$\forall \Omega \in \mathcal{K}^2, \quad \lambda_1(\Omega) < \frac{\pi^2}{4}\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^2, \tag{12}$$

where equality is asymptotically attained by any family of convex sets $(\Omega_k)_{k\in\mathbb{N}}$ such as $|\Omega_k| = V_0$ for any $k \in \mathbb{N}$ (where V_0 is a positive constant) and $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$.

Proof. We have for every $\Omega \in \mathcal{K}^2$:

$$\lambda_1(\Omega) < \frac{\pi^2}{4}h(\Omega)^2 \leq \frac{\pi^2}{4}\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^2,$$

where the first inequality is the reverse Cheeger inequality (also called Buser inequality) proved by E. Parini in [27, Proposition 4.1] and the second inequality corresponds to the upper bound given in (4).

Let us now prove the sharpness inequality (12). Let $V_0 > 0$ and $(\Omega_k)_{k\in\mathbb{N}}$ a family of convex sets such as $|\Omega_k| = V_0$ for any $k \in \mathbb{N}$ and $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$. We have on the one hand:

$$\forall k \in \mathbb{N}^*, \quad \frac{\pi^2}{4r(\Omega_k)^2} < \lambda_1(\Omega_k) < \frac{\pi^2}{4}\left(\frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{|\Omega_k|}}\right)^2 = \frac{\pi^2}{4}\left(\frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{V_0}}\right)^2,$$

on the other hand, we have:

$$\frac{1}{r(\Omega_k)} \geq \frac{P(\Omega_k)}{2|\Omega_k|} \geq \frac{d(\Omega_k)}{V_0} \underset{k\to+\infty}{\longrightarrow} +\infty,$$

thus:

$$\lambda_1(\Omega_k) \underset{k\to+\infty}{\sim} \frac{\pi^2}{4r(\Omega_k)^2}, \tag{13}$$

which proofs the sharpness of inequality (12). □

Remark 5.2. *We note that one can use inequalities* (4)*, to provide a similar equivalence as* (13) *for the Cheeger constant. Indeed, let us consider $V_0 > 0$ and $(\Omega_k)_{k\in\mathbb{N}}$ a family of convex sets such as $|\Omega_k| = V_0$ for any $k \in \mathbb{N}$ and $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$. We have by* (4)*:*

$$\forall k \in \mathbb{N}, \quad \frac{1}{r(\Omega_k)} + \frac{\pi r(\Omega_k)}{V_0} \leq h(\Omega_k) \leq \frac{1}{r(\Omega_k)} + \sqrt{\frac{\pi}{|V_0|}},$$

and since $\frac{1}{r(\Omega_k)} \underset{k\to+\infty}{\longrightarrow} +\infty$, we have the following equivalence:

$$h(\Omega_k) \underset{k\to+\infty}{\sim} \frac{1}{r(\Omega_k)}. \tag{14}$$

By combining (13) *and* (14)*, we retrieve (with an alternative method) the asymptotic result of [27, Proposition 4.1]:*

$$\lim_{k\to+\infty} J(\Omega_k) = \lim_{k\to+\infty} \frac{\lambda_1(\Omega_k)}{h(\Omega_k)^2} = \frac{\pi^2}{4}.$$

It is interesting to compare inequality (12) with other inequalities involving the inradius and the area. One immediate estimate can be obtained by considering the inclusion $B_{r(\Omega)} \subset \Omega$ (where $B_{r(\Omega)}$ is an inscribed ball of Ω (with radius $r(\Omega)$). We have by the monotonicity of λ_1:

$$\lambda_1(\Omega) \leq \lambda_1\big(B_{r(\Omega)}\big) = \frac{j_{01}^2}{r(\Omega)^2}, \tag{15}$$

where j_{01} denotes the first zero of the first Bessel function. This inequality was already stated in [27, inequality (3)] and in [8, inequality (1.5)] in higher dimensions and for a more general setting. In Figure 7, we plot the curves corresponding to the latter inequalities and an approximation of the Blaschke-Santaló diagram corresponding to the functionals λ_1, the inradius r and the area $|\cdot|$, obtained by generating 10^4 random convex sets. The diagram corresponds to the set of points:

$$\mathcal{D} := \left\{ \left(\frac{1}{r(\Omega)}, \lambda_1(\Omega) \right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\}.$$

