Quantitative fluid approximation in transport theory: a unified approach

Emeric Bouin, Clément Mouhot

QUANTITATIVE FLUID APPROXIMATION IN TRANSPORT THEORY: A UNIFIED APPROACH

ÉMERIC BOUIN AND CLÉMENT MOUHOT

ABSTRACT. We propose a unified method for the large space-time scaling limit of *linear* collisional kinetic equations in the whole space. The limit is of *fractional* diffusion type for heavy tail equilibria with slow enough decay, and of diffusive type otherwise. The proof is constructive and the fractional/standard diffusion matrix is obtained. The equilibria satisfy a *generalised* weighted mass condition. The method combines energy estimates and quantitative spectral methods to construct a 'fluid mode'. The method is applied to scattering models (without assuming detailed balance conditions), Fokker-Planck operators and Lévy-Fokker-Planck operators. It proves a series of new results, including the fractional diffusive limit for Fokker-Planck operators in any dimension, for which the formulas for the diffusion coefficient were not known, for Lévy-Fokker-Planck operators with general equilibria, and for scattering operators with infinite mass equilibria. It also unifies and generalises the results of previous papers with a quantitative method, and our estimates on the fluid approximation error also seem novel.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

The study of *transport processes*, i.e. linear collisional kinetic equations, is theoretically rooted in the mean-free path argument of Maxwell [39] and the kinetic theory of gases of Maxwell and Boltzmann [40, 10]. A linear version of the Maxwell-Boltzmann equation can be written for the movement of a tagged particle within a rarefied gas, but the study of such transport processes was given a crucial new impetus in the twentieth century with:

(1) the *radiative transfer theory* [46], where the kinetic distribution models the flux of photons that are transported in the plasma making up the internal layers of the sun,
(2) the *nuclear reactor theory* (see [52], the collection [7] and in particular its fifth chapter [53]) where the kinetic distribution models the neutrons transported and scattered inside the reactor, whose flux is used to initiate and maintain the chain reaction,
(3) the *semi-conductor theory* [37] where the kinetic distribution models the flow of charge carriers in semiconductors, i.e. the evolution of the position-momentum distribution

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of negatively charged conduction electrons or of positively charged holes, which are responsible for the current flow in semiconductor crystals.

The main mathematical object of study in *transport theory* is the linear equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f \tag{1.1}$$

on the time-dependent density of particles $f = f(t,x,v) \geq 0$ over $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, for $t \geq 0$. The left hand side accounts for free motion and the right hand side accounts for the interaction with a background, for instance scatterers, with an operator $\mathcal{L}$ that only acts on the kinetic variable v. Several forms are possible. In nuclear reactor, radiative transfer and semi-conductor theories it is common to consider *scattering operators*, sometimes also called *linear Boltzmann operators*, of the form

$$\mathcal{L} f(v) = \left(\int_{\mathbb{R}^d} b(v,v') f(v') \, \mathrm{d}v' \right) \mathcal{M}(v) - \nu(v) f(v) \tag{1.2}$$

$$\text{for a } \textit{collision frequency} \quad \nu(v) := \int_{\mathbb{R}^d} b(v,v') \mathcal{M}(v') \, \mathrm{d}v',$$

some *collisional kernel* $b = b(v,v')$ and an *equilibrium distribution* $\mathcal{M}(v)$. In astrophysics and sometimes in semi-conductor theory, one also considers *Fokker-Planck operators*,

$$\mathcal{L} f := \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right). \tag{1.3}$$

Finally, as a simplified model of long-range collisional interactions in a gas of charged particles, we also consider *Lévy-Fokker-Planck operators* (given $s \in (0,1)$):

$$\begin{cases} \mathcal{L}(f) = \Delta_v^s f + \nabla_v \cdot (U f) & \text{with } U(v) = U(|v|) \text{ radially symmetric so that} \\ \Delta_v^s \mathcal{M} + \nabla_v \cdot (U \mathcal{M}) = 0. \end{cases} \tag{1.4}$$

Denoting $\mathcal{F}$ the Fourier transform, the fractional Laplacian is defined as

$$\Delta_v^s f(v) := -\mathcal{F}^{-1} \left[|\cdot|^{2s} \mathcal{F} f(\cdot) \right](v). \tag{1.5}$$

These three operators are discussed respectively in Sections 6-7-8. Extensions, such as Fokker-Planck operators with non-gradient force, are discussed in Section 9.

The equation (1.1) is too intricate for many applications. When the relevant time and space scales of observation are much larger than the mean free time and mean free path, it is thus natural to search for a simplified regime. The so-called *diffusion theory* was born out of this endeavour, and in the words of Wigner [53], 'this [diffusion] theory gives the spatial variation of the [neutron transport] flux quite accurately in regions well removed from interfaces'. We also refer to [52, Chap. IX] for the diffusion theory of monoenergetic neutrons, to [46, Chap. III.2] for the so-called *Eddington approximation* in radiative transfer theory, and to [12, Chap. 2] for a modern mathematical review. Note that anomalous diffusions and Levy flights are observed by biologists and physicists [3, 50, 5, 38, 48].

We rewrite the equation (1.1) by changing the unknown to $h := \frac{f}{\mathcal{M}}$:

$$\partial_t h + v \cdot \nabla_v h = L h \quad \text{where} \quad L h := \mathcal{M}^{-1} \mathcal{L} (\mathcal{M} h). \tag{1.6}$$

This change of unknown is convenient since asymptotic estimates compare f with the equilibrium $\mathcal{M}$. Consider the complex Hilbert spaces $L^2(\mathbb{R}^d; \mathcal{M} \, \mathrm{d}v) =: L^2_v(\mathcal{M})$ and $L^2(\mathbb{R}^d \times \mathbb{R}^d; \mathcal{M} \, \mathrm{d}x \, \mathrm{d}v) =: L^2_{x,v}(\mathcal{M})$ and denote $\|h\|_k := \|(1+|\cdot|^2)^{\frac{k}{2}} h\|_{L^2(\mathcal{M})}$ (the integration variable(s) will be emphasized when there is ambiguity). We omit the index when $k = 0$. The scalar product $\langle \cdot, \cdot \rangle$ refers to $L^2_v(\mathcal{M})$ or $L^2_{x,v}(\mathcal{M})$ depending on context.

We assume, for some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$, and some $\lambda \in \mathbb{R}_+^*$:

Hypothesis 1 (Equilibria). *The equilibrium $\mathcal{M}$ takes one of the following two forms.*

(i) *Either it is given by*

$$\mathcal{M}(v) = c_{\alpha,\beta} \lfloor v \rceil^{-(d+\alpha)} \text{ with } c_{\alpha,\beta} := \left(\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha-\beta} \, \mathrm{d}v \right)^{-1} \text{ and } \lfloor v \rceil := \sqrt{1+|v|^2}. \tag{1.7}$$

(ii) *Or it is a smooth positive radially symmetric function decaying faster than any polynomial. The latter case is denoted by '$\alpha = +\infty$' in the sequel.*

Note that the normalisation implies the following **generalised mass condition**

$$\int_{\mathbb{R}^d} \mathcal{M}_\beta(v)\,\mathrm{d}v = 1 \quad \text{with} \quad \mathcal{M}_\beta := \lfloor\cdot\rceil^{-\beta}\mathcal{M}. \tag{1.8}$$

We present our main results assuming that the equilibrium $\mathcal{M}$ is given by the exact formula (1.7) in the case of a polynomial decay because it leads to a neater treatment. However, as discussed in Section 9, our results remain true with an equilibrium $\mathcal{M}$ that is not an explicit power-law or even symmetric or centered, but only comparable to $\lfloor\cdot\rceil^{-(d+\alpha)}$ (see equation (9.1) and Subsections 9.1-9.2); this requires a few technical changes in the proofs that we present separately in this last section so as not to clutter the paper.

Hypothesis 2 (Weighted coercivity)**.** *The operator L is linear, independent of time t and space x, commutes with rotations in v, is closed densely defined on $\mathrm{Dom}(L) \subset L^2_v(\mathcal{M})$ and satisfies $L(1) = L^*(1) = 0$, where L^* is the $L^2_v(\mathcal{M})$-adjoint. Finally $\tilde{L} := \lfloor\cdot\rceil^{\frac{\beta}{2}} L(\lfloor\cdot\rceil^{\frac{\beta}{2}}\cdot)$ is closed densely defined on $\mathrm{Dom}(\tilde{L}) \subset L^2_v(\mathcal{M})$, with the spectral gap estimate*

$$\forall\, g \in \mathrm{Dom}(\tilde{L}), \quad g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}, \quad -\mathrm{Re}\,\left\langle \tilde{L}g, g\right\rangle \geq \lambda\, \|g\|^2\,.$$

The latter means, translating back to L,

$$\forall\, h \in \mathrm{Dom}(L), \quad -\mathrm{Re}\,\left\langle Lh, h\right\rangle \geq \lambda\, \|h - \mathcal{P}h\|^2_{-\beta} \quad \text{with} \quad \mathcal{P}h := \left(\int_{\mathbb{R}^d} h(v')\mathcal{M}_\beta(v')\,\mathrm{d}v'\right).$$

The assumption that $\mathcal{L}$ commutes with rotations in v is convenient (and satisfied for most physical models), but in fact only $\mathcal{M}(v) = \mathcal{M}(-v)$ is really used in the proof. The latter could in turn be relaxed at the price of a few technical changes in the proofs discussed in Section 9.

Hypothesis 3 (Amplitude of collisions at large velocities)**.** *Given $0 \leq \chi \leq 1$ a smooth function that is 1 on $B(0,1)$ and 0 outside $B(0,2)$, and $\chi_R = \chi(\frac{\cdot}{R})$ for $R \geq 1$, one has*

$$\|L\,(\chi_R)\|_\beta \lesssim R^{-\frac{\alpha+\beta}{2}}.$$

Our first result, on the basis of the three previous hypothesis, is a quantitative construction of a branch of 'fluid eigenmode' in the asymptotic of large time and small spatial frequencies, i.e. a unique eigenvalue branching from zero for $\tilde{L}^* + i\eta\lfloor v\rceil^\beta(v\cdot\sigma)$ for small η (see Figure 1):

Lemma 1.1 (Construction of the fluid mode)**.** *Given Hypothesis 1–2–3, there are $\eta_0 > 0$ and $r_0 \in (0,\lambda)$, explicit in terms of the constants in these hypothesis, such that for any $\eta \in (0,\eta_0)$ and any $\sigma \in \mathbb{S}^{d-1}$, there is a unique solution $\phi_\eta = \phi_\eta(v) \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $\mu(\eta) \in B(0,r_0)$ to*

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 1.$$

Moreover, the branch $(\phi_\eta, \mu(\eta))$ connects to $(1,0)$ as $\eta \to 0$, with $\mu(\eta) > 0$ and the asymptotics

$$\|\phi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac{1}{2}} \quad \text{and} \quad \mu(\eta) \in (\mathrm{R}_0\Theta(\eta), \mathrm{R}_1\Theta(\eta)) \tag{1.9}$$

for some $0 < \mathrm{R}_0 < \mathrm{R}_1$, where the function Θ is defined by

$$\Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > 2+\beta, \\ \eta^2|\ln(\eta)| & \text{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha+\beta}{1+\beta}} & \text{when } -\beta < \alpha < 2+\beta. \end{cases} \tag{1.10}$$

Note that Θ is well-defined in the case $\alpha \in (-\beta, 2+\beta)$ since $(1+\beta) > (\alpha+\beta)/2 > 0$. In this lemma and in the rest of the paper the dependency in σ is kept implicit rather than explicit in order to lighten notation. In fact, ϕ_η also depends on σ, but $\mu(\eta)$ does not if L is invariant by rotations in v. To identify the macroscopic limit with quantitative rates and constants, it is necessary to estimate the leading order of $\mu(\eta)$, and this requires estimates on the eigenvector, which is our last hypothesis. We denote $|u|_\eta := (\eta^{\frac{2}{1+\beta}} + |u|^2)^{\frac{1}{2}}$.

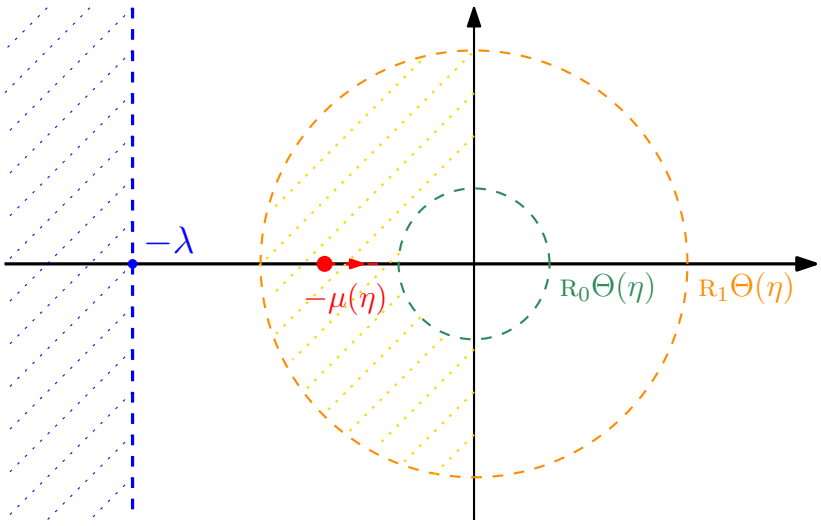

FIGURE 1. The blue dashed zone on the left of $\mathrm{Re}\, z = -\lambda$ corresponds to the spectral gap estimates on $\tilde{L}^* + i\eta\lfloor v\rceil^\beta(v\cdot\sigma)$ for $g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}$ (Hypothesis 2). The yellow dashed zone is where Lemmas 1.1-1.2 construct a unique real eigenvalue $-\mu(\eta) \sim -\mu_0\Theta(\eta)$ of the latter operator, that goes to zero as $\eta \to 0$.

Hypothesis 4 (Scaling of the fluid mode)**.** *We make different assumptions depending on α:*

(i) *Case $\alpha > 2+\beta$: The fluid mode ϕ_η constructed in Lemma 1.1 satisfies*

$$\forall\, \ell < \alpha, \quad \|\phi_\eta\|_\ell \lesssim_\ell 1.$$

(ii) *Case $\alpha \in (-\beta, 2+\beta]$: The rescaled fluid mode $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}}\cdot)$ is converging in $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash 0)$ as $\eta \to 0$ to a limit Φ and satisfies the pointwise controls*

$$\forall\, \eta \in (0,\eta_1),\ \forall\, u \in \mathbb{R}^d, \quad \begin{cases} |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)}, \\ |\mathrm{Im}\,\Phi_\eta(u)| \lesssim |u|_\eta^{\beta+\min(\alpha,1)-\delta} \end{cases} \tag{1.11}$$

for some $\eta_1 \in (0,\eta_0)$ and $C > 0$ and $\delta < \beta + \min(2-\alpha, 1)$. We also make the following additional assumptions depending in the two following subcases:

(ii)-(a) *Case $\alpha = 2+\beta$: There are $\mathfrak{a} : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\lim_{\eta\to 0}\mathfrak{a}(\eta) = 0$ and $\Omega : \mathbb{R}^d \to \mathbb{R}$ locally integrable such that*

$$\begin{cases} \left| \displaystyle\int_{1\geq|u|\geq\eta^{\frac{1}{1+\beta}}} (u\cdot\sigma)\Big[\mathrm{Im}\,\Phi_\eta(u) - \mathrm{Im}\,\Phi(u)\Big]|u|_\eta^{-d-\alpha}\,\mathrm{d}u \right| \leq \mathfrak{a}(\eta)|\ln(\eta)|, \\ \forall\, \sigma' \in \mathbb{S}^{d-1}, \quad \dfrac{\mathrm{Im}\,\Phi(\lambda\sigma')}{\lambda^{1+\beta}} \xrightarrow[\lambda\to 0]{\lambda\neq 0} \Omega(\sigma') \quad \text{in} \quad L^1(\mathbb{S}^{d-1}). \end{cases}$$

(ii)-(b) *Case $\alpha \in [0,\beta]$: The additional following integral control holds:*

$$\int_{|u|\geq 1} |\Phi_\eta(u)|^2\,|u|_\eta^{-d-\alpha+\beta}\,\mathrm{d}u \lesssim 1. \tag{1.12}$$

(ii)-(c) *Case $\alpha \in (-\beta, 0)$: The integral control (1.12) holds, and furthermore the limit rescaled mass is positive:*

$$\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u > 0. \tag{1.13}$$

Note that in (1.11), $|u|_\eta^{C\mu(\eta)} \sim 1$ as $\eta \to 0$ in the region $|u| \lesssim \eta^{\frac{1}{1+\beta}}$. Note also that (1.11) and (1.9) imply $\Phi(0) = 1$. The second part of point (ii) above is subtle and made necessary by the fact that the case $\alpha = 2+\beta$ is borderline between two different regimes (standard diffusion

vs. fractional diffusion) as well as borderline between two different scalings for obtaining the diffusion coefficient (fluid mode in variable v vs. fluid mode in the rescaled variable $u = \eta^{-\frac{1}{1+\beta}} v$).

With these four hypothesis we can characterise the precise scaling of the fluid eigenvalue:

Lemma 1.2 (Rescaled limit of the fluid eigenvalue)**.** *Assume Hypothesis 1–2–3–4. The eigenvalue $\mu(\eta)$ constructed in Lemma 1.1 satisfies (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypothesis)*

$$\mu(\eta) \sim_{\eta\to 0} \mu_0 \Theta(\eta), \tag{1.14}$$

where the constant $\mu_0 \in (\mathrm{R}_0, \mathrm{R}_1)$ is positive and determined as follows:

$$\begin{cases} \mu_0 := \displaystyle\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v) \mathcal{M}(v)\,\mathrm{d}v \quad \text{when } \alpha > 2+\beta, \\ \qquad \text{where } \ F = \displaystyle\lim_{\eta\to 0} \frac{\operatorname{Im}\phi_\eta}{\eta} \text{ is solution to } LF = -(v\cdot\sigma) \ \text{ and } \ \int_{\mathbb{R}^d} F(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 0, \\ \mu_0 := \dfrac{c_{2+\beta,\beta}}{1+\beta} \displaystyle\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\,\mathrm{d}\sigma' \quad \text{when } \alpha = 2+\beta, \\ \qquad \text{where } \ \Omega(u) = \displaystyle\lim_{\lambda\to 0,\ \lambda\neq 0} \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}} \ \text{ and } \ \Phi = \lim_{\eta\to 0}\Phi_\eta = \lim_{\eta\to 0}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}\cdot\right), \\ \mu_0 := c_{\alpha,\beta} \displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma) \operatorname{Im}\Phi(u) |u|^{-d-\alpha}\,\mathrm{d}u \quad \text{when } \alpha \in (-\beta, 2+\beta). \end{cases}$$

Note how in the previous statement, when $\alpha > 2+\beta$, the function F used in the previous works on standard diffusive limit (usually with $\beta = 0$) is recovered here as a limit of our fluid mode; this allows our proof to track the convergence rate. Define the *diffusion exponent*

$$\zeta = \zeta(\alpha,\beta) := \begin{cases} 2 & \text{when } \alpha \in [2+\beta, +\infty] \\ \dfrac{\alpha_+ + \beta}{1+\beta} & \text{when } \alpha \in (-\beta, 2+\beta), \end{cases} \tag{1.15}$$

with $\alpha_+ := \max(\alpha, 0)$, and the *scaling function*

$$\theta(\varepsilon) := \begin{cases} \varepsilon^\zeta & \text{when } \alpha \in (-\beta, +\infty] \setminus \{0, 2+\beta\}, \\ \varepsilon^2 |\ln\varepsilon| & \text{when } \alpha = 2+\beta, \\ \dfrac{\varepsilon^{\frac{\beta}{1+\beta}}}{|\ln\varepsilon|} & \text{when } \alpha = 0. \end{cases} \tag{1.16}$$

Note that the threshold $\alpha = 2+\beta$ between standard and fractional diffusion corresponds to whether or not $\mathcal{M}_\beta$ has finite variance. We finally derive the *diffusion coefficient*:

Lemma 1.3 (Diffusion coefficient)**.** *Assume Hypothesis 1–2–3–4. Then the following limit holds true (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypothesis)*

$$\kappa := \lim_{\eta\to 0} \frac{\mu(\eta)|\xi|^{-\zeta}}{\theta(\varepsilon)\,\langle 1, \phi_\eta\rangle} = \mu_0 \times \begin{cases} \|\mathcal{M}\|_{L^1(\mathbb{R}^d)}^{-1} & \text{when } \alpha > 0, \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha = 0, \\ \left[c_{\alpha,\beta}\displaystyle\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u\right]^{-1} & \text{when } \alpha \in (-\beta, 0). \end{cases} \tag{1.17}$$

Note that the last line is well-defined thanks to Hypothesis 4-(ii)-(c) (equation (1.13)). The diffusion coefficient thus emerges from ratios between (rescaled) integrals as follows:

$$\kappa := \begin{cases} \displaystyle\frac{\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v)\mathcal{M}(v)\,\mathrm{d}v}{\|\mathcal{M}\|_{L^1(\mathbb{R}^d)}} & \boxed{\text{when } \alpha > 2+\beta} \\[2ex] \displaystyle\frac{1}{1+\beta}\,\frac{\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\,\mathrm{d}\sigma'}{\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha = 2+\beta} \\[2ex] \displaystyle\frac{\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha \in (0, 2+\beta)} \\[2ex] \displaystyle\frac{1+\beta}{|\mathbb{S}^{d-1}|}\,\frac{\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha-\beta}\,\mathrm{d}v} & \boxed{\text{when } \alpha = 0} \\[2ex] \displaystyle\frac{\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u} & \boxed{\text{when } \alpha \in (-\beta, 0)} \end{cases} \tag{1.18}$$

where we recall

$$F = \lim_{\eta\to 0} \frac{\mathrm{Im}\,\phi_\eta}{\eta}, \qquad \Phi = \lim_{\eta\to 0} \Phi_\eta = \lim_{\eta\to 0} \phi_\eta\left(\eta^{-\frac{1}{1+\beta}}\cdot\right), \qquad \Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0} \frac{\mathrm{Im}\,\Phi\,(\lambda u)}{\lambda^{1+\beta}},$$

and (when $\alpha > 2+\beta$) F is also the unique solution to $LF = -(v\cdot\sigma)$ with $\int_{\mathbb{R}^d} F(v)\,\lfloor v\rceil^{-d-\alpha-\beta}\,\mathrm{d}v = 0$. For legibility again, we wrote, in the cases $\alpha \in (-\beta, 2+\beta]$, the formula for κ with $\mathcal{M}$ given by (1.7), and we refer to Section 9 for more general $\mathcal{M}$'s.

The proof of Lemma 1.3 is done in Section 5; it requires the estimating of $\langle 1, \phi_\eta\rangle$, which is done in Lemma 5.1.

