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# QUANTITATIVE FLUID APPROXIMATION IN TRANSPORT THEORY: A UNIFIED APPROACH

ÉMERIC BOUIN AND CLÉMENT MOUHOT

ABSTRACT. We propose a unified method for the large space-time scaling limit of *linear* collisional kinetic equations in the whole space. The limit is of *fractional* diffusion type for heavy tail equilibria with slow enough decay, and of diffusive type otherwise. The proof is constructive and the fractional/standard diffusion matrix is obtained. The method combines energy estimates and quantitative spectral methods to construct a ‘fluid mode’. The method is applied to scattering models (without assuming detailed balance conditions), Fokker-Planck operators and Lévy-Fokker-Planck operators. It proves a series of new results, including the fractional diffusive limit for Fokker-Planck operators in any dimension, for which the formulas for the diffusion coefficient were not known, for Lévy-Fokker-Planck operators with general equilibria, and for scattering operators including some cases of infinite mass equilibria. It also unifies and generalises the results of previous papers with a quantitative method, and our estimates on the fluid approximation error also seem novel.

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## 1. INTRODUCTION AND MAIN RESULTS

The study of *transport processes*, i.e. linear collisional kinetic equations, is theoretically rooted in the mean-free path argument of Maxwell [39] and the kinetic theory of gases of Maxwell and Boltzmann [40, 10]. A linear version of the Maxwell-Boltzmann equation can be written for the movement of a tagged particle within a rarefied gas, but the study of such transport processes was given a crucial new impetus in the twentieth century with:

- (1) the *radiative transfer theory* [46], where the kinetic distribution models the flux of photons that are transported in the plasma making up the internal layers of the sun,
- (2) the *nuclear reactor theory* (see [52], the collection [7] and in particular its fifth chapter [53]) where the kinetic distribution models the neutrons transported and scattered inside the reactor, whose flux is used to initiate and maintain the chain reaction,
- (3) the *semi-conductor theory* [37] where the kinetic distribution models the flow of charge carriers in semiconductors, i.e. the evolution of the position-momentum distribution

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of negatively charged conduction electrons or of positively charged holes, which are responsible for the current flow in semiconductor crystals.

The main mathematical object of study in *transport theory* is the linear equation

$$(1.1) \quad \partial_t f + v \cdot \nabla_x f = \mathcal{L}f$$

on the time-dependent density of particles  $f = f(t, x, v) \geq 0$  over  $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ , for  $t \geq 0$ . The left hand side accounts for free motion and the right hand side accounts for the interaction with a background, for instance scatterers, with an operator  $\mathcal{L}$  that only acts on the kinetic variable  $v$ . Several forms are possible. In nuclear reactor, radiative transfer and semi-conductor theories it is common to consider *scattering operators*, sometimes also called *linear Boltzmann operators*, of the form

$$(1.2) \quad \mathcal{L}f(v) = \left( \int_{\mathbb{R}^d} b(v, v') f(v') dv' \right) \mathcal{M}(v) - \nu(v) f(v)$$

for a *collision frequency*  $\nu(v) := \int_{\mathbb{R}^d} b(v, v') \mathcal{M}(v') dv'$ ,

some *collisional kernel*  $b = b(v, v')$  and an *equilibrium distribution*  $\mathcal{M}(v)$ . In astrophysics and sometimes in semi-conductor theory, one also considers *Fokker-Planck operators*,

$$(1.3) \quad \mathcal{L}f := \nabla_v \cdot \left( \mathcal{M} \nabla_v \left( \frac{f}{\mathcal{M}} \right) \right).$$

Finally, as a simplified model of long-range collisional interactions in a gas of charged particles, we also consider *Lévy-Fokker-Planck operators* (given  $s \in (0, 1)$ ):

$$(1.4) \quad \begin{cases} \mathcal{L}(f) = \Delta_v^s f + \nabla_v \cdot (U f) & \text{with } U(v) = U(|v|) \text{ radially symmetric so that} \\ \Delta_v^s \mathcal{M} + \nabla_v \cdot (U \mathcal{M}) = 0. \end{cases}$$

Denoting by  $\mathcal{F}$  the Fourier transform, the fractional Laplacian is defined as

$$(1.5) \quad \Delta_v^s f(v) := -\mathcal{F}^{-1} [|\cdot|^{2s} \mathcal{F}f(\cdot)](v).$$

These three operators are discussed respectively in Sections 6-7-8. Extensions, such as Fokker-Planck operators with non-gradient force, are discussed in Section 9.

The equation (1.1) is too intricate for many applications. When the relevant time and space scales of observation are much larger than the mean free time and mean free path, it is thus natural to search for a simplified regime. The so-called *diffusion theory* was born out of this endeavour, and in the words of Wigner [53], ‘this [diffusion] theory gives the spatial variation of the [neutron transport] flux quite accurately in regions well removed from interfaces’. We also refer to [52, Chapter IX] for the diffusion theory of monoenergetic neutrons, to [46, Chapter III.2] for the so-called *Eddington approximation* in radiative transfer theory, and to [12, Chapter 2] for a modern mathematical review. Note that anomalous diffusions and Lévy flights are observed by biologists and physicists [3, 50, 5, 38, 48].

We rewrite (1.1) by changing the unknown to  $h := \frac{f}{\mathcal{M}}$ :

$$(1.6) \quad \partial_t h + v \cdot \nabla_v h = Lh \quad \text{where} \quad Lh := \mathcal{M}^{-1} \mathcal{L}(\mathcal{M}h).$$

This change of unknown is convenient since asymptotic estimates compare  $f$  with the equilibrium  $\mathcal{M}$ . Consider the complex Hilbert spaces  $L^2(\mathbb{R}^d; \mathcal{M} dv) =: L_v^2(\mathcal{M})$  and  $L^2(\mathbb{R}^d \times \mathbb{R}^d; \mathcal{M} dx dv) =: L_{x,v}^2(\mathcal{M})$  and denote  $\|h\|_k := \|(1 + |\cdot|^2)^{\frac{k}{2}} h\|_{L^2(\mathcal{M})}$  (the integration variable(s) will be emphasized when there is ambiguity). We omit the index when  $k = 0$ . The scalar product  $\langle \cdot, \cdot \rangle$  refers to  $L_v^2(\mathcal{M})$  or  $L_{x,v}^2(\mathcal{M})$  depending on context.

We assume the following hypotheses for some  $\alpha, \beta \in \mathbb{R}$  with  $\alpha + \beta > 0$ , and some  $\lambda \in \mathbb{R}_+^*$ :

**Hypothesis 1** (Equilibria). *The equilibrium  $\mathcal{M}$  takes one of the following two forms.*

(i) *Either it is given by*

$$(1.7) \quad \mathcal{M}(v) = c_{\alpha, \beta} [v]^{-(d+\alpha)} \quad \text{with} \quad c_{\alpha, \beta} := \left( \int_{\mathbb{R}^d} [v]^{-d-\alpha-\beta} dv \right)^{-1} \quad \text{and} \quad [v] := \sqrt{1 + |v|^2}.$$

- (ii) Or it is a smooth positive radially symmetric function decaying faster than any polynomial. This case is denoted by ‘ $\alpha = +\infty$ ’ in the sequel.

Note that the normalisation implies the **generalised mass condition**

$$(1.8) \quad \int_{\mathbb{R}^d} \mathcal{M}_\beta(v) \, dv = 1 \quad \text{with} \quad \mathcal{M}_\beta := [\cdot]^{-\beta} \mathcal{M}.$$

We present our main results assuming that the equilibrium  $\mathcal{M}$  is given by the exact formula (1.7) in the case of a polynomial decay because it leads to a neater treatment. However, as discussed in Section 9, our results remain true with an equilibrium  $\mathcal{M}$  that is not an explicit power-law or even symmetric or centered, but only comparable to  $[\cdot]^{-(d+\alpha)}$  (see (9.1) and Subsections 9.1 and 9.2); this requires a few technical changes in the proofs that we present separately in this last section so as not to clutter the paper.

**Hypothesis 2** (Weighted coercivity). *The operator  $L$  is linear, independent of time  $t$  and space  $x$ , commutes with rotations in  $v$ , is closed densely defined on  $\text{Dom}(L) \subset L_v^2(\mathcal{M})$  and satisfies  $L(1) = L^*(1) = 0$ , where  $L^*$  is the  $L_v^2(\mathcal{M})$ -adjoint. Finally  $\tilde{L} := [\cdot]^{\frac{\beta}{2}} L([\cdot]^{\frac{\beta}{2}} \cdot)$  is closed densely defined on  $\text{Dom}(\tilde{L}) \subset L_v^2(\mathcal{M})$ , with the spectral gap estimate*

$$\forall g \in \text{Dom}(\tilde{L}), \quad g \perp [\cdot]^{-\frac{\beta}{2}}, \quad -\text{Re} \langle \tilde{L}g, g \rangle \geq \lambda \|g\|^2.$$

This means, translating back to  $L$ ,

$$\forall h \in \text{Dom}(L), \quad -\text{Re} \langle Lh, h \rangle \geq \lambda \|h - \mathcal{P}h\|_{-\beta}^2 \quad \text{with} \quad \mathcal{P}h := \left( \int_{\mathbb{R}^d} h(v') \mathcal{M}_\beta(v') \, dv' \right).$$

The assumption that  $\mathcal{L}$  commutes with rotations in  $v$  is convenient (and satisfied for most physical models), but in fact only  $\mathcal{M}(v) = \mathcal{M}(-v)$  is really used in the proof. The latter could in turn be relaxed at the price of a few technical changes in the proofs discussed in Section 9.

**Hypothesis 3** (Amplitude of collisions at large velocities). *Given  $0 \leq \chi \leq 1$  a smooth function that is 1 on  $B(0, 1)$  and 0 outside  $B(0, 2)$ , and  $\chi_R = \chi(\frac{\cdot}{R})$  and  $\tilde{\chi}_R = (v \cdot \sigma)[v]^\beta \chi_R$  for  $R \geq 1$ ,*

$$\|L(\chi_R)\|_\beta \lesssim R^{-\frac{\alpha+\beta}{2}} \quad \text{and} \quad \|L(\tilde{\chi}_R)\|_\beta \lesssim \begin{cases} R^{1+\beta-\frac{\alpha+\beta}{2}} & \text{when } \alpha \in (-\beta, 2+\beta) \\ (\ln R)^{\frac{1}{2}} & \text{when } \alpha = 2+\beta. \end{cases}$$

Our first result, on the basis of the three previous hypotheses, is a quantitative construction of a branch of ‘fluid eigenmode’ in the asymptotic of large time and small spatial frequencies, i.e. a unique eigenvalue branching from zero for  $\tilde{L}^* + i\eta[v]^\beta(v \cdot \sigma)$  for small  $\eta$  (see Figure 1):

**Lemma 1.1** (Construction of the fluid mode). *Given Hypotheses 1–2–3, there are  $\eta_0 > 0$  and  $r_0 \in (0, \lambda)$ , explicit in terms of the constants in these hypotheses, such that for any  $\eta \in (0, \eta_0)$  and any  $\sigma \in \mathbb{S}^{d-1}$ , there is a unique solution  $\phi_\eta = \phi_\eta(v) \in L_v^2([\cdot]^{-\beta} \mathcal{M})$  and  $\mu(\eta) \in B(0, r_0)$  to*

$$-L^* \phi_\eta - i\eta(v \cdot \sigma) \phi_\eta = \mu(\eta) [v]^{-\beta} \phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}_\beta(v) \, dv = 1.$$

Moreover, the branch  $(\phi_\eta, \mu(\eta))$  connects to  $(1, 0)$  as  $\eta \rightarrow 0$ , with  $\mu(\eta) > 0$  and the asymptotics

$$(1.9) \quad \|\phi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac{1}{2}} \quad \text{and} \quad \mu(\eta) \in (R_0 \Theta(\eta), R_1 \Theta(\eta))$$

for some  $0 < R_0 < R_1$ , where the function  $\Theta$  is defined by

$$(1.10) \quad \Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > 2 + \beta, \\ \eta^2 |\ln(\eta)| & \text{when } \alpha = 2 + \beta, \\ \eta^{\frac{\alpha+\beta}{1+\beta}} & \text{when } -\beta < \alpha < 2 + \beta. \end{cases}$$

Note that  $\Theta$  is well-defined in the case  $\alpha \in (-\beta, 2+\beta)$  since  $(1+\beta) > (\alpha+\beta)/2 > 0$ . In this lemma and in the rest of the paper the dependency in  $\sigma$  is kept implicit rather than explicit in order to lighten notation. In fact,  $\phi_\eta$  also depends on  $\sigma$ , but  $\mu(\eta)$  does not if  $L$  is invariant by rotations in  $v$ . To identify the macroscopic limit with quantitative rates and constants, it is

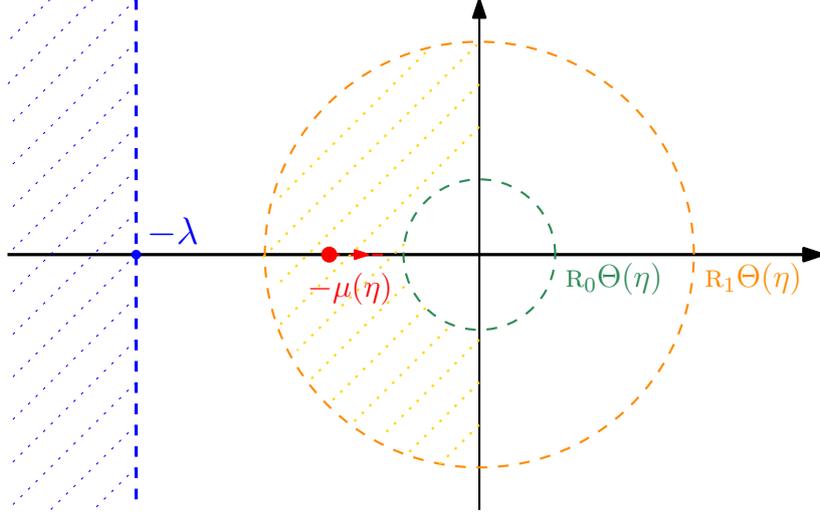


FIGURE 1. The blue dashed zone on the left of  $\operatorname{Re} z = -\lambda$  corresponds to the spectral gap estimates on  $\tilde{L}^* + i\eta[v]^\beta(v \cdot \sigma)$  for  $g \perp [\cdot]^{-\frac{\beta}{2}}$  (Hypothesis 2). The yellow dashed zone is where Lemmas 1.1-1.2 construct a unique real eigenvalue  $-\mu(\eta) \sim -\mu_0\Theta(\eta)$  of the operator  $\tilde{L}^* + i\eta[v]^\beta(v \cdot \sigma)$ , that goes to zero as  $\eta \rightarrow 0$ .

necessary to estimate the leading order of  $\mu(\eta)$ , and this requires estimates on the eigenvector, which is our last hypothesis. We denote  $|u|_\eta := (\eta^{\frac{2}{1+\beta}} + |u|^2)^{\frac{1}{2}}$ .