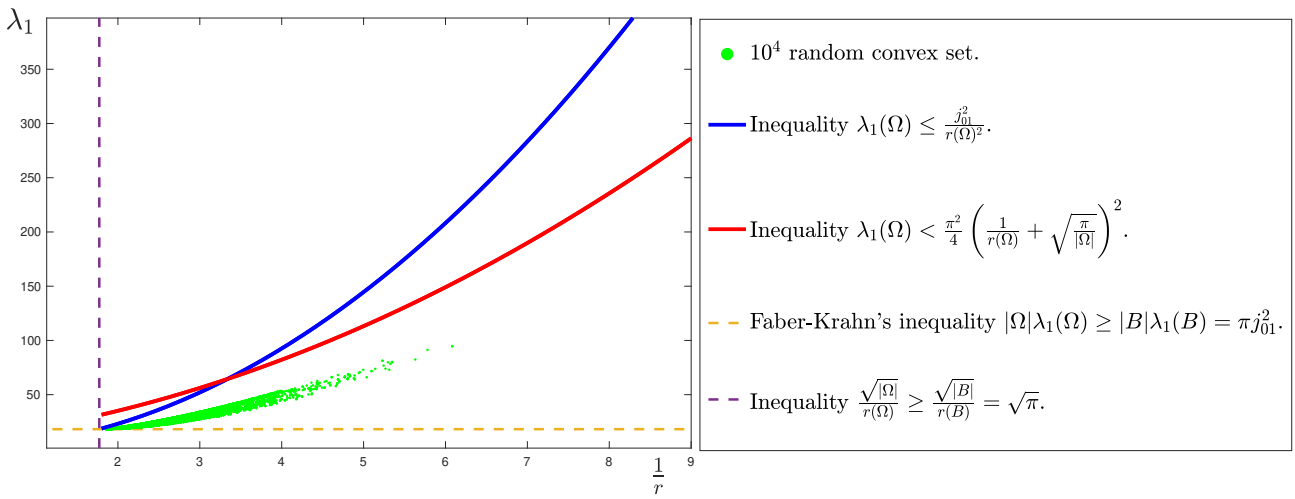

Figure 7: Inequality (12) improves (15) for convex sets with small inradius (ie. large $\frac{1}{r}$).

Remark 5.3. *We note that the use of numerical simulations can be very helpful to have an idea on the shape and the properties of Blaschke-Santaló diagrams and state possible conjectures. It is then common to generate a large number of random domains, compute the values of the involved functionals and then obtain a cloud of dots that approximates the Blaschke-Santaló diagram. For various examples we refer to [1, 14, 15, 24]..*

5.1.2 Sets that are homothetic to their form bodies: in particular "triangles"

We recall that in the case of sets that are homothetic to their form bodies, one has $\frac{1}{2}P(\Omega)r(\Omega) = |\Omega|$ and:

$$h(\Omega) = \frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} = \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}.$$

Thus one can write the following result, which is an immediate Corollary of the reverse Cheeger's inequality of [27, Proposition 4.1]:

Corollary 1. *For every set $\Omega \in \mathcal{K}^2$, that is homothetic to its form body (in particular triangles), we have the following inequality:*

$$\lambda_1(\Omega) < \frac{\pi^2}{4} \times \left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}} \right)^2 = \frac{\pi^2}{16} \times \left(\frac{P(\Omega)}{|\Omega|} + 2\sqrt{\frac{\pi}{|\Omega|}} \right)^2. \tag{16}$$

The inequality is sharp as it is asymptotically attained by any sequence of convex sets (Ω_k) of unit area that are homothetic to their form bodies such that $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$.