Given a solution f in $L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ to equation (1.1) we denote

$$f_\varepsilon(t,x,v) := f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) \in L^\infty_t\left([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1})\right) \text{ and } r_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\lfloor v\rceil^{-\beta}\,\mathrm{d}v,$$

where $\varepsilon > 0$, and $\theta(\varepsilon)$ is defined in (1.16). The equation satisfied by f_ε is

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon. \tag{1.19}$$

Theorem 1.4 (Unified second fluid approximation, see Figure 2)**.** *Assume Hypothesis 1–2–3–4, and consider $f_\varepsilon \in L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ solving (1.1) in the weak sense with initially*

$$\left\|\frac{f_\varepsilon(0,\cdot,\cdot)}{\mathcal{M}}\right\|_0 = 1, \quad \left\|\frac{f_\varepsilon}{\mathcal{M}}(0,\cdot,\cdot) - r_\varepsilon(0,\cdot)\right\|_{-\beta} \lesssim \theta(\varepsilon) \quad \text{and} \quad r_\varepsilon(0,\cdot) \xrightarrow[\varepsilon\to 0]{H^{-\zeta}(\mathbb{R}^d)} r(0,\cdot). \tag{1.20}$$

Then for any $T > 0$ (and recalling the definition of ζ in (1.15))

$$\left\|\frac{f_\varepsilon}{\mathcal{M}} - r\right\|_{L^2_t([0,T]; H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

when $\alpha > \beta$ and

$$\left\|\left|\ln\frac{2|\nabla_x|}{1+|\nabla_x|}\right|\left(\frac{f_\varepsilon}{\mathcal{M}} - r\right)\right\|_{L^2_t([0,T]; H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

when $\alpha = \beta$ *and*

$$\left\| |\nabla_x|^{\frac{\beta-|\alpha|}{2(1+\beta)}} \lfloor \nabla_x \rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}} \left(\frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

when $\alpha \in (-\beta, \beta)$, *where* $r = r(t,x)$ *solves*

$$\partial_t r = \kappa \, \Delta_x^{\frac{\zeta}{2}} r, \quad t > 0, \qquad \text{with initial data } r(0,\cdot) \text{ defined in (1.20).}$$

The rates of convergence are estimated in terms of T, *the constants, error terms and convergence rates in Hypothesis 1–2–3–4, and the initial convergence rate on* r_ε *in* (1.20). *Apart from the latter (that depends on the initial data), the errors we obtain are polynomial in* ε *for* $\alpha \in (-\beta, +\infty) \setminus \{0, 2+\beta\}$ *and logarithmic for* $\alpha \in \{0, 2+\beta\}$.

Note that when the second inequality in (1.20) is not satisfied at $t = 0$, the energy estimate implies that it is for any later time $\tau > 0$, and we could still deduce the fluid approximation for $t \in [\tau, T]$ (at the price of an initial time layer). We however chose to keep the assumption (1.20) since we are interested in tracking precisely the rate of convergence. Note also that when $\alpha < 0$, one could relax the first equality in (1.20) and allow for initial data with $L^2_{x,v}(\mathcal{M}^{-1})$ norm diverging slowly enough as $\varepsilon \to 0$; the modifications in the proofs are straightforward.

This theorem is the core contribution of the paper, and is used to obtain results on concrete models in the corollaries below. Together with Lemmas 1.1–1.2–1.3, it reveals the relevant macroscopic scales for a large class of operators in any dimension and provides a unified theoretical framework to answer questions of the last decades on the topic. The diffusive limit is reduced to a spectral problem –the construction of the fluid mode– that we solve in a general setting. The proof is constructive and the key constants governing the macroscopic behaviours are derived. The fractional Laplacian in the space variable is defined as in (1.5), and $r(t,x)$ is the limit (in the topology of the above theorem) of the weighted velocity average

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \lfloor v \rceil^{-\beta} \, \mathrm{d}v.$$

When $\alpha > 0$, the density $\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) \mathrm{d}v$ exists and also converges to $r(t,x)$.

We now apply the previous abstract theorem to particular models:

Corollary 1.5 (Scattering equation). *Assume that* $\mathcal{L}$ *is the scattering operator* (1.2) *with* $b \in \mathcal{C}^1$ *and* $\mathcal{M}$ *satisfying Hypothesis 1 and that, for some constant* $\nu_0 > 0$ *and* $\beta > -\alpha$,

$$\left\{ \begin{array}{ll} \forall\, v \in \mathbb{R}^d, & \lfloor v \rceil^{-\beta} \lesssim \nu(v) \lesssim \lfloor v \rceil^{-\beta} \\ \forall\, v \in \mathbb{R}^d \setminus \{0\}, & \lambda^\beta \nu(\lambda v) \sim_{\lambda \to \infty} \nu_0 |v|^{-\beta} \\ \forall\, v \in \mathbb{R}^d, & \|b(v,\cdot)\|_\beta + \|b(\cdot,v)\|_\beta \lesssim \lfloor v \rceil^{-\beta}. \end{array} \right. \tag{1.21}$$

This includes $b(v,v') = \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta}$ *for any* $\alpha + \beta > 0$, *and* $b(v,v') = \lfloor v - v' \rceil^{-\beta}$ *when* $\beta < 0$ *and* $\alpha + \beta > 0$ *or when* $\beta \geq 0$ *and* $\alpha > 3\beta$. *Then Theorem 1.4 applies with* α, β *given in Hypothesis 1 and* (1.21). *This proves the diffusive limit for solutions to* (1.19) *with quantitative rate, diffusion exponent* $\zeta = \frac{\alpha_+ + \beta}{1+\beta}$, *scaling function* (1.16) *and diffusion coefficient* (1.18). *Apart from the convergence of the initial data* (1.20), *the errors we obtain are polynomial in* ε *for* $\alpha \in (-\beta, +\infty) \setminus \{0, 2+\beta\}$ *and logarithmic for* $\alpha \in \{0, 2+\beta\}$. *Moreover the diffusion coefficient can be computed explicitly with* $F(u) = \nu(v)^{-1}(v \cdot \sigma)$ *when* $\alpha > 2 + \beta$, $\Omega(u) = \nu_0^{-1} |u|^\beta (u \cdot \sigma)$

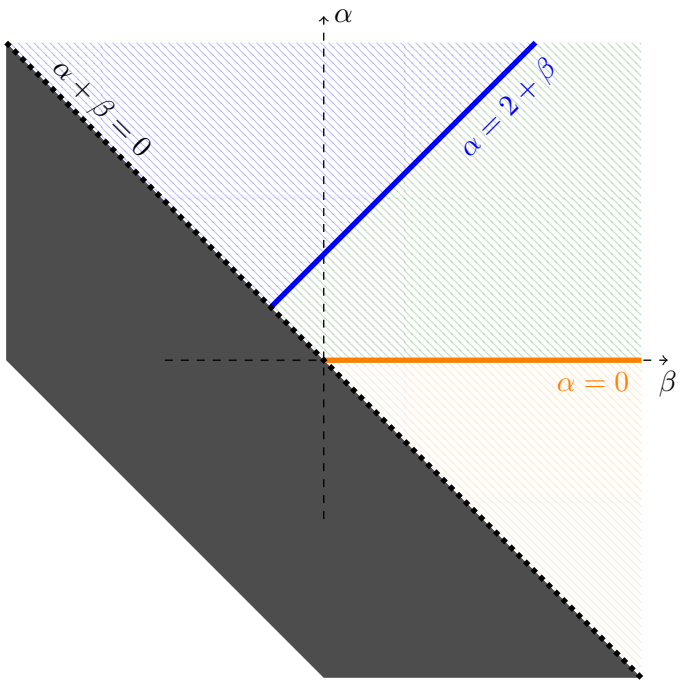

FIGURE 2. Summary of the results in the (α,β) plane. Admissible parameters are in half-plane $\alpha+\beta>0$. The blue hatched area leads to $\theta(\varepsilon)=\varepsilon^2$ and a standard diffusive limit with symbol $\kappa|\xi|^2$. The blue line is the set of parameters yielding the anomalous scaling $\theta(\varepsilon)=\varepsilon^2|\ln(\varepsilon)|$ but still a standard diffusive limit with symbol $\kappa|\xi|^2$. The green hatched area results into the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\alpha+\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\alpha+\beta}{1+\beta}}$. The orange bold line yields the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\beta}{1+\beta}}|\ln(\varepsilon)|^{-1}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$. Finally, the orange hatched area yields the fractional scaling $\theta(\varepsilon)=\varepsilon^{\frac{\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$.

when $\alpha=2+\beta$ *and* $\Phi(u)=\frac{\nu_0}{\nu_0-i|u|^\beta(u\cdot\sigma)}$ *when* $\alpha\in(-\beta,2+\beta)$, *resulting in*

$$\kappa:=\begin{cases}
\displaystyle\frac{\int_{\mathbb{R}^d}(v\cdot\sigma)^2\,\nu(v)^{-1}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v}{\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha\in(2+\beta,+\infty)}\\[2ex]
\displaystyle\frac{1}{\nu_0(1+\beta)}\,\frac{\int_{\mathbb{S}^{d-1}}(\sigma\cdot\sigma')^2\,\mathrm{d}\sigma'}{\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha=2+\beta}\\[2ex]
\displaystyle\frac{\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha\in(0,2+\beta)}\\[2ex]
\displaystyle\frac{(1+\beta)}{|\mathbb{S}^{d-1}|}\,\frac{\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d}}}{\int_{\mathbb{R}^d}\lfloor v\rceil^{-d-\beta}\,\mathrm{d}v} & \boxed{\text{when } \alpha=0}\\[2ex]
\displaystyle\frac{\int_{\mathbb{R}^d}\frac{\nu_0|u|^\beta(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\int_{\mathbb{R}^d}\frac{\nu_0^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{\mathrm{d}u}{|u|^{d+\alpha}}} & \boxed{\text{when } \alpha\in(-\beta,0)}
\end{cases}$$

as well as $\kappa := \|\mathcal{M}\|_{L^1(\mathbb{R}^d)}^{-1} \int_{\mathbb{R}^d} (v\cdot\sigma)^2 \nu^{-1} \mathcal{M}(v)\,\mathrm{d}v$ *is the case "*$\alpha = +\infty$*".*

This recovers and unifies the results in [32, 42, 6, 21, 41] (except for the case of space-dependent collision kernels in [21]) and extend them to new cases such as $\alpha \in (-\beta, 0)$ (infinite mass). The convergence rate is also new. Our approach shares common points with, but differs from the probabilistic method in [32], the Hilbert expansions in [6, 21], the moment method in [41] and the Fourier-Laplace method in [42].

Corollary 1.6 (Kinetic Fokker-Planck equation)**.** *Assume that* $\mathcal{L}$ *is the Fokker-Planck operator* (1.3) *with* $\mathcal{M}$ *satisfying Hypothesis 1 with* $\alpha \geq 0$*. Then Theorem 1.4 applies with* α *given in Hypothesis 1 and* $\beta = 2$*. This proves the diffusive limit for solutions to* (1.19) *with quantitative rate, diffusion exponent* $\zeta = \min\left(2, \frac{\alpha+2}{3}\right)$*, scaling function* (1.16) *and diffusion coefficient* (1.18)*. Apart from the convergence of the initial data* (1.20)*, the errors we obtain are polynomial in* ε *for* $\alpha \in (0,4)\cup(4,+\infty)$ *and logarithmic for* $\alpha = 4$*. The diffusion coefficient can be precised when* $\alpha \in (0,4]$ *using that* Φ *solves the Schrödinger-type equation*

$$-|u|^2\Delta_u\Phi + (d+\alpha)u\cdot\nabla_u\Phi - i(u\cdot\sigma)|u|^2\Phi = 0 \quad \text{with the normalisation } \Phi(0) = 1.$$

In particular when $\alpha = 4$*, the function* Ω *solves*

$$-|u|^2\Delta_u\Omega + (d+\alpha)u\cdot\nabla_u\Omega = (u\cdot\sigma)|u|^2 \quad \text{with} \quad \Omega(0) = 0 \quad \Longrightarrow \quad \Omega(u) := \frac{|u|^2(u\cdot\sigma)}{d+8}.$$

This recovers and unifies the fractional diffusive limit results in [15, 43, 27, 26, 36]. Novel contributions include formulas for the diffusion coefficient in dimension higher than 1, and the quantitative argument providing a convergence rate.

Corollary 1.7 (Kinetic Lévy-Fokker-Planck equation)**.** *Assume that* $\mathcal{L}$ *is the Lévy-Fokker-Planck operator* (1.4) *with parameter* $s \in (\frac12, 1)$ *and with* $\mathcal{M}$ *satisfying Hypothesis 1 with* $\alpha > s$*. Then Theorem 1.4 applies with* $\beta := 2s-\alpha$*. This proves the diffusive limit for solutions to* (1.19) *with quantitative rate and diffusion exponent*

$$\zeta = \begin{cases} 2 & \text{when } \alpha \geq 1+s \\ \dfrac{2s}{1+2s-\alpha} & \text{when } \alpha \in (s, 1+s), \end{cases}$$

and scaling function (1.16) *and diffusion coefficient* (1.18)*. Apart from the convergence of the initial data* (1.20)*, the errors we obtain are polynomial in* ε *when* $\alpha \in (s,1+s)\cup(1+s,+\infty)$ *and logarithmic for* $\alpha = 1+s$*. Moreover* $\Phi(u) := \exp\left(i\frac{2sc_{\alpha,0}}{c_{\alpha,\beta}}\frac{|u|^\beta(u\cdot\sigma)}{1+\beta}\right)$ *when* $\alpha \in (s, 1+s]$ *and* $\Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}}\frac{|u|^\beta(u\cdot\sigma)}{1+\beta}$ *when* $\alpha = 1+s$*, which yields for the diffusion coefficient*

$$\kappa := \begin{cases} \dfrac{2sc_{\alpha,0}^2}{c_{\alpha,\beta}(1+\beta)^2}\displaystyle\int_{\mathbb{S}^{d-1}} (\sigma'\cdot\sigma)^2\,\mathrm{d}\sigma' & \boxed{\text{when } \alpha = 1+s} \\[2ex] \dfrac{c_{\alpha,0}}{1+\beta}\left(\dfrac{2sc_{\alpha,0}}{c_{\alpha,\beta}(1+\beta)}\right)^{\frac{\alpha-1}{1+\beta}}\displaystyle\int_{\mathbb{R}^d} (w\cdot\sigma)\sin(w\cdot\sigma)\frac{dw}{|w|^{d+\frac{\alpha+\beta}{1+\beta}}} & \boxed{\text{when } \alpha \in (s,1+s)} \end{cases}$$

This recovers and extends the qualitative results in [1, 18] to general equilibria, with quantitative error estimates and formulas for the diffusion coefficient. In the latter papers, the moment method initiated by Mellet is used to derive a fractional limit in the case $\beta = 0$. It raises several interesting questions: (1) can our approach be extended to $s \in (0, \frac12)$? (this seems to be a technical difficulty), (2) is the fractional diffusive limit possible for infinite mass equilibria? (i.e. $\alpha < 0$), (3) can the connexion between the kinetic Lévy-Fokker-Planck equation with $\alpha = 2s$ (for which the $\mathcal{L}$ is the generator of a Lévy process) and the standard kinetic Fokker-Planck equation with Gaussian equilibrium be clarified as $s \to 1$? (our diffusion constant κ above diverges as $s \to 1$ so the two limits in $\varepsilon \to 0$ and $s \to 1$ do not commute which calls for further investigation).

Let us summarise our contributions. Theorem 1.4 and Corollaries 1.5–1.6–1.7 recover the results of [1, 6, 15, 21, 27, 26, 36, 41, 42, 43] with a shorter and unified constructive method and prove new results for (1) Lévy-Fokker-Planck operators, (2) scattering operators with decaying collision kernel and infinite mass equilibria and importantly (3) Fokker-Planck operators in any dimension (for which formulas for the diffusion coefficient were not known). The quantitative error in this fluid approximation seems to also be novel for all equations considered. Note finally that like the abstract Theorem 1.4, Corollaries 1.5–1.6–1.7 are stated with the exact equilibrium of Hypothesis 1, but can be extended to more general equilibria, see Section 9. Moreover, it would be interesting to try and apply this method in other settings such as [4, 28] (radiative transfer theory), [11, 22, 30] (rarefied gas in a region between two parallel plates), [35, 16, 17] (domains with interface or boundaries), [2] (scattering with external acceleration field), [45] (models for chemotaxis) and [33, 31, 34] (adding a local conservation of momentum).

The method of the present paper extends to the fractional diffusive limit the approach pioneered in [44, 24] of constructing exact dispersion laws in the regime of parabolic time-space scaling and small eigenvalues; this extension is inspired by the recent one-dimensional result [36] and in particular we use and generalise the idea of rescaling velocities to obtain a non-trivial dispersion law in the latter paper. In comparison with [36], the main novelty of the present paper is a quantitative spectral method for constructing the branch of fluid eigenvalue: in [36] it was done by a one-dimensional argument connecting two infinite series on $\mathbb{R}_-$ and $\mathbb{R}_+$ (and it was done by fixed points in the simpler case of classical diffusive limit in the older works [44, 24]).

Let us now compare our paper with the previous recent works by probabilists [27, 26]. In probabilistic terms, we try to describe particles moving in the full d-dimensional space along $\mathrm{d}X_t = V_t\,\mathrm{d}t$ with velocities V_t following a reversible process with invariant measure of the form given in (1). The velocity process is typically of scattering type, or Langevin type with drift and Brownian or non-Gaussian Lévy-type noises. We show in (1.4) that the rescaled process $\varepsilon X_{\theta(\varepsilon)^{-1}t}$ converges, with explicit rates and multiplicative constants, towards a Brownian motion when $\alpha \geq 2+\beta$, and towards a radially symmetric ζ-stable process when $\alpha \in (0, \alpha+\beta)$. In spite of using quite different languages, the common point between [27, 26] and the present paper is the use of a scaling in velocity, which corresponds to applying some power function to the random variable in the probabilistic viewpoint and corresponds to the study of the rescaled fluid mode $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}}\cdot)$ in our study. Note however that in the case $\alpha \in (-\beta, 0)$, the scaling considered in [26] does not correspond to a fractional diffusion scaling and seems more akin to a large deviation scaling. And the authors obtain a kinetic (rather than fluid) limit equation with a Bessel process in velocity. Note finally that the eigenvalue problem we study to compute the limit diffusion coefficient does not seem to have a counterpart in [27, 26].

The rest of the paper is structured as follows. Section 2 is devoted to the proof of Theorem 1.4 assuming Lemmas 1.1, 1.2 and 1.3. We then prove Lemma 1.1 (construction of the fluid mode) in Section 3, Lemma 1.2 (scaling of the fluid mode) in Section 4, and Lemma 1.3 (derivation of the diffusion coefficient) in Section 5. Sections 6-7-8 prove the abstract hypothesis on the three concrete models; one argument of independent interest is a tightness estimate for the Schrödinger-type equation satisfied by the rescaled fluid mode in the cases of Fokker-Planck operators, see Lemma 7.3. Finally, Section 9 briefly discusses extensions of our results to more general equilibrium distributions and operators.

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2. Proof of Theorem 1.4 (convergence)

In this section we assume Lemmas 1.1, 1.2 and 1.3 and prove Theorem 1.4. Consider equation (1.6) and the rescaling

$$h_\varepsilon(t,x,v) := h\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) = \frac{f_\varepsilon(t,x,v)}{\mathcal{M}(v)} = \frac{f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right)}{\mathcal{M}(v)}.$$

It satisfies the equation

$$\theta(\varepsilon)\partial_t h_\varepsilon + \varepsilon v\cdot\nabla_x h_\varepsilon = Lh_\varepsilon. \tag{2.1}$$

2.1. The energy estimate.

Integrate (2.1) against $h_\varepsilon\mathcal{M}$ in t,x,v, and take the real part:

$$\begin{aligned}\frac{\theta(\varepsilon)}{2}\|h_\varepsilon(t)\|^2 &= \frac{\theta(\varepsilon)}{2}\|h_\varepsilon(0)\|^2 + \int_0^t \mathrm{Re}\,\left\langle Lh_\varepsilon(\tau), h_\varepsilon(\tau)\right\rangle \mathrm{d}\tau\\ &\le \frac{\theta(\varepsilon)}{2}\|h_\varepsilon(0)\|^2 - \lambda\int_0^t \|h_\varepsilon(\tau)-r_\varepsilon(\tau)\|_{-\beta}^2\,\mathrm{d}\tau\end{aligned}$$

where we have used Hypothesis 2 and

$$r_\varepsilon(t,x) := \int_{\mathbb{R}^d} h_\varepsilon(t,x,v)\mathcal{M}_\beta(v)\,\mathrm{d}v.$$

This proves

$$\forall\, t\ge 0,\quad \|h_\varepsilon(t)\|^2 \le \|h_\varepsilon(0)\|^2 = 1 \ \text{ and } \ \int_0^t \|h_\varepsilon(\tau)-r_\varepsilon(\tau)\|_{-\beta}^2\,\mathrm{d}\tau \le \frac{\theta(\varepsilon)}{2\lambda}\|h_\varepsilon(0)\|^2 = \frac{\theta(\varepsilon)}{2\lambda}. \tag{2.2}$$

2.2. Framework of the calculations.

Denote ξ the Fourier variable of x, and Fourier-transform equation (2.1) in x to get on $\hat h_\varepsilon(t,\xi,v)$

$$\theta(\varepsilon)\partial_t\hat h_\varepsilon = L\hat h_\varepsilon - i\varepsilon(v\cdot\xi)\hat h_\varepsilon. \tag{2.3}$$

Note that (2.2) and the Plancherel theorem imply $\hat h_\varepsilon \in L^\infty_t(\mathbb{R}^+; L^2_{\xi,v}(\mathcal{M}))$ and

$$\left\|\hat h_\varepsilon - \hat r_\varepsilon\right\|_{L^2_t(\mathbb{R}^+;L^2_{\xi,v}(\mathcal{M}_\beta))} \lesssim \theta(\varepsilon)^{\frac12}. \tag{2.4}$$

Denote $\xi =: |\xi|\sigma$ and $\eta := \varepsilon|\xi|$. Test (2.3) against $\mathcal{M}\phi_\eta$ with ϕ_η constructed in Lemma 1.1:

$$\begin{aligned}\theta(\varepsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\hat h_\varepsilon,\phi_\eta\right\rangle &= \left\langle L\hat h_\varepsilon - i\varepsilon(v\cdot\xi)\hat h_\varepsilon,\phi_\eta\right\rangle = \left\langle \hat h_\varepsilon, L^*(\phi_\eta) + i\varepsilon(v\cdot\xi)\phi_\eta\right\rangle\\ &= -\mu(\eta)\left\langle\hat h_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta\right\rangle.\end{aligned} \tag{2.5}$$

We then split the integrals as follows:

$$\left\langle\hat h_\varepsilon,\phi_\eta\right\rangle = \hat r_\varepsilon\left\langle 1,\phi_\eta\right\rangle + \left\langle\hat h_\varepsilon-\hat r_\varepsilon,\phi_\eta\right\rangle =: \left\langle 1,\phi_\eta\right\rangle[\hat r_\varepsilon - E_1]$$

$$\left\langle\hat h_\varepsilon,\lfloor v\rceil^{-\beta}\phi_\eta\right\rangle = \hat r_\varepsilon\left\langle 1,\lfloor v\rceil^{-\beta}\phi_\eta\right\rangle + \left\langle\hat h_\varepsilon-\hat r_\varepsilon,\lfloor v\rceil^{-\beta}\phi_\eta\right\rangle =: \left\langle 1,\phi_\eta\right\rangle\frac{\theta(\varepsilon)}{\mu(\eta)}[\kappa_\eta\hat r_\varepsilon - E_2]$$

with the definitions (using the normalisation $\langle 1,\lfloor v\rceil^{-\beta}\phi_\eta\rangle = 1$)

$$\kappa_\eta := \frac{\mu(\eta)\left\langle 1,\lfloor v\rceil^{-\beta}\phi_\eta\right\rangle}{\theta(\varepsilon)\left\langle 1,\phi_\eta\right\rangle} = \frac{\mu(\eta)}{\theta(\varepsilon)\left\langle 1,\phi_\eta\right\rangle},$$

$$E_1 := -\frac{\left\langle\hat h_\varepsilon-\hat r_\varepsilon,\phi_\eta\right\rangle}{\left\langle 1,\phi_\eta\right\rangle} \quad\text{and}\quad E_2 := -\frac{\mu(\eta)\left\langle\hat h_\varepsilon-\hat r_\varepsilon,\lfloor v\rceil^{-\beta}\phi_\eta\right\rangle}{\theta(\varepsilon)\left\langle 1,\phi_\eta\right\rangle}.$$

Consequently, equation (2.5) writes

$$\partial_t\hat r_\varepsilon + \kappa_\eta\hat r_\varepsilon = \partial_t E_1 + E_2.$$

We then want to pass to the limit $\varepsilon\to 0$ (hence $\eta\to 0$ for each frequency ξ).