**Hypothesis 4** (Scaling of the fluid mode). *We make different assumptions depending on  $\alpha$ :*

- (i) Case  $\alpha > 2 + \beta$ : The fluid mode  $\phi_\eta$  constructed in Lemma 1.1 satisfies

$$\forall \ell < \alpha, \quad \|\phi_\eta\|_\ell \lesssim_\ell 1.$$

- (ii) Case  $\alpha \in (-\beta, 2 + \beta]$ : The rescaled fluid mode  $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}} \cdot)$  is converging in  $L^2_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$  as  $\eta \rightarrow 0$  to a limit  $\Phi$  and satisfies the pointwise controls

$$(1.11) \quad \forall \eta \in (0, \eta_1), \quad \forall u \in \mathbb{R}^d, \quad \begin{cases} |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)}, \\ |\operatorname{Im} \Phi_\eta(u)| \lesssim |u|_\eta^{\beta + \min(\alpha, 1) - \delta} \end{cases}$$

for some  $\eta_1 \in (0, \eta_0)$  and  $C > 0$  and  $\delta < \beta + \min(2 - \alpha, 1)$ . We also make the following additional assumptions in the two following subcases:

- (ii)-(a) Case  $\alpha = 2 + \beta$ : There are  $\mathbf{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , satisfying  $\lim_{\eta \rightarrow 0} \mathbf{a}(\eta) = 0$  and  $\Omega : \mathbb{R}^d \rightarrow \mathbb{R}$  locally integrable such that

$$\begin{cases} \left| \int_{1 \geq |u| \geq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \left[ \operatorname{Im} \Phi_\eta(u) - \operatorname{Im} \Phi(u) \right] |u|_\eta^{-d-\alpha} du \right| \leq \mathbf{a}(\eta) |\ln(\eta)|, \\ \forall \sigma' \in \mathbb{S}^{d-1}, \quad \frac{\operatorname{Im} \Phi(\lambda \sigma')}{\lambda^{1+\beta}} \xrightarrow{\lambda \neq 0} \Omega(\sigma') \quad \text{in } L^1(\mathbb{S}^{d-1}). \end{cases}$$

- (ii)-(b) Case  $\alpha \in (-\beta, \beta]$ : The additional following integral control holds:

$$(1.12) \quad \int_{|u| \geq 1} |\Phi_\eta(u)|^2 |u|_\eta^{-d-\alpha+\beta} du \lesssim 1.$$

Note that in (1.11),  $|u|_\eta^{C\mu(\eta)} \sim 1$  as  $\eta \rightarrow 0$  in the region  $|u| \lesssim \eta^{\frac{1}{1+\beta}}$ . Note also that (1.11) and (1.9) imply  $\Phi(0) = 1$ . The case (a) in (ii) above is subtle and made necessary by the fact that the case  $\alpha = 2 + \beta$  is borderline between two different regimes (standard diffusion vs. fractional diffusion) as well as borderline between two different scalings for obtaining the diffusion coefficient (fluid mode in variable  $v$  vs. fluid mode in the rescaled variable  $u = \eta^{-\frac{1}{1+\beta}} v$ ).

With these four hypotheses we can characterise the precise scaling of the fluid eigenvalue:

**Lemma 1.2** (Rescaled limit of the fluid eigenvalue). *Assume Hypotheses 1–2–3–4. The eigenvalue  $\mu(\eta)$  constructed in Lemma 1.1 satisfies (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypotheses)*

$$(1.13) \quad \mu(\eta) \sim_{\eta \rightarrow 0} \mu_0 \Theta(\eta),$$

where the constant  $\mu_0 \in (\mathbb{R}_0, \mathbb{R}_1)$  is positive and determined as follows:

$$\left\{ \begin{array}{l} \mu_0 := \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \mathcal{M}(v) \, dv \quad \text{when } \alpha > 2 + \beta, \\ \text{where } F = \lim_{\eta \rightarrow 0} \frac{\text{Im } \phi_\eta}{\eta} \text{ is solution to } LF = -(v \cdot \sigma) \text{ and } \int_{\mathbb{R}^d} F(v) \mathcal{M}_\beta(v) \, dv = 0, \\ \mu_0 := \frac{c_{2+\beta, \beta}}{1 + \beta} \int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma') \, d\sigma' \quad \text{when } \alpha = 2 + \beta, \\ \text{where } \Omega(u) = \lim_{\lambda \rightarrow 0, \lambda \neq 0} \frac{\text{Im } \Phi(\lambda u)}{\lambda^{1+\beta}} \text{ and } \Phi = \lim_{\eta \rightarrow 0} \Phi_\eta = \lim_{\eta \rightarrow 0} \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} \cdot \right), \\ \mu_0 := c_{\alpha, \beta} \int_{\mathbb{R}^d} (u \cdot \sigma) \text{Im } \Phi(u) |u|^{-d-\alpha} \, du \quad \text{when } \alpha \in (-\beta, 2 + \beta). \end{array} \right.$$

Note how, when  $\alpha > 2 + \beta$ , the function  $F$  used in the previous works on standard diffusive limit (usually with  $\beta = 0$ ) is recovered here as a limit of our fluid mode; this allows our proof to track the convergence rate.

We now assume  $\alpha \geq 0$  and define the *diffusion exponent*

$$(1.14) \quad \zeta = \zeta(\alpha, \beta) := \begin{cases} 2 & \text{when } \alpha \geq 2 + \beta \\ \frac{\alpha + \beta}{1 + \beta} & \text{when } \alpha < 2 + \beta, \end{cases}$$

and the *scaling function*

$$(1.15) \quad \theta(\varepsilon) := \begin{cases} \varepsilon^\zeta & \text{when } \alpha \in (0, +\infty] \setminus \{2 + \beta\}, \\ \varepsilon^2 |\ln \varepsilon| & \text{when } \alpha = 2 + \beta, \\ \frac{\varepsilon^{\frac{\beta}{1+\beta}}}{|\ln \varepsilon|} & \text{when } \alpha = 0. \end{cases}$$

Note that the threshold  $\alpha = 2 + \beta$  between standard and fractional diffusion corresponds to whether or not  $\mathcal{M}_\beta$  has finite variance. We finally derive the *diffusion coefficient*:

**Lemma 1.3** (Diffusion coefficient). *Assume Hypotheses 1–2–3–4 and  $\alpha \geq 0$ . Then the following limit holds true with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypotheses: for any  $\xi \in \mathbb{R}^d \setminus \{0\}$ ,*

$$(1.16) \quad \kappa := \lim_{\varepsilon \rightarrow 0} \left( \frac{\mu(\varepsilon|\xi)|\xi|^{-\zeta}}{\theta(\varepsilon) \langle 1, \phi_{\varepsilon|\xi} \rangle} \right) = \mu_0 \times \begin{cases} \|\mathcal{M}\|_{L^1(\mathbb{R}^d)}^{-1} & \text{when } \alpha > 0, \\ \frac{1 + \beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha = 0. \end{cases}$$

The diffusion coefficient thus emerges from ratios between (rescaled) integrals as follows:

$$(1.17) \quad \kappa := \begin{cases} \frac{\int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \mathcal{M}(v) dv}{\|\mathcal{M}\|_{L^1(\mathbb{R}^d)}} & \text{when } \alpha > 2 + \beta \\ \frac{1}{1 + \beta} \frac{\int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma') d\sigma'}{\int_{\mathbb{R}^d} [v]^{-d-\alpha} dv} & \text{when } \alpha = 2 + \beta \\ \frac{\int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha} du}{\int_{\mathbb{R}^d} [v]^{-d-\alpha} dv} & \text{when } \alpha \in (0, 2 + \beta) \\ \frac{1 + \beta}{|\mathbb{S}^{d-1}|} \frac{\int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha} du}{\int_{\mathbb{R}^d} [v]^{-d-\alpha-\beta} dv} & \text{when } \alpha = 0 \end{cases}$$

where we recall

$$F = \lim_{\eta \rightarrow 0} \frac{\operatorname{Im} \phi_\eta}{\eta}, \quad \Phi = \lim_{\eta \rightarrow 0} \Phi_\eta = \lim_{\eta \rightarrow 0} \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} \cdot \right), \quad \Omega(u) = \lim_{\lambda \rightarrow 0, \lambda \neq 0} \frac{\operatorname{Im} \Phi(\lambda u)}{\lambda^{1+\beta}},$$

and (when  $\alpha > 2 + \beta$ )  $F$  is also the unique solution to  $LF = -(v \cdot \sigma)$  with  $\int_{\mathbb{R}^d} F(v) [v]^{-d-\alpha-\beta} dv = 0$ . For legibility again, we wrote, in the cases  $\alpha \in [0, 2 + \beta]$ , the formula for  $\kappa$  with  $\mathcal{M}$  given by (1.7), and we refer to Section 9 for more general  $\mathcal{M}$ 's. The proof of Lemma 1.3 is done in Section 5; it requires the estimating of  $\langle 1, \phi_\eta \rangle$ , which is done in Lemma 5.1.

Consider a solution  $f$  in  $L_t^\infty([0, +\infty); L_{x,v}^2(\mathcal{M}^{-1}))$  to (1.1) with initial data  $f_{\text{in}}^{(\varepsilon)}$ . Note that the initial data  $f_{\text{in}}^{(\varepsilon)}$ , before the rescaling, is allowed to depend on  $\varepsilon$ . Given  $\varepsilon > 0$  and  $\theta(\varepsilon)$  defined in (1.15), we rescale the solution and define a weighted rescaled spatial density:

$$f_\varepsilon(t, x, v) := \frac{1}{\varepsilon^d} f \left( \frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \in L_t^\infty([0, +\infty); L_{x,v}^2(\mathcal{M}^{-1}))$$

$$r_\varepsilon(t, x) := \int_{\mathbb{R}^d} f_\varepsilon(t, x, v) [v]^{-\beta} dv.$$

The equation satisfied by  $f_\varepsilon$  is

$$(1.18) \quad \theta(\varepsilon) \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon.$$

The *rescaled* initial data is then  $f_\varepsilon(0, x, v) = \varepsilon^{-d} f_{\text{in}}^{(\varepsilon)}(\varepsilon^{-1} x, v)$ , and in the following theorem we assume the original initial data  $f_{\text{in}}^{(\varepsilon)}$  to be *well-prepared* (see (1.19)-(1.20)-(1.21)): this means that the fluid limit holds at time zero with  $f_\varepsilon(0, \cdot) \sim r_\varepsilon(0, \cdot) \mathcal{M}$  and  $r_\varepsilon \sim r(0, \cdot)$  which provides the initial data for the limit equation; this is standard in the literature. We however note that when (1.19) is satisfied but (1.20)-(1.21) are not imposed at  $t = 0$ , the energy estimate and compactness arguments on  $r_\varepsilon$  would imply that (1.20)-(1.21) are satisfied at any later positive time  $\tau > 0$  (without information on the rate though), and our method would prove the fluid approximation for  $t \geq \tau$ . This would allow us for instance to choose  $f_{\text{in}}^{(\varepsilon)} = r \mathcal{M}$  independent of  $\varepsilon$ . We however kept the assumptions (1.20)-(1.21) in order to precisely track the rate of convergence and the initial data of the limit equation.

**Theorem 1.4** (Unified second fluid approximation, see Figure 2). *Assume Hypotheses 1–2–3–4,  $\alpha \geq 0$ , and consider  $f_\varepsilon \in L_t^\infty([0, +\infty); L_{x,v}^2(\mathcal{M}^{-1}))$  solving (1.1) in the weak sense with initially*

$$(1.19) \quad \left\| \frac{f_\varepsilon(0, \cdot, \cdot)}{\mathcal{M}} \right\|_0 = o \left( \begin{cases} \theta(\varepsilon)^{-\frac{1}{2}}, & \text{when } \alpha > \beta, \\ \varepsilon^{-\frac{\beta}{1+\beta}} |\ln(\varepsilon)|^{-\frac{1}{2}} & \text{when } \alpha = \beta, \\ \varepsilon^{-\frac{\alpha}{1+\beta}} & \text{when } \alpha \in (0, \beta), \\ |\ln(\varepsilon)|^{\frac{3}{2}} & \text{when } \alpha = 0, \end{cases} \right)$$

and

$$(1.20) \quad \left\| \frac{f_\varepsilon}{\mathcal{M}}(0, \cdot, \cdot) - r_\varepsilon(0, \cdot) \right\|_{-\beta} = o \left( \begin{cases} 1, & \text{when } \alpha > \beta, \\ |\ln(\varepsilon)|^{-\frac{1}{2}} & \text{when } \alpha = \beta, \\ \varepsilon^{\frac{\beta-\alpha}{2(1+\beta)}} & \text{when } \alpha \in (0, \beta), \\ \varepsilon^{\frac{\beta}{2(1+\beta)}} |\ln(\varepsilon)| & \text{when } \alpha = 0, \end{cases} \right),$$

and (recalling the definition of  $\zeta$  in (1.14))

$$(1.21) \quad r_\varepsilon(0, \cdot) \xrightarrow[\varepsilon \rightarrow 0]{H^{-\zeta}(\mathbb{R}^d)} r(0, \cdot).$$

Then for any  $T > 0$

$$(1.22) \quad \left\| \frac{f_\varepsilon}{\mathcal{M}} - r \right\|_{L_t^2([0, T]; H_x^{-\zeta} L_v^2(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

when  $\alpha > \beta$  and

$$\left\| \left| \ln \frac{2|\nabla_x|}{1+|\nabla_x|} \right| \left( \frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L_t^2([0, T]; H_x^{-\zeta} L_v^2(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

when  $\alpha = \beta$  and

$$\left\| |\nabla_x|^{\frac{\beta-|\alpha|}{2(1+\beta)}} [\nabla_x]^{-\frac{\beta-|\alpha|}{2(1+\beta)}} \left( \frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L_t^2([0, T]; H_x^{-\zeta} L_v^2(\mathcal{M}_\beta))} \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

when  $\alpha \in (0, \beta)$ , where  $r = r(t, x)$  solves

$$\partial_t r = \kappa \Delta_x^{\frac{\zeta}{2}} r, \quad t > 0, \quad \text{with initial data } r(0, \cdot) \text{ defined in (1.21)}.$$

The rates of convergence are estimated in terms of  $T$ , the constants, error terms and convergence rates in Hypotheses 1–2–3–4, and the initial convergence rates (1.19)–(1.20)–(1.21). Apart from (1.21), the errors we obtain are polynomial in  $\varepsilon$  for  $\alpha \in (-\beta, +\infty) \setminus \{0, 2 + \beta\}$  and logarithmic for  $\alpha \in \{0, 2 + \beta\}$ .