The most important thing about this upper bound is that in the case of triangles, inequality (16) is better than the following bound obtained by B. Siudeja in [32, Theorem 1.1] for "thin" triangles:

$$\lambda_1(T) \leq \frac{\pi^2}{9} \times \left(\frac{P(T)}{|T|} \right)^2. \tag{17}$$

It is also interesting to note that inequality (16) is even better (also for thin triangles) than the following upper bound stated in [32, Conjecture 1.2]:

Conjecture 1. *For every triangle T, one has:*

$$\lambda_1(T) \leq \frac{\pi^2}{12} \times \left(\frac{P(T)}{|T|}\right)^2 + \frac{\sqrt{3}\pi^2}{3|T|}. \tag{18}$$

Here also, let us compare the different estimates in a Blaschke-Santaló diagram: we consider the one involving the perimeter, the area and λ_1 in the class of triangles, that is the set of points:

$$\mathcal{T} := \big\{\big(P(T), \lambda_1(T)\big) \mid T \text{ is a triangle such that } |T| = 1\big\}.$$

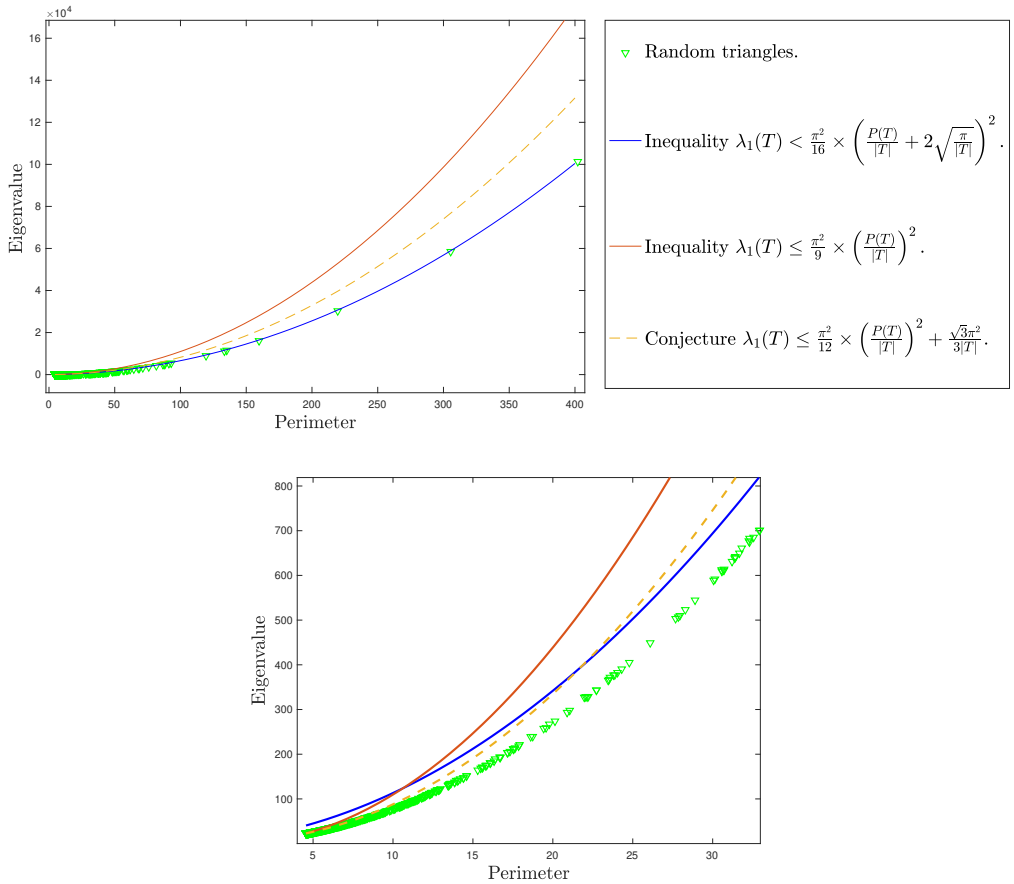

Figure 8: Comparison between inequalities (16) and (17) and Conjecture (18) with a zoom on smaller values of the perimeter.

5.2 A sharp Cheeger-type inequality

Proposition 5.4. *We have the following sharp Cheeger-type inequality:*

$$\forall \Omega \in \mathcal{K}^2, \qquad \lambda_1(\Omega) > \frac{\pi^2}{4}\left(h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}}\right)^2, \tag{19}$$

where equality is asymptotically attained by any family of convex sets $(\Omega_k)_{k\in\mathbb{N}}$ such as $|\Omega_k| = V_0$ for any $k \in \mathbb{N}$ (where V_0 is a positive constant) and $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$.