2.3. **Estimating κ_η, $\partial_t E_1$ and E_2.** Lemma 1.1- 1.2 yield

$$\lim_{\varepsilon\to 0}\frac{\mu(\eta)}{\theta(\varepsilon)}=\mu_0|\xi|^\zeta \quad \text{with} \quad \zeta:=\frac{\alpha_+ +\beta}{1+\beta},$$

with constructive rate, for each frequency $\xi\in\mathbb{R}^d$ (note that in the cases $\alpha=0$ or $\alpha=2+\beta$, the error in the convergence includes a loss of frequency weight $|\ln|\xi||$). Lemma 1.3 implies

$$\lim_{\eta\to 0}\kappa_\eta=\kappa|\xi|^\zeta=\mu_0|\xi|^\zeta\times\begin{cases}\|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} & \text{when } \alpha>0,\\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha=0,\\ \left[c_{\alpha,\beta}\displaystyle\int_{\mathbb{R}^d}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u\right]^{-1} & \text{when } \alpha\in(-\beta,0),\end{cases} \tag{2.6}$$

with constructive convergence rate and

$$\frac{\mu(\eta)}{\theta(\varepsilon)|\langle 1,\phi_\eta\rangle|}\lesssim|\xi|^\zeta.$$

To estimate E_2, write

$$\left|\left\langle\hat h_\varepsilon-\hat r_\varepsilon,\lfloor\cdot\rceil^{-\beta}\phi_\eta\right\rangle\right|\lesssim\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}$$

where we have used $\|\phi_\eta\|_{-\beta}=1$. All in all, we get, using again Lemmas 1.3 and 5.1,

$$|E_2|\lesssim\frac{\mu(\eta)}{\theta(\varepsilon)|\langle 1,\phi_\eta\rangle|}\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}\lesssim\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}|\xi|^\zeta.$$

To estimate E_1, compute first

$$\left|\left\langle\hat h_\varepsilon-\hat r_\varepsilon,\phi_\eta\right\rangle\right|\le\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}\|\phi_\eta\|_\beta,$$

to get

$$|E_1|\lesssim\frac{\|\phi_\eta\|_\beta}{|\langle 1,\phi_\eta\rangle|}\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}.$$

One then estimates $\|\phi_\eta\|_\beta$. When $\alpha>\beta$, it is bounded by construction, and when $\alpha\le\beta$,

$$\|\phi_\eta\|_\beta^2=\eta^{\frac{\alpha-\beta}{1+\beta}}\int_{\mathbb{R}^d}|\Phi_\eta(u)|^2\,|u|_\eta^{-d-\alpha+\beta}\,\mathrm{d}u.$$

Using the pointwise bound (1.11) and the moment bound (1.12) from Hypothesis 4, the latter integral exists and is uniformly bounded in η for $\alpha\in(-\beta,\beta)$ and is bounded by $|\ln\eta|$ when $\alpha=\beta$. Thus we get, using Lemma 5.1 to estimate $\langle 1,\phi_\eta\rangle$ again,

$$|E_1|\lesssim\theta(\varepsilon)^{-\frac12}\left\|\hat h_\varepsilon-\hat r_\varepsilon\right\|_{-\beta}\times\begin{cases}\theta(\varepsilon)^{\frac12}, & \text{when } \alpha>\beta,\\ \varepsilon^{\frac{\beta}{1+\beta}}\,|\ln(\varepsilon|\xi|)| & \text{when } \alpha=\beta,\\ \varepsilon^{\frac{\alpha}{1+\beta}}|\xi|^{\frac{\alpha-\beta}{2(1+\beta)}} & \text{when } \alpha\in(0,\beta),\\ |\ln(\varepsilon)|^{-1}(\ln|\xi|)^{-1}|\xi|^{-\frac{\beta}{2(1+\beta)}} & \text{when } \alpha=0,\\ \varepsilon^{-\frac{\alpha}{2(1+\beta)}}|\xi|^{-\frac{\alpha+\beta}{2(1+\beta)}} & \text{when } \alpha\in(-\beta,0).\end{cases}$$

We then define $r:=r(t,x)$ solution to $\partial_t r+\kappa|\xi|^\zeta r=0$ with initial data $r(0,\cdot)$ defined in (1.20) and deduce that $\omega_\varepsilon:=\hat r_\varepsilon-\hat r$ satisfies

$$\partial_t\omega_\varepsilon+\kappa|\xi|^\zeta\omega_\varepsilon=\partial_t E_1+E_2+(\kappa-\kappa_\eta)\,\hat r_\varepsilon$$

which implies

$$\begin{aligned}\omega_\varepsilon(t,\xi) &= e^{-\kappa|\xi|^\zeta t}\omega_\varepsilon(0,\xi) + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)}\left[\partial_t E_1(s,\xi) + E_2(s,\xi) + (\kappa-\kappa_\eta)\,\hat r_\varepsilon(s,\xi)\right]\mathrm{d}s\\ &= \omega_\varepsilon(0,\xi)e^{-\kappa|\xi|^\zeta t} + E_1(t,\xi) - e^{-\kappa|\xi|^\zeta t}E_1(0,\xi)\\ &\quad + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)}\left[\kappa|\xi|^\zeta E_1(s,\xi) + E_2(s,\xi) + (\kappa-\kappa_\eta)\,\hat r_\varepsilon(s,\xi)\right]\mathrm{d}s.\end{aligned}$$

Define then

$$W(\xi) := \lfloor\xi\rceil^{-\zeta} \times \begin{cases} 1 & \text{when} \quad \alpha > \beta,\\ \left|\ln\frac{2|\xi|}{1+|\xi|}\right|^{-1} & \text{when} \quad \alpha = \beta,\\ |\xi|^{\frac{\beta-|\alpha|}{2(1+\beta)}}\lfloor\xi\rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}} & \text{when} \quad \alpha\in(-\beta,\beta).\end{cases}$$

and integrate in $L^2_\xi(W)$ and then in $L^2_t([0,T])$ to get, using again (2.4) as well as (1.20), $\|\omega_\varepsilon\|_{L^2_t([0,T];L^2_\xi(W))}$ which concludes the proof.

3. Proof of Lemma 1.1 (construction of the fluid mode)

In this section we prove Lemma 1.1, assuming Hypothesis 1–2–3. Denote

$$\tilde L^*_\eta\psi := \lfloor v\rceil^{\frac\beta2}L^*_\eta\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right) = \lfloor v\rceil^{\frac\beta2}L^*\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right) + i\eta\lfloor v\rceil^\beta(v\cdot\sigma)\psi.$$

As before, the dependency in σ is omitted from the subscripts for readability.

3.1. Existence of the resolvent.

3.1.1. *Near zero.* We first prove that when $z\in B(0,\mathrm{R}_0\Theta(\eta))$ with R_0 small enough and $\eta\in(0,\eta_0)$ with η_0 small enough, the operator $\tilde L^*_\eta - z$ has a bounded inverse in $L^2_v(\mathcal M)$. Given $G\in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal M)$ and $z\in B(0,\mathrm{R}_0\Theta(\eta))$, consider an *a priori* solution $F\in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal M)$ to

$$-L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = \lfloor v\rceil^{-\beta}G. \tag{3.1}$$

Recall the decomposition

$$F = \mathcal PF + \mathcal P^\perp F \quad \text{with} \quad \mathcal PF := \int_{\mathbb R^d} F(v)\mathcal M_\beta(v)\,\mathrm{d}v, \tag{3.2}$$

which is orthogonal for the scalar product associated with $\|\cdot\|_{-\beta}$. Integrate (3.1) against $\bar F\mathcal M$ and take the real part to get, using Hypothesis 2 and denoting $r := |z|$,

$$\begin{aligned}&\lambda\left\|\mathcal P^\perp F\right\|^2_{-\beta} - r\|F\|^2_{-\beta} \le \|G\|_{-\beta}\|F\|_{-\beta}\\ \Longrightarrow\quad &(\lambda-r)\left\|\mathcal P^\perp F\right\|^2_{-\beta} \le \|G\|_{-\beta}\|F\|_{-\beta} + r\,|\mathcal PF|^2\\ \Longrightarrow\quad &(\lambda-r)\left\|\mathcal P^\perp F\right\|^2_{-\beta} \le \frac r2\|F\|^2_{-\beta} + \frac{1}{2r}\|G\|^2_{-\beta} + r\,|\mathcal PF|^2\\ \Longrightarrow\quad &(\lambda-r)\left\|\mathcal P^\perp F\right\|^2_{-\beta} \le \frac r2\|\mathcal P^\perp F\|^2_{-\beta} + \frac{1}{2r}\|G\|^2_{-\beta} + \frac{3r}{2}\,|\mathcal PF|^2\end{aligned}$$

which implies finally for r small enough

$$\left\|\mathcal P^\perp F\right\|_{-\beta} \lesssim r^{\frac12}\,|\mathcal PF| + r^{-\frac12}\|G\|_{-\beta}. \tag{3.3}$$

Consider then a function $0\le\mathfrak K\le1$ smooth radially symmetric and such that $\mathfrak K\equiv1$ on $B(0,3)\setminus B(0,2)$ and $\mathfrak K\equiv0$ on $B(0,1)$ and outside $B(0,4)$, and denote $\mathfrak K_R(v) := \mathfrak K(\frac vR)$ for $R>0$.

Denote $\tilde{\mathfrak{K}}_R := (v\cdot\sigma)\lfloor v\rceil^\beta \mathfrak{K}_R$ and integrate (3.1) against $\tilde{\mathfrak{K}}_R\mathcal{M}$:

$$-\left\langle L^*F, \tilde{\mathfrak{K}}_R\right\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)^2F(v)\mathfrak{K}_R(v)\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v - z\int_{\mathbb{R}^d}F(v)\tilde{\mathfrak{K}}_R(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^d}G(v)\tilde{\mathfrak{K}}_R(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v. \tag{3.4}$$

Using the decomposition (3.2), $L^*1=0$ and Hypothesis 3:

$$\left|\left\langle L^*F, \tilde{\mathfrak{K}}_R\right\rangle\right| = \left|\left\langle L^*\left(\mathcal{P}^\perp F\right), \tilde{\mathfrak{K}}_R\right\rangle\right| \le \left\|L\left(\tilde{\mathfrak{K}}_R\right)\right\|_\beta \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim R^{1+\beta-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta}. \tag{3.5}$$

Observe also that, for $\alpha\in(-\beta,2+\beta)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}(v\cdot\sigma)^2F(v)\mathfrak{K}_R(v)\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right| \\
&\gtrsim |\mathcal{P}F|\left|\int_{\mathbb{R}^d}(v\cdot\sigma)^2\mathfrak{K}_R(v)\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right| - \left|\int_{\mathbb{R}^d}(v\cdot\sigma)^2\mathcal{P}^\perp F(v)\mathfrak{K}_R(v)\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right| \\
&\gtrsim R^{2+\beta-\alpha}|\mathcal{P}F| - \left(\int_{\mathbb{R}^d}(v\cdot\sigma)^4\mathfrak{K}_R(v)^2\lfloor v\rceil^{3\beta}\mathcal{M}(v)\,\mathrm{d}v\right)^{\frac12}\left\|\mathcal{P}^\perp F\right\|_{-\beta} \\
&\le R^{2+\beta-\alpha}|\mathcal{P}F| - R^{2+\frac{3\beta}{2}-\frac{\alpha}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta}.
\end{aligned} \tag{3.6}$$

Finally we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}F(v)\tilde{\mathfrak{K}}_R(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v\right| &= \left|\int_{\mathbb{R}^d}(v\cdot\sigma)^2\mathcal{P}^\perp F(v)\mathfrak{K}_R(v)\mathcal{M}(v)\,\mathrm{d}v\right| \\
&\lesssim \left\|\mathcal{P}^\perp F\right\|_{-\beta}\left(\int_{\mathbb{R}^d}(v\cdot\sigma)^2\mathfrak{K}_R(v)^2\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right)^{\frac12} \\
&\lesssim R^{1+\beta-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta}.
\end{aligned} \tag{3.7}$$

Combining (3.3)–(3.4)–(3.5)–(3.6)–(3.7) yields

$$\begin{aligned}
|\mathcal{P}F| &\lesssim \left[r^{\frac12}R^{\frac{\alpha+\beta}{2}} + r^{\frac12}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}} + r^{\frac32}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}}\right]|m[F]| \\
&\quad + \left[\eta^{-1}R^{-2-\beta+\alpha} + r^{-\frac12}R^{\frac{\alpha+\beta}{2}} + r^{-\frac12}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}} + r^{\frac12}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}}\right]R^{1+\frac{\beta}{2}-\frac{\alpha}{2}}\|G\|_{-\beta} \\
&\lesssim r^{\frac12}\left(R^{\frac{\alpha+\beta}{2}} + \eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}}\right)|m[F]| + \left[r^{-\frac12}\eta^{-1} + \eta^{-1}R^{-1-\frac{\beta}{2}+\frac{\alpha}{2}} + r^{-\frac12}R^{1+\beta}\right]\|G\|_{-\beta}
\end{aligned} \tag{3.8}$$

Observe that in the first bracket, the third term is negligeable in front of the second term when r is small, and optimising the choice of R so that the first and second term are equal yields $R=\eta^{-\frac{1}{1+\beta}}$. Then for $r\le \mathrm{R}_0\Theta(\eta)$ with $\Theta(\eta)$ defined in (1.10)) and $\mathrm{R}_0>0$, we have

$$\left[r^{\frac12}R^{\frac{\alpha+\beta}{2}} + r^{\frac12}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}} + r^{\frac32}\eta^{-1}R^{-1+\frac{\alpha}{2}-\frac{\beta}{2}}\right] \le 2\mathrm{R}_0^{\frac12} + \mathrm{R}_0^{\frac32}$$

which is small when R_0 is small enough, yielding therefore after calculations (and using (3.3))

$$|\mathcal{P}F| \lesssim r^{-\frac12}\eta^{-1}\|G\|_{-\beta} \quad\text{and}\quad \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \left(r^{-\frac12}+\eta^{-1}\right)\|G\|_{-\beta}. \tag{3.9}$$

When $\alpha=2+\beta$, the calculation is slightly modified as follows: we define $\tilde{\chi}_R := (v\cdot\sigma)\lfloor v\rceil^\beta\chi_R$ and integrate (3.1) against $\tilde{\chi}_R\mathcal{M}$ and have

$$\begin{aligned}
&|\langle L^*F, \tilde{\chi}_R\rangle| \lesssim \left\|\mathcal{P}^\perp F\right\|_{-\beta} \\
&\left|\int_{\mathbb{R}^d}(v\cdot\sigma)^2F(v)\chi_R(v)\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right| \le (\ln R)\,|\mathcal{P}[F]| - R^{1+\beta}\left\|\mathcal{P}^\perp F\right\|_{-\beta}, \\
&\left|\int_{\mathbb{R}^d}F(v)\tilde{\chi}_R(v)\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v\right| \lesssim (\ln R)\left\|\mathcal{P}^\perp F\right\|_{-\beta},
\end{aligned}$$

which results into the new estimate (with the same choices $R = \eta^{-\frac{1}{1+\beta}}$)

$$|\mathcal{P}F| \lesssim \frac{r^{\frac12}}{\eta|\ln\eta|}\,|\mathcal{P}F| + \frac{1}{r^{\frac12}\eta|\ln\eta|}\,\|G\|_{-\beta}$$

and, with $r \le \mathrm{R}_0\Theta(\eta)$ and $\Theta(\eta) = \eta^2(\ln\eta)^2$ and R_0 small enough we deduce

$$|\mathcal{P}F| \lesssim r^{-\frac12}\Theta(\eta)^{-\frac12}\,\|G\|_{-\beta} \qquad \text{and} \qquad \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \left(r^{-\frac12} + \Theta(\eta)^{-1}\right)\|G\|_{-\beta}\,.$$

Given $z \in B(0, \mathrm{R}_0\Theta(\eta))$ with R_0 small enough and $\eta \in (0,\eta_0)$ with η_0 small enough, we now deduce from the latter *a priori* estimate on the *a priori* inverse of $\tilde L^*_\eta - z$ the existence of a solution to (3.1) with the uniform estimate $\|F\|_{-\beta} \lesssim \Theta(\eta)^{-1}\|G\|_{-\beta}$. The equation (3.1) re-writes

$$-\tilde L^*\tilde F - i\eta(v\cdot\sigma)\lfloor v\rceil^\beta\tilde F - z\tilde F = \lfloor v\rceil^{-\frac\beta2}G \in L^2_v(\mathcal{M}), \tag{3.10}$$

with $\tilde F := \lfloor\cdot\rceil^{-\frac\beta2}F$. Since (Hypothesis 2) $\tilde L^*$ generates a contraction semigroup in $L^2_v(\mathcal{M})$, it is a standard result (see [25, Theorem II.3.15]) that $\tilde L^*$ is maximal dissipative. Therefore, given any $M \ge 1$, the operator $\tilde L^*_{\eta,M} := \tilde L^* + i\eta(v\cdot\sigma)\lfloor v\rceil^\beta\chi_M(v)$ is maximal dissipative (perturbation by a bounded purely imaginary multiplicative operator). Observe that the previous a priori estimate (3.9) holds for $\tilde L^*_{\eta,M}$ by the same calculation, and uniformly as $M \to +\infty$. This implies that for each $M \ge 1$ and $z \in S(0,r)$, there is $\tilde F_M \in L^2_v(\mathcal{M})$ that solves $-\tilde L^*_M\tilde F_M - z\tilde F_M = \lfloor v\rceil^{-\frac\beta2}G$, and that $\tilde F_M$ is uniformly bounded in $L^2_v(\mathcal{M})$ as $M \to \infty$. Taking a subsequence weakly converging to some $\tilde F \in L^2_v(\mathcal{M})$ as $M\to\infty$ gives a solution to (3.10) and thus to (3.1).

3.1.2. *Away from zero.* We now prove that when $z \in \mathbb{C}$ with $|z| \in (\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and r_0 *small* enough and η *small* enough, the operator $\tilde L^*_\eta - z$ has a bounded inverse in $L^2_v(\mathcal{M})$, and the bound is uniform in $|z| \in (\mathrm{R}_1\Theta(\eta), r_0)$. Given $G \in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $z \in S(0,r)$, consider an *a priori* solution $F \in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ to (3.1). Integrating (3.1) against $\bar F\mathcal{M}$, taking the real part to get and using Hypothesis 2 yields (3.3) again. Consider then $0 \le \chi \le 1$ smooth radially symmetric and such that $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ outside $B(0,2)$, and denote $\chi_R(v) := \chi(\frac{v}{R})$ for $R > 0$. Integrate (3.1) against $\chi_R\mathcal{M}$:

$$-\langle L^*F, \chi_R\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v - z\mathcal{P}_RF = \mathcal{P}_RG \tag{3.11}$$

where we denote the truncated average

$$\mathcal{P}_RF := \int_{\mathbb{R}^d} F(v)\chi_R(v)\mathcal{M}_\beta(v)\,\mathrm{d}v.$$

Using the decomposition (3.2), $L^*1 = 0$ and Hypothesis 3:

$$|\langle L^*F, \chi_R\rangle| = \left|\left\langle L^*\left(\mathcal{P}^\perp F\right), \chi_R\right\rangle\right| \le \|L\left(\chi_R\right)\|_\beta\left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta}. \tag{3.12}$$

Observe also that

$$\begin{aligned}\left|\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v\right| &= \left|\int_{\mathbb{R}^d}(v\cdot\sigma)\left[\mathcal{P}^\perp F(v)\right]\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v\right| \\ &\le \left(\int_{|v|\le 2R}(v\cdot\sigma)^2\lfloor v\rceil^\beta\mathcal{M}(v)\,dv\right)^{\frac12}\left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \ell(R)\left\|\mathcal{P}^\perp F\right\|_{-\beta}\end{aligned} \tag{3.13}$$

with

$$\ell(R) = \left(\int_{|v|\le 2R}(v\cdot\sigma)^2\lfloor v\rceil^\beta\mathcal{M}(v)\,\mathrm{d}v\right)^{\frac12} \lesssim \begin{cases} 1 & \text{when } \alpha > 2+\beta,\\ \sqrt{\ln(R)} & \text{when } \alpha = 2+\beta,\\ R^{1-\frac{\alpha-\beta}{2}} & \text{when } \alpha < 2+\beta.\end{cases} \tag{3.14}$$

Combining (3.11)–(3.12)–(3.13) yields the following estimate on the truncated average:

$$|\mathcal{P}_R F| \leq \frac{1}{r}\left[\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}\right]\left\|\mathcal{P}^\perp F\right\|_{-\beta} + \frac{1}{r}\|G\|_{-\beta}\,. \tag{3.15}$$

We next estimate the difference between $\mathcal{P}F$ and $\mathcal{P}_R F$:

$$\begin{aligned}|\mathcal{P}F - \mathcal{P}_R F| &\leq \int_{\mathbb{R}^d} |F|\,|1-\chi_R|\,\mathcal{M}_\beta(v)\,\mathrm{d}v \\ &\leq \left(\int_{\mathbb{R}^d} |1-\chi_R|^2\,\mathcal{M}_\beta\,\mathrm{d}v\right)^{\frac12}\|F\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}}\|F\|_{-\beta},\end{aligned}$$

which implies for R large enough

$$|\mathcal{P}_R F| \lesssim |\mathcal{P}_R F| + R^{-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta}\,. \tag{3.16}$$

Combining (3.15) and (3.16), we deduce

$$\begin{aligned}|\mathcal{P}F| &\lesssim \left(\frac{\eta\ell(R)+R^{-\frac{\alpha+\beta}{2}}}{r}\right)\left\|\mathcal{P}^\perp F\right\|_{-\beta} + R^{-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp F\right\|_{-\beta} + \frac{1}{r}\|G\|_{-\beta}\,, \\ &\lesssim \left(\frac{\eta\ell(R)+R^{-\frac{\alpha+\beta}{2}}}{r}\right)\left\|\mathcal{P}^\perp F\right\|_{-\beta} + \frac{1}{r}\|G\|_{-\beta}\,. \qquad (3.17)\end{aligned}$$

Optimising R so that the two terms in the parenthesis are equal yields again $R = \eta^{-\frac{1}{1+\beta}}$ (with η small enough so that R is large enough to obtain (3.16)) and therefore $\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}} \sim \Theta(\eta)^{\frac12}$ where Θ was defined in (1.10). Combining (3.3) and (3.17) we get then

$$|\mathcal{P}F| \lesssim \frac{\Theta(\eta)^{\frac12}}{r^{\frac12}}\,|m[F]| + \left(1 + \frac{\Theta(\eta)^{\frac12}}{r^{\frac12}}\right)\frac{1}{r}\|G\|_{-\beta}\,.$$

When $r > \mathrm{R}_1\Theta(\eta)$ with R_1 *large* enough, $\frac{\Theta(\eta)^{\frac12}}{r^{\frac12}}$ is small and we deduce

$$|\mathcal{P}F| \lesssim \frac{1}{r}\|G\|_{-\beta} \quad \text{and} \quad \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim r^{-\frac12}\|G\|_{-\beta}\,.$$

Arguing as before, given $|z| \in (\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and r_0 *small* enough and η *small* enough, we deduce from the latter *a priori* estimate the construction of a solution F to (3.1) with the uniform estimate $\|F\|_{-\beta} \lesssim \Theta(\eta)^{-1}\|G\|_{-\beta}$. Together with the previous subsubsection, we have thus proved that $\tilde{L}^*$ has no eigenvalues in $|z| < \mathrm{R}_0\Theta(\eta)$ and $\mathrm{R}_1\Theta(\eta) < |z| < r_0$. We now prove the existence of a unique eigenvalue in $|z| \in (\mathrm{R}_0\Theta(\eta), \mathrm{R}_1\Theta(\eta))$.