This theorem is the core contribution of the paper, and is used to obtain results on concrete models in the corollaries below. Together with Lemmas 1.1–1.2–1.3, it reveals the relevant macroscopic scales for a large class of operators in any dimension and provides a unified theoretical framework to answer questions of the last decades on the topic. The diffusive limit is reduced to a spectral problem—the construction of the fluid mode—that we solve in a general setting. The proof is constructive and the key constants governing the macroscopic behaviours are derived. The fractional Laplacian in the space variable is defined as in (1.5), and  $r(t, x)$  is the limit (in the topology of the above theorem) of the weighted velocity average

$$r_\varepsilon(t, x) = \int_{\mathbb{R}^d} f \left( \frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) [v]^{-\beta} dv.$$

When  $\alpha > 0$ , the density  $\rho_\varepsilon(t, x) := \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) dv$  exists. When  $\alpha > \beta$ , it is straightforward that  $\rho_\varepsilon \rightarrow \|\mathcal{M}\|_{L_v^1} r$  by Hölder's inequality and the convergence (1.22). When  $\alpha \in (0, \beta)$ , the latter still holds under the slightly stronger assumption  $|f_\varepsilon(0, x, \cdot)| \leq C\mathcal{M}$  for some  $C > 0$  on the initial data. Indeed then  $|f_\varepsilon(t, x, \cdot)| \leq C\mathcal{M}$  by the comparison principle, and

$$\forall t \geq 0, x \in \mathbb{R}^d, \quad \left\| \frac{f_\varepsilon(t, x, \cdot)}{\mathcal{M}(\cdot)} - r \right\|_{L_v^2(\mathcal{M})} \lesssim 1$$

so we have by Hölder's inequality

$$\begin{aligned} |\rho_\varepsilon(t, x) - \|\mathcal{M}\|_{L_v^1} r(t, x)| &= \left| \int_{\mathbb{R}^d} (f_\varepsilon(t, x, v) - r\mathcal{M}) dv \right| \\ &\leq \left\| \mathcal{M}(\cdot) [\cdot]^{\frac{\alpha}{2}} \right\| \times \left\| \frac{f_\varepsilon(t, x, \cdot)}{\mathcal{M}(\cdot)} - r \right\|_{L_v^2(\mathcal{M}_{\frac{\alpha}{2}})} \\ &\leq \left\| \mathcal{M}(\cdot) [\cdot]^{\frac{\alpha}{2}} \right\| \left\| \frac{f_\varepsilon(t, x, \cdot)}{\mathcal{M}(\cdot)} - r \right\|_{L_v^2(\mathcal{M})}^{1-\frac{2\beta}{\alpha}} \left\| \frac{f_\varepsilon(t, x, \cdot)}{\mathcal{M}(\cdot)} - r \right\|_{L_v^2(\mathcal{M}_\beta)}^{\frac{2\beta}{\alpha}} \\ &\lesssim \left\| \frac{f_\varepsilon(t, x, \cdot)}{\mathcal{M}(\cdot)} - r \right\|_{L_v^2(\mathcal{M}_\beta)}^{\frac{2\beta}{\alpha}} \end{aligned}$$

which implies  $\rho_\varepsilon \rightarrow \|\mathcal{M}\|_{L_v^1} r$  by integrating against decaying test functions in  $x$  and using (1.22).

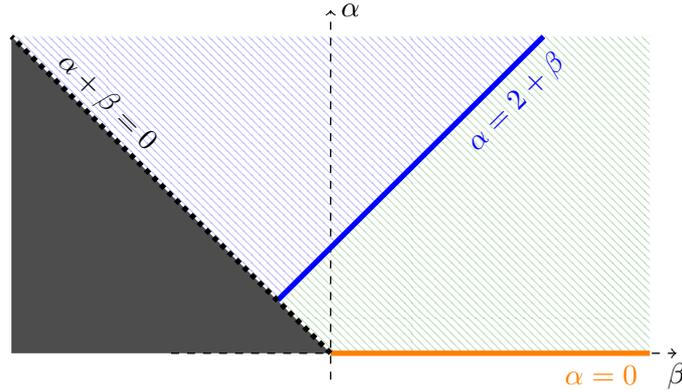


FIGURE 2. Summary of the results in the  $(\alpha, \beta)$  plane. Admissible parameters are in the half-plane  $\alpha + \beta > 0$ . The blue hatched area leads to  $\theta(\varepsilon) = \varepsilon^2$  and a standard diffusive limit with symbol  $\kappa|\xi|^2$ . The blue line is the set of parameters yielding the anomalous scaling  $\theta(\varepsilon) = \varepsilon^2 |\ln(\varepsilon)|$  but still a standard diffusive limit with symbol  $\kappa|\xi|^2$ . The green hatched area results into the fractional scaling  $\theta(\varepsilon) = \varepsilon^{\frac{\alpha+\beta}{1+\beta}}$  and a fractional diffusive limit with symbol  $\kappa|\xi|^{\frac{\alpha+\beta}{1+\beta}}$ . The orange bold line yields the fractional scaling  $\theta(\varepsilon) = \varepsilon^{\frac{\beta}{1+\beta}} |\ln(\varepsilon)|^{-1}$  and a fractional diffusive limit with symbol  $\kappa|\xi|^{\frac{\beta}{1+\beta}}$ .

We now apply the abstract Theorem 1.4 to particular models:

**Corollary 1.5** (Scattering equation). *Assume that  $\mathcal{L}$  is the scattering operator (1.2) with  $b \in C^1$  and  $\mathcal{M}$  satisfying Hypothesis 1 and that, for some constant  $\nu_0 > 0$  and  $\alpha \geq 0$  and  $\alpha + \beta > 0$ ,*

$$(1.23) \quad \begin{cases} \forall v \in \mathbb{R}^d, & [v]^{-\beta} \lesssim \nu(v) \lesssim [v]^{-\beta} \\ \forall v \in \mathbb{R}^d \setminus \{0\}, & \lambda^\beta \nu(\lambda v) \sim_{\lambda \rightarrow \infty} \nu_0 |v|^{-\beta} \\ \forall v \in \mathbb{R}^d, & \|b(v, \cdot)\|_\beta + \|b(\cdot, v)\|_\beta \lesssim [v]^{-\beta}. \end{cases}$$

This includes  $b(v, v') = [v]^{-\beta} [v']^{-\beta}$ , and  $b(v, v') = [v - v']^{-\beta}$  when  $\beta < 0$  and  $\alpha + \beta > 0$  or when  $\beta \geq 0$  and  $\alpha > 3\beta$ . Then Theorem 1.4 applies with  $\alpha, \beta$  given in Hypotheses 1 and (1.23). This proves the diffusive limit for solutions to (1.18) satisfying (1.19)-(1.20)-(1.21) with quantitative rate, diffusion exponent  $\zeta = \frac{\alpha + \beta}{1 + \beta}$ , scaling function (1.15) and diffusion coefficient (1.17). Apart from (1.21), the errors we obtain are polynomial in  $\varepsilon$  for  $\alpha \in (0, +\infty) \setminus \{2 + \beta\}$  and logarithmic for  $\alpha \in \{0, 2 + \beta\}$ . Moreover the diffusion coefficient can be computed explicitly with  $F(u) = \nu(v)^{-1} (v \cdot \sigma)$  when  $\alpha > 2 + \beta$ ,  $\Omega(u) = \nu_0^{-1} |u|^\beta (u \cdot \sigma)$  when  $\alpha = 2 + \beta$  and

$$\Phi(u) = \frac{\nu_0}{\nu_0 - i|u|^\beta (u \cdot \sigma)}$$

when  $\alpha \in (-\beta, 2 + \beta)$ , resulting in

$$\kappa := \begin{cases} \frac{\int_{\mathbb{R}^d} (v \cdot \sigma)^2 \nu(v)^{-1} [v]^{-d-\alpha} dv}{\int_{\mathbb{R}^d} [v]^{-d-\alpha} dv} & \text{when } \alpha \in (2 + \beta, +\infty) \\ \frac{1}{\nu_0(1 + \beta)} \frac{\int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma')^2 d\sigma'}{\int_{\mathbb{R}^d} [v]^{-d-\alpha} dv} & \text{when } \alpha = 2 + \beta \\ \frac{\int_{\mathbb{R}^d} \frac{\nu_0 |u|^\beta (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta} (u \cdot \sigma)^2} |u|^{d+\alpha} du}{\int_{\mathbb{R}^d} [v]^{-d-\alpha} dv} & \text{when } \alpha \in (0, 2 + \beta) \\ \frac{(1 + \beta)}{|\mathbb{S}^{d-1}|} \frac{\int_{\mathbb{R}^d} \frac{\nu_0 |u|^\beta (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta} (u \cdot \sigma)^2} |u|^d du}{\int_{\mathbb{R}^d} [v]^{-d-\beta} dv} & \text{when } \alpha = 0 \end{cases}$$

as well as  $\kappa := \|\mathcal{M}\|_{L^1(\mathbb{R}^d)}^{-1} \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \nu^{-1} \mathcal{M}(v) dv$  for the case “ $\alpha = +\infty$ ”.

This recovers and unifies the results in [32, 42, 6, 21, 41] (except for the case of space-dependent collision kernels in [21]) and extends them to new cases such as  $\alpha = 0$  (infinite mass). The convergence rate is also new. Our approach shares common points with, but differs from the probabilistic method in [32], the Hilbert expansions in [6, 21], the moment method in [41] and the Fourier-Laplace method in [42].

**Corollary 1.6** (Kinetic Fokker-Planck equation). *Assume that  $\mathcal{L}$  is the Fokker-Planck operator (1.3) with  $\mathcal{M}$  satisfying Hypothesis 1 with  $\alpha \geq 0$ . Then Theorem 1.4 applies with  $\alpha$  given in Hypothesis 1 and  $\beta = 2$ . This proves the diffusive limit for solutions to (1.18) satisfying (1.19)-(1.20)-(1.21) with quantitative rate, diffusion exponent  $\zeta = \min(2, \frac{\alpha+2}{3})$ , scaling function (1.15) and diffusion coefficient (1.17). Apart from (1.21), the errors we obtain are polynomial in  $\varepsilon$  for  $\alpha \in (0, +\infty) \setminus \{4\}$  and logarithmic for  $\alpha = 4$ . The diffusion coefficient can be made precise when  $\alpha \in (0, 4]$  using that  $\Phi$  solves the Schrödinger-type equation*

$$-|u|^2 \Delta_u \Phi + (d + \alpha) u \cdot \nabla_u \Phi - i(u \cdot \sigma) |u|^2 \Phi = 0 \quad \text{with the normalisation } \Phi(0) = 1.$$

In particular when  $\alpha = 4$ , the function  $\Omega$  solves

$$-|u|^2 \Delta_u \Omega + (d + \alpha) u \cdot \nabla_u \Omega = (u \cdot \sigma) |u|^2 \quad \text{with} \quad \Omega(0) = 0 \quad \implies \quad \Omega(u) := \frac{|u|^2 (u \cdot \sigma)}{d + 8}.$$

This recovers and unifies the fractional diffusive limit results in [15, 43, 27, 26, 36]. Novel contributions include formulas for the diffusion coefficient in dimension higher than 1, and the quantitative argument providing a convergence rate.

**Corollary 1.7** (Kinetic Lévy-Fokker-Planck equation). *Assume that  $\mathcal{L}$  is the Lévy-Fokker-Planck operator (1.4) with parameter  $s \in (\frac{1}{2}, 1)$  and with  $\mathcal{M}$  satisfying Hypothesis 1 with  $\alpha > s$ . Then Theorem 1.4 applies with  $\beta := 2s - \alpha$ . This proves the diffusive limit for solutions to (1.18) satisfying (1.19)-(1.20)-(1.21) with quantitative rate and diffusion exponent*

$$\zeta = \begin{cases} 2 & \text{when } \alpha \geq 1 + s \\ \frac{2s}{1 + 2s - \alpha} & \text{when } \alpha \in (s, 1 + s), \end{cases}$$

and scaling function (1.15) and diffusion coefficient (1.17). Apart from (1.21), the errors we obtain are polynomial in  $\varepsilon$  when  $\alpha \in (s, 1 + s) \cup (1 + s, +\infty)$  and logarithmic for  $\alpha = 1 + s$ . Moreover

$$\Phi(u) := \exp\left(i \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u \cdot \sigma)}{1 + \beta}\right)$$

when  $\alpha \in (s, 1 + s]$  and

$$\Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u \cdot \sigma)}{1 + \beta}$$

when  $\alpha = 1 + s$ , which yields for the diffusion coefficient

$$\kappa := \begin{cases} \frac{2sc_{\alpha,0}^2}{c_{\alpha,\beta}(1 + \beta)^2} \int_{\mathbb{S}^{d-1}} (\sigma' \cdot \sigma)^2 d\sigma' & \boxed{\text{when } \alpha = 1 + s} \\ \frac{c_{\alpha,0}}{1 + \beta} \left( \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}(1 + \beta)} \right)^{\frac{\alpha-1}{1+\beta}} \int_{\mathbb{R}^d} (w \cdot \sigma) \sin(w \cdot \sigma) \frac{dw}{|w|^{d + \frac{\alpha+\beta}{1+\beta}}} & \boxed{\text{when } \alpha \in (s, 1 + s)} \end{cases}$$

This recovers and extends the qualitative results in [1, 18] to general equilibria, with quantitative error estimates and formulas for the diffusion coefficient. In [1, 18], the moment method initiated by Mellet is used to derive a fractional limit in the case  $\beta = 0$ . It raises several interesting questions: (1) can our approach be extended to  $s \in (0, \frac{1}{2})$ ? (this seems to be a technical difficulty), (2) is the fractional diffusive limit possible for infinite mass equilibria? (i.e.  $\alpha < 0$ ), (3) can the connexion between the kinetic Lévy-Fokker-Planck equation with  $\alpha = 2s$  (for which the  $\mathcal{L}$  is the generator of a Lévy process) and the standard kinetic Fokker-Planck equation with Gaussian equilibrium be clarified as  $s \rightarrow 1$ ? (our diffusion constant  $\kappa$  above diverges as  $s \rightarrow 1$  so the two limits in  $\varepsilon \rightarrow 0$  and  $s \rightarrow 1$  do not commute which calls for further investigation).

Let us summarise our contributions. Theorem 1.4 and Corollaries 1.5–1.6–1.7 recover the results of [1, 6, 15, 21, 27, 26, 36, 41, 42, 43] with a shorter and unified constructive method and prove new results for (1) Lévy-Fokker-Planck operators, (2) scattering operators with decaying collision kernel and infinite mass equilibria and importantly (3) Fokker-Planck operators in any dimension (for which formulas for the diffusion coefficient were not known). The quantitative error in this fluid approximation seems to also be novel for all equations considered. Note finally that like the abstract Theorem 1.4, Corollaries 1.5–1.6–1.7 are stated with the exact equilibrium of Hypothesis 1, but can be extended to more general equilibria, see Section 9. Moreover, it would be interesting to try and apply this method in other settings such as [4, 28] (radiative transfer theory), [11, 22, 30] (rarefied gas in a region between two parallel plates), [35, 16, 17] (domains with interface or boundaries), [2] (scattering with external acceleration field), [45] (models for chemotaxis) and [33, 31, 34] (adding a local conservation of momentum).