Proof. Let $\Omega \in \mathcal{K}^2$, we have:

$$\lambda_1(\Omega) > \frac{\pi^2}{4} \times \frac{1}{r(\Omega)^2} \geq \frac{\pi^2}{4}\left(h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}}\right)^2,$$

where the first inequality is the classical Hersch's inequality [20] and the second follows from is the upper estimate of (4).

As for the equality case, let (Ω_k) a family of convex sets $(\Omega_k)_{k\in\mathbb{N}}$ such as $|\Omega_k| = V_0$ for any $k \in \mathbb{N}$ and $d(\Omega_k) \underset{k\to+\infty}{\longrightarrow} +\infty$. By [27, Proposition 4.1], we have: $\lambda_1(\Omega_k) \underset{k\to+\infty}{\sim} \frac{\pi^2}{4} h(\Omega_k)^2$ and by the equivalence (14) and $\lim\limits_{k\longrightarrow+\infty} \frac{1}{r(\Omega_k)} = +\infty$ (see the proof of Proposition 5.1), we have $\lim\limits_{k\longrightarrow+\infty} h(\Omega_k) = +\infty$ which implies the equivalence: $\lambda_1(\Omega) \underset{k\to+\infty}{\sim} \frac{\pi^2}{4}\left(h(\Omega) - \sqrt{\frac{\pi}{|\Omega|}}\right)^2$. □

We note that inequality (19) is better than the improved Cheeger inequality of Theorem 1.1 (and even the conjecture $J(\Omega) \geq J((0,1)^2)$) for thin planar convex domains, see Figure 9, where we provide an approximation of the following Blaschke-Santaló diagram relating λ_1, the Cheeger constant and the area:

$$\mathcal{C} := \left\{ \left(h(\Omega), \lambda_1(\Omega)\right) \mid \Omega \in \mathcal{K}^2 \text{ and } |\Omega| = 1 \right\}.$$

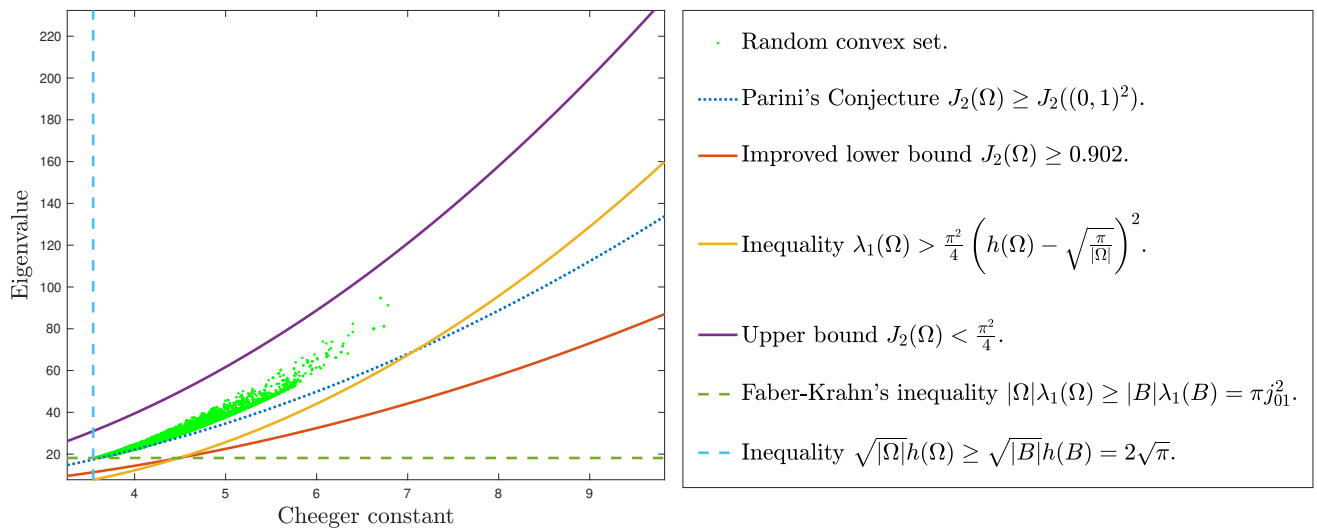

Figure 9: Approximation of the Blaschke-Santaló diagram $\mathcal{C}$ and relevant inequalities.

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