3.2. **The spectral projections.** We define the spectral projections

$$\Pi_{r,\eta} := \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde{L}^*_\eta - z\right]^{-1}\mathrm{d}z$$

for $r \in (\mathrm{R}_1\Theta(\eta), r_0)$ for r_0, R_1 and η as above; it is well-defined since we proved above that $(\tilde{L}^*_\eta - z)^{-1}$ then exists. In this section, we first estimate the difference between the projections $\Pi_{r,\eta}$ and $\Pi_{r,0}$ when acting on $\psi_0 := \lfloor v\rceil^{-\frac{\beta}{2}}$ (the kernel of $\tilde{L}_0$) and projected on $\mathrm{Span}(\psi_0)$ and proves that it goes to zero as $\eta \to 0$; this implies that $\Pi_{r,\eta}$ is non-zero for r and η small enough and thus proves the existence of an eigenvalue. Second, we amplify the previous estimate and prove that $|||\Pi_{r,\eta} - \Pi_{r,0}||| \to 0$ as $\eta \to 0$, which implies that the dimensions of these two projections are the same for η small enough. This implies the existence and uniqueness of the eigenvalue in $|z| \in (\mathrm{R}_0\Theta(\eta), \mathrm{R}_1\Theta(\eta))$ and quantitative convergence estimates as $\eta \to 0$.

3.3. **Preparation for the first scalar estimate.** Recall $\psi_0 = \lfloor\cdot\rceil^{-\frac{\beta}{2}}$, then

$$\begin{aligned}\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0 &= \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde{L}_\eta^* - z\right]^{-1}\left[\tilde{L}_0^* - \tilde{L}_\eta\right]\left[\tilde{L}_0^* - z\right]^{-1}\psi_0\,\mathrm{d}z\\ &= -\frac{\eta}{2\pi}\int_{S(0,r)}\left[\tilde{L}_\eta^* - z\right]^{-1}\left\{(v\cdot\sigma)\lfloor v\rceil^{\beta}\left[\tilde{L}_0^* - z\right]^{-1}\psi_0\right\}\mathrm{d}z\\ &= \frac{\eta}{2\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac{\beta}{2}}F\,\frac{\mathrm{d}z}{z}\end{aligned}$$

where we have used

$$\left(\tilde{L}_0^* - z\right)^{-1}\psi_0 = \left[\lfloor\cdot\rceil^{\frac{\beta}{2}}L\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\cdot\right) - z\right]^{-1}\psi_0 = -\frac{1}{z}\psi_0$$

and we have defined F through

$$\left[\tilde{L}_\eta^* - z\right]^{-1}\left[v'\mapsto (v'\cdot\sigma)\lfloor v'\rceil^{\frac{\beta}{2}}\right](v) =: \lfloor v\rceil^{-\frac{\beta}{2}}F(v),$$

that is

$$-L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = (v\cdot\sigma) \tag{3.18}$$

(the dependency of F on η, z and σ is omitted for readability).

Since $\Pi_{r,0}\psi_0 = \psi_0$ and

$$\int_{\mathbb{R}^d}\Pi_{r,0}\psi_0(v)\lfloor v\rceil^{-\frac{\beta}{2}}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\lfloor v\rceil^{-\beta}\mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\mathcal{M}_\beta(v)\,\mathrm{d}v = 1,$$

to prove the existence of an eigenvalue, it is enough to prove that for r and η small enough

$$A_{r,\eta} := \left|\int_{\mathbb{R}^d}\left(\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0\right)\lfloor v\rceil^{-\frac{\beta}{2}}\mathcal{M}(v)\,\mathrm{d}v\right| < 1.$$

Using the decomposition (3.2) one gets

$$A_{r,\eta} = \left|\frac{\eta}{2\pi}\int_{S(0,r)}\frac{\mathcal{P}F}{z}\,\mathrm{d}z\right|. \tag{3.19}$$

The next three steps are devoted to estimating $\mathcal{P}F$.

3.4. **Localised average estimate.** Integrate (3.18) against $\chi_R\mathcal{M}$: the right hand side vanishes since $\mathcal{M}$ and χ_R are even and one gets

$$-\langle L^*F, \chi_R\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v - z\mathcal{P}_RF = 0.$$

Using the same argument as for (3.12) and (3.13), we get

$$|\mathcal{P}_RF| \le \frac{1}{r}\left[\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}\right]\left\|\mathcal{P}^\perp F\right\|_{-\beta} \tag{3.20}$$

and using (3.16) we deduce, for $R = \eta^{-\frac{1}{1+\beta}}$ large enough,

$$|\mathcal{P}F| \lesssim \frac{\left(\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}\right)}{r}\left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{r}\left\|\mathcal{P}^\perp F\right\|_{-\beta}.$$

3.5. L^2 **estimate.** Re-organise (3.18) as

$$-L^*F - i\eta(v\cdot\sigma)\left(F - \frac{1}{i\eta}\right) = z\lfloor v\rceil^{-\beta}F,$$

integrate it against

$$\overline{\left(F - \frac{1}{i\eta}\right)}\mathcal{M}$$

and take the real part to obtain

$$-\operatorname{Re}\left\langle L^*F, \left(F - \frac{1}{i\eta}\right)\right\rangle = \operatorname{Re}\left(z\int_{\mathbb{R}^d}\lfloor v\rceil^{-\beta}F\overline{\left(F - \frac{1}{i\eta}\right)}\mathcal{M}\,\mathrm{d}v\right).$$

The left hand side satisfies (using $L1 = 0$ and Hypothesis 2)

$$-\operatorname{Re}\left\langle L^*F, \left(F - \frac{1}{i\eta}\right)\right\rangle = -\operatorname{Re}\langle L^*F, F\rangle \geq \lambda\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2,$$

and the right hand side is bounded by

$$\operatorname{Re}\left(z\int_{\mathbb{R}^d}\lfloor v\rceil^{-\beta}F\overline{\left(F - \frac{1}{i\eta}\right)}\mathcal{M}\,\mathrm{d}v\right) \leq r\|F\|_{-\beta}^2 + \frac{r}{\eta}\left|\mathcal{P}F\right|.$$

This results in the estimate (using again the orthogonal decomposition)

$$\begin{aligned}\lambda\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 &\leq r\|F\|_{-\beta}^2 + \frac{r}{\eta}\left|\mathcal{P}F\right| \\ &\leq r\left|\mathcal{P}F\right|^2 + r\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + \frac{r}{\eta}\left|\mathcal{P}F\right|,\end{aligned}$$

which implies, when r small,

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim r\left|\mathcal{P}F\right|^2 + \frac{r}{\eta}\left|\mathcal{P}F\right|. \tag{3.21}$$

3.6. **Synthesis and the first scalar estimate.** The two previous steps lead to

$$\begin{cases} \left|\mathcal{P}F\right|^2 \lesssim \dfrac{\Theta(\eta)}{r^2}\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2, \\ \left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim r\left|\mathcal{P}F\right|^2 + \dfrac{r}{\eta}\left|\mathcal{P}F\right|. \end{cases}$$

Plugging the second estimate into the first one, we obtain

$$\left|\mathcal{P}F\right| \lesssim \frac{\Theta(\eta)}{r}\left|\mathcal{P}F\right| + \frac{\Theta(\eta)}{\eta r}. \tag{3.22}$$

Given $r \in [\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and η *small* enough so that $\frac{\Theta(\eta)}{r}$ is small we get

$$\left|\mathcal{P}F\right| \lesssim \frac{\Theta(\eta)}{\eta r}. \tag{3.23}$$

Plugging the latter into (3.19) finally yields

$$A_{r,\eta} \lesssim \frac{\eta}{2\pi}\int_{S(0,r)}\frac{\Theta(\eta)}{\eta r^2}\,\mathrm{d}z \lesssim \frac{\Theta(\eta)}{r},$$

which is as small as wanted for $r \in (\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and η *small* enough.

3.7. **Estimating the full norm of the difference of the projections at ψ_0.** Combining (3.21) and (3.23) yields

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \frac{\Theta(\eta)}{r^{\frac12}\eta} + \frac{\Theta(\eta)^{\frac12}}{\eta}.$$

This implies

$$\begin{aligned}\|\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0\| &\lesssim \frac{\eta}{2\pi}\int_{S(0,r)} \frac1r \|F\|_{-\beta}\,\mathrm{d}z \\ &\lesssim \frac{\eta}{2\pi}\int_{S(0,r)} \frac1r \left(|\mathcal{P}F| + \left\|\mathcal{P}^\perp F\right\|_{-\beta}\right)\mathrm{d}z \\ &\lesssim \frac1r\Theta(\eta) + \frac{1}{r^{\frac12}}\Theta(\eta) + \Theta(\eta)^{\frac12} \lesssim \frac{\Theta(\eta)}{r} \end{aligned} \tag{3.24}$$

which is as small as wanted for $r \in (\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and η *small* enough.

3.8. **Estimating the full norm of the difference of projections.** Take now any $\psi \in L^2(\mathcal{M})$. Then $\lfloor\cdot\rceil^{\frac\beta2}\psi \in L^2(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and the following decomposition holds

$$\psi = \lfloor v\rceil^{-\frac\beta2}\mathcal{P}\left[\lfloor\cdot\rceil^{\frac\beta2}\psi\right] + \lfloor v\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right).$$

As a consequence,

$$\|(\Pi_{r,\eta}-\Pi_{r,0})\psi\| \le \left|\mathcal{P}\left[\lfloor\cdot\rceil^{\frac\beta2}\psi\right]\right| \|(\Pi_{r,\eta}-\Pi_{r,0})\psi_0\| + \left\|(\Pi_{r,\eta}-\Pi_{r,0})\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right]\right\|.$$

The first term in the right hand side is estimated by (3.24). We estimate the second term in the right hand side by the triangle inequality:

$$\begin{aligned}&\left\|(\Pi_{r,\eta}-\Pi_{r,0})\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right]\right\| \\ &\quad\le \left\|\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right]\right\| + \left\|\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right]\right\|\end{aligned}$$

and now consider each term separately. Start with

$$\begin{aligned}\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right] &= \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde L_\eta - z\right]^{-1}\left[\lfloor\cdot\rceil^{-\frac\beta2}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right]\mathrm{d}z \\ &= \frac{1}{2i\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac\beta2}F\,\mathrm{d}z \end{aligned} \tag{3.25}$$

where F satisfies this time (as before we omit writing the dependency in η, z, σ)

$$-L^*F - i\eta(v\cdot\sigma)F - z\lfloor v\rceil^{-\beta}F = \lfloor\cdot\rceil^{-\beta}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right). \tag{3.26}$$

First, test (3.26) on $\overline{F}\mathcal{M}$, take the real part and use $\mathcal{P}[\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac\beta2}\psi)] = 0$:

$$\begin{aligned}\lambda\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 &\le (\mathrm{Re}\ z)\,\|F\|_{-\beta}^2 + \mathrm{Re}\left\langle\lfloor\cdot\rceil^{-\beta}\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac\beta2}\psi), F\right\rangle \\ &= (\mathrm{Re}\ z)\,\|F\|_{-\beta}^2 + \mathrm{Re}\left\langle\lfloor\cdot\rceil^{-\beta}\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac\beta2}\psi), \mathcal{P}^\perp F\right\rangle \\ &\le r|\mathcal{P}F|^2 + r\|\mathcal{P}^\perp F\|_{-\beta}^2 + \left\|\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right\|_{-\beta}\left\|\mathcal{P}^\perp F\right\|_{-\beta},\end{aligned}$$

which implies, since $r < r_0 < \lambda$ stays away from λ,

$$\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim r\ |\mathcal{P}F|^2 + \left\|\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right\|_{-\beta}^2 \lesssim r\ |\mathcal{P}F|^2 + \|\psi\|^2. \tag{3.27}$$

We now estimate $\mathcal{P}F$. Integrate (3.26) against $\chi_R\mathcal{M}$ with $R = \eta^{-\frac{1}{1+\beta}}$

$$-\langle L^*F, \chi_R\rangle - i\eta\int_{\mathbb{R}^d}(v\cdot\sigma)F(v)\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v - z\mathcal{P}_R F = \mathcal{P}_R\left[\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac\beta2}\psi\right)\right].$$

Using the same arguments as in Subsections 3.1 and 3.4 we obtain

$$|\mathcal{P}_R F| \lesssim \frac{\Theta(\eta)^{\frac12}}{r} \left\|\mathcal{P}^\perp F\right\|_{-\beta} + \frac{1}{r}\left|\mathcal{P}_R\left[\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right|.$$

Since $\mathcal{P}[\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)] = 0$, we can estimate $|\mathcal{P}_R[\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)]|$ as follows:

$$\begin{aligned}\left|\mathcal{P}_R\left[\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right| &= \left|\mathcal{P}\left[\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right] - \mathcal{P}_R\left[\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right| \\ &\lesssim R^{-\frac{\alpha+\beta}{2}}\left\|\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}}\|\psi\| \lesssim \Theta(\eta)^{\frac12}\|\psi\|.\end{aligned}$$

We deduce

$$|\mathcal{P}_R F| \lesssim \frac{\Theta(\eta)^{\frac12}}{r}\left(\left\|\mathcal{P}^\perp F\right\|_{-\beta} + \|\psi\|\right),$$

and using

$$|\mathcal{P}F|^2 \lesssim |\mathcal{P}_R F|^2 + R^{-(\alpha+\beta)}\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \le |\mathcal{P}_R F|^2 + \Theta(\eta)\left\|\mathcal{P}^\perp G\right\|_{-\beta}^2$$

we finally get

$$|\mathcal{P}F|^2 \lesssim \frac{\Theta(\eta)}{r^2}\left(\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + \|\psi\|^2\right) + \Theta(\eta)\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r^2}\left(\left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 + \|\psi\|^2\right). \tag{3.28}$$

Combining (3.27)–(3.28) implies for $r \in [\mathrm{R}_1\Theta(\eta), r_0)$ with R_1 *large* enough and η *small* enough

$$|\mathcal{P}F|^2 \lesssim \frac{\Theta(\eta)}{r^2}\|\psi\|^2 \quad \text{and thus} \quad \left\|\mathcal{P}^\perp F\right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r}\|\psi\|^2 + \|\psi\|^2 \lesssim \|\psi\|^2.$$

Plugging the latter estimates into (3.25) yields

$$\left\|\Pi_{r,\eta}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \le r\|F\|_{-\beta} \lesssim r|\mathcal{P}F| + r\|\mathcal{P}^\perp F\|_{-\beta} \lesssim \Theta(\eta)^{\frac12}\|\psi\| + r\|\psi\|.$$

We now come to the estimate of

$$\begin{aligned}\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right] &= \frac{1}{2i\pi}\int_{S(0,r)}\left[\tilde{L}_0 - z\right]^{-1}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\mathrm{d}z \\ &= \frac{1}{2i\pi}\int_{S(0,r)}\lfloor v\rceil^{-\frac{\beta}{2}}F\,\mathrm{d}z\end{aligned}$$

where F satisfies this time

$$-L^*F - z\lfloor v\rceil^{-\beta}F = \lfloor v\rceil^{-\beta}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right). \tag{3.29}$$

Integrating this equation against $\mathcal{M}$ implies $\mathcal{P}F = 0$ since $\langle L^*F, 1\rangle = \mathcal{P}[\mathcal{P}^\perp(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi)] = 0$ and $z \neq 0$. Hypothesis 2 then implies, since $r < r_0 < \lambda$ is away from λ:

$$\|F\|_{-\beta} = \left\|\mathcal{P}^\perp F\right\|_{-\beta} \lesssim \|\psi\|,$$

and thus

$$\left\|\Pi_{r,0}\left[\lfloor\cdot\rceil^{-\frac{\beta}{2}}\mathcal{P}^\perp\left(\lfloor\cdot\rceil^{\frac{\beta}{2}}\psi\right)\right]\right\| \lesssim r\|\psi\|.$$

The conclusion is that for any $\psi \in L^2(\mathcal{M})$,

$$\|(\Pi_{r,\eta} - \Pi_{r,0})\,\psi\| \lesssim \Theta(\eta)^{\frac12}\|\psi\| + r\|\psi\|$$

which implies (combining all the previous conditions), for $r \in (\mathrm{R}_1\Theta(\eta), r_0)$ with $r_0 > 0$ *small* enough *independently* of η and R_1 *large* enough *independently* of η and η *small* enough,

$$\|\Pi_{r,\eta} - \Pi_{r,0}\|_{L^2(\mathcal{M})\to L^2(\mathcal{M})} < 1.$$

It implies that, for $r \in (\mathrm{R}_1\Theta(\eta), r_0)$ and η small enough, the projections $\Pi_{r,\eta}$ and $\Pi_{r,0}$ both exist thanks to Subsection 3.1 and their dimensions are the same, i.e. 1, which proves existence and uniqueness of an eigenvalue $\mu(\eta) \in B(0, r_0)$ with $|\mu(\eta)| \in (\mathrm{R}_0\Theta(\eta), \mathrm{R}_1\Theta(\eta))$. This implies

that this eigenvalue is real: if $(\psi_\eta, -\mu(\eta))$ is an eigenpair of $\tilde{L}_\eta^*$ with $\mu(\eta) \in B(0, r_0)$, then so is $(\overline{\psi_\eta(-\cdot)}, -\overline{\mu(\eta)})$. Since $\tilde{L}_\eta^* \le 0$ and 0 is not an eigenvalue for $\eta \neq 0$, this proves that $\mu(\eta) > 0$.

3.9. **Estimate on the branch as $\eta \to 0$.** Denote $\phi_\eta := \lfloor\cdot\rceil^{\frac{\beta}{2}} \psi_\eta$ and normalize

$$\int_{\mathbb{R}^d} \psi_\eta(v) \lfloor v\rceil^{-\frac{\beta}{2}} \mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} \phi_\eta(v) \lfloor v\rceil^{-\beta} \mathcal{M}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}_\beta(v)\,\mathrm{d}v = m\left[\phi_\eta\right] = 1.$$

Then integrating the equation against $\overline{\psi_\eta}\mathcal{M}$, taking the real part and using Hypothesis 2 yields

$$\lambda \left\|\psi_\eta - \psi_0\right\|^2 \le \mu(\eta) \left\|\psi_\eta\right\|^2 \lesssim \mu(\eta) \left\|\psi_\eta - \psi_0\right\|^2 + \mu(\eta)$$

where we have used $\|\psi_0\| = 1$. Hence for η small enough we deduce

$$\left\|\phi_\eta - 1\right\|_{-\beta} = \left\|\psi_\eta - \psi_0\right\| \lesssim \mu(\eta)^{\frac{1}{2}}.$$

This concludes the proof of Lemma 1.1.

4. Proof of Lemma 1.2 (scaling of the eigenvalue)

In this section we prove Lemma 1.2, assuming all Hypothesis 1–2–3–4. Consider the unique eigenpair $(\phi_\eta, \mu(\eta))$ that satisfies $\mu(\eta) \in B(0, r_1)$ and

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad \text{and} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\mathcal{M}_\beta(v)\,\mathrm{d}v = 1. \tag{4.1}$$

4.1. **Proof in the case $\alpha > 2 + \beta$.** The function $F_\eta := \frac{\operatorname{Im}\phi_\eta}{\eta}$ satisfies

$$-L^*F_\eta - \mu(\eta)\lfloor v\rceil^{-\beta}F_\eta = (v\cdot\sigma)\operatorname{Re}\phi_\eta \quad \text{and} \quad \int_{\mathbb{R}^d} F_\eta(v)\mathcal{M}_\beta\,\mathrm{d}v = 0.$$

Since (Hypothesis 2) $\tilde{L}^*$ is invertible on the $L^2_v(\mathcal{M})$-orthogonal of $\lfloor\cdot\rceil^{-\frac{\beta}{2}}$, and $(v \mapsto \lfloor v\rceil^{\frac{\beta}{2}}(v\cdot\sigma)$ belongs to $L^2_v(\mathcal{M})$ when $\alpha > 2+\beta$, we can define then $F \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ solution to

$$-L^*F = (v\cdot\sigma) \quad \text{with} \quad \int_{\mathbb{R}^d} F(v)\mathcal{M}_\beta\,\mathrm{d}v = 0.$$

The difference $F_\eta - F$ satisfies

$$-L^*\left(F_\eta - F\right) - \mu(\eta)\lfloor v\rceil^{-\beta}\left(F_\eta - F\right) = (v\cdot\sigma)\left[\operatorname{Re}\phi_\eta - 1\right] + \mu(\eta)\lfloor v\rceil^{-\beta}F.$$

Integrate the latter against $(F_\eta - F)\mathcal{M}$ and use Hypothesis 2:

$$\begin{aligned}\left[\lambda - \mu\left(\eta\right)\right]\left\|F_\eta - F\right\|^2_{-\beta} &\le \int_{\mathbb{R}^d} (v\cdot\sigma)\left(\operatorname{Re}\phi_\eta - 1\right)\left(F_\eta - F\right)\mathcal{M}\,\mathrm{d}v + \mu(\eta)\int_{\mathbb{R}^d} F\left(F_\eta - F\right)\mathcal{M}_\beta\,\mathrm{d}v \\ &\le \left\|\operatorname{Re}\phi_\eta - 1\right\|_{2+\beta}\left\|F_\eta - F\right\|_{-\beta} + \mu(\eta)\|F\|_{-\beta}\left\|F_\eta - F\right\|_{-\beta}.\end{aligned}$$

Write for any $\ell \in (2+\beta, \alpha)$

$$\left\|\operatorname{Re}\phi_\eta - 1\right\|_{2+\beta} \le \left\|\operatorname{Re}\phi_\eta - 1\right\|^{\zeta}_{-\beta} \left\|\operatorname{Re}\phi_\eta - 1\right\|^{1-\zeta}_{\ell} \le \mu(\eta)^{\frac{\zeta}{2}} \left\|\operatorname{Re}\phi_\eta - 1\right\|^{1-\zeta}_{\ell}$$

with $\mathsf{z} = \frac{\ell-(2+\beta)}{\ell+\beta} \in (0,1)$, then Hypothesis 4-(i) implies

$$\left\|\operatorname{Re}\phi_\eta - 1\right\|_{2+\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \quad \text{as} \quad \eta \to 0,$$

and thus, since $\alpha > 1$ (combining $\alpha > 2+\beta$ and $\alpha + \beta > 0$) and

$$\|F\|_{-\beta} \lesssim \|(v\cdot\sigma)\| \lesssim 1,$$

we deduce

$$\left\|F_\eta - F\right\|_{-\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \quad \text{as} \quad \eta \to 0.$$

Finally,

$$\begin{aligned}\left|\frac{\mu(\eta)}{\eta^2} - \int_{\mathbb{R}^d} (v\cdot\sigma)F(v)\mathcal{M}(v)\,\mathrm{d}v\right| &\le \left|\int_{\mathbb{R}^d} (v\cdot\sigma)(F_\eta(v) - F(v))\mathcal{M}(v)\,\mathrm{d}v\right| \\ &\lesssim \|1\|_{2+\beta}\|F_\eta - F\|_{-\beta} \lesssim \mu(\eta)^{\frac{\mathsf{z}}{2}} \to 0 \text{ as } \eta \to 0,\end{aligned}$$

which identifies the limit of $\mu(\eta)$ and provides a rate.