The method of the present paper extends to the fractional diffusive limit the approach pioneered in [44, 24] of constructing exact dispersion laws in the regime of parabolic time-space scaling and small eigenvalues; this extension is inspired by the recent one-dimensional result [36] and in particular we use and generalise the idea of rescaling velocities to obtain a non-trivial dispersion law in [36]. In comparison with [36], the main novelty of the present paper is a quantitative spectral method for constructing the branch of fluid eigenvalue: in [36] it was done by a one-dimensional argument connecting two infinite series on  $\mathbb{R}_-$  and  $\mathbb{R}_+$  (and it was done by fixed points in the simpler case of classical diffusive limit in the older works [44, 24]).

Let us now compare our paper with the previous recent works by probabilists [27, 26]. In probabilistic terms, we try to describe particles moving in the full  $d$ -dimensional space along  $dX_t = V_t dt$  with velocities  $V_t$  following a reversible process with invariant measure of the form given in (1). The velocity process is typically of scattering type, or Langevin type with drift and Brownian or non-Gaussian Lévy-type noises. We show in Theorem 1.4 that the rescaled process  $\varepsilon X_{\theta(\varepsilon)^{-1}t}$  converges, with explicit rates and multiplicative constants, towards a Brownian motion when  $\alpha \geq 2 + \beta$ , and towards a radially symmetric  $\zeta$ -stable process when  $\alpha \in [0, 2 + \beta)$ . In spite of using quite different languages, the common point between [27, 26] and the present paper is the use of a scaling in velocity, which corresponds to applying some power function to the random variable in the probabilistic viewpoint and corresponds to the study of the rescaled fluid mode  $\Phi_\eta := \phi_\eta(\eta^{-\frac{1}{1+\beta}} \cdot)$  in our study. Note finally that the eigenvalue problem we study to compute the limit diffusion coefficient does not seem to have a counterpart in [27, 26].

The rest of the paper is structured as follows. Section 2 is devoted to the proof of Theorem 1.4 assuming Lemmas 1.1, 1.2 and 1.3. We then prove Lemma 1.1 (construction of the fluid mode) in Section 3, Lemma 1.2 (scaling of the fluid mode) in Section 4, and Lemma 1.3 (derivation of the diffusion coefficient) in Section 5. Sections 6-7-8 prove the abstract hypotheses on the three concrete models; one argument of independent interest is a tightness estimate for the Schrödinger-type equation satisfied by the rescaled fluid mode in the cases of Fokker-Planck operators, see Lemma 7.3. Finally, Section 9 briefly discusses extensions of our results to more general equilibrium distributions and operators.

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## 2. PROOF OF THEOREM 1.4 (CONVERGENCE)

In this section we assume the Lemmas 1.1, 1.2 and 1.3 to hold,  $\alpha \geq 0$ , and prove Theorem 1.4. Consider (1.6) and the rescaling

$$h_\varepsilon(t, x, v) := h\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) = \frac{f_\varepsilon(t, x, v)}{\mathcal{M}(v)} = \frac{f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right)}{\mathcal{M}(v)}.$$

It satisfies the equation

$$(2.1) \quad \theta(\varepsilon)\partial_t h_\varepsilon + \varepsilon v \cdot \nabla_x h_\varepsilon = Lh_\varepsilon.$$

**2.1. The energy estimate.** Integrate (2.1) against  $h_\varepsilon \mathcal{M}$  in  $t, x, v$ , and take the real part:

$$\begin{aligned} \frac{\theta(\varepsilon)}{2} \|h_\varepsilon(t)\|^2 &= \frac{\theta(\varepsilon)}{2} \|h_\varepsilon(0)\|^2 + \int_0^t \operatorname{Re} \langle Lh_\varepsilon(\tau), h_\varepsilon(\tau) \rangle d\tau \\ &\leq \frac{\theta(\varepsilon)}{2} \|h_\varepsilon(0)\|^2 - \lambda \int_0^t \|h_\varepsilon(\tau) - r_\varepsilon(\tau)\|_{-\beta}^2 d\tau \end{aligned}$$

where we have used Hypothesis 2 and

$$r_\varepsilon(t, x) := \int_{\mathbb{R}^d} h_\varepsilon(t, x, v) \mathcal{M}_\beta(v) dv.$$

This proves

$$(2.2) \quad \forall t \geq 0, \quad \|h_\varepsilon(t)\|^2 \leq \|h_\varepsilon(0)\|^2 \quad \text{and} \quad \int_0^t \|h_\varepsilon(\tau) - r_\varepsilon(\tau)\|_{-\beta}^2 d\tau \leq \frac{\theta(\varepsilon)}{2\lambda} \|h_\varepsilon(0)\|^2.$$

**2.2. Framework of the calculations.** Denote  $\xi$  to be the Fourier variable of  $x$ , and Fourier-transform (2.1) in  $x$  to get on  $\hat{h}_\varepsilon(t, \xi, v)$

$$(2.3) \quad \theta(\varepsilon) \partial_t \hat{h}_\varepsilon = L \hat{h}_\varepsilon - i\varepsilon(v \cdot \xi) \hat{h}_\varepsilon.$$

Note that (2.2) and the Plancherel theorem imply  $\hat{h}_\varepsilon \in L_t^\infty(\mathbb{R}^+; L_{\xi, v}^2(\mathcal{M}))$  and

$$(2.4) \quad \|\hat{h}_\varepsilon - \hat{r}_\varepsilon\|_{L_t^2(\mathbb{R}^+; L_{\xi, v}^2(\mathcal{M}_\beta))} \lesssim \theta(\varepsilon)^{\frac{1}{2}} \|h_\varepsilon(0)\|.$$

Denote  $\xi =: |\xi|\sigma$  and  $\eta := \varepsilon|\xi|$ . Test (2.3) against  $\mathcal{M}\phi_\eta$  with  $\phi_\eta$  constructed in Lemma 1.1:

$$(2.5) \quad \begin{aligned} \theta(\varepsilon) \frac{d}{dt} \langle \hat{h}_\varepsilon, \phi_\eta \rangle &= \langle L \hat{h}_\varepsilon - i\varepsilon(v \cdot \xi) \hat{h}_\varepsilon, \phi_\eta \rangle = \langle \hat{h}_\varepsilon, L^*(\phi_\eta) + i\varepsilon(v \cdot \xi) \phi_\eta \rangle \\ &= -\mu(\eta) \langle \hat{h}_\varepsilon, [v]^{-\beta} \phi_\eta \rangle. \end{aligned}$$

We then split the integrals as follows:

$$\langle \hat{h}_\varepsilon, \phi_\eta \rangle = \hat{r}_\varepsilon \langle 1, \phi_\eta \rangle + \langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \phi_\eta \rangle =: \langle 1, \phi_\eta \rangle [\hat{r}_\varepsilon - E_1]$$

$$\langle \hat{h}_\varepsilon, [v]^{-\beta} \phi_\eta \rangle = \hat{r}_\varepsilon \langle 1, [v]^{-\beta} \phi_\eta \rangle + \langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, [v]^{-\beta} \phi_\eta \rangle =: \langle 1, \phi_\eta \rangle \frac{\theta(\varepsilon)}{\mu(\eta)} [\kappa_\eta \hat{r}_\varepsilon - E_2]$$

with the definitions (using the normalisation  $\langle 1, [v]^{-\beta} \phi_\eta \rangle = 1$ )

$$\begin{aligned} \kappa_\eta &:= \frac{\mu(\eta) \langle 1, [v]^{-\beta} \phi_\eta \rangle}{\theta(\varepsilon) \langle 1, \phi_\eta \rangle} = \frac{\mu(\eta)}{\theta(\varepsilon) \langle 1, \phi_\eta \rangle}, \\ E_1 &:= -\frac{\langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \phi_\eta \rangle}{\langle 1, \phi_\eta \rangle} \quad \text{and} \quad E_2 := -\frac{\mu(\eta) \langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, [v]^{-\beta} \phi_\eta \rangle}{\theta(\varepsilon) \langle 1, \phi_\eta \rangle}. \end{aligned}$$

Consequently, (2.5) gives

$$\partial_t \hat{r}_\varepsilon + \kappa_\eta \hat{r}_\varepsilon = \partial_t E_1 + E_2.$$

We then want to pass to the limit  $\varepsilon \rightarrow 0$  (hence  $\eta \rightarrow 0$  for each frequency  $\xi$ ).

**2.3. Estimating  $\kappa_\eta$ ,  $\partial_t E_1$  and  $E_2$ .** Lemma 1.1-1.2 yield

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu(\eta)}{\theta(\varepsilon)} = \mu_0 |\xi|^\zeta \quad \text{with} \quad \zeta := \frac{\alpha + \beta}{1 + \beta},$$

with constructive rate, for each frequency  $\xi \in \mathbb{R}^d$  (note that in the cases  $\alpha = 0$  or  $\alpha = 2 + \beta$ , the error in the convergence includes a loss of frequency weight  $|\ln |\xi||$ ). Lemma 1.3 implies

$$(2.6) \quad \lim_{\eta \rightarrow 0} \kappa_\eta = \kappa |\xi|^\zeta = \mu_0 |\xi|^\zeta \times \begin{cases} \|\mathcal{M}\|_{L^1(\mathbb{R}^d)}^{-1} & \text{when } \alpha > 0, \\ \frac{1 + \beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha = 0, \end{cases}$$

with constructive convergence rate and

$$\frac{\mu(\eta)}{\theta(\varepsilon) |\langle 1, \phi_\eta \rangle|} \lesssim |\xi|^\zeta.$$

To estimate  $E_2$ , write

$$\left| \langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, [v]^{-\beta} \phi_\eta \rangle \right| \lesssim \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta}$$

where we have used  $\|\phi_\eta\|_{-\beta} = 1$ . All in all, we get, using again Lemmas 1.3 and 5.1,

$$(2.7) \quad |E_2| \lesssim \frac{\mu(\eta)}{\theta(\varepsilon) |\langle 1, \phi_\eta \rangle|} \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta} \lesssim \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta} |\xi|^\zeta.$$

To estimate  $E_1$ , compute first

$$\left| \langle \hat{h}_\varepsilon - \hat{r}_\varepsilon, \phi_\eta \rangle \right| \leq \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta} \|\phi_\eta\|_\beta,$$

to get

$$|E_1| \lesssim \frac{\|\phi_\eta\|_\beta}{|\langle 1, \phi_\eta \rangle|} \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta}.$$

One then estimates  $\|\phi_\eta\|_\beta$ . When  $\alpha > \beta$ , it is bounded by construction, and when  $\alpha \leq \beta$ ,

$$\|\phi_\eta\|_\beta^2 = \eta^{\frac{\alpha-\beta}{1+\beta}} \int_{\mathbb{R}^d} |\Phi_\eta(u)|^2 |u|_\eta^{-d-\alpha+\beta} du.$$

Using the pointwise bound (1.11) and the moment bound (1.12) from Hypothesis 4, the latter integral exists and is uniformly bounded in  $\eta$  for  $\alpha \in [0, \beta)$  and is bounded by  $|\ln \eta|$  when  $\alpha = \beta$ . Thus we get, using Lemma 5.1 to estimate  $\langle 1, \phi_\eta \rangle$  again,

$$(2.8) \quad |E_1| \lesssim \left\| \hat{h}_\varepsilon - \hat{r}_\varepsilon \right\|_{-\beta} \times \begin{cases} 1, & \text{when } \alpha > \beta, \\ |\ln(\varepsilon|\xi|)|^{\frac{1}{2}} & \text{when } \alpha = \beta, \\ (\varepsilon|\xi|)^{\frac{\alpha-\beta}{2(1+\beta)}} & \text{when } \alpha \in (0, \beta), \\ |\ln(\varepsilon|\xi|)|^{-1} (\varepsilon|\xi|)^{-\frac{\beta}{2(1+\beta)}} & \text{when } \alpha = 0. \end{cases}$$

Set  $r := r(t, x)$  to be solution to  $\partial_t r + \kappa|\xi|^\zeta r = 0$  with initial data  $r(0, \cdot)$  defined in (1.21) and deduce that  $\omega_\varepsilon := \hat{r}_\varepsilon - \hat{r}$  satisfies

$$\partial_t \omega_\varepsilon + \kappa|\xi|^\zeta \omega_\varepsilon = \partial_t E_1 + E_2 + (\kappa - \kappa_\eta) \hat{r}_\varepsilon.$$

We assume that (1.19)-(1.20) hold. By Duhamel's formula,

$$\begin{aligned} \omega_\varepsilon(t, \xi) &= \omega_\varepsilon(0, \xi) e^{-\kappa|\xi|^\zeta t} + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)} [\partial_t E_1(s, \xi) + E_2(s, \xi) + (\kappa - \kappa_\eta) \hat{r}_\varepsilon(s, \xi)] ds \\ &= \omega_\varepsilon(0, \xi) e^{-\kappa|\xi|^\zeta t} + E_1(t, \xi) - e^{-\kappa|\xi|^\zeta t} E_1(0, \xi) \\ &\quad + \int_0^t e^{-\kappa|\xi|^\zeta(t-s)} \left[ \kappa|\xi|^\zeta E_1(s, \xi) + E_2(s, \xi) + (\kappa - \kappa_\eta) \hat{r}_\varepsilon(s, \xi) \right] ds. \end{aligned}$$

Define the weight

$$W(\xi) := |\xi|^{-\zeta} \times \begin{cases} 1 & \text{when } \alpha > \beta, \\ \left| \ln \frac{2|\xi|}{1+|\xi|} \right|^{-1} & \text{when } \alpha = \beta, \\ |\xi|^{\frac{\beta-\alpha}{2(1+\beta)}} |\xi|^{-\frac{\beta-\alpha}{2(1+\beta)}} & \text{when } \alpha \in [0, \beta). \end{cases}$$

Then we integrate in  $L_\xi^2(W)$  and then in  $L_t^2([0, T])$ : the estimates (2.2)-(2.7)-(2.8) imply that the errors  $E_1$  and  $E_2$  go to zero in  $L_t^2([0, T]; L_\xi^2(W))$ , and (1.21) ensures that  $\omega_\varepsilon(0, \xi)$  goes to zero in  $L_\xi^2(W)$ , which concludes the proof.

### 3. PROOF OF LEMMA 1.1 (CONSTRUCTION OF THE FLUID MODE)

In this section we prove Lemma 1.1, assuming Hypotheses 1–2–3 with  $\alpha + \beta > 0$ . Denote

$$\tilde{L}_\eta^* \psi := [v]^{\frac{\beta}{2}} L_\eta^* \left( [v]^{\frac{\beta}{2}} \psi \right) = [v]^{\frac{\beta}{2}} L^* \left( [v]^{\frac{\beta}{2}} \psi \right) + i\eta [v]^\beta (v \cdot \sigma) \psi.$$

As before, the dependency in  $\sigma$  is omitted from the subscripts for readability.

#### 3.1. Existence of the resolvent.

3.1.1. *Near zero.* We first prove that when  $z \in B(0, R_0\Theta(\eta))$  with  $R_0$  small enough and  $\eta \in (0, \eta_0)$  with  $\eta_0$  small enough, the operator  $\tilde{L}_\eta^* - z$  has a bounded inverse in  $L_v^2(\mathcal{M})$ . Given  $G \in L^2([\cdot]^{-\beta}\mathcal{M})$  and  $z \in B(0, R_0\Theta(\eta))$ , consider an *a priori* solution  $F \in L^2([\cdot]^{-\beta}\mathcal{M})$  to

$$(3.1) \quad -L^*F - i\eta(v \cdot \sigma)F - z[v]^{-\beta}F = [v]^{-\beta}G.$$

Recall the decomposition

$$(3.2) \quad F = \mathcal{P}F + \mathcal{P}^\perp F \quad \text{with} \quad \mathcal{P}F := \int_{\mathbb{R}^d} F(v)\mathcal{M}_\beta(v) dv,$$

which is orthogonal for the scalar product associated with  $\|\cdot\|_{-\beta}$ . Integrate (3.1) against  $\bar{F}\mathcal{M}$  and take the real part to get, using Hypothesis 2 and denoting  $r := |z|$ ,

$$\begin{aligned} & \lambda \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 - r \|F\|_{-\beta}^2 \leq \|G\|_{-\beta} \|F\|_{-\beta} \\ \implies & (\lambda - r) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \leq \|G\|_{-\beta} \|F\|_{-\beta} + r |\mathcal{P}F|^2 \\ \implies & (\lambda - r) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \leq \frac{R_0\Theta(\eta)}{2} \|F\|_{-\beta}^2 + \frac{1}{2R_0\Theta(\eta)} \|G\|_{-\beta}^2 + r |\mathcal{P}F|^2 \\ \implies & (\lambda - r) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \leq \frac{R_0\Theta(\eta)}{2} \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 + \frac{1}{2R_0\Theta(\eta)} \|G\|_{-\beta}^2 + \frac{3R_0\Theta(\eta)}{2} |\mathcal{P}F|^2 \\ \implies & \left( \frac{\lambda}{2} - r \right) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \leq \frac{1}{2R_0\Theta(\eta)} \|G\|_{-\beta}^2 + \frac{3R_0\Theta(\eta)}{2} |\mathcal{P}F|^2 \end{aligned}$$

which implies finally for  $r$  small enough (say for instance  $r < \frac{\lambda}{4}$ ),

$$(3.3) \quad \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim R_0^{\frac{1}{2}} \Theta(\eta)^{\frac{1}{2}} |\mathcal{P}F| + R_0^{-\frac{1}{2}} \Theta(\eta)^{-\frac{1}{2}} \|G\|_{-\beta}.$$

Consider then a smooth function  $0 \leq \mathfrak{K} \leq 1$ , radially symmetric, and such that  $\mathfrak{K} \equiv 1$  on  $B(0, 3) \setminus B(0, 2)$  and  $\mathfrak{K} \equiv 0$  on  $B(0, 1)$  and outside  $B(0, 4)$ , and denote  $\mathfrak{K}_R(v) := \mathfrak{K}(\frac{v}{R})$  for  $R > 0$ . Denote  $\tilde{\mathfrak{K}}_R := (v \cdot \sigma)[v]^\beta \mathfrak{K}_R$  and integrate (3.1) against  $\tilde{\mathfrak{K}}_R \mathcal{M}$ :

$$(3.4) \quad \begin{aligned} & -\left\langle L^*F, \tilde{\mathfrak{K}}_R \right\rangle - i\eta \int_{\mathbb{R}^d} (v \cdot \sigma)^2 F(v) \mathfrak{K}_R(v) [v]^\beta \mathcal{M}(v) dv - z \int_{\mathbb{R}^d} F(v) \tilde{\mathfrak{K}}_R(v) [v]^{-\beta} \mathcal{M}(v) dv \\ & = \int_{\mathbb{R}^d} G(v) \tilde{\mathfrak{K}}_R(v) [v]^{-\beta} \mathcal{M}(v) dv. \end{aligned}$$

Using the decomposition (3.2),  $L^*1 = 0$  and Hypothesis 3, we have, for  $\alpha \in (-\beta, 2 + \beta)$ ,

$$(3.5) \quad \left| \left\langle L^*F, \tilde{\mathfrak{K}}_R \right\rangle \right| = \left| \left\langle L^* \left( \mathcal{P}^\perp F \right), \tilde{\mathfrak{K}}_R \right\rangle \right| \leq \left\| L \left( \tilde{\mathfrak{K}}_R \right) \right\|_\beta \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim R^{1+\beta-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.$$

Observe also that

$$(3.6) \quad \begin{aligned} & \left| \int_{\mathbb{R}^d} (v \cdot \sigma)^2 F(v) \mathfrak{K}_R(v) [v]^\beta \mathcal{M}(v) dv \right| \\ & \gtrsim |\mathcal{P}F| \left| \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \mathfrak{K}_R(v) [v]^\beta \mathcal{M}(v) dv \right| - \left| \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \mathcal{P}^\perp F(v) \mathfrak{K}_R(v) [v]^\beta \mathcal{M}(v) dv \right| \\ & \gtrsim R^{2+\beta-\alpha} |\mathcal{P}F| - \left( \int_{\mathbb{R}^d} (v \cdot \sigma)^4 \mathfrak{K}_R(v)^2 [v]^{3\beta} \mathcal{M}(v) dv \right)^{\frac{1}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} \\ & \geq R^{2+\beta-\alpha} |\mathcal{P}F| - R^{2+\frac{3\beta}{2}-\frac{\alpha}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}. \end{aligned}$$

Then, we have

$$\begin{aligned}
 \left| \int_{\mathbb{R}^d} F(v) \tilde{\mathfrak{K}}_R(v) [v]^{-\beta} \mathcal{M}(v) \, dv \right| &= \left| \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \mathcal{P}^\perp F(v) \tilde{\mathfrak{K}}_R(v) \mathcal{M}(v) \, dv \right| \\
 &\lesssim \left\| \mathcal{P}^\perp F \right\|_{-\beta} \left( \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \tilde{\mathfrak{K}}_R(v)^2 [v]^\beta \mathcal{M}(v) \, dv \right)^{\frac{1}{2}} \\
 (3.7) \quad &\lesssim R^{1+\beta-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 \left| \int_{\mathbb{R}^d} G(v) \tilde{\mathfrak{K}}_R(v) [v]^{-\beta} \mathcal{M}(v) \, dv \right| &\lesssim \|G\|_{-\beta} \left( \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \tilde{\mathfrak{K}}_R(v)^2 [v]^\beta \mathcal{M}(v) \, dv \right)^{\frac{1}{2}} \\
 (3.8) \quad &\lesssim R^{1+\beta-\frac{\alpha+\beta}{2}} \|G\|_{-\beta}.
 \end{aligned}$$

Combining (3.4)–(3.5)–(3.6)–(3.8) yields

$$\begin{aligned}
 \eta R^{2+\beta-\alpha} |\mathcal{P}F| &\leq \eta R^{2+\frac{3\beta}{2}-\frac{\alpha}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} + R^{1+\beta-\frac{\alpha+\beta}{2}} \|G\|_{-\beta} \\
 &\quad + r R^{1+\beta-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} + R^{1+\beta-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.
 \end{aligned}$$

Observe that the last but one term is negligible in front of the last one when  $r$  is small, giving

$$|\mathcal{P}F| \lesssim \left( R^{\frac{\alpha+\beta}{2}} + \eta^{-1} R^{-1+\frac{\alpha-\beta}{2}} \right) \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \eta^{-1} R^{-1+\frac{\alpha-\beta}{2}} \|G\|_{-\beta}.$$

Taking  $R = \eta^{-\frac{1}{1+\beta}}$ , we have

$$(3.9) \quad |\mathcal{P}F| \lesssim \Theta(\eta)^{-\frac{1}{2}} \left( \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \|G\|_{-\beta} \right).$$

Combining the latter with (3.3), and for  $r \leq R_0 \Theta(\eta)$  with  $\Theta(\eta)$  defined in (1.10) and  $R_0 > 0$  small enough, we have,

$$(3.10) \quad |\mathcal{P}F| \lesssim \Theta(\eta)^{-1} \|G\|_{-\beta} \quad \text{and} \quad \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim \Theta(\eta)^{-\frac{1}{2}} \|G\|_{-\beta}.$$

When  $\alpha = 2 + \beta$ , the calculation is slightly modified as follows: we recall  $\chi_R$  and  $\tilde{\chi}_R := (v \cdot \sigma) [v]^\beta \chi_R$  defined in Hypothesis 3, and integrate (3.1) against  $\tilde{\chi}_R \mathcal{M}$  to get

$$\begin{aligned}
 |\langle L^* F, \tilde{\chi}_R \rangle| &\lesssim (\ln R)^{\frac{1}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} \\
 \left| \int_{\mathbb{R}^d} (v \cdot \sigma)^2 F(v) \chi_R(v) [v]^\beta \mathcal{M}(v) \, dv \right| &\geq (\ln R) |\mathcal{P}[F]| - R^{1+\beta} \left\| \mathcal{P}^\perp F \right\|_{-\beta}, \\
 \left| \int_{\mathbb{R}^d} F(v) \tilde{\chi}_R(v) [v]^{-\beta} \mathcal{M}(v) \, dv \right| &\lesssim (\ln R)^{\frac{1}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}, \\
 \left| \int_{\mathbb{R}^d} G(v) \tilde{\chi}_R(v) [v]^{-\beta} \mathcal{M}(v) \, dv \right| &\lesssim (\ln R)^{\frac{1}{2}} \|G\|_{-\beta},
 \end{aligned}$$

which results in the same estimate (3.10) (with the choices  $R = \eta^{-\frac{1}{1+\beta}}$ ,  $r \leq R_0 \Theta(\eta)$  and  $\Theta(\eta) = \eta^2 |\ln \eta|$  with  $R_0$  small enough).

Given  $z \in B(0, R_0 \Theta(\eta))$  with  $R_0$  small enough and  $\eta \in (0, \eta_0)$  with  $\eta_0$  small enough, we deduce from (3.10) on the *a priori* inverse of  $\tilde{L}_\eta^* - z$  the existence of a solution to (3.1) with the uniform estimate  $\|F\|_{-\beta} \lesssim \Theta(\eta)^{-1} \|G\|_{-\beta}$ . The equation (3.1) re-writes

$$(3.11) \quad -\tilde{L}^* \tilde{F} - i\eta(v \cdot \sigma) [v]^\beta \tilde{F} - z \tilde{F} = [v]^{-\frac{\beta}{2}} G \in L_v^2(\mathcal{M}),$$

with  $\tilde{F} := [\cdot]^{-\frac{\beta}{2}} F$ . Since (by Hypothesis 2)  $\tilde{L}^*$  generates a contraction semigroup in  $L_v^2(\mathcal{M})$ , it is a standard result (see [25, Theorem II.3.15]) that  $\tilde{L}^*$  is maximal dissipative. Therefore, given any  $M \geq 1$ , the operator  $\tilde{L}_{\eta, M}^* := \tilde{L}^* + i\eta(v \cdot \sigma) [v]^\beta \chi_M(v)$  is maximal dissipative (perturbation

by a bounded purely imaginary multiplicative operator). Observe that the previous a priori estimate (3.10) holds for  $\tilde{L}_{\eta, M}^*$  by the same calculation, and uniformly as  $M \rightarrow +\infty$ . This implies that for each  $M \geq 1$  and  $z \in S(0, r)$ , there is  $\tilde{F}_M \in L_v^2(\mathcal{M})$  that solves  $-\tilde{L}_M^* \tilde{F}_M - z \tilde{F}_M = [v]^{-\frac{\beta}{2}} G$ , and that  $\tilde{F}_M$  is uniformly bounded in  $L_v^2(\mathcal{M})$  as  $M \rightarrow \infty$ . Taking a subsequence weakly converging to some  $\tilde{F} \in L_v^2(\mathcal{M})$  as  $M \rightarrow \infty$  gives a solution to (3.11) and thus to (3.1).

**3.1.2. Away from zero.** We now prove that when  $z \in \mathbb{C}$  with  $|z| \in (R_1 \Theta(\eta), r_0)$  with  $R_1$  large enough and  $r_0$  small enough and  $\eta$  small enough, the operator  $\tilde{L}_\eta^* - z$  has a bounded inverse in  $L_v^2(\mathcal{M})$ , and the bound is uniform in  $|z| \in (R_1 \Theta(\eta), r_0)$ . Given  $G \in L^2([\cdot]^{-\beta} \mathcal{M})$  and  $z \in S(0, r)$ , consider an a priori solution  $F \in L^2([\cdot]^{-\beta} \mathcal{M})$  to (3.1). Integrating (3.1) against  $\bar{F} \mathcal{M}$ , taking the real part and using Hypothesis 2 yields (3.3) again. Consider then  $0 \leq \chi \leq 1$  smooth radially symmetric and such that  $\chi \equiv 1$  on  $B(0, 1)$  and  $\chi \equiv 0$  outside  $B(0, 2)$ , and denote  $\chi_R(v) := \chi(\frac{v}{R})$  for  $R > 0$ . Integrate (3.1) against  $\chi_R \mathcal{M}$ :

$$(3.12) \quad -\langle L^* F, \chi_R \rangle - i\eta \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \chi_R(v) \mathcal{M}(v) dv - z \mathcal{P}_R F = \mathcal{P}_R G$$

where we denote the truncated average

$$\mathcal{P}_R F := \int_{\mathbb{R}^d} F(v) \chi_R(v) \mathcal{M}_\beta(v) dv.$$

Using the decomposition (3.2),  $L^* 1 = 0$  and Hypothesis 3:

$$(3.13) \quad |\langle L^* F, \chi_R \rangle| = \left| \langle L^* (\mathcal{P}^\perp F), \chi_R \rangle \right| \leq \|L(\chi_R)\|_\beta \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.$$

Observe also that

$$(3.14) \quad \left| \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \chi_R(v) \mathcal{M}(v) dv \right| = \left| \int_{\mathbb{R}^d} (v \cdot \sigma) [\mathcal{P}^\perp F(v)] \chi_R(v) \mathcal{M}(v) dv \right| \\ \leq \left( \int_{|v| \leq 2R} (v \cdot \sigma)^2 [v]^\beta \mathcal{M}(v) dv \right)^{\frac{1}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim \ell(R) \left\| \mathcal{P}^\perp F \right\|_{-\beta}$$

with

$$(3.15) \quad \ell(R) = \left( \int_{|v| \leq 2R} (v \cdot \sigma)^2 [v]^\beta \mathcal{M}(v) dv \right)^{\frac{1}{2}} \lesssim \begin{cases} 1 & \text{when } \alpha > 2 + \beta, \\ \sqrt{\ln(R)} & \text{when } \alpha = 2 + \beta, \\ R^{1-\frac{\alpha-\beta}{2}} & \text{when } \alpha < 2 + \beta. \end{cases}$$

Combining (3.12)–(3.13)–(3.14) yields the following estimate on the truncated average:

$$(3.16) \quad |\mathcal{P}_R F| \leq \frac{1}{r} \left[ \eta \ell(R) + R^{-\frac{\alpha+\beta}{2}} \right] \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \frac{1}{r} \|G\|_{-\beta}.$$