4.2. **Proof in the case $\alpha < 2+\beta$.** Take $0 \le \chi \le 1$ a smooth test function that is 1 on $B(0,R_0)$ and zero outside $B(0,2R_0)$. Integrate (4.1) against $\Theta(\eta)^{-1}\chi(\cdot\eta^{\frac{1}{1+\beta}})\mathcal{M}$ and take the real part:

$$\begin{aligned}(4.2)\qquad &\frac{\mu(\eta)}{\Theta(\eta)} + \frac{1}{\Theta(\eta)}\eta\left\langle (v\cdot\sigma)\operatorname{Im}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)}\left(\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle - 1\right) - \frac{1}{\Theta(\eta)}\left\langle L^*(\operatorname{Re}\phi_\eta - 1), \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle - \frac{1}{\Theta(\eta)}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right)\right\rangle.\end{aligned}$$

The first term in the right hand side is controlled by

$$\left|\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \left|\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}}\eta^{\frac{\alpha+\beta}{2(1+\beta)}}$$

and the second term is controlled by

$$\begin{aligned}\left|\frac{1}{\Theta(\eta)}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi(\cdot\eta^{\frac{1}{1+\beta}})\right)\right\rangle\right| &\le \frac{1}{\Theta(\eta)}\left\|\phi_\eta - 1\right\|_{-\beta}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac12}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac12}\eta^{\frac{\alpha+\beta}{2(1+\beta)}}R_0^{-\frac{\alpha+\beta}{2(1+\beta)}} \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}}.\end{aligned}$$

The second term in the left hand side satisfies (changing variable to $u = v\eta^{\frac{1}{1+\beta}}$)

$$\frac{\eta}{\Theta(\eta)}\left\langle (v\cdot\sigma)\operatorname{Im}\phi_\eta, \chi(\cdot\eta^{\frac{1}{1+\beta}})\right\rangle = c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u$$

and we deduce

$$\left|\frac{\mu(\eta)}{\Theta(\eta)} + c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right| \lesssim R_0^{-\frac{\alpha+\beta}{2}}.$$

Then observe that assumption (1.11) in Hypothesis 4 implies the uniform integrability of the integrand on the support of χ and the convergence of the integral as $\eta\to 0$ for a given χ.

All in all we have the double limit

$$\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi_\eta(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u \xrightarrow[R_0\to\infty]{\eta\to 0} \int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u.$$

This double limit thus proves that $\frac{\mu(\eta)}{\Theta(\eta)}$ converges and

$$\lim_{\eta\to 0}\frac{\mu(\eta)}{\Theta(\eta)} = c_{\alpha,\beta}\int_{\mathbb{R}^d}(u\cdot\sigma)\operatorname{Im}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u.$$

This limit then belongs to $(\mathrm{R}_0,\mathrm{R}_1)$ because of estimates on $\mu(\eta)$ already established.

4.3. **Proof in the case $\alpha = 2+\beta$.** Take $0\le\chi\le 1$ a smooth test function that is 1 on $B(0,1)$ and zero outside $B(0,2)$. Consider again (4.2) (with now $\Theta(\eta) = \eta^2|\ln\eta|$) and estimate

$$\left|\frac{\mu(\eta)}{\Theta(\eta)}\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \left|\left\langle \lfloor v\rceil^{-\beta}\operatorname{Re}\phi_\eta, \chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right) - 1\right\rangle\right| \lesssim \eta$$

and

$$\begin{aligned}\left|\frac{1}{\eta^2|\ln\eta|}\left\langle \operatorname{Re}\phi_\eta - 1, L\left(\chi(\cdot\eta^{\frac{1}{1+\beta}})\right)\right\rangle\right| &\le \frac{1}{\eta^2|\ln\eta|}\left\|\phi_\eta - 1\right\|_{-\beta}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \\ &\lesssim \frac{1}{\eta|\ln(\eta)|^{\frac12}}\left\|L\left[\chi\left(\cdot\eta^{\frac{1}{1+\beta}}\right)\right]\right\|_\beta \lesssim \frac{1}{|\ln(\eta)|^{\frac12}}.\end{aligned}$$

We have also

$$\frac{1}{\eta|\ln\eta|}\left\langle (v\cdot\sigma)\,\mathrm{Im}\,\phi_\eta, \chi\left(\cdot\eta^{-\frac{1}{1+\beta}}\right)\right\rangle = \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{\mathbb{R}^d}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u$$

which gives

$$\left|\frac{\mu(\eta)}{\Theta(\eta)} + \frac{c_{\alpha,\beta}}{|\ln(\eta)|}\int_{\mathbb{R}^d}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right| \lesssim \frac{1}{|\ln(\eta)|^{\frac12}} + \eta. \tag{4.3}$$

Let us decompose

$$\begin{aligned}&\frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{\mathbb{R}^d}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\\ &= \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\le\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u + \frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u.\end{aligned}$$

The first term is bounded by

$$\begin{aligned}\frac{c_{\alpha,\beta}}{|\ln\eta|}\int_{|u|\le\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi_\eta(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u &= \frac{c_{\alpha,\beta}}{\eta|\ln\eta|}\int_{|v|\le1}(v\cdot\sigma)\,\mathrm{Im}\,\phi_\eta(v)\mathcal{M}(v)\,\mathrm{d}v\\ &\lesssim \frac{\Theta(\eta)^{\frac12}}{\eta|\ln\eta|}\lesssim\frac{1}{|\ln(\eta)|^{\frac12}}.\end{aligned}$$

We approximate, using the second part of Hypothesis 4-(ii)

$$\frac{1}{|\ln(\eta)|}\left|\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\Big[\,\mathrm{Im}\,\Phi_\eta(u)-\mathrm{Im}\,\Phi(u)\Big]|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right|\lesssim\mathfrak{a}(\eta).$$

Define,

$$N(\eta):=\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\chi(u)\,\mathrm{d}u.$$

Observe that since $|\mathrm{Im}\,\Phi(u)|\lesssim|u|^{1+\beta}$, and $\alpha=2+\beta$,

$$\begin{aligned}\left|N(\eta)-\int_{|u|\ge\eta^{\frac{1}{1+\beta}}}(u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|_\eta^{-d-\alpha}\chi(u)\,\mathrm{d}u\right| &\le \int_{2\ge|u|\ge\eta^{\frac{1}{1+\beta}}}|u|^{2+\beta}\left||u|_\eta^{-d-\alpha}-|u|^{-d-\alpha}\right|\mathrm{d}u\\ &\le \int_{1\le|v|\le2\eta^{-\frac{1}{1+\beta}}}|v|^{-d}\left||v|^{d+\alpha}\lfloor v\rceil^{-d-\alpha}-1\right|\mathrm{d}v\\ &\lesssim 1,\end{aligned}$$

since $\left||v|^{d+\alpha}\lfloor v\rceil^{-d-\alpha}-1\right|\sim_{v\to\infty}\frac{d+\alpha}{2}\frac{1}{1+|v|^2}$. We get, using Hypothesis 4-(ii)-(a),

$$-\eta N'(\eta)\sim\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\frac{\mathrm{Im}\,\Phi\left(\eta^{\frac{1}{1+\beta}}\sigma'\right)}{\eta}\,\mathrm{d}\sigma'\sim\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\Omega(\sigma')\,\mathrm{d}\sigma'.$$

Apply then L'Hôpital's rule to deduce

$$\lim_{\eta\to0}\frac{N(\eta)}{|\ln\eta|}=\frac{1}{1+\beta}\int_{\sigma'\in\mathbb{S}^{d-1}}(\sigma\cdot\sigma')\,\Omega(\sigma')d\sigma'.$$

We conclude by taking $\eta\to0$ in (4.3).

5. Proof of Lemma 1.3 (the diffusion coefficient)

Lemma 1.3 follows Lemma 1.2, the definition (1.16) of θ, and:

Lemma 5.1. *Assume Hypothesis 1–2–3–4. Then, the following convergence holds*

$$\langle 1, \phi_\eta \rangle \sim_{\eta \to 0} \begin{cases} \|\mathcal{M}\|_{L^1(\mathbb{R}^d)} & \text{when } \alpha > 0 \\ \dfrac{|\mathbb{S}^{d-1}|}{1+\beta} |\ln(\eta)| & \text{when } \alpha = 0 \\ \eta^{\frac{\alpha}{1+\beta}} c_{\alpha,\beta} \displaystyle\int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} \, \mathrm{d}u & \text{when } \alpha \in (-\beta, 0). \end{cases}$$

with explicit convergence rate.

Proof. When $\alpha > 0$, the integral

$$\frac{c_{\alpha,\beta}}{c_{\alpha,0}} = \langle 1, 1 \rangle = \int_{\mathbb{R}^d} \mathcal{M} \, \mathrm{d}v < +\infty$$

is well defined and, choosing $\ell \in (0, \alpha)$,

$$\begin{aligned} |\langle 1, \phi_\eta \rangle - \langle 1, 1 \rangle| \le |\langle 1, \phi_\eta - 1 \rangle| &\le \|1\|_{\min(\ell,\beta)} \, \|\phi_\eta - 1\|_{-\min(\ell,\beta)} \\ &\lesssim \|\phi_\eta - 1\|_{-\beta}^{a} \, \|\phi_\eta - 1\|_0^{1-a} \lesssim \mu(\eta)^{\frac{a}{2}} \end{aligned}$$

with $a = \min(\frac{\ell}{\beta}, 1) \in (0, 1]$, which shows (with explicit rate)

$$\langle 1, \phi_\eta \rangle \xrightarrow{\eta \to 0} \langle 1, 1 \rangle = \frac{c_{\alpha,\beta}}{c_{\alpha,0}} = \|\mathcal{M}\|_{L^1(\mathbb{R}^d)} \quad \text{when } \alpha > 0.$$

In the case $\alpha = 0$,

$$\int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v = \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v + \int_{|v| \ge \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v.$$

The second term is estimated by

$$\begin{aligned} \int_{|v| \ge \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v &= c_{0,\beta} \int_{|u| \ge 1} \Phi_\eta(u) |u|_\eta^{-d} \, \mathrm{d}u \\ &= c_{0,\beta} \left(\int_{|u| \ge 1} |\Phi_\eta(u)|^2 |u|_\eta^{-d+\beta} \, \mathrm{d}u \right)^{\frac{1}{2}} \left(\int_{|u| \ge 1} |u|_\eta^{-d-\beta} \, \mathrm{d}u \right)^{\frac{1}{2}} \lesssim 1, \end{aligned}$$

using the moment bounds (1.12). The first term is decomposed into

$$\int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v = \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1) \, \mathcal{M}(v) \, \mathrm{d}v + \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v) \, \mathrm{d}v.$$

Since

$$\left| \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1) \, \mathcal{M}(v) \, \mathrm{d}v \right| \le \|\phi_\eta(v) - 1\|_{-\beta} \left\| \mathbf{1}_{|\cdot| \le \eta^{-\frac{1}{1+\beta}}} \right\|_\beta \lesssim \mu(\eta)^{\frac{1}{2}} \eta^{-\frac{\beta}{2(1+\beta)}} \lesssim 1,$$

we deduce

$$\int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v \sim \int_{|v| \le \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v) \, \mathrm{d}v \sim c_{0,\beta} \frac{|\mathbb{S}^{d-1}|}{1+\beta} |\ln \eta|$$

with explicit error term.

We finally consider, in the case $\alpha \in (-\beta, 0)$, the convergence of the integral

$$\eta^{-\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v \xrightarrow{\eta \to 0} c_{\alpha,\beta} \int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} \, \mathrm{d}u.$$

Observe that the left hand side is

$$\eta^{-\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, \mathrm{d}v = c_{\alpha,\beta} \int_{\mathbb{R}^d} \Phi_\eta(u) |u|_\eta^{-d-\alpha} \, \mathrm{d}u.$$

The bound (1.11) implies that the integrand is uniformly integrable near zero:

$$\int_{|u|\le\epsilon} \Phi_\eta(u)|u|_\eta^{-d-\alpha}\,\mathrm{d}u \xrightarrow{\epsilon\to 0} 0$$

uniformly as $\eta\to 0$. On the region $|u|\ge R_0$ the integral bound (1.12) implies

$$\int_{|u|\ge R_0} |\Phi_\eta(u)|\,|u|_\eta^{-d-\alpha}\,\mathrm{d}u \xrightarrow{R_0\to\infty} 0$$

uniformly as $\eta\to 0$. We finally use the L^2-convergence $\Phi_\eta\to\Phi$ on $\{\epsilon\le|u|\le R_0\}$. □

6. Proof of the hypothesis for scattering equations

In this section we consider the scattering operator, written in the "f" or "h" scaling:

$$\begin{cases} \mathcal{L}f(v) = \displaystyle\int_{\mathbb{R}^d} b(v,v')\,\left[f(v')\,\mathcal{M}(v) - f(v)\,\mathcal{M}(v')\right]\,\mathrm{d}v', \\ Lh(v) = \displaystyle\int_{\mathbb{R}^d} b(v,v')\,\mathcal{M}(v')\,\left[h(v') - h(v)\right]\,\mathrm{d}v'. \end{cases}$$

We assume that b is C^1, that the operator conserves the local mass

$$\int_{\mathbb{R}^d} \left[\mathrm{b}(v,v') - \mathrm{b}(v',v)\right]\mathcal{M}(v')\,\mathrm{d}v' = 0$$

and that the *collision kernel* b and *collision frequency* $\nu(v) := \int_{\mathbb{R}^d} b(v,v')\,\mathcal{M}(v')\,\mathrm{d}v'$ satisfy, for some constant $\nu_0>0$,

$$\lfloor v\rceil^{-\beta} \lesssim \nu(v) \lesssim \lfloor v\rceil^{-\beta}, \qquad \lambda^\beta\nu(\lambda u) \sim_{\lambda\to\infty} \nu_0|u|^{-\beta} \quad \text{and} \quad \|b(v,\cdot)\|_\beta + \|b(\cdot,v)\|_\beta \lesssim \lfloor v\rceil^{-\beta}.$$

This includes $b(v,v') = \lfloor v\rceil^{-\beta}\lfloor v'\rceil^{-\beta}$ for any $\alpha+\beta>0$, $b(v,v') = \lfloor v-v'\rceil^{-\beta}$ when $\beta\ge 0$ and $\alpha>3\beta$, and even $b(v,v') = |v-v'|^{-\beta}$ when $\beta<0$ and $\alpha+\beta>0$.

6.1. Proof of Hypothesis 2.

Hypothesis 2 is standard and proved for instance in [21].

6.2. Proof of Hypothesis 3.

We perform the following calculations:

$$\begin{aligned}
\|L(\chi_R)\|_\beta^2 &= \int_{\mathbb{R}^d} \lfloor v\rceil^\beta |L(\chi_R)|^2 \mathcal{M}(v)\,\mathrm{d}v \\
&\le \int_{\mathbb{R}^d} \lfloor v\rceil^\beta \nu(v) \int_{\mathbb{R}^d} |\chi_R(v)-\chi_R(v')|^2\,\mathrm{b}(v,v')\,\mathcal{M}(v')\,\mathcal{M}(v)\,\mathrm{d}v'\,\mathrm{d}v \\
&\lesssim \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\chi_R(v)-\chi_R(v')|^2\,\mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\lesssim \iint_{\{|v|<R\}\times\mathbb{R}^d} \cdots + \iint_{\{|v|>R\}\times\mathbb{R}^d} \cdots \\
&\lesssim \iint_{\mathbb{R}^d\times\{|v'|>R\}} \mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v\,\mathrm{d}v' + \iint_{\{|v|>R\}\times\mathbb{R}^d} \mathrm{b}(v,v')\,\mathcal{M}(v)\,\mathcal{M}(v')\,\mathrm{d}v'\,\mathrm{d}v \\
&\lesssim \|\chi_R^c\mathcal{M}\|_{-\beta} \lesssim R^{-\frac{(\alpha+\beta)}{2}} \xrightarrow[R\to\infty]{} 0.
\end{aligned}$$

6.3. Proof of Hypothesis 4.

The eigenvalue problem can be written

$$\left(L^{*,+}\phi_\eta\right)(v) := \int_{\mathbb{R}^d} b(v',v)\mathcal{M}(v')\phi_\eta(v')\,\mathrm{d}v' = \left(\nu(v) - i\eta(v\cdot\sigma) - \mu(\eta)\lfloor v\rceil^{-\beta}\right)\phi_\eta(v)$$

with the normalization $\int_{\mathbb{R}^d}\mathcal{M}_\beta(v')\phi_\eta(v')\,\mathrm{d}v' = 1$. Observe first that Hypothesis 2 implies

$$\|\phi_\eta - 1\|_{-\beta}^2 \le \mu(\eta)\,\|\phi_\eta\|_{-\beta}^2$$

and thus, for η small enough

$$\|\phi_\eta\|_{-\beta}^2 \le \frac{\lambda}{\lambda-\mu(\eta)}$$

is uniformly bounded as $\eta \to 0$. Observe second that

$$\left|L^{*,+}(\phi_\eta)(v)\right| \leq \|b(\cdot,v)\|_\beta \, \|\phi_\eta\|_{-\beta} \lesssim \lfloor v \rceil^{-\beta}$$

which yields, for η small enough,

$$|\phi_\eta(v)| \lesssim \frac{\lfloor v \rceil^{-\beta}}{\left[(\nu(v) - \mu(\eta)\lfloor v \rceil^{-\beta})^2 + \eta^2 (v\cdot\sigma)^2\right]^{\frac12}} \lesssim \frac{1}{\lfloor v \rceil^\beta \nu(v) - \mu(\eta)} \lesssim 1,$$

i.e. ϕ_η is uniformly bounded in $L^\infty(\mathbb{R}^d)$ as $\eta \to 0$, and Hypothesis 4-(i) when $\alpha > 2+\beta$ follows.

The rescaled eigenvector Φ_η satisfies

$$\Phi_\eta(u) := \phi_\eta\left(\eta^{-\frac{1}{1+\beta}} u\right) = \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}} u\right)}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.$$

We turn to the case $\alpha \leq 2+\beta$. Estimate (1.11) in Hypothesis 4-(ii) follows from Φ_η uniformly bounded and for η small and $|u| \leq 1$ (using $|L^{*,+}(\phi_\eta)(v)| \lesssim \lfloor v \rceil^{-\beta}$),

$$|\operatorname{Im}\Phi_\eta(u)| = \left|\frac{(u\cdot\sigma)\left[\eta^{\frac{\beta}{1+\beta}} L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}} u\right)\right]}{\left(\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - \mu(\eta)|u|_\eta^{-\beta}\right)^2 + (u\cdot\sigma)^2}\right| \lesssim \frac{|u\cdot\sigma|}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right)} \lesssim |u|_\eta^{1+\beta}.$$

When $\alpha \in (-\beta, \beta]$, the integral moment bound (1.12) in Hypothesis 4-(ii)-(b) and (ii)-(c) follows from (for small η and large u and using again $|L^{*,+}(\phi_\eta)(v)| \lesssim \lfloor v \rceil^{-\beta}$)

$$|\Phi_\eta(u)| \lesssim \frac{1}{1 + |u|^\beta |u\cdot\sigma|}$$

which implies

$$\|\Phi_\eta\|_\beta^2 \lesssim \int_0^\pi \int_1^{+\infty} \frac{r^{-1-\alpha+\beta}}{1 + r^{2+2\beta}\cos\theta^2} \, \mathrm{d}r \, \mathrm{d}\theta < +\infty.$$

To prove the remaining points we use $L^*1 = 0$ to write

$$\begin{aligned}\Phi_\eta(u) - \frac{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right)}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}} &= \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}} u\right) - \eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right)}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}} \\ &= \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+}\left(\phi_\eta - 1\right)\left(\eta^{-\frac{1}{1+\beta}} u\right)}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.\end{aligned}$$

Since then

$$\left|\eta^{\frac{\beta}{1+\beta}} L^{*,+}\left(\phi_\eta - 1\right)\left(\eta^{-\frac{1}{1+\beta}} u\right)\right| \lesssim \eta^{\frac{\beta}{1+\beta}} \left\lfloor \eta^{-\frac{1}{1+\beta}} u \right\rceil^{-\beta} \|\phi_\eta - 1\|_{-\beta} \lesssim \sqrt{\mu(\eta)}\, \eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right),$$

we deduce

$$\left\| \frac{\Phi_\eta}{\Phi_{\eta,0}} - 1 \right\|_{L^\infty(\mathbb{R}^d)} \lesssim \sqrt{\mu(\eta)} \xrightarrow[\eta\to 0]{} 0$$

with the simpler function

$$\Phi_{\eta,0}(u) := \frac{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right)}{\eta^{\frac{\beta}{1+\beta}} \nu\left(\eta^{-\frac{1}{1+\beta}} u\right) - i(u\cdot\sigma) - \mu(\eta)|u|_\eta^{-\beta}}.$$

To prove the convergence of Φ_η it is thus enough to check the convergence of $\Phi_{\eta,0}$:

$$\lim_{\eta\to 0} \Phi_\eta(u) = \lim_{\eta\to 0} \Phi_{\eta,0}(u) = \frac{\nu_0}{\nu_0 - i|u|^\beta (u\cdot\sigma)} =: \Phi(u)$$

and in the case $\alpha = 2+\beta$ we also have

$$\Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0} \lambda^{-(1+\beta)} \frac{\nu_0|\lambda u|^\beta(\lambda u\cdot\sigma)}{\nu_0^2+|\lambda u|^{2\beta}(\lambda u\cdot\sigma)^2} = \nu_0^{-1}|u|^\beta(u\cdot\sigma),$$

and the corresponding diffusion coefficients are given in the statement of Corollary 1.5.

Moreover, since

$$\mathrm{Im}\,\Phi_{\eta,0}(u) := \frac{|u|_\eta^\beta \nu_\eta(u)|u|_\eta^\beta(u\cdot\sigma)}{\Big(|u|_\eta^\beta\nu_\eta(u)-\mu(\eta)\Big)^2+|u|_\eta^{2\beta}(u\cdot\sigma)^2},$$

and

$$\mathrm{Im}\,\Phi(u) := \frac{\nu_0|u|^\beta(u\cdot\sigma)}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2},$$

we deduce

$$\begin{aligned}\frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_{\eta,0}(u)} &= \frac{\nu_0}{|u|_\eta^\beta\nu_\eta(u)}\frac{|u|^\beta}{|u|_\eta^\beta}\frac{\Big(|u|_\eta^\beta\nu_\eta(u)-\mu(\eta)\Big)^2+|u|_\eta^{2\beta}(u\cdot\sigma)^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\\ &= (1+o(1))\frac{|u|^\beta}{|u|_\eta^\beta}.\end{aligned}$$

Since $|\mathrm{Im}\,\Phi_\eta - \mathrm{Im}\,\Phi_{\eta,0}| \le \sqrt{\mu(\eta)}\,|\mathrm{Im}\,\Phi_{\eta,0}|$,

$$\frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_\eta(u)} = \frac{\mathrm{Im}\,\Phi(u)}{\mathrm{Im}\,\Phi_{\eta,0}(u)}\frac{\mathrm{Im}\,\Phi_{\eta,0}(u)}{\mathrm{Im}\,\Phi(u)} = (1+o(1))\frac{|u|^\beta}{|u|_\eta^\beta}$$

which proves the remaining part of Hypothesis 4-(ii)-(a).

Finally, we check that (1.13) in Hypothesis 4-(ii)-(c) is satisfied when $\alpha\in(-\beta,0)$:

$$\int_{\mathbb{R}^d}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u = \int_{\mathbb{R}^d}\mathrm{Re}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u = \int_{\mathbb{R}^d}\frac{\nu_0^2}{\nu_0^2+|u|^{2\beta}(u\cdot\sigma)^2}\frac{1}{|u|^{d+\alpha}}\,\mathrm{d}u > 0$$

by inspection and we also prove that it is finite:

$$\begin{aligned}\int_{\mathbb{R}^d}\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u &= \int_{r\ge0}\left(\int_{\theta\in(-\frac{\pi}{2},\frac{\pi}{2})}\frac{\nu_0^2|\sin^{d-2}\theta|}{\nu_0^2+r^{2(1+\beta)}\sin^2\theta}\,\mathrm{d}\theta\right)\frac{1}{r^{1+\alpha}}\,\mathrm{d}r\\ &= \int_{r=0}^1\cdots + \int_{r\ge1}\int_{|\theta|\in(\frac{\pi}{4},\frac{\pi}{2})}\cdots + \int_{r\ge1}\int_{|\theta|\le\frac{\pi}{4}}\cdots =: I_1+I_2+I_3.\end{aligned}$$

Then $I_1 \lesssim \int_{r\sim0} r^{-1-\alpha}\,\mathrm{d}r < +\infty$ and $I_2 \lesssim \int_{r\ge1} r^{-3-\alpha-2\beta}\,\mathrm{d}r < +\infty$ and finally

$$\begin{aligned}I_3 &\lesssim \int_{r\ge1}\left(\int_{|z|\le\frac{1}{\sqrt{2}}}\frac{\nu_0^2}{\nu_0^2+r^{2(1+\beta)}z^2}\,\mathrm{d}z\right)\frac{1}{r^{1+\alpha}}\,\mathrm{d}r\\ &\lesssim \int_{r\ge1}\left(\int_{|w|\le\frac{r^{1+\beta}}{\sqrt{2}}}\frac{\nu_0^2}{\nu_0^2+w^2}\,\mathrm{d}w\right)\frac{1}{r^{2+\alpha+\beta}}\,\mathrm{d}r < +\infty.\end{aligned}$$

7. Proof of the hypothesis for kinetic Fokker-Planck equations

In this section we have $\beta = 2$, $\mathcal{M}$ given by Hypothesis 1, and the operators

$$\mathcal{L}(f) := \nabla_v\cdot\left(\mathcal{M}\nabla_v\left(\frac{f}{\mathcal{M}}\right)\right) \quad\text{and}\quad Lh := \mathcal{M}^{-1}\nabla_v\cdot(\mathcal{M}\nabla_v h),$$

are self-adjoint in respectively $L^2_v(\mathcal{M}^{-1})$ and $L^2_v(\mathcal{M})$.

7.1. **Proof of Hypothesis 2.** This hypothesis is:

$$\int_{\mathbb{R}^d} |\nabla_v h|^2 \, \mathcal{M}(v) \, \mathrm{d}v \geq \lambda \, \|h - \mathcal{P}h\|_{-2} \text{ with } \mathcal{P}h := \int_{\mathbb{R}^d} h(v') \lfloor v' \rceil^{-2} \mathcal{M}(v') \, \mathrm{d}v'$$

for some $\lambda > 0$ (recall that $\int \lfloor \cdot \rceil^{-2} \mathcal{M} = 1$ as per Hypothesis 1). It is a form of the so-called *Hardy-Poincaré inequality*, see for instance [8, equation (1)] where references are collected for proving it for $d \geq 3$ and $\alpha > -2$, [23, Corollary 1] and [9, Appendix A] where it is proved in all dimensions $d \geq 1$ under the condition $d + \alpha > 0$ (for instance the "α" in [23, Corollary 1] corresponds to our "$-(d+\alpha)$"). Note that the case when $d \geq 3$ and $\alpha \in (-d, -2)$ would correspond in [23, 9] to situations where the Hardy-Poincaré inequality holds without the need of the zero-average condition. These cases are however excluded by our assumption $\alpha \geq 0$.

7.2. **Proof of Hypothesis 3.** It is proved via the following computation

$$\begin{aligned}
\|L(\chi_R)\|_2^2 &= \int_{\mathbb{R}^d} |\nabla \cdot (\mathcal{M} \nabla_v \chi_R)|^2 \, \lfloor \cdot \rceil^2 \, \frac{\mathrm{d}v}{\mathcal{M}} \\
&= \int_{\mathbb{R}^d} \left| \Delta \chi_R + \frac{\nabla_v \mathcal{M}}{\mathcal{M}} \cdot \nabla \chi_R \right|^2 \lfloor \cdot \rceil^2 \, \mathcal{M} \, \mathrm{d}v \\
&= \int_{B_{2R} \setminus B_R} \left| \Delta \chi_R + \frac{\nabla_v \mathcal{M}}{\mathcal{M}} \cdot \nabla \chi_R \right|^2 \lfloor \cdot \rceil^2 \, \mathcal{M} \, \mathrm{d}v \\
&= \int_{B_{2R} \setminus B_R} \lfloor \cdot \rceil^{-2} \, \mathcal{M} \, \mathrm{d}v \lesssim_\chi R^{-(2+\alpha)} = R^{-(\beta+\alpha)}.
\end{aligned}$$

7.3. **Proof of Hypothesis 4.** The equation satisfied by Φ_η is

$$-|u|_\eta^2 \Delta_u \Phi_\eta + (d+\alpha) u \cdot \nabla_u \Phi_\eta - i(u \cdot \sigma)|u|_\eta^2 \Phi_\eta = \mu(\eta) \Phi_\eta. \tag{7.1}$$

We now prove (1.11) in Hypothesis 4, but estimate first the non-rescaled eigenfunction.

Lemma 7.1. *The unique solution to*

$$-L\phi_\eta - i\eta(v \cdot \sigma)\phi_\eta = \mu(\eta) \lfloor v \rceil^{-2} \phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) \, \lfloor v \rceil^{-2} \, \mathcal{M}(v) \, \mathrm{d}v = 1$$

satisfies for any $R \geq 1$

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1 \quad \text{and} \quad \|\operatorname{Im} \phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta, \mu(\eta))$$

with constants depending only on R *but uniform in* $\eta \to 0$.

Proof of Lemma 7.1. As for the scattering equation, Hypothesis 2 implies, for η small enough

$$\lambda \, \|\phi_\eta - 1\|_{-2}^2 \leq \mu(\eta) \, \|\phi_\eta\|_{-2} \quad \Rightarrow \quad \|\phi_\eta\|_{-2}^2 \leq \frac{\lambda}{\lambda - \mu(\eta)} \lesssim 1.$$

The elliptic regularity of the operator $L = \Delta - (d+\alpha)\langle v \rangle^{-2} v \cdot \nabla_v$, with uniform ellipticity constant, then classically implies that

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1.$$

Since $\mathcal{P}\phi_\eta = 1$ in the decomposition $\phi_\eta = \mathcal{P}\phi_\eta + \mathcal{P}^\perp \phi_\eta$, one deduces

$$\|\operatorname{Im} \phi_\eta\|_{-2} \leq \left\|\mathcal{P}^\perp \phi_\eta\right\|_{-2} \lesssim \mu(\eta)$$

and the imaginary part satisfies the equation

$$-L(\operatorname{Im} \phi_\eta) - \mu(\eta) \lfloor v \rceil^{-2} \operatorname{Im} \phi_\eta = \eta (v \cdot \sigma) \operatorname{Re} \phi_\eta.$$

Therefore the elliptic regularity combined with the integral bound on $\operatorname{Im} \phi_\eta$ and the bound $\|\eta(v \cdot \sigma) \operatorname{Re} \phi_\eta\|_{L^2(B(0,R))} \lesssim \eta$ on the right hand side implies that

$$\|\operatorname{Im} \phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta, \mu(\eta))$$

which concludes the proof. □

The following lemma proves (1.11).

Lemma 7.2. *There are $\eta_1 \in (0,\eta_0)$ small enough and $\delta \in (0, \min(4-\alpha, 3))$ and C large enough so that*

$$\forall\, \eta \in (0,\eta_1),\ \forall\, u \in \mathbb{R}^d, \qquad |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)}, \quad |\operatorname{Im}\Phi_\eta(u)| \lesssim |u|_\eta^{\min(2+\alpha,3)-\delta}.$$

Proof of Lemma 7.2. Multiply (7.1) by $\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}$ and take the real part:

$$-|u|_\eta^2 \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\Delta_u\Phi_\eta\right) + (d+\alpha)u\cdot \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\nabla_u\Phi_\eta\right) = \mu(\eta)|\Phi_\eta|.$$

Since

$$\nabla_u|\Phi_\eta| = \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\nabla_u\Phi_\eta\right), \qquad \Delta_u|\Phi_\eta| \geq \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\Delta_u\Phi_\eta\right),$$

one gets

$$-|u|_\eta^2\Delta_u|\Phi_\eta| + (d+\alpha)u\cdot\nabla_u|\Phi_\eta| - \mu(\eta)|\Phi_\eta| \leq 0.$$

Then observe that the real function $F(u) = |u|_\eta^{C\mu(\eta)}$ satisfies for $|u| \geq A\eta^{\frac{1}{3}}$ with A large:

$$\begin{aligned}
&-|u|_\eta^2\Delta_u F + (d+\alpha)u\cdot\nabla_u F - \mu(\eta)F \\
&\geq \mu(\eta)F\left[-Cd\frac{|u|_\eta^2}{|u|^2} - C(C\mu(\eta)-2)\frac{|u|_\eta^2}{|u|^2} + C(d+\alpha) - 1\right] \\
&\geq \mu(\eta)F\left[-Cd(1+\varepsilon) - C(C\mu(\eta)-2)(1+\varepsilon) + C(d+\alpha) - 1\right] \\
&\geq \mu(\eta)F\left[C(2+\alpha) - 1 - \varepsilon C(d-2) - C^2\mu(\eta)(1+\varepsilon)\right]
\end{aligned}$$

where we have used that $\frac{|u|_\eta^2}{|u|^2} \leq 1+\varepsilon$ with ε small for $|u| \geq A\eta^{\frac{1}{3}}$ when A large enough. The right hand side is thus positive for C large enough and ε and η small enough, since $2+\alpha > 0$:

$$\forall\, |u| \geq A\eta^{\frac{1}{3}}, \quad -|u|_\eta^2\Delta_u F + (d+\alpha)u\cdot\nabla_u F - \mu(\eta)F \geq 0$$

i.e. F is a super-solution in this region. Moreover the previous lemma shows that

$$\sup_{|u|\leq A\eta^{\frac{1}{3}}} |\Phi_\eta(u)| \leq \|\phi_\eta\|_{L^\infty(B(0,A))} \lesssim_A 1$$

and we can therefore compare Φ_η and F on the ball $|u| \leq A\eta^{\frac{1}{3}}$ with a bound uniform in η. The maximum principle thus implies that $|\Phi_\eta| \lesssim |u|_\eta^{C\mu(\eta)}$ for all $|u| \geq A\eta^{\frac{1}{3}}$ with a bound uniform in η. Finally, since $\eta^{C\mu(\eta)} \sim 1$ as $\eta \to 0$, this bound extends to any $u \in \mathbb{R}^d$ up to enlarging the comparison constant (independently of $\eta \to 0$).

Take then the imaginary part of equation (7.1)

$$-|u|_\eta^2\Delta_u \operatorname{Im}\Phi_\eta + (d+\alpha)u\cdot\nabla_u \operatorname{Im}\Phi_\eta - \mu(\eta)\operatorname{Im}\Phi_\eta = (u\cdot\sigma)|u|_\eta^2 \operatorname{Re}\Phi_\eta,$$

multiply by $\frac{\operatorname{Im}\Phi_\eta}{|\operatorname{Im}\Phi_\eta|}$ and use the previous estimate to get for $|u| \geq A\eta^{\frac{1}{3}}$

$$-|u|_\eta^2\Delta_u|\operatorname{Im}\Phi_\eta| + (d+\alpha)u\cdot\nabla_u|\operatorname{Im}\Phi_\eta| - \mu(\eta)|\operatorname{Im}\Phi_\eta| \leq |u|_\eta^{3+C\mu(\eta)}.$$

Define $G(u) := |u|_\eta^{\mathsf{e}}$ with $\mathsf{e} := 2 + \min(\alpha,1) - \delta$ and compute for $|u| \in [A\eta^{\frac{1}{3}}, 1]$:

$$-|u|_\eta^2\Delta_u G + (d+\alpha)u\cdot\nabla_u G - \mu(\eta)G \quad \geq G\left[\mathsf{e}\delta - \mu(\eta) - \mathcal{O}\left(A^{-2}\right)\right] \gtrsim G \gtrsim |u|_\eta^{3+C\mu(\eta)}$$

for A large enough and η small enough. The maximum principle then shows that $|\operatorname{Im}\Phi_\eta| \lesssim |u|_\eta^{\mathsf{e}}$ on $|u| \in [A\eta^{\frac{1}{3}}, 1]$ by comparing $\operatorname{Im}\Phi_\eta$ and G on $|u| = A\eta^{\frac{1}{3}}$ thanks to the second inequality in the previous lemma. Again the bound extends to any $|u| \leq A\eta^{\frac{1}{3}}$ using the second inequality in the previous lemma, since $\max(\eta, \mu(\eta)) \lesssim \eta^{\frac{\mathsf{e}}{3}}$ uniformly as $\eta \to 0$ (examining separately the cases $\alpha \in [0,1]$ and $\alpha \in (1,4)$). □

The next lemma allows to prove the integral moment estimate (1.12) in Hypothesis 4-(iv).

Lemma 7.3. *There is* $\mathsf{g} > 0$ *such that for any* $q \geq -2$ *and* $G, H \in L^2(\lfloor u \rceil^{q-d-\alpha})$ *such that*

$$-|u|_\eta^2 \Delta_u H + (d+\alpha) u \cdot \nabla_u H - i(u\cdot\sigma)|u|_\eta^2 H = G, \tag{7.2}$$

the following gain of decay at infinity holds

$$\int_{\mathbb{R}^d} |H(u)|^2 \lfloor u \rceil^{q+\mathsf{g}-d-\alpha} \, \mathrm{d}u \lesssim_{q,\zeta} \int_{\mathbb{R}^d} |G(u)|^2 \lfloor u \rceil^{q-d-\alpha} \, \mathrm{d}u + \int_{\mathbb{R}^d} |H(u)|^2 \lfloor u \rceil^{q-d-\alpha} \, \mathrm{d}u.$$

Proof of Lemma 7.3. Consider a real-valued smooth function $\chi_0(u)$ that is zero on $|u| \leq \frac{1}{2}$ and equal to 1 on $|u| \geq 1$, and integrate (7.2) against $\overline{H} \chi_0^2 |u|_\eta^{q-d-\alpha}$ and take the real part:

$$\begin{aligned}
&\int_{|u|\geq 1} |u|_\eta^2 \, |\nabla H(u)|^2 \, |u|_\eta^{q-d-\alpha} \, \mathrm{d}u \\
&\lesssim \int_{\mathbb{R}^d} \left(\chi_0^2 \, |H|^2 + \left|\Delta\left(\chi_0^2\right)\right| |u|_\eta^2 \, |H|^2 + |\nabla(\chi_0)| \, |u|_\eta |H|^2 + \chi_0^2 \, |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u \\
&\lesssim \int_{|u|\geq \frac{1}{2}} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u.
\end{aligned}$$

Integrate then (7.2) against $\overline{H}(u\cdot\sigma)\chi_1^2 |u|_\eta^{q-d-\alpha-1}$ where χ_1 is a real-valued smooth function that is zero on $|u| \leq 1$ and equal to 1 on $|u| \geq 2$, and take the imaginary part:

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u\cdot\sigma)^2 \chi_1(u)^2 \, |H(u)|^2 \, |u|_\eta^{1+q-d-\alpha} \, \mathrm{d}u \\
&\lesssim \int_{|u|\geq 1} |u|_\eta^2 \, |\nabla H(u)|^2 \, |u|_\eta^{q-d-\alpha} \, \mathrm{d}u + \int_{|u|\geq 1} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u \\
&\lesssim \int_{|u|\geq \frac{1}{2}} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u
\end{aligned}$$

where we have used the previous real part estimate in the last line. This yields

$$\int_{\mathbb{R}^d} (u\cdot\sigma)^2 |u| \, |H(u)|^2 \, \lfloor u \rceil^{q-d-\alpha} \, \mathrm{d}u \lesssim \int_{\mathbb{R}^d} \left(|H|^2 + |G|^2 \right) \lfloor u \rceil^{q-d-\alpha} \, \mathrm{d}u.$$

This first estimate improves the decay at infinity outside a cone around $u \perp \sigma$. We now use the ellipticity of the equation to control this latter region. The operator writes $\mathsf{L}_\eta = -|u|_\eta^{d+\alpha} \nabla_u \cdot [|u|_\eta^{-d-\alpha} \nabla_u]$ and we deduce by simple commutator estimates that

$$\int_{|u|\geq 2} \left| \nabla_u \left(H(u) |u|^{\frac{q-d-\alpha+2}{2}} \right) \right|^2 \mathrm{d}u \lesssim \int_{|u|\geq 1} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u.$$

Consider first the case $d > 2$. The Caffarelli-Kohn-Nirenberg inequality yields

$$\left\| H(\cdot) |\cdot|^{\frac{q-d-\alpha+2}{2}} 1_{|\cdot|\geq 2} \right\|_{L^{\frac{2d}{d-2}}}^2 \lesssim \int_{|u|\geq 1} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u.$$

Consider the cone $\mathcal{C} := \{ |\frac{u}{|u|} \cdot \sigma| \leq \lfloor u \rceil^{-\frac{\delta}{2}}, |u| \geq 2 \}$ for some $\delta > 0$, and a gain of weight $\lfloor u \rceil^{\mathsf{g}}$ for some $\mathsf{g} > 0$ to be chosen later. The Hölder inequality then yields

$$\int_{\mathcal{C}} |H(u)|^2 \, |u|^{q-d-\alpha+\mathsf{g}} \, \mathrm{d}u \leq \left\| H(u) |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \left(\int_{\mathcal{C}} |u|^{\frac{d(\mathsf{g}-2)}{2}} \, \mathrm{d}u \right)^{\frac{2}{d}}.$$

The extra volume integral may be estimated as follows using spherical coordinates,

$$\int_{\mathcal{C}} |u|^{\frac{d(\mathsf{g}-2)}{2}} \, \mathrm{d}u \lesssim \int_1^{+\infty} r^{d-1} \langle r \rangle^{\frac{d(\mathsf{g}-2)}{2}} \langle r \rangle^{-\frac{\delta}{2}} \, \mathrm{d}r \lesssim \int_1^{+\infty} r^{-1+\frac{d\mathsf{g}}{2}-\frac{\delta}{2}} \, \mathrm{d}r$$

which is finite as soon as $\mathsf{g} < \frac{\delta}{d}$ (which defines and restricts δ). Outside the cone we use the first estimate:

$$\int_{\mathcal{C}^c \cap \{|\cdot|\geq 2\}} |H(u)|^2 \, |u|^{q-d-\alpha+\mathsf{g}} \, \mathrm{d}u \lesssim \int_{\mathbb{R}^d} |H(u)|^2 \, (u\cdot\sigma)^2 |u|^{-2+\delta+q-d-\alpha+\mathsf{g}} \, \mathrm{d}u$$

which is controlled as soon as $\mathsf{g} \leq 3 - \delta$. The constraints are compatible for $\mathsf{g} \in (0, \frac{3}{d+1})$.

In the case $d = 1$, the gain of decay is immediate from the first estimate alone. In the case $d = 2$, we follow a similar argument but replace the Caffarelli-Kohn-Nirenberg inequality with the Onofri inequality:

$$\left\| H(u)|u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|_{L^{2p}}^2 \lesssim_p \int_{|u|\geq 1} \left(|H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} \, \mathrm{d}u$$

for any $p < \infty$. The Hölder inequality then gives

$$\int_{\mathcal{C}} |H(u)|^2 \, |u|^{q-d-\alpha+\mathsf{g}} \, \mathrm{d}u \leq \left\| H(u)|u|^{\frac{q-d-\alpha+2}{2}} 1_{|u|\geq 2} \right\|_{L^{2p}(\mathbb{R}^d)}^2 \left(\int_{\mathcal{C}} |u|^{q(\mathsf{g}-2)} \, \mathrm{d}u \right)^{\frac{1}{q}}$$

where $q = \frac{1}{1-1/p}$ is the exponent conjugate to p. The conclusion follows as before by taking p large enough. □

We now deduce the moment bound (1.12) from Lemma 7.3. Observe first that the pointwise bound $|\Phi_\eta(u)| \lesssim |u|^{C\mu(\eta)}$ proved in the Lemma 7.2 implies that

$$\int_{|u|\geq 1} |\Phi_\eta(u)|^2 |u|^{q-d-\alpha} \, \mathrm{d}u < +\infty$$

for $q = -2$ and η small enough with bound uniform in η, since $\alpha + 2 > 0$. We then repeatedly apply Lemma 7.3 with $H = \Phi_\eta$ and $G = \mu(\eta)\Phi_\eta$ to obtain that Φ_η decays faster than any polynomial at infinity, with constants uniform in η (note that g is independent of q in the lemma).

To prove the asymptotic convergence we consider the solution $\Phi : \mathbb{R}^d \to \mathbb{C}$ to

$$-|u|^2 \Delta\Phi + (d+\alpha) u \cdot \nabla\Phi + i(u\cdot\sigma)\Phi = 0, \quad \Phi(0) = 1, \quad \Phi \in C^0, \quad \Phi \in L^2(\langle u \rangle^\infty). \tag{7.3}$$

This solution exists by weak limit $\Phi_\eta \to \Phi$. The regularity of Φ follows from the equation outside of zero, and the fact that Φ goes to 1 as u goes to zero comes from the fact that the normalisation of ϕ_η yields $\int_{\mathbb{R}^d} |\Phi(u) - 1|^2 |u|^{-d-2-\alpha} \, \mathrm{d}u < \infty$ (a different conclusion near zero would make the latter integral infinite). The solution Φ is unique since integrating the equation for the difference of two solutions $\Phi := \Phi_1 - \Phi_2$ against $\overline{\Phi}|u|^{-d}$ yields $\int_{\mathbb{R}^d} |\nabla\Phi|^2 |u|^{2-d} \, \mathrm{d}u = 0$ with all boundary terms near zero behaving like $|\Phi|^2(u)$, $u \sim 0$. Note that this solution can computed semi-explicitely as an entire series $\Phi(u) = \sum_{k,l\geq 0} a_{k,l} |u|^{2k} (u\cdot\sigma)^l$ by solving the two-parameters induction (see the discussion later and Figure 3).