We next estimate the difference between  $\mathcal{P}F$  and  $\mathcal{P}_R F$ :

$$|\mathcal{P}F - \mathcal{P}_R F| \leq \int_{\mathbb{R}^d} |F| |1 - \chi_R| \mathcal{M}_\beta(v) dv \\ \leq \left( \int_{\mathbb{R}^d} |1 - \chi_R|^2 \mathcal{M}_\beta dv \right)^{\frac{1}{2}} \|F\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}} \|F\|_{-\beta},$$

which implies for  $R$  large enough

$$(3.17) \quad |\mathcal{P}_R F| \lesssim |\mathcal{P}_R F| + R^{-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.$$

Combining (3.16) and (3.17), we deduce

$$(3.18) \quad \begin{aligned} |\mathcal{P}F| &\lesssim \left( \frac{\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}}{r} \right) \left\| \mathcal{P}^\perp F \right\|_{-\beta} + R^{-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \frac{1}{r} \|G\|_{-\beta}, \\ &\lesssim \left( \frac{\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}}}{r} \right) \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \frac{1}{r} \|G\|_{-\beta}. \end{aligned}$$

Optimising  $R$  so that the two terms in the parenthesis are equal yields again  $R = \eta^{-\frac{1}{1+\beta}}$  (with  $\eta$  small enough so that  $R$  is large enough to obtain (3.17)) and therefore  $\eta\ell(R) + R^{-\frac{\alpha+\beta}{2}} \sim \Theta(\eta)^{\frac{1}{2}}$  where  $\Theta$  was defined in (1.10). Combining (3.3) and (3.18) we get then

$$|\mathcal{P}F| \lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{r^{\frac{1}{2}}} |\mathcal{P}F| + \left( 1 + \frac{\Theta(\eta)^{\frac{1}{2}}}{r^{\frac{1}{2}}} \right) \frac{1}{r} \|G\|_{-\beta}.$$

When  $r > R_1\Theta(\eta)$  with  $R_1$  large enough,  $\frac{\Theta(\eta)^{\frac{1}{2}}}{r^{\frac{1}{2}}}$  is small and we deduce

$$(3.19) \quad |\mathcal{P}F| \lesssim \frac{1}{r} \|G\|_{-\beta} \quad \text{and} \quad \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim r^{-\frac{1}{2}} \|G\|_{-\beta}.$$

Arguing as before, given  $|z| \in (R_1\Theta(\eta), r_0)$  with  $R_1$  large enough and  $r_0$  small enough and  $\eta$  small enough, we deduce from (3.19) the construction of a solution  $F$  to (3.1) with the uniform estimate  $\|F\|_{-\beta} \lesssim \Theta(\eta)^{-1} \|G\|_{-\beta}$ . Together with Subsubsection 3.1.1, we have thus proved that  $\tilde{L}^*$  has no eigenvalues in  $|z| < R_0\Theta(\eta)$  and  $R_1\Theta(\eta) < |z| < r_0$ . We now prove the existence of a unique eigenvalue in  $|z| \in (R_0\Theta(\eta), R_1\Theta(\eta))$ .

**3.2. The spectral projections.** We define the spectral projections

$$\Pi_{r,\eta} := \frac{1}{2i\pi} \int_{S(0,r)} \left[ \tilde{L}_\eta^* - z \right]^{-1} dz$$

for  $r \in (R_1\Theta(\eta), r_0)$  for  $r_0$ ,  $R_1$  and  $\eta$  as above; it is well-defined since we proved above that  $(\tilde{L}_\eta^* - z)^{-1}$  then exists. In this section, we first estimate the difference between the projections  $\Pi_{r,\eta}$  and  $\Pi_{r,0}$  when acting on  $\psi_0 := [v]^{-\frac{\beta}{2}}$  (the kernel of  $\tilde{L}_0$ ) and projected on  $\text{Span}(\psi_0)$  and prove that it goes to zero as  $\eta \rightarrow 0$ ; this implies that  $\Pi_{r,\eta}$  is non-zero for  $r$  and  $\eta$  small enough and thus proves the existence of an eigenvalue. Second, we amplify the previous estimate and prove that  $\|\Pi_{r,\eta} - \Pi_{r,0}\| \rightarrow 0$  as  $\eta \rightarrow 0$ , which implies that the dimensions of these two projections are the same for  $\eta$  small enough. This implies the existence and uniqueness of the eigenvalue in  $|z| \in (R_0\Theta(\eta), R_1\Theta(\eta))$  and quantitative convergence estimates as  $\eta \rightarrow 0$ .

**3.3. Preparation for the first scalar estimate.** Recall  $\psi_0 = [\cdot]^{-\frac{\beta}{2}}$ , then

$$\begin{aligned} \Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0 &= \frac{1}{2i\pi} \int_{S(0,r)} \left[ \tilde{L}_\eta^* - z \right]^{-1} \left[ \tilde{L}_0^* - \tilde{L}_\eta \right] \left[ \tilde{L}_0^* - z \right]^{-1} \psi_0 dz \\ &= -\frac{\eta}{2\pi} \int_{S(0,r)} \left[ \tilde{L}_\eta^* - z \right]^{-1} \left\{ (v \cdot \sigma) [v]^\beta \left[ \tilde{L}_0^* - z \right]^{-1} \psi_0 \right\} dz \\ &= \frac{\eta}{2\pi} \int_{S(0,r)} [v]^{-\frac{\beta}{2}} F \frac{dz}{z} \end{aligned}$$

where we have used

$$\left( \tilde{L}_0^* - z \right)^{-1} \psi_0 = \left[ [\cdot]^{\frac{\beta}{2}} L \left( [\cdot]^{\frac{\beta}{2}} \cdot \right) - z \right]^{-1} \psi_0 = -\frac{1}{z} \psi_0$$

and we have defined  $F$  through

$$\left[ \tilde{L}_\eta^* - z \right]^{-1} \left[ v' \mapsto (v' \cdot \sigma) [v']^{\frac{\beta}{2}} \right] (v) =: [v]^{-\frac{\beta}{2}} F(v),$$

that is

$$(3.20) \quad -L^*F - i\eta(v \cdot \sigma)F - z[v]^{-\beta}F = (v \cdot \sigma)$$

(the dependency of  $F$  on  $\eta$ ,  $z$  and  $\sigma$  is omitted for readability).

Since  $\Pi_{r,0}\psi_0 = \psi_0$  and

$$\int_{\mathbb{R}^d} \Pi_{r,0}\psi_0(v) [v]^{-\frac{\beta}{2}} \mathcal{M}(v) dv = \int_{\mathbb{R}^d} [v]^{-\beta} \mathcal{M}(v) dv = \int_{\mathbb{R}^d} \mathcal{M}_\beta(v) dv = 1,$$

to prove the existence of an eigenvalue, it is enough to prove that for  $r$  and  $\eta$  small enough

$$A_{r,\eta} := \left| \int_{\mathbb{R}^d} (\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0) [v]^{-\frac{\beta}{2}} \mathcal{M}(v) dv \right| < 1.$$

Using the decomposition (3.2) one gets

$$(3.21) \quad A_{r,\eta} = \left| \frac{\eta}{2\pi} \int_{S(0,r)} \frac{\mathcal{P}F}{z} dz \right|.$$

The next three steps are devoted to estimating  $\mathcal{P}F$ .

**3.4. Localised average estimate.** Integrate (3.20) against  $\chi_R \mathcal{M}$ : the right hand side vanishes since  $\mathcal{M}$  and  $\chi_R$  are even and one gets

$$-\langle L^*F, \chi_R \rangle - i\eta \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \chi_R(v) \mathcal{M}(v) dv - z \mathcal{P}_R F = 0.$$

Using the same argument as for (3.13) and (3.14), we get

$$(3.22) \quad |\mathcal{P}_R F| \leq \frac{1}{r} \left[ \eta \ell(R) + R^{-\frac{\alpha+\beta}{2}} \right] \left\| \mathcal{P}^\perp F \right\|_{-\beta}$$

and using (3.17) we deduce, for  $R = \eta^{-\frac{1}{1+\beta}}$  large enough,

$$|\mathcal{P}F| \lesssim \frac{\left( \eta \ell(R) + R^{-\frac{\alpha+\beta}{2}} \right)}{r} \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{r} \left\| \mathcal{P}^\perp F \right\|_{-\beta}.$$

**3.5.  $L^2$  estimate.** Re-organise (3.20) as

$$-L^*F - i\eta(v \cdot \sigma) \left( F - \frac{1}{i\eta} \right) = z[v]^{-\beta}F,$$

integrate it against

$$\overline{\left( F - \frac{1}{i\eta} \right)} \mathcal{M}$$

and take the real part to obtain

$$-\operatorname{Re} \left\langle L^*F, \left( F - \frac{1}{i\eta} \right) \right\rangle = \operatorname{Re} \left( z \int_{\mathbb{R}^d} [v]^{-\beta} F \overline{\left( F - \frac{1}{i\eta} \right)} \mathcal{M} dv \right).$$

The left hand side satisfies (using  $L1 = 0$  and Hypothesis 2)

$$-\operatorname{Re} \left\langle L^*F, \left( F - \frac{1}{i\eta} \right) \right\rangle = -\operatorname{Re} \langle L^*F, F \rangle \geq \lambda \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2,$$

and the right hand side is bounded by

$$\operatorname{Re} \left( z \int_{\mathbb{R}^d} [v]^{-\beta} F \overline{\left( F - \frac{1}{i\eta} \right)} \mathcal{M} dv \right) \leq r \|F\|_{-\beta}^2 + \frac{r}{\eta} |\mathcal{P}F|.$$

This results in the estimate (using again the orthogonal decomposition)

$$\begin{aligned} \lambda \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 &\leq r \|F\|_{-\beta}^2 + \frac{r}{\eta} |\mathcal{P}F| \\ &\leq r |\mathcal{P}F|^2 + r \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 + \frac{r}{\eta} |\mathcal{P}F|, \end{aligned}$$

which implies, when  $r$  is small,

$$(3.23) \quad \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \lesssim r |\mathcal{P}F|^2 + \frac{r}{\eta} |\mathcal{P}F|.$$

**3.6. Synthesis and the first scalar estimate.** The two previous steps lead to

$$\begin{cases} |\mathcal{P}F|^2 \lesssim \frac{\Theta(\eta)}{r^2} \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2, \\ \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \lesssim r |\mathcal{P}F|^2 + \frac{r}{\eta} |\mathcal{P}F|. \end{cases}$$

Plugging the second estimate into the first one, we obtain

$$(3.24) \quad |\mathcal{P}F| \lesssim \frac{\Theta(\eta)}{r} |\mathcal{P}F| + \frac{\Theta(\eta)}{\eta r}.$$

Given  $r \in [R_1\Theta(\eta), r_0)$  with  $R_1$  *large* enough and  $\eta$  *small* enough so that  $\frac{\Theta(\eta)}{r}$  is small we get

$$(3.25) \quad |\mathcal{P}F| \lesssim \frac{\Theta(\eta)}{\eta r}.$$

Plugging this into (3.21) finally yields

$$A_{r,\eta} \lesssim \frac{\eta}{2\pi} \int_{S(0,r)} \frac{\Theta(\eta)}{\eta r^2} dz \lesssim \frac{\Theta(\eta)}{r},$$

which is as small as wanted for  $r \in (R_1\Theta(\eta), r_0)$  with  $R_1$  *large* enough and  $\eta$  *small* enough.

**3.7. Estimating the full norm of the difference of the projections at  $\psi_0$ .** Combining (3.23) and (3.25) yields

$$\left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim \frac{\Theta(\eta)}{r^{\frac{1}{2}}\eta} + \frac{\Theta(\eta)^{\frac{1}{2}}}{\eta}.$$

This implies

$$\begin{aligned} \|\Pi_{r,\eta}\psi_0 - \Pi_{r,0}\psi_0\| &\lesssim \frac{\eta}{2\pi} \int_{S(0,r)} \frac{1}{r} \|F\|_{-\beta} dz \\ &\lesssim \frac{\eta}{2\pi} \int_{S(0,r)} \frac{1}{r} \left( |\mathcal{P}F| + \left\| \mathcal{P}^\perp F \right\|_{-\beta} \right) dz \\ (3.26) \quad &\lesssim \frac{1}{r} \Theta(\eta) + \frac{1}{r^{\frac{1}{2}}} \Theta(\eta) + \Theta(\eta)^{\frac{1}{2}} \lesssim \frac{\Theta(\eta)}{r} \end{aligned}$$

which is as small as wanted for  $r \in (R_1\Theta(\eta), r_0)$  with  $R_1$  *large* enough and  $\eta$  *small* enough.

**3.8. Estimating the full norm of the difference of projections.** Take now any  $\psi \in L^2(\mathcal{M})$ . Then  $[\cdot]^{\frac{\beta}{2}}\psi \in L^2([\cdot]^{-\beta}\mathcal{M})$  and the following decomposition holds

$$\psi = [v]^{-\frac{\beta}{2}} \mathcal{P} \left[ [\cdot]^{\frac{\beta}{2}} \psi \right] + [v]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right).$$

As a consequence,

$$\|(\Pi_{r,\eta} - \Pi_{r,0})\psi\| \leq \left\| \mathcal{P} \left[ [\cdot]^{\frac{\beta}{2}} \psi \right] \right\| \|(\Pi_{r,\eta} - \Pi_{r,0})\psi_0\| + \left\| (\Pi_{r,\eta} - \Pi_{r,0}) \left[ [\cdot]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right) \right] \right\|.$$

The first term in the right hand side is estimated by (3.26). We estimate the second term in the right hand side by the triangle inequality:

$$\begin{aligned} & \left\| (\Pi_{r,\eta} - \Pi_{r,0}) \left[ \cdot \right]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\| \\ & \leq \left\| \Pi_{r,\eta} \left[ \cdot \right]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\| + \left\| \Pi_{r,0} \left[ \cdot \right]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\| \end{aligned}$$

and now consider each term separately. Start with

$$\begin{aligned} \Pi_{r,\eta} \left[ \cdot \right]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) &= \frac{1}{2i\pi} \int_{S(0,r)} \left[ \tilde{L}_\eta - z \right]^{-1} \left[ \cdot \right]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) dz \\ (3.27) \qquad \qquad \qquad &= \frac{1}{2i\pi} \int_{S(0,r)} [v]^{-\frac{\beta}{2}} F dz \end{aligned}$$

where this time  $F$  satisfies (as before we omit writing the dependency in  $\eta, z, \sigma$ )

$$(3.28) \qquad -L^*F - i\eta(v \cdot \sigma)F - z[v]^{-\beta}F = \left[ \cdot \right]^{-\beta} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right).$$

First, test (3.28) on  $\overline{F}\mathcal{M}$ , take the real part and use  $\mathcal{P}[\mathcal{P}^\perp(\left[ \cdot \right]^{\frac{\beta}{2}}\psi)] = 0$ :

$$\begin{aligned} \lambda \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 &\leq (\operatorname{Re} z) \|F\|_{-\beta}^2 + \operatorname{Re} \left\langle \left[ \cdot \right]^{-\beta} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right), F \right\rangle \\ &= (\operatorname{Re} z) \|F\|_{-\beta}^2 + \operatorname{Re} \left\langle \left[ \cdot \right]^{-\beta} \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right), \mathcal{P}^\perp F \right\rangle \\ &\leq r|\mathcal{P}F|^2 + r\|\mathcal{P}^\perp F\|_{-\beta}^2 + \left\| \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\|_{-\beta} \left\| \mathcal{P}^\perp F \right\|_{-\beta}, \end{aligned}$$

which implies, since  $r < r_0 < \lambda$  stays away from  $\lambda$ ,

$$(3.29) \qquad \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \lesssim r |\mathcal{P}F|^2 + \left\| \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\|_{-\beta}^2 \lesssim r |\mathcal{P}F|^2 + \|\psi\|^2.$$

We now estimate  $\mathcal{P}F$ . Integrate (3.28) against  $\chi_R\mathcal{M}$  with  $R = \eta^{-\frac{1}{1+\beta}}$

$$-\langle L^*F, \chi_R \rangle - i\eta \int_{\mathbb{R}^d} (v \cdot \sigma)F(v)\chi_R(v)\mathcal{M}(v) dv - z\mathcal{P}_R F = \mathcal{P}_R \left[ \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right].$$

Using the same arguments as in Subsections 3.1 and 3.4 we obtain

$$|\mathcal{P}_R F| \lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{r} \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \frac{1}{r} \left| \mathcal{P}_R \left[ \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right] \right|.$$

Since  $\mathcal{P}[\mathcal{P}^\perp(\left[ \cdot \right]^{\frac{\beta}{2}}\psi)] = 0$ , we can estimate  $|\mathcal{P}_R[\mathcal{P}^\perp(\left[ \cdot \right]^{\frac{\beta}{2}}\psi)]|$  as follows:

$$\begin{aligned} \left| \mathcal{P}_R \left[ \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right] \right| &= \left| \mathcal{P} \left[ \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right] - \mathcal{P}_R \left[ \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right] \right| \\ &\lesssim R^{-\frac{\alpha+\beta}{2}} \left\| \mathcal{P}^\perp \left( \left[ \cdot \right]^{\frac{\beta}{2}} \psi \right) \right\|_{-\beta} \lesssim R^{-\frac{\alpha+\beta}{2}} \|\psi\| \lesssim \Theta(\eta)^{\frac{1}{2}} \|\psi\|. \end{aligned}$$

We deduce

$$|\mathcal{P}_R F| \lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{r} \left( \left\| \mathcal{P}^\perp F \right\|_{-\beta} + \|\psi\| \right),$$

and using

$$|\mathcal{P}F|^2 \lesssim |\mathcal{P}_R F|^2 + R^{-(\alpha+\beta)} \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \leq |\mathcal{P}_R F|^2 + \Theta(\eta) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2$$

we finally get

$$(3.30) \quad |\mathcal{P}F|^2 \lesssim \frac{\Theta(\eta)}{r^2} \left( \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 + \|\psi\|^2 \right) + \Theta(\eta) \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r^2} \left( \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 + \|\psi\|^2 \right).$$

Combining (3.29)–(3.30) implies for  $r \in [R_1\Theta(\eta), r_0)$  with  $R_1$  large enough and  $\eta$  small enough

$$|\mathcal{P}F|^2 \lesssim \frac{\Theta(\eta)}{r^2} \|\psi\|^2 \quad \text{and thus} \quad \left\| \mathcal{P}^\perp F \right\|_{-\beta}^2 \lesssim \frac{\Theta(\eta)}{r} \|\psi\|^2 + \|\psi\|^2 \lesssim \|\psi\|^2.$$

Plugging these estimates into (3.27) yields

$$\left\| \Pi_{r,\eta} \left[ [\cdot]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right) \right] \right\| \leq r \|F\|_{-\beta} \lesssim r |\mathcal{P}F| + r \|\mathcal{P}^\perp F\|_{-\beta} \lesssim \Theta(\eta)^{\frac{1}{2}} \|\psi\| + r \|\psi\|.$$

We now come to the estimate of

$$\begin{aligned} \Pi_{r,0} \left[ [\cdot]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right) \right] &= \frac{1}{2i\pi} \int_{S(0,r)} \left[ \tilde{L}_0 - z \right]^{-1} \left[ [\cdot]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right) \right] dz \\ &= \frac{1}{2i\pi} \int_{S(0,r)} [v]^{-\frac{\beta}{2}} F dz \end{aligned}$$

where  $F$  satisfies this time

$$(3.31) \quad -L^*F - z[v]^{-\beta}F = [v]^{-\beta} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right).$$

Integrating this equation against  $\mathcal{M}$  implies  $\mathcal{P}F = 0$  since  $\langle L^*F, 1 \rangle = \mathcal{P}[\mathcal{P}^\perp([\cdot]^{\frac{\beta}{2}}\psi)] = 0$  and  $z \neq 0$ . Hypothesis 2 then implies, since  $r < r_0 < \lambda$  is away from  $\lambda$ :

$$\|F\|_{-\beta} = \left\| \mathcal{P}^\perp F \right\|_{-\beta} \lesssim \|\psi\|,$$

and thus

$$\left\| \Pi_{r,0} \left[ [\cdot]^{-\frac{\beta}{2}} \mathcal{P}^\perp \left( [\cdot]^{\frac{\beta}{2}} \psi \right) \right] \right\| \lesssim r \|\psi\|.$$

The conclusion is that for any  $\psi \in L^2(\mathcal{M})$ ,

$$\|(\Pi_{r,\eta} - \Pi_{r,0})\psi\| \lesssim \Theta(\eta)^{\frac{1}{2}} \|\psi\| + r \|\psi\|$$

which implies (combining all the previous conditions), for  $r \in (R_1\Theta(\eta), r_0)$  with  $r_0 > 0$  *small* enough *independently* of  $\eta$  and  $R_1$  *large* enough *independently* of  $\eta$  and  $\eta$  *small* enough,

$$\|\Pi_{r,\eta} - \Pi_{r,0}\|_{L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})} < 1.$$

It implies that, for  $r \in (R_1\Theta(\eta), r_0)$  and  $\eta$  small enough, the projections  $\Pi_{r,\eta}$  and  $\Pi_{r,0}$  both exist thanks to Subsection 3.1 and their dimensions are the same, i.e. 1, which proves existence and uniqueness of an eigenvalue  $\mu(\eta) \in B(0, r_0)$  with  $|\mu(\eta)| \in (R_0\Theta(\eta), R_1\Theta(\eta))$ . This implies that this eigenvalue is real: if  $(\psi_\eta, -\mu(\eta))$  is an eigenpair of  $\tilde{L}_\eta^*$  with  $\mu(\eta) \in B(0, r_0)$ , then so is  $(\overline{\psi_\eta(\cdot)}, -\overline{\mu(\eta)})$ . Since  $\tilde{L}_\eta^* \leq 0$  and 0 is not an eigenvalue for  $\eta \neq 0$ , this proves that  $\mu(\eta) > 0$ .

**3.9. Estimate on the branch as  $\eta \rightarrow 0$ .** Denote  $\phi_\eta := [\cdot]^{\frac{\beta}{2}} \psi_\eta$  and normalize

$$\int_{\mathbb{R}^d} \psi_\eta(v) [v]^{-\frac{\beta}{2}} \mathcal{M}(v) dv = \int_{\mathbb{R}^d} \phi_\eta(v) [v]^{-\beta} \mathcal{M}(v) dv = \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}_\beta(v) dv = m[\phi_\eta] = 1.$$

Then integrating the equation against  $\overline{\psi_\eta} \mathcal{M}$ , taking the real part and using Hypothesis 2 yields

$$\lambda \|\psi_\eta - \psi_0\|^2 \leq \mu(\eta) \|\psi_\eta\|^2 \lesssim \mu(\eta) \|\psi_\eta - \psi_0\|^2 + \mu(\eta)$$

where we have used  $\|\psi_0\| = 1$ . Hence for  $\eta$  small enough we deduce

$$\|\phi_\eta - 1\|_{-\beta} = \|\psi_\eta - \psi_0\| \lesssim \mu(\eta)^{\frac{1}{2}}.$$

This concludes the proof of Lemma 1.1.

#### 4. PROOF OF LEMMA 1.2 (SCALING OF THE EIGENVALUE)

In this section we prove Lemma 1.2, assuming all Hypotheses 1–2–3–4. Consider the unique eigenpair  $(\phi_\eta, \mu(\eta))$  that satisfies  $\mu(\eta) \in B(0, r_1)$  and

$$(4.1) \quad -L^*\phi_\eta - i\eta(v \cdot \sigma)\phi_\eta = \mu(\eta)[v]^{-\beta}\phi_\eta \quad \text{and} \quad \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}_\beta(v) dv = 1.$$

4.1. **Proof in the case  $\alpha > 2 + \beta$ .** The function  $F_\eta := \frac{\text{Im } \phi_\eta}{\eta}$  satisfies

$$-L^* F_\eta - \mu(\eta) [v]^{-\beta} F_\eta = (v \cdot \sigma) \text{Re } \phi_\eta \quad \text{and} \quad \int_{\mathbb{R}^d} F_\eta(v) \mathcal{M}_\beta \, dv = 0.$$

Since (by Hypothesis 2)  $\tilde{L}^*$  is invertible on the  $L_v^2(\mathcal{M})$ -orthogonal of  $[\cdot]^{-\frac{\beta}{2}}$ , and  $v \mapsto [v]^{\frac{\beta}{2}}(v \cdot \sigma)$  belongs to  $L_v^2(\mathcal{M})$  when  $\alpha > 2 + \beta$ , we can then define a solution  $F \in L_v^2([\cdot]^{-\beta} \mathcal{M})$  to

$$-L^* F = (v \cdot \sigma) \quad \text{with} \quad \int_{\mathbb{R}^d} F(v) \mathcal{M}_\beta \, dv = 0.$$

The difference  $F_\eta - F$  satisfies

$$-L^* (F_\eta - F) - \mu(\eta) [v]^{-\beta} (F_\eta - F) = (v \cdot \sigma) [\text{Re } \phi_\eta - 1] + \mu(\eta) [v]^{-\beta} F.$$

Integrate this against  $(F_\eta - F) \mathcal{M}$  and use Hypothesis 2:

$$\begin{aligned} [\lambda - \mu(\eta)] \|F_\eta - F\|_{-\beta}^2 &\leq \int_{\mathbb{R}^d} (v \cdot \sigma) (\text{Re } \phi_\eta - 1) (F_\eta - F) \mathcal{M} \, dv + \mu(\eta) \int_{\mathbb{R}^d} F (F_\eta - F) \mathcal{M}_\beta \, dv \\ &\leq \|\text{Re } \phi_\eta - 1\|_{2+\beta} \|F_\eta - F\|_{-\beta} + \mu(\eta) \|F\|_{-\beta} \|F_\eta - F\|_{-\beta}. \end{aligned}$$

Write for any  $\ell \in (2 + \beta, \alpha)$

$$\|\text{Re } \phi_\eta - 1\|_{2+\beta} \leq \|\text{Re } \phi_\eta - 1\|_{-\beta}^\zeta \|\text{Re } \phi_\eta - 1\|_\ell^{1-\zeta} \leq \mu(\eta)^{\frac{\zeta}{2}} \|\text{Re } \phi_\eta - 1\|_\ell^{1-\zeta}$$

with  $\mathbf{z} = \frac{\ell - (2+\beta)}{\ell + \beta} \in (0, 1)$ , then Hypothesis 4-(i) implies

$$\|\text{Re } \phi_\eta - 1\|_{2+\beta} \lesssim \mu(\eta)^{\frac{\zeta}{2}} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0,$$

and thus, since  $\alpha > 1$  (combining  $\alpha > 2 + \beta$  and  $\alpha + \beta > 0$ ) and

$$\|F\|_{-\beta} \lesssim \|(v \cdot \sigma)\| \lesssim 1,$$

we deduce

$$\|F_\eta - F\|_{-\beta} \lesssim \mu(\eta)^{\frac{\zeta}{2}} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0.$$

Finally,

$$\begin{aligned} \left| \frac{\mu(\eta)}{\eta^2} - \int_{\mathbb{R}^d} (v \cdot \sigma) F(v) \mathcal{M}(v) \, dv \right| &\leq \left| \int_{\mathbb{R}^d} (v \cdot \sigma) (F_\eta(v) - F(v)) \mathcal{M}(v) \, dv \right| \\ &\lesssim \|1\|_{2+\beta} \|F_\eta - F\|_{-\beta} \lesssim \mu(\eta)^{\frac{\zeta}{2}} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0, \end{aligned}$$

which identifies the limit of  $\mu(\eta)$  and provides a rate.

4.2. **Proof in the case  $\alpha \in (-\beta, 2 + \beta)$ .** Take  $0 \leq \chi \leq 1$  smooth that is 1 on  $B(0, R_0)$  and 0 outside  $B(0, 2R_0)$ . Integrate (4.1) against  $\Theta(\eta)^{-1} \chi(\cdot \eta^{\frac{1}{1+\beta}}) \mathcal{M}$  and take the real part:

$$\begin{aligned} (4.2) \quad &\frac{\mu(\eta)}{\Theta(\eta)} + \frac{1}{\Theta(\eta)} \eta \left\langle (v \cdot \sigma) \text{Im } \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)} \left( \left\langle [v]^{-\beta} \text{Re } \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right\rangle - 1 \right) - \frac{1}{\Theta(\eta)} \left\langle L^* (\text{Re } \phi_\eta - 1), \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right\rangle \\ &= -\frac{\mu(\eta)}{\Theta(\eta)} \left\langle [v]^{-\beta} \text{Re } \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) - 1 \right\rangle - \frac{1}{\Theta(\eta)} \left\langle \text{Re } \phi_\eta - 1, L \left( \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right) \right\rangle. \end{aligned}$$

The first term in the right hand side is controlled by

$$\left| \frac{\mu(\eta)}{\Theta(\eta)} \left\langle [v]^{-\beta} \text{Re } \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) - 1 \right\rangle \right| \lesssim \left| \left\langle [v]^{-\beta} \text{Re } \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) - 1 \right\rangle \right| \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}} \eta^{\frac{\alpha+\beta}{2(1+\beta)}}$$

and the second term is controlled by

$$\begin{aligned} \left| \frac{1}{\Theta(\eta)} \left\langle \operatorname{Re} \phi_\eta - 1, L \left( \chi(\cdot \eta^{\frac{1}{1+\beta}}) \right) \right\rangle \right| &\leq \frac{1}{\Theta(\eta)} \|\phi_\eta - 1\|_{-\beta} \left\| L \left[ \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right] \right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac{1}{2}} \left\| L \left[ \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right] \right\|_\beta \\ &\lesssim \Theta(\eta)^{-\frac{1}{2}} \eta^{\frac{\alpha+\beta}{2(1+\beta)}} R_0^{-\frac{\alpha+\beta}{2(1+\beta)}} \lesssim R_0^{-\frac{\alpha+\beta}{2(1+\beta)}}. \end{aligned}$$

The second term in the left hand side satisfies (changing variable to  $u = v\eta^{\frac{1}{1+\beta}}$ )

$$\frac{\eta}{\Theta(\eta)} \left\langle (v \cdot \sigma) \operatorname{Im} \phi_\eta, \chi(\cdot \eta^{\frac{1}{1+\beta}}) \right\rangle = c_{\alpha,\beta} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta(u) |u|_\eta^{-d-\alpha} \chi(u) \, du$$

and we deduce

$$\left| \frac{\mu(\eta)}{\Theta(\eta)} + c_{\alpha,\beta} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta(u) |u|_\eta^{-d-\alpha} \chi(u) \, du \right| \lesssim R_0^{-\frac{\alpha+\beta}{2}}.$$

Then observe that assumption (1.11) in Hypothesis 4-(ii) implies the uniform integrability of the integrand on the support of  $\chi$  and the convergence of the integral as  $\eta \rightarrow 0$  for a given  $\chi$ .