With the solution Φ above at hand, then the convergence $\Phi_\eta \to \Phi$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d \backslash \{0\})$ follows easily from the bounds established and the convergence of the coefficients of the equation satisfied by Φ_η: one can prove that $\eta \mapsto \Phi_\eta$ is Cauchy in L^2 on any such compact set as $\eta \to 0$, and such convergence has a polynomial rate and is uniform on any compact set in $\mathbb{R}^d \setminus \{0\}$.

We now prove Hypothesis 4-(ii)-(a). The equation for $W_\eta := \Phi_\eta - \Phi$ is

$$-|u|_\eta^2 \Delta_u W_\eta + (d+\alpha) u \cdot \nabla_u W_\eta - i(u\cdot\sigma)|u|_\eta^2 W_\eta - \mu(\eta) \, W_\eta = \eta^{\frac{2}{3}} (d+\alpha) \frac{u}{|u|^2} \cdot \nabla_u \Phi + \mu(\eta)\Phi.$$

We derive a bound on $\nabla_u \Phi$. For this, differentiate the limit equation for Φ,

$$-|u|^2 \Delta\left(\nabla\Phi\right) + (d+\alpha) u \cdot \nabla\left(\nabla\Phi\right) + (d+\alpha)\nabla\Phi - 2(\Delta\Phi)u = i(u\cdot\sigma)|u|^2 \nabla\Phi + i\nabla\left((u\cdot\sigma)|u|^2\right)\Phi.$$

Test against $\frac{\overline{\nabla\Phi}}{|\nabla\Phi|}$, use

$$\nabla|\nabla\Phi| = \mathrm{Re}\left(\frac{\overline{\nabla\Phi}}{|\nabla\Phi|}\nabla\nabla\Phi\right), \qquad \mathrm{Re}\left(\frac{\overline{\nabla\Phi}}{|\nabla\Phi|}\Delta\left(\nabla\Phi\right)\right) \leq \Delta\left(|\nabla\Phi|\right),$$

$$u \cdot \mathrm{Re}\left(\Delta\Phi \frac{\overline{\nabla\Phi}}{|\nabla\Phi|}\right) = u \cdot \nabla\left(|\nabla\Phi|\right), \qquad |\nabla\left((u\cdot\sigma)|u|^2\right)| \leq 3|u|^2,$$

and take the real part to get

$$-|u|^2 \Delta\left(|\nabla\Phi|\right) + (d+\alpha-2) u \cdot \nabla\left(|\nabla\Phi|\right) + (d+\alpha)\,|\nabla\Phi| \leq 3|u|^2 \|\Phi\|_\infty.$$

The comparison principle (with the same sort of computations as for Lemma 7.2) then yields that $|\nabla\Phi| \lesssim |u|^2$ since we know from the equation that $\nabla\Phi(0) = 0$. As a consequence,

$$-|u|_\eta^2 \Delta_u |W_\eta| + (d+\alpha) u \cdot \nabla_u |W_\eta| - \mu(\eta)|W_\eta| \lesssim \eta^{\frac{2}{3}}|u| + \mu(\eta) \lesssim \eta^{\frac{2}{3}}|u|_\eta,$$

since Φ is uniformly bounded. From this, one deduces $|W_\eta| \lesssim \eta^{\frac{2}{3}}|u|_\eta$ (since W_η is of order η near zero). This implies the hypothesis since then, recalling $\alpha = 4$ and $\beta = 2$,

$$\left| \int_{1 \geq |u| \geq \eta^{\frac{1}{3}}} (u\cdot\sigma) \Big[\operatorname{Im} \Phi_\eta(u) - \operatorname{Im} \Phi(u) \Big] |u|_\eta^{-d-4} \, \mathrm{d}u \right|$$
$$\leq \int_{1 \geq |u| \geq \eta^{\frac{1}{3}}} \eta^{\frac{2}{3}} |u|_\eta^{-d-2} \, \mathrm{d}u = \int_{\eta^{-\frac{1}{3}} \geq |u| \geq 1} \frac{\eta^{\frac{d+2}{3}} \, \mathrm{d}u}{\eta^{\frac{d+2}{3}} |1+|w|^2|^{\frac{d+2}{2}}} \lesssim 1.$$

We now prove the second part of Hypothesis 4-(ii)-(a), regarding the existence of a scaling limit of Φ. The rescaled limit Φ satisfies $\Phi(0) = 1$ and the Schrödinger equation

$$-|u|^2 \Delta_u \Phi + (d+\alpha) u \cdot \nabla_u \Phi - i(u\cdot\sigma)|u|^2 \Phi = 0.$$

Therefore, $\Omega_\lambda(u) := \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^3}$ satisfies (using that Φ is continuous at $\Phi(0) = 1$)

$$-|u|^2 \Delta_u \Omega_\lambda(u) + (d+\alpha) u \cdot \nabla_u \Omega_\lambda(u) = (u\cdot\sigma)|u|^2 \operatorname{Re}\Phi(\lambda u) \xrightarrow[\lambda \to 0]{} (u\cdot\sigma)|u|^2.$$

Since the limit equation $-|u|^2 \Delta\Omega + (d+\alpha) u \cdot \nabla\Omega = (u\cdot\sigma)|u|^2$ has unique continuous solution satisfying $\Omega(0) = 0$ given by $\Omega(u) = \frac{(u\cdot\sigma)|u|^2}{d+8}$, elliptic estimates imply $\Omega_\lambda \to \Omega$ in $L^1(\mathbb{S}^{d-1})$.

Finally let us explain why the estimate (1.13) is **not** satisfied for the Fokker-Planck operator when $\alpha \in (-\beta, 0)$, which explains why our fractional diffusive limit does not apply in this case. Let us prove that when $\alpha \in (-\beta, 0)$ the solution to (7.3) satisfies $\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha} \, \mathrm{d}u = 0$, hence violating (1.13). To prove the latter, it is enough to construct $\mathfrak{G} : \mathbb{R}^d \to \mathbb{C}$ that is C^2 so that $\mathfrak{G}\Phi$ and $\Phi\nabla\mathfrak{G}$ decay faster than any polynomials at infinity,

$$-|u|^{d+\alpha} \nabla \cdot \Big[|u|^{-(d+\alpha)} \nabla \left(|u|^2 \mathfrak{G} \right) \Big] - i(u\cdot\sigma)|u|^2 \mathfrak{G} = 1 \tag{7.4}$$

and $\mathfrak{G}(0)$ and $\nabla\mathfrak{G}(0) = 0$. Then the integration by parts are justifed near zero and infinity and

$$\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha} \, \mathrm{d}u = \int_{\mathbb{R}^d} \Phi(u) \left\{ -|u|^{d+\alpha} \nabla \cdot \Big[|u|^{-(d+\alpha)} \nabla \left(|u|^2 \mathfrak{G} \right) \Big] - i(u\cdot\sigma)|u|^2 \mathfrak{G} \right\} |u|^{-d-\alpha} \, \mathrm{d}u$$
$$= \int_{\mathbb{R}^d} \left\{ -|u|^2 \Delta_u \Phi + (d+\alpha) u \cdot \nabla_u \Phi - i(u\cdot\sigma)|u|^2 \Phi \right\} \mathfrak{G}(u) |u|^{-d-\alpha} \, \mathrm{d}u = 0.$$

Define a priori $\mathfrak{G}(u) = \sum_{k,l \geq 0} a_{k,l} B_{k,l}(u)$ with $B_{k,l}(u) := |u|^{2k} (u\cdot\sigma)^l$ with $a_{0,l} = 0$ for all $l \in \mathbb{N}$, $a_{k \neq 1,0} = 0$ and $a_{1,0} = 1$, and solve a priori the equation (7.4) to get the sufficient condition

$$[2(k+1)(\alpha - 2l - 2k) + (d+\alpha) l] \, a_{k,l} = i a_{k-1,l-1} + l(l-1) a_{k-1,l+2}.$$

This relation forms a triangle in the (l,k) quadrant with horizontal base, and given the conditions above one can solve by induction on l for each k and then on k. Moreover one can prove by induction (only large k, l matter) that $|a_{k.l}| \lesssim \frac{1}{k!l!}$ so that $\mathfrak{G}$ is well-defined, see Figure 3.

Finally $\mathfrak{H}(u) := |u|^2 \mathfrak{G}(u)$ satisfies, arguing as before, $-|u|^{d+\alpha} \nabla \cdot [|u|^{-(d+\alpha)} \nabla (|\mathfrak{H}|)] \leq 1$. Then $|\cdot|^{2+\alpha+0}$ provides a super-solution at infinity, and $\mathfrak{G}$ bounded for u near zero, so this proves that $\mathfrak{G}\Phi$ decays faster than any polynomial (see the discussion following Lemma 7.3 giving the decay of Φ). Similar arguments can be performed on $|\nabla\mathfrak{H}|$ and $(\nabla\mathfrak{G})\Phi$ by differentiating (7.4).

8. Proof of the hypothesis for kinetic Lévy-Fokker-Planck equations

In this section, we consider, given $s \in (\frac{1}{2}, 1)$ and $\mathcal{M}$ is given by Hypothesis 1, the operator

$$\mathcal{L}(f) := \Delta_v^s f + \nabla_v \cdot (U f).$$

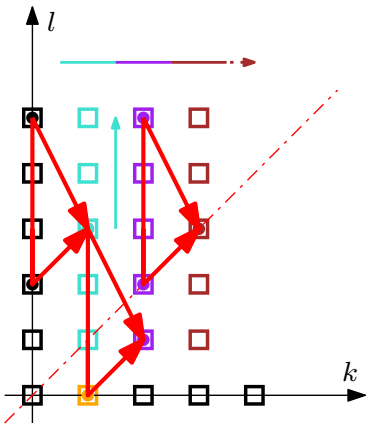

FIGURE 3. The black squares are zero values given to start the induction. The orange square has value 1. The black values for $k = 0$ allow to compute the turquoise blue squares (that are all zero then). The turquoise blue squares together with the (non-zero) orange square give the values held by the brown squares, etc. On each column, values are computed from bottom to top, sliding the triangular red planer on the previous column.

The fractional Laplacian is defined as in (1.5) but we use the equivalent definition (see for example [20] and the references therein)

$$\Delta_v^s f(v) := -C_{d,s} \int_{\mathbb{R}^d} \frac{\big[f(v) - f(v')\big]}{|v - v'|^{d+2s}} \, \mathrm{d}v' \quad \text{with} \quad C_{d,s} := \frac{4^s \Gamma\left(\frac{d}{2} + s\right)}{\pi^{\frac{d}{2}} |\Gamma(-s)|}.$$

The drift force U solves $\Delta_v^s \mathcal{M} + \nabla_v \cdot (U\,\mathcal{M}) = 0$. We restrict ourselves to $\alpha > s$. [13, Proposition 7] shows that the explicit radial solution U to the previous equation satisfies $U(v) = \mathbb{U}(v) \lfloor v \rceil^{-\beta} v$ with $\beta := 2s - \alpha$ and $\mathbb{U}$ a uniformly positive function bounded from above. The operator L is

$$Lh := \mathcal{M}^{-1} \Delta_v^s (\mathcal{M} h) + \mathcal{M}^{-1} \nabla_v \cdot (U\,\mathcal{M} h) = \mathcal{M}^{-1} \Big[\Delta_v^s (\mathcal{M} h) - (\Delta_v^s \mathcal{M})\, h \Big] + U \cdot \nabla_v h.$$

8.1. **Proof of Hypothesis 2.** This hypothesis is implied by the fractional Hardy-Poincaré inequalities proved in [13] and earlier in [51]:

Proposition 8.1 ([13, 51])**.** *Let $d \geq 1$, $s \in (0,1)$, $\alpha > s$ and $\beta := 2s - \alpha$. Then there is $\lambda > 0$ (depending on s) such that*

$$- \operatorname{Re} \big\langle Lh, h \big\rangle \geq \lambda \| h - \mathcal{P} h \|_{-\beta}^2.$$

Proof. Compute

$$\begin{aligned}
- \operatorname{Re} \big\langle Lh, h \big\rangle &= - \operatorname{Re} \int_{\mathbb{R}^d} \Big[\Delta_v^s (\mathcal{M} h) + \nabla_v \cdot (U\,\mathcal{M} h) \Big] \overline{h} \, \mathrm{d}v' \\
&= - \operatorname{Re} \int_{\mathbb{R}^d} \big(\Delta_v^s \overline{h} \big)\, h\, \mathcal{M} \, \mathrm{d}v + \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} U \cdot \nabla_v \big(|h|^2 \big)\, \mathcal{M} \, \mathrm{d}v \\
&= - \operatorname{Re} \int_{\mathbb{R}^d} \big(\Delta_v^s \overline{h} \big)\, h\, \mathcal{M} \, \mathrm{d}v - \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} \nabla_v \cdot (U\,\mathcal{M})\, |h|^2 \, \mathrm{d}v \\
&= - \operatorname{Re} \int_{\mathbb{R}^d} \big(\Delta_v^s \overline{h} \big)\, h\, \mathcal{M} \, \mathrm{d}v + \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} (\Delta_v^s \mathcal{M})\, |h|^2 \, \mathrm{d}v \\
&= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|h - h'|^2}{|v - v'|^{d+2s}} \, \mathcal{M} \, \mathrm{d}v \, \mathrm{d}v'
\end{aligned}$$

and thus

$$- \operatorname{Re} \big\langle Lh, h \big\rangle = \frac{C_{d,s}}{4} \int_{\mathbb{R}^d} \frac{|h - h'|^2}{|v - v'|^{d+2s}} \, (\mathcal{M} + \mathcal{M}') \, \mathrm{d}v \, \mathrm{d}v'.$$

Note that there is $\kappa > 0$ such that

$$\forall\, (v, v') \in \mathbb{R}^d \times \mathbb{R}^d, \quad \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta} \mathcal{M}\, \mathcal{M}' \leq \kappa\, \frac{\mathcal{M} + \mathcal{M}'}{|v - v'|^{d+2s}},$$

by matching the asymptotics at large v and v'. Hence we get that

$$-\operatorname{Re} \left\langle Lh, h \right\rangle \gtrsim \int_{\mathbb{R}^d} |h - h'|^2 \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta} \mathcal{M}\, \mathcal{M}' \, \mathrm{d}v \, \mathrm{d}v' \gtrsim \|h - \mathcal{P}h\|^2_{-\beta},$$

where we used in the last line the classical coercivity for scattering operator discussed above. □

Note that in the previous coercivity inequality $\lambda(s) \to 0$ as $s \to 1$ since $C_{d,s} \to 0$ as $s \to 1$. This explains why the coercivity weight $\beta = 2$ of the Fokker-Planck operator differs from the coercivity weight $\beta = 2 - \alpha$ of the Lévy-Fokker-Planck operator when $s \to 1$. In fact when $\alpha = 2s$ and $s \to 1$ the correct formal limit is the Fokker-Planck operator with Gaussian equilibrium, in view of the general theory of Lévy processes, for which $\beta = 0$ is indeed the limit of $\beta = 2 - \alpha = 2 - 2s$ as $s \to 1$.

8.2. **Proof of Hypothesis 3.** We estimate

$$\|L(\chi_R)\|_\beta = \left\| \mathcal{M}^{-1} \Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] + U \cdot \nabla_v \chi_R \right\|_\beta$$

in several steps. Write first

$$\begin{aligned} \|U \cdot \nabla_v \chi_R\|^2_\beta &= \int_{\mathbb{R}^d} |U \cdot \nabla_v \chi_R|^2 \lfloor v \rceil^\beta \mathcal{M}(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} \Big| \mathbb{U}(v) \lfloor v \rceil^{-\beta} v \cdot \nabla_v \chi_R \Big|^2 \lfloor v \rceil^\beta \mathcal{M}(v)\, \mathrm{d}v \\ &\leq \|\mathbb{U}\|_\infty \int_{\mathbb{R}^d} |v \cdot \nabla_v \chi_R|^2 \, \mathcal{M}_\beta(v)\, \mathrm{d}v \lesssim R^{-\alpha-\beta}. \end{aligned}$$

Then split the other term into

$$\begin{aligned} &\left\| \mathcal{M}^{-1} \Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] \right\|^2_\beta \\ &= \left\| \mathcal{M}^{-1} \Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] \mathbf{1}_{|v| \leq R} \right\|^2_\beta + \left\| \mathcal{M}^{-1} \Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] \mathbf{1}_{|v| \geq R} \right\|^2_\beta. \end{aligned}$$

When $|v| \leq R$, write $v = Rw$ with $|w| \leq 1$ and observe that $\chi(w) = \chi(w')$ when $|w'| \leq 1$ to get,

$$\begin{aligned} \left| \left[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \right] (Rw) \right| &= C_{d,s} \left| \int_{|w'| \geq 1} \frac{\chi(w) - \chi(w')}{R^{d+a} \, |w - w'|^{d+2s}} \mathcal{M}(Rw')\, R^d \, \mathrm{d}w' \right| \\ &\lesssim \int_{|w'| \geq 1} \frac{|\chi(w) - \chi(w')|}{R^{d+a+\alpha} \, |w - w'|^{d+2s}} \, \frac{\mathrm{d}w'}{|w'|^{d+\alpha}} \lesssim R^{-(d+2s+\alpha)}, \end{aligned}$$

which yields (using $\beta = 2s - \alpha$ and $\alpha > 0$)

$$\left\| \mathcal{M}^{-1} \Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] \mathbf{1}_{|v| \leq R} \right\|^2_\beta \lesssim R^{-4s+\beta-\alpha} \lesssim R^{-\alpha-\beta}.$$

When $|v| \geq R$, we write

$$\Big[\Delta_v^s \left(\mathcal{M} \chi_R \right) - \left(\Delta_v^s \mathcal{M} \right) \chi_R \Big] (v) = \int_{\mathbb{R}^d} \frac{\left[\chi_R(v) - \chi_R(v') \right]}{|v - v'|^{d+2s}} \mathcal{M}(v')\, \mathrm{d}v' = \int_{|v-v'| \leq \frac{|v|}{2}} \cdots + \int_{|v-v'| \geq \frac{|v|}{2}} \cdots$$

Start with the first integral in the right hand side:

$$\begin{aligned} &\left| \int_{|v-v'| \leq \frac{|v|}{2}} \frac{\left[\chi_R(v) - \chi_R(v') \right]}{|v - v'|^{d+2s}} \mathcal{M}(v')\, \mathrm{d}v' \right| \\ &\qquad \lesssim \int_{|v-v'| \leq \frac{|v|}{2}} \frac{\sup_{B\left(v, \frac{|v|}{2}\right)} \left| \nabla^2_{v'} \left[\left(\chi_R(v) - \chi_R(v') \right) \mathcal{M}(v') \right] \right|}{|v - v'|^{d+2s-2}} \, \mathrm{d}v' \\ &\qquad \lesssim |v|^{2-2s} \sup_{B\left(v, \frac{|v|}{2}\right)} \left| \nabla^2_{v'} \left[\left(\chi_R(v) - \chi_R(v') \right) \mathcal{M}(v') \right] \right|. \end{aligned}$$

One has

$$\sup_{B\left(v,\frac{|v|}{2}\right)} \left|D^2_{v'}\left((\chi_R(v)-\chi_R(v'))\mathcal{M}(v')\right)\right|$$
$$\lesssim \frac{|v|^{-d-\alpha}}{R^2}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right| + \frac{|v|^{-d-\alpha-1}}{R}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right| + |v|^{-d-\alpha-2}.$$

Consequently,

$$\left|\int_{|v-v'|\le\frac{|v|}{2}} \frac{[\chi_R(v)-\chi_R(v')]}{|v-v'|^{d+2s}}\mathcal{M}(v')\,dv'\right|$$
$$\lesssim |v|^{2-2s}\left[\frac{|v|^{-d-\alpha}}{R^2}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right| + \frac{|v|^{-d-\alpha-1}}{R}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right| + |v|^{-d-\alpha-2}\right]$$
$$\lesssim |v|^{-d-\alpha-2s}\left[\frac{|v|^2}{R^2}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi''\left(\frac{v'}{R}\right)\right| + \frac{|v|}{R}\sup_{v'\in B(v,\frac{|v|}{2})}\left|\chi'\left(\frac{v'}{R}\right)\right| + 1\right] \lesssim |v|^{-d-\alpha-2s},$$

where we have used that χ' ans χ'' have compact support and $|v|\le 2|v'|$ in this region.

Focus now on the second integral (using $\alpha>0$)

$$\left|\int_{|v-v'|\ge\frac{|v|}{2}} \frac{[\chi_R(v)-\chi_R(v')]}{|v-v'|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v'\right| \le \int_{|v-v'|\ge\frac{|v|}{2}} \frac{|\chi_R(v)-\chi_R(v')|}{|v-v'|^{d+2s}}\mathcal{M}(v')\,\mathrm{d}v' \lesssim |v|^{-d-2s}.$$

As a conclusion,

$$\left\|\mathcal{M}^{-1}\left[\Delta_v^s(\mathcal{M}\chi_R) - (\Delta_v^s\mathcal{M})\chi_R\right]\mathbf{1}_{|v|\ge R}\right\|_\beta^2 \lesssim \int_{|v'|>R} |v'|^{2\alpha-4s}\lfloor v'\rceil^{\beta-d-\alpha}\,\mathrm{d}v' \lesssim R^{-\alpha-\beta},$$

since $\beta=2s-\alpha$. This concludes the proof.

8.3. **Proof of Hypothesis 4.** The adjoint of L is $L^*=\Delta_v^s - U\cdot\nabla_v$ and following exactly the same arguments as in the proof of Lemma 7.1 for the Fokker-Planck operator yields

Lemma 8.2. *The unique solution to the eigenvalue equation*

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d}\phi_\eta(v)\,\lfloor v\rceil^{-\beta}\,\mathcal{M}(v)\,\mathrm{d}v = 1$$

satisfies for any $R\ge 1$,

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1 \quad \text{and} \quad \|\operatorname{Im}\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta,\mu(\eta)),$$

with constants depending only on R and uniform in $\eta\to 0$.

Note that local fractional ellipticity results are present in [47]. We now come to the pointwise estimates on the rescaled eigenvector. This is when $\alpha\le 2+\beta$, that is $\alpha\le 1+s$. Observe indeed that when $\alpha>1+s$, the scaling is diffusive, and the diffusion coefficient is obtained by solving

$$\Delta_v^s(\mathcal{M}F) + \nabla_v\cdot(U\,\mathcal{M}F) = -(v\cdot\sigma)\mathcal{M}(v),$$

with $\int_{\mathbb{R}^d} F(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 0$.

Lemma 8.3. *Assume that $s\in(\frac12,1)$. There is $\eta_1\in(0,\eta_0)$ small enough and A and C large enough so that*

$$\forall\,\eta\in(0,\eta_1),\ \forall\,u\in\mathbb{R}^d,\quad |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)} \quad \text{and} \quad |\operatorname{Im}\Phi_\eta(u)| \lesssim |u|_\eta^{\min(1,\alpha)+\beta-C\mu(\eta)}.$$

Proof. The rescaled equation for Φ_η is, using $U_\eta(u) := \eta^{\frac{1-\beta}{1+\beta}} U(u\eta^{-\frac{1}{1+\beta}})$ and $2s-\beta=\alpha$,

$$-\eta^{\frac{\alpha}{1+\beta}} \Delta_u^s \Phi_\eta + U_\eta(u)\cdot\nabla_u\Phi_\eta - i(u\cdot\sigma)\Phi_\eta = \mu(\eta)|u|_\eta^{-\beta}\Phi_\eta.$$

Multiply the latter equation by $\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}$ and take the real part:

$$-\eta^{\frac{\alpha}{1+\beta}} \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\Delta_u^s\Phi_\eta\right) + U_\eta(u)\cdot \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\nabla_u\Phi_\eta\right) = \mu(\eta)|u|_\eta^{-\beta}|\Phi_\eta|.$$

Using the Kato inequality $\Delta_u^s|\Phi_\eta| \geq \operatorname{Re}\left(\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}\Delta_u^s\Phi_\eta\right)$ (see [14] for the Laplacian and [19] for the fractional Laplacian), one gets

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s|\Phi_\eta| + |u|_\eta^\beta U_\eta(u)\cdot\nabla_u|\Phi_\eta| - \mu(\eta)|\Phi_\eta| \leq 0.$$

Then observe that the real function $F(u) = |u|_\eta^{C\mu(\eta)}$ satisfies for $|u| \geq A\eta^{\frac{1}{3}}$:

$$\begin{aligned}
&-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s F + |u|_\eta^\beta U_\eta(u)\cdot\nabla_u F - \mu(\eta)F \\
&= -\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s F + C\mu(\eta)|u|_\eta^{C\mu(\eta)-2}\mathbb{U}_\eta(u)\,|u|^2 - \mu(\eta)|u|_\eta^{C\mu(\eta)}
\end{aligned}$$

where we have used that $U_\eta(u) = |u|_\eta^{-\beta}\mathbb{U}_\eta(u)u$ with some $\mathbb{U}_\eta$ positive bounded from below (independently of η). We now estimate $\Delta_u^s\left(|\cdot|_\eta^{C\mu(\eta)}\right)(u)$. By scaling:

$$\forall\, u\in\mathbb{R}^d, \quad \Delta_u^s\left(|\cdot|_\eta^{C\mu(\eta)}\right)(u) = \eta^{\frac{C\mu(\eta)-2s}{1+\beta}}\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)\left(u\eta^{-\frac{1}{1+\beta}}\right).$$

We then estimate $\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)$ using

$$\begin{aligned}
\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)(v) &= C_{d,s}\int_{\mathbb{R}^d}\frac{\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v', \\
&= C_{d,s}\int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v' + C_{d,s}\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v'.
\end{aligned}$$

To control the first term in the right hand side, use that

$$\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)} - \nabla_v\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)\cdot(v'-v) \lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2}|v'-v|^2$$

to get

$$\begin{aligned}
\int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v' &\lesssim \int_{|v-v'|<\frac{|v|}{2}}\frac{C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2}|v'-v|^2}{|v'-v|^{d+2s}}\,\mathrm{d}v' \\
&\lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2}\int_{|v-v'|<\frac{|v|}{2}}|v'-v|^{2-2s-d}\,\mathrm{d}v' \lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2s}.
\end{aligned}$$

To control the second term, use that

$$\left(\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}\right) \lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-1}|v'-v|$$

to get (using here $s>\frac{1}{2}$)

$$\begin{aligned}
\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^{C\mu(\eta)} - \lfloor v\rceil^{C\mu(\eta)}}{|v'-v|^{d+2s}}\,\mathrm{d}v' &\lesssim \int_{|v-v'|>\frac{|v|}{2}}\frac{C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-1}|v'-v|}{|v'-v|^{d+2s}}\,\mathrm{d}v' \\
&\lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-1}\int_{|v-v'|>\frac{|v|}{2}}\frac{\mathrm{d}v'}{|v'-v|^{d+2s-1}} \\
&\lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2s}.
\end{aligned}$$

We therefore have (using the scaling)

$$\Delta_v^s\left(\lfloor\cdot\rceil^{C\mu(\eta)}\right)(v) \lesssim C\mu(\eta)\lfloor v\rceil^{C\mu(\eta)-2s} \quad\Longrightarrow\quad \Delta_u^s\left(|\cdot|_\eta^{C\mu(\eta)}\right)(u) \lesssim C\mu(\eta)|u|_\eta^{C\mu(\eta)-2s}.$$

This estimate implies, for some absolute constant $C_0 > 0$,

$$\begin{aligned}\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s F &\le C_0 C\mu(\eta)\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^{C\mu(\eta)+\beta-2s}\\ &\le C_0 C\mu(\eta)|u|_\eta^{C\mu(\eta)}\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^{-\alpha} \lesssim C_0 C\mu(\eta)|u|_\eta^{C\mu(\eta)}\left(1+A^2\right)^{-\frac{\alpha}{2}}\end{aligned}$$

in the region $|u| \ge A\eta^{\frac{1}{1+\beta}}$. As a consequence

$$\begin{aligned}&-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s F + \mathbb{U}_\eta(u)u\cdot\nabla_u F - \mu(\eta)F\\ &= -\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s F + C\mu(\eta)|u|_\eta^{C\mu(\eta)-2}\mathbb{U}_\eta(u)\,|u|^2 - \mu(\eta)|u|_\eta^{C\mu(\eta)}\\ &\ge -C_0 C\mu(\eta)|u|_\eta^{C\mu(\eta)}\left(1+A^2\right)^{-\frac{\alpha}{2}} + C\mu(\eta)|u|_\eta^{C\mu(\eta)-2}\left(\inf\mathbb{U}_\eta\right)\,|u|^2 - \mu(\eta)|u|_\eta^{C\mu(\eta)}\\ &\ge C\mu(\eta)|u|_\eta^{C\mu(\eta)}\left[-C_0\left(1+A^2\right)^{-\frac{\alpha}{2}} + |u|_\eta^{-2}\left(\inf\mathbb{U}_\eta\right)\,|u|^2 - C^{-1}\right]\\ &\ge C\mu(\eta)|u|_\eta^{C\mu(\eta)}\left[-C_0\left(1+A^2\right)^{-\frac{\alpha}{2}} + \left(1+A^{-2}\right)^{-1}\left(\inf\mathbb{U}_\eta\right)\,- C^{-1}\right] \ge 0\end{aligned}$$

for A and C sufficiently large, and we deduce $|\Phi_\eta| \lesssim F$ on $|u| \ge A\eta^{\frac{1}{1+\beta}}$ and, for the same reasons as for the Fokker-Planck operator, the bound extends to any $u \in \mathbb{R}^d$.

Taking now the imaginary part of the equation, one gets

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s|\operatorname{Im}\Phi_\eta| + \mathbb{U}_\eta(u)u\cdot\nabla_u|\operatorname{Im}\Phi_\eta| - \mu(\eta)|\operatorname{Im}\Phi_\eta| \lesssim |u|_\eta^{1+\beta+C\mu(\eta)}.$$

Define then $\gamma := \min(\alpha,1)+\beta-C\mu(\eta) \in (0,2s)$ and the real function $G(u) := |u|_\eta^\gamma$. Note that $\gamma \in (0,2s)$ for η small enough, which implies that $\Delta_u^s G$ makes sense. Write for $|u| \ge A\eta^{\frac{1}{3}}$

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s G + |u|_\eta^\beta U_\eta(u)\cdot\nabla_u G - \mu(\eta)G = -\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta \Delta_u^s G + \gamma|u|_\eta^{\gamma-2}\mathbb{U}_\eta(u)\,|u|^2 - \mu(\eta)G.$$

Let us estimate $\Delta_u^s\left(|\cdot|_\eta^\gamma\right)(u)$. Note that by scaling

$$\forall\, u\in\mathbb{R}^d,\quad \Delta_u^s\left(|\cdot|_\eta^\gamma\right)(u) = \eta^{\frac{\gamma-2s}{1+\beta}}\Delta_v^s\left(\lfloor\cdot\rceil^\gamma\right)(u\eta^{-\frac{1}{1+\beta}}).$$

One then estimates $\Delta_v^s\left(\lfloor\cdot\rceil^\gamma\right)$ using

$$\begin{aligned}\Delta_v^s\left(\lfloor\cdot\rceil^\gamma\right)(v) &= C_{d,s}\int_{\mathbb{R}^d}\frac{\left(\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma\right)}{|v'-v|^{d+2s}}\,\mathrm{d}v',\\ &= C_{d,s}\int_{|v-v'|<\frac{|v|}{2}}\frac{\left(\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma\right)}{|v'-v|^{d+2s}}\,\mathrm{d}v' + C_{d,s}\int_{|v-v'|>\frac{|v|}{2}}\frac{\left(\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma\right)}{|v'-v|^{d+2s}}\,\mathrm{d}v'.\end{aligned}$$

Small v's are fine since $\Delta_v^s\left(\lfloor\cdot\rceil^\gamma\right)$ is locally bounded. Continue with large v. In the first integral,

$$\begin{aligned}\int_{|v-v'|<\frac{|v|}{2}}\frac{\left(\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma\right)}{|v'-v|^{d+2s}}\,\mathrm{d}v' &= \int_{|v-v'|<\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma - \nabla_v\left(\lfloor\cdot\rceil^\gamma\right)(v)(v'-v)}{|v'-v|^{d+2s}}\,\mathrm{d}v'\\ &\lesssim \int_{|v-v'|<\frac{|v|}{2}}\frac{\sup_{z\in B(v,\frac{|v|}{2})}\left|\nabla^2\left(\lfloor\cdot\rceil^\gamma\right)(z)\right||v'-v|^2}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim |v|^{\gamma-2s}.\end{aligned}$$

The second integral may be estimated from above using that $|v-v'| > \frac{|v|}{2}$ implies $|v-v'| > \frac{|v'|}{3}$,

$$\int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v'\rceil^\gamma - \lfloor v\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim \int_{|v-v'|>\frac{|v|}{2}}\frac{\lfloor v-v'\rceil^\gamma}{|v'-v|^{d+2s}}\,\mathrm{d}v' \lesssim |v|^{\gamma-2s}.$$

From this, we deduce $\Delta_u^s(|\cdot|_\eta^\gamma)(u) \lesssim \eta^{\frac{\gamma-2s}{1+\beta}}\lfloor u\eta^{-\frac{1}{1+\beta}}\rceil^{\gamma-2s} = |u|_\eta^{\gamma-2s}$ which implies

$$\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s G \lesssim \eta^{\frac{\alpha}{1+\beta}}|u|_\eta^{\gamma+\beta-2s} \lesssim \left(1+A^2\right)^{-\frac{\alpha}{2}}|u|_\eta^\gamma$$

in the region $|u| \ge A\eta^{\frac{1}{1+\beta}}$. As a consequence, as previously,

$$-\eta^{\frac{\alpha}{1+\beta}}|u|_\eta^\beta\Delta_u^s G + |u|_\eta^\beta U_\eta(u)\cdot\nabla_u G - \mu(\eta)G \gtrsim |u|_\eta^\gamma$$

for A sufficiently large and we deduce $|\operatorname{Im}\Phi_\eta| \lesssim G$ on $|u| \ge A\eta^{\frac{1}{1+\beta}}$ and, for the same reasons as for the Fokker-Planck operator, the bound extends to any $u \in \mathbb{R}^d$. □

8.4. **Rescaled drift force and limit equation.** We formally discuss the behaviour of the force U_η when η goes to 0: setting $v = u\eta^{-\frac{1}{1+\beta}}$ gives the equation

$$\eta^{\frac{a-\beta}{1+\beta}} \Delta_v^s \mathcal{M}_\eta + \nabla_v \cdot (U_\eta \, \mathcal{M}_\eta) = 0.$$

Observe that when $u \neq 0$,

$$\eta^{\frac{\alpha}{1+\beta}} \Delta_v^s \mathcal{M}_\eta(u) = -c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \frac{\big(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}\big)}{|u-u'|^{d+2s}} \, \mathrm{d}u'$$

$$= -c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{\big(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}\big)}{|u-u'|^{d+2s}} \, \mathrm{d}u' - c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)^c} \frac{\big(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}\big)}{|u-u'|^{d+2s}} \, \mathrm{d}u'.$$

The second term in the right hand side goes to zero as $\eta \to 0$ since the singularity around zero has been removed from the integration domain. To deal with the first term, decompose

$$\eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{\big(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}\big)}{|u-u'|^{d+2s}} \, \mathrm{d}u' = \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}} \, \mathrm{d}u' - \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}} \, \mathrm{d}u'.$$

The first part goes to zero if $\varepsilon < |u|$. The second part writes

$$\eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}} \, \mathrm{d}u' \sim_\varepsilon |u|^{-d-2s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} |u'|_\eta^{-d-\alpha} \, \mathrm{d}u'$$

$$= |u|^{-d-2s} \int_{B\left(0,\varepsilon\eta^{-\frac{\alpha}{1+\beta}}\right)} (1+|v'|^2)^{-d-\alpha} \, \mathrm{d}v'.$$

Taking η small then ε small yields

$$\lim_{\eta\to 0, \varepsilon \to 0} \left(-c_{\alpha,\beta} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}}{|u-u'|^{d+2s}} \, \mathrm{d}u' \right) = \frac{c_{\alpha,\beta}}{c_{\alpha,0}} \cdot \frac{1}{|u|^{d+2s}}.$$

Since $\nabla_v (|v|^{-d-2s} v) = -2s|v|^{-d-2s}$, we deduce that

$$\lim_{\eta\to 0} \eta^{\frac{1-\beta}{1+\beta}} U\left(u\eta^{-\frac{1}{1+\beta}}\right) = U_\infty(u) = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} \frac{u}{|u|^{d+2s}} |u|^{d+\alpha} = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} |u|^{-\beta} u.$$

This proves the scaling limit of the drift force.

From the rescaled equation for Φ_η, we deduce that Φ_η goes to Φ, where Φ solves,

$$\frac{c_{\alpha,\beta}}{2asc_{\alpha,0}} \frac{u}{|u|^\beta} \cdot \nabla_u \Phi - i(u\cdot\sigma)\Phi = 0 \quad \text{with} \quad \Phi(0) = 1 \quad \Longrightarrow \quad \Phi(u) := \exp\left(i \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta} \right).$$

Thus, in the limit case $\alpha = 1+s = 2+\beta$, $\Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0} \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}}$ satisfies,

$$\frac{c_{\alpha,\beta}}{2ac_{\alpha,0}} u \cdot \nabla_u \Omega = (u\cdot\sigma)|u|^\beta \quad \Longrightarrow \quad \Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u\cdot\sigma)}{1+\beta}.$$

8.5. **The particular case $\alpha = 2s$.** More explicit calculations are available when $\alpha = 2s$. In this case $\beta = 0$, $U(v) = c_0 v$ for some constant $c_0 > 0$, and the eigenproblem is

$$-\Delta_v^s \phi + c_0 v \cdot \nabla_v \phi - i\eta(v\cdot\sigma)\phi = \mu(\eta)\phi.$$

Taking the Fourier transform (in the dual of the Schwarz space) gives

$$-|\xi|^{2s}\hat\phi - c_0 \xi \cdot \nabla_\xi \hat\phi + \eta\sigma\cdot\nabla_\xi\hat\phi = (\mu(\eta)+c_0)\hat\phi$$

or equivalently

$$(\eta\sigma - c_0\xi)\cdot\nabla_\xi\hat\phi = \big(\mu(\eta) + c_0 + |\xi|^{2s}\big)\,\hat\phi.$$

The solution to this equation is given by $\hat\phi = \delta_{c_0^{-1}\eta\sigma}$ and $\mu(\eta) = -|c_0^{-1}\eta\sigma|^{2s} = c_0^{-2s}\eta^{2s}$, which yields by inverse Fourier transform $\phi_\eta(v) := \exp\big(ic_0^{-1}\eta(v\cdot\sigma)\big)$. This agrees with the expression of Φ given above, and allows to compute $c_0 = \frac{1}{2s}$.

9. Remarks and extensions

In Hypothesis 1, the equilibrium $\mathcal{M}$ is an explicit power law, and in particular is centered and even. We discuss in this section the changes required for our proofs to deal with more general $\mathcal{M}$ that are (i) characterised by *asymptotic* power-law estimates rather than exact formula, and (ii) non necessarily even or centered. This means replacing Hypothesis 1 with:

Hypothesis 1' (Equilibria)**.** *The equilibrium distribution satisfies*

$$\mathcal{M} = \lfloor \cdot \rceil^{-(d+\alpha)} \mathcal{S}(v), \tag{9.1}$$

where $\mathcal{S}$ is a slowly varying function, and the ***generalised mass condition*** (1.8).

Slowly varying functions are non-vanishing measurable functions that satisfy $\mathcal{S}(ax) \sim \mathcal{S}(x)$ as x goes to infinity, for any $a > 0$. Some examples of slowly varying functions are positive constants, functions that converge to positive constants, logarithms and iterated logarithms.

9.1. **Equilibria characterised only asymptotically.** If one considers a *centered* equilibrium $\mathcal{M}$ that satisfies Hypothesis 1', the proof of Theorem 1.4 in Section 2 and the proof of Lemma 1.1 in Section 3 are essentially unchanged. The formulas for μ_0 and κ in Lemmas 1.2 and 1.3 are slightly modified, and rely on the existence of a scaling limit of $\eta^{-\frac{d+\alpha}{1+\beta}} \mathcal{M}(u\eta^{-\frac{1}{1+\beta}})$ as $\eta \to 0$, which follows from Hypothesis 1'. Everything else remains unchanged and the structures of the proofs in Sections 4 and 5 are the same. Rates of convergence will depend on the form of $\mathcal{S}$.

9.2. **Non-centered equilibria.** When the microscopic equilibrium $\mathcal{M}(v)$ is not centered, it results in a drift in the macroscopic equation. Our approach however allows to tackle such a situation, with the following changes depending on whether this macroscopic drift is of higher, comparable or smaller order than the resulting (fractional) macroscopic diffusion. In view of Theorem 1.4 in the centered situation, we expect a macroscopic diffusion of order $\zeta(\alpha, \beta) = \min\left(2, \frac{\alpha_+ + \beta}{1+\beta}\right)$, and therefore we expect the drift to be dominant when $\alpha > 1$ and dominated when $\alpha < 1$, with a borderline case at $\alpha = 1$. Observe that $\alpha = 1$ is also the threshold for the absolute convergence of the integral $\int_{\mathbb{R}^d} (v \cdot \sigma) \mathcal{M}(v)\, \mathrm{d}v$ defining the macroscopic drift.

Consider a solution f in $L^\infty([0, +\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ to equation (1.1) and denote

$$f_\varepsilon(t, x, v) := f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon} + \frac{\bar{v}_\varepsilon t}{\theta(\varepsilon)}, v\right) \in L^\infty_t([0, +\infty); L^2_{x,v}(\mathcal{M}^{-1}))$$

where $\varepsilon > 0$ and $\theta(\varepsilon)$ is defined in (1.16), and where the *velocity corrector* $\bar{v}_\varepsilon$ is defined by

$$\bar{v}_\varepsilon := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} v\mathcal{M}(v)\, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d} \mathcal{M}(v)\, \mathrm{d}v} & \text{when } \alpha > 1, \\ \left(\displaystyle\lim_{R \to \infty} \frac{1}{\ln(R)} \frac{\displaystyle\int_{\mathbb{R}^d} v\chi_R(v)\mathcal{M}(v)\, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d} \chi_R(v)\mathcal{M}(v)\, \mathrm{d}v} \right) \frac{|\ln(\varepsilon)|}{1+\beta} & \text{when } \alpha = 1, \\ 0 & \text{when } \alpha \in (-\beta, 1). \end{cases} \tag{9.2}$$

The equation satisfied by f_ε is

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon(v - \bar{v}_\varepsilon) \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon. \tag{9.3}$$

With this definition of f_ε, Theorem 1.4 holds and yields the (fractional) diffusive limit of f_ε. The changes in the proofs are as follows. The arguments presented in Section 2 are essentially unchanged with a few modifications to obtain the scaling of the eigenvalue resulting from (9.3).

We chose $\bar v_\varepsilon$ in such a way that the dominant eigenmode has the scaling obtained in Lemmas 1.2 and 1.3. The new spectral problem to be considered in the modified Lemma 1.1 is

$$-L^*\phi_\eta - i\eta\,[(v-\bar v_\varepsilon)\cdot\sigma]\,\phi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\phi_\eta \qquad \text{with} \qquad \int_{\mathbb{R}^d}\phi_\eta(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 1.$$

Line-by-line technical modifications are needed in the proof of Lemmas 1.2 and 1.3 due to the additional drift but the procedure and method are preserved and we do not repeat the arguments. Let us just explain why we define the correction velocity $\bar v_\varepsilon$ in this way. The spectral projector estimate of Section 3 follows the same procedure, with (3.18) replaced by

$$-L^*F - i\eta\,[(v-\bar v_\varepsilon)\cdot\sigma]\,F - z\lfloor v\rceil^{-\beta}F = (v-\bar v_\varepsilon)\cdot\sigma. \tag{9.4}$$

The L^2 estimate is unchanged and the crucial estimate (3.23) remains true as long as

$$q(R) := \int_{\mathbb{R}^d}[v-\bar v_\varepsilon]\,\chi_R(v)\mathcal{M}(v)\,\mathrm{d}v \qquad \text{at} \qquad R := \eta^{-\frac{1}{1+\beta}}$$

is small compared with $\mathrm{R}_1\eta^{-1}\Theta(\eta)$, when R_1 is large enough. This implies that the influence of the drift is smaller than the size of the fluid mode, which is of order $\eta^{-1}\Theta(\eta)$. Recall that

$$\frac{\Theta(\eta)}{\eta} := \begin{cases} \eta & \text{when } \alpha > 2+\beta, \\ \eta|\ln(\eta)| & \text{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha-1}{1+\beta}} & \text{when } -\beta < \alpha < 2+\beta \end{cases} \tag{9.5}$$

One can then prove that for all $\alpha > -\beta$, one has $q(\eta^{-\frac{1}{1+\beta}}) \lesssim \eta^{\frac{\alpha-1}{1+\beta}}$, which proves that $q(\eta^{-\frac{1}{1+\beta}})$ is small compared with $\mathrm{R}_1\eta^{-1}\Theta(\eta)$ when R_1 is large enough.

9.3. **More general velocity fields.** One could replace the transport operator $v\cdot\nabla_x$ by a more general $a(v)\cdot\nabla_x$, where a is odd. All our results and proofs can be extended, even though the scalings found may be changed since the scaling of $\ell(R)$ in (3.14) will be different. If $a(v)$ scales like $|v|^\delta$, redoing the computations as in Section 2 and Section 3 then one would find

$$\Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > 2\delta+\beta, \\ \eta^2|\ln(\eta)| & \text{when } \alpha = 2\delta+\beta, \\ \eta^{\frac{\alpha+\beta}{\delta+\beta}} & \text{when } -\beta < \alpha < 2\delta+\beta. \end{cases} \tag{9.6}$$

An example is given by relativistic particles, for which $a(v) := c\frac{v}{\sqrt{c^2+v^2}}$, where c is the speed of light. Such transport operators are relevant to special relativity, see for instance [49] in physics and [29] in mathematics. There, Θ is given by (9.6) with $\delta = 0$.

9.4. **Kinetic Fokker-Planck equation with non gradient confining force.** All the results we obtain for the Fokker-Planck equation with gradient force can be extended to Fokker-Planck operators with non-gradient confining force at little expense. We chose not to present this more general setting in the core of the paper to stay consistent with the clean and simple Hypothesis 1 and to help with readability. It is however possible to consider

$$\mathcal{L}(f) = \Delta_v f + \nabla_v\cdot(U\,f) \qquad \text{where } U \text{ satisfies} \qquad \Delta_v\mathcal{M} + \nabla_v\cdot(U\,\mathcal{M}) = 0,$$

provided that quantitative bounds are available on U to ensure it is comparable to the drift in the Fokker-Planck operator. The analysis is then similar.

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Email address: `bouin@ceremade.dauphine`

Université Paris-Dauphine, France

Email address: `c.mouhot@dpmms.cam.ac.uk`

University of Cambridge