All in all we have the double limit

$$\int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta(u) |u|_\eta^{-d-\alpha} \chi(u) \, du \xrightarrow[R_0 \rightarrow \infty]{\eta \rightarrow 0} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha} \, du.$$

This double limit thus proves that  $\frac{\mu(\eta)}{\Theta(\eta)}$  converges and

$$\lim_{\eta \rightarrow 0} \frac{\mu(\eta)}{\Theta(\eta)} = c_{\alpha,\beta} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha} \, du.$$

This limit then belongs to  $(R_0, R_1)$  because of estimates on  $\mu(\eta)$  already established.

**4.3. Proof in the case  $\alpha = 2 + \beta$ .** Take  $0 \leq \chi \leq 1$  smooth that is 1 on  $B(0, 1)$  and 0 outside  $B(0, 2)$ . Consider again (4.2) (with now  $\Theta(\eta) = \eta^2 |\ln \eta|$ ) and estimate

$$\left| \frac{\mu(\eta)}{\Theta(\eta)} \left\langle [v]^{-\beta} \operatorname{Re} \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) - 1 \right\rangle \right| \lesssim \left| \left\langle [v]^{-\beta} \operatorname{Re} \phi_\eta, \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) - 1 \right\rangle \right| \lesssim \eta$$

and

$$\begin{aligned} \left| \frac{1}{\eta^2 |\ln \eta|} \left\langle \operatorname{Re} \phi_\eta - 1, L \left( \chi(\cdot \eta^{\frac{1}{1+\beta}}) \right) \right\rangle \right| &\leq \frac{1}{\eta^2 |\ln \eta|} \|\phi_\eta - 1\|_{-\beta} \left\| L \left[ \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right] \right\|_\beta \\ &\lesssim \frac{1}{\eta |\ln(\eta)|^{\frac{1}{2}}} \left\| L \left[ \chi \left( \cdot \eta^{\frac{1}{1+\beta}} \right) \right] \right\|_\beta \lesssim \frac{1}{|\ln(\eta)|^{\frac{1}{2}}}. \end{aligned}$$

We have also

$$\frac{1}{\eta |\ln \eta|} \left\langle (v \cdot \sigma) \operatorname{Im} \phi_\eta, \chi \left( \cdot \eta^{-\frac{1}{1+\beta}} \right) \right\rangle = \frac{c_{\alpha,\beta}}{|\ln \eta|} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta |u|_\eta^{-d-\alpha} \chi(u) \, du$$

which gives

$$(4.3) \quad \left| \frac{\mu(\eta)}{\Theta(\eta)} + \frac{c_{\alpha,\beta}}{|\ln(\eta)|} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta |u|_\eta^{-d-\alpha} \chi(u) \, du \right| \lesssim \frac{1}{|\ln(\eta)|^{\frac{1}{2}}} + \eta.$$

Let us decompose

$$\begin{aligned} &\frac{c_{\alpha,\beta}}{|\ln \eta|} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi_\eta |u|_\eta^{-d-\alpha} \chi(u) \, du \\ &= \frac{c_{\alpha,\beta}}{|\ln \eta|} \int_{|u| \leq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \operatorname{Im} \Phi_\eta |u|_\eta^{-d-\alpha} \chi(u) \, du + \frac{c_{\alpha,\beta}}{|\ln \eta|} \int_{|u| \geq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \operatorname{Im} \Phi_\eta |u|_\eta^{-d-\alpha} \chi(u) \, du. \end{aligned}$$

The first term is bounded by

$$\begin{aligned} \frac{c_{\alpha,\beta}}{|\ln \eta|} \int_{|u| \leq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \operatorname{Im} \Phi_\eta(u) |u|_\eta^{-d-\alpha} \chi(u) \, du &= \frac{c_{\alpha,\beta}}{\eta |\ln \eta|} \int_{|v| \leq 1} (v \cdot \sigma) \operatorname{Im} \phi_\eta(v) \mathcal{M}(v) \, dv \\ &\lesssim \frac{\Theta(\eta)^{\frac{1}{2}}}{\eta |\ln \eta|} \lesssim \frac{1}{|\ln(\eta)|^{\frac{1}{2}}}. \end{aligned}$$

We approximate, using Hypothesis 4-(ii)-(a),

$$\frac{1}{|\ln(\eta)|} \left| \int_{|u| \geq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \left[ \operatorname{Im} \Phi_\eta(u) - \operatorname{Im} \Phi(u) \right] |u|_\eta^{-d-\alpha} \chi(u) \, du \right| \lesssim \mathbf{a}(\eta).$$

Define

$$N(\eta) := \int_{|u| \geq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha} \chi(u) \, du.$$

Observe that since  $|\operatorname{Im} \Phi(u)| \lesssim |u|^{1+\beta}$ , and  $\alpha = 2 + \beta$ ,

$$\begin{aligned} \left| N(\eta) - \int_{|u| \geq \eta^{\frac{1}{1+\beta}}} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|_\eta^{-d-\alpha} \chi(u) \, du \right| &\leq \int_{2 \geq |u| \geq \eta^{\frac{1}{1+\beta}}} |u|^{2+\beta} \left| |u|_\eta^{-d-\alpha} - |u|^{-d-\alpha} \right| \, du \\ &\leq \int_{1 \leq |v| \leq 2\eta^{-\frac{1}{1+\beta}}} |v|^{-d} \left| |v|^{d+\alpha} [v]^{-d-\alpha} - 1 \right| \, dv \\ &\lesssim 1, \end{aligned}$$

since  $\left| |v|^{d+\alpha} [v]^{-d-\alpha} - 1 \right| \sim_{v \rightarrow \infty} \frac{d+\alpha}{2} \frac{1}{1+|v|^2}$ . We get, using Hypothesis 4-(ii)-(a),

$$-\eta N'(\eta) \sim \frac{1}{1+\beta} \int_{\sigma' \in \mathbb{S}^{d-1}} (\sigma \cdot \sigma') \frac{\operatorname{Im} \Phi\left(\eta^{\frac{1}{1+\beta}} \sigma'\right)}{\eta} \, d\sigma' \sim \frac{1}{1+\beta} \int_{\sigma' \in \mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma') \, d\sigma'.$$

Apply then L'Hôpital's rule to deduce

$$\lim_{\eta \rightarrow 0} \frac{N(\eta)}{|\ln \eta|} = \frac{1}{1+\beta} \int_{\sigma' \in \mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma') \, d\sigma'.$$

We conclude by taking  $\eta \rightarrow 0$  in (4.3).

## 5. PROOF OF LEMMA 1.3 (THE DIFFUSION COEFFICIENT)

We assume  $\alpha \geq 0$ . Lemma 1.3 follows from Lemma 1.2, the definition (1.15) of  $\theta$ , and the

**Lemma 5.1.** *Assume Hypotheses 1-2-3-4. Then, the convergence*

$$\langle 1, \phi_\eta \rangle \sim_{\eta \rightarrow 0} \begin{cases} \|\mathcal{M}\|_{L^1(\mathbb{R}^d)} & \text{when } \alpha > 0 \\ \frac{|\mathbb{S}^{d-1}|}{1+\beta} |\ln(\eta)| & \text{when } \alpha = 0 \end{cases}$$

holds, with explicit convergence rate.

*Proof.* When  $\alpha > 0$ , the integral

$$\frac{c_{\alpha,\beta}}{c_{\alpha,0}} = \langle 1, 1 \rangle = \int_{\mathbb{R}^d} \mathcal{M} \, dv < +\infty$$

is well defined and, choosing  $\ell \in (0, \alpha)$ ,

$$\begin{aligned} |\langle 1, \phi_\eta \rangle - \langle 1, 1 \rangle| &\leq |\langle 1, \phi_\eta - 1 \rangle| \leq \|1\|_{\min(\ell, \beta)} \|\phi_\eta - 1\|_{-\min(\ell, \beta)} \\ &\lesssim \|\phi_\eta - 1\|_{-\beta}^a \|\phi_\eta - 1\|_0^{1-a} \lesssim \mu(\eta)^{\frac{a}{2}} \end{aligned}$$

with  $a = \min(\frac{\ell}{\beta}, 1) \in (0, 1]$ , which shows (with explicit rate)

$$\langle 1, \phi_\eta \rangle \xrightarrow{\eta \rightarrow 0} \langle 1, 1 \rangle = \frac{c_{\alpha,\beta}}{c_{\alpha,0}} = \|\mathcal{M}\|_{L^1(\mathbb{R}^d)} \quad \text{when } \alpha > 0.$$

In the case  $\alpha = 0$ ,

$$\int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, dv = \int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, dv + \int_{|v| \geq \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, dv.$$

The second term is estimated by

$$\begin{aligned} \int_{|v| \geq \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, dv &= c_{0,\beta} \int_{|u| \geq 1} \Phi_\eta(u) |u|_\eta^{-d} \, du \\ &= c_{0,\beta} \left( \int_{|u| \geq 1} |\Phi_\eta(u)|^2 |u|_\eta^{-d+\beta} \, du \right)^{\frac{1}{2}} \left( \int_{|u| \geq 1} |u|_\eta^{-d-\beta} \, du \right)^{\frac{1}{2}} \lesssim 1, \end{aligned}$$

using the moment bounds (1.12). The first term is decomposed into

$$\int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} \phi_\eta(v) \mathcal{M}(v) \, dv = \int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1) \mathcal{M}(v) \, dv + \int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v) \, dv.$$

Since

$$\left| \int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} (\phi_\eta(v) - 1) \mathcal{M}(v) \, dv \right| \leq \|\phi_\eta(v) - 1\|_{-\beta} \left\| \mathbf{1}_{|v| \leq \eta^{-\frac{1}{1+\beta}}} \right\|_\beta \lesssim \mu(\eta)^{\frac{1}{2}} \eta^{-\frac{\beta}{2(1+\beta)}} \lesssim 1,$$

we deduce

$$\int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}(v) \, dv \sim \int_{|v| \leq \eta^{-\frac{1}{1+\beta}}} \mathcal{M}(v) \, dv \sim c_{0,\beta} \frac{|\mathbb{S}^{d-1}|}{1+\beta} |\ln \eta|$$

with explicit error term. □

## 6. PROOF OF THE HYPOTHESES FOR SCATTERING EQUATIONS

Let us consider  $\alpha \geq 0$  and the scattering operator, written both on “ $f$ ” or “ $h = \frac{f}{\mathcal{M}}$ ”:

$$\begin{cases} \mathcal{L}f(v) = \int_{\mathbb{R}^d} b(v, v') [f(v') \mathcal{M}(v) - f(v) \mathcal{M}(v')] \, dv', \\ \mathcal{L}h(v) = \int_{\mathbb{R}^d} b(v, v') \mathcal{M}(v') [h(v') - h(v)] \, dv'. \end{cases}$$

We assume that  $b$  is  $C^1$ , that the operator conserves the local mass

$$\int_{\mathbb{R}^d} [b(v, v') - b(v', v)] \mathcal{M}(v') \, dv' = 0$$

and that the *collision kernel*  $b$  and *collision frequency*  $\nu(v) := \int_{\mathbb{R}^d} b(v, v') \mathcal{M}(v') \, dv'$  satisfy, for some constant  $\nu_0 > 0$ ,

$$[v]^{-\beta} \lesssim \nu(v) \lesssim [v]^{-\beta}, \quad \lambda^\beta \nu(\lambda u) \sim_{\lambda \rightarrow \infty} \nu_0 |u|^{-\beta} \quad \text{and} \quad \|b(v, \cdot)\|_\beta + \|b(\cdot, v)\|_\beta \lesssim [v]^{-\beta}.$$

This includes  $b(v, v') = [v]^{-\beta} [v']^{-\beta}$  for any  $\alpha + \beta > 0$ ,  $b(v, v') = [v - v']^{-\beta}$  when  $\beta \geq 0$  and  $\alpha > 3\beta$ , and even  $b(v, v') = |v - v'|^{-\beta}$  when  $\beta < 0$  and  $\alpha + \beta > 0$ .

**6.1. Proof of Hypothesis 2.** Hypothesis 2 is standard and proved for instance in [21].

**6.2. Proof of Hypothesis 3.** We perform the following calculations (the case of  $\tilde{\chi}_R$  is similar):

$$\begin{aligned}
\|L(\chi_R)\|_\beta^2 &= \int_{\mathbb{R}^d} [v]^\beta |L(\chi_R)|^2 \mathcal{M}(v) dv \\
&\leq \int_{\mathbb{R}^d} [v]^\beta \nu(v) \int_{\mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2 b(v, v') \mathcal{M}(v') \mathcal{M}(v) dv' dv \\
&\lesssim \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2 b(v, v') \mathcal{M}(v) \mathcal{M}(v') dv dv' \\
&\lesssim \iint_{\{|v| < R\} \times \mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2 b(v, v') \mathcal{M}(v) \mathcal{M}(v') dv dv' \\
&\quad + \iint_{\{|v| > R\} \times \mathbb{R}^d} |\chi_R(v) - \chi_R(v')|^2 b(v, v') \mathcal{M}(v) \mathcal{M}(v') dv dv' \\
&\lesssim \iint_{\mathbb{R}^d \times \{|v'| > R\}} b(v, v') \mathcal{M}(v) \mathcal{M}(v') dv dv' + \iint_{\{|v| > R\} \times \mathbb{R}^d} b(v, v') \mathcal{M}(v) \mathcal{M}(v') dv' dv \\
&\lesssim \|\chi_R^c \mathcal{M}\|_{-\beta} \lesssim R^{-\frac{(\alpha+\beta)}{2}} \xrightarrow{R \rightarrow \infty} 0.
\end{aligned}$$

**6.3. Proof of Hypothesis 4.** The eigenvalue problem can be written

$$(L^{*,+} \phi_\eta)(v) := \int_{\mathbb{R}^d} b(v', v) \mathcal{M}(v') \phi_\eta(v') dv' = \left( \nu(v) - i\eta(v \cdot \sigma) - \mu(\eta) [v]^{-\beta} \right) \phi_\eta(v)$$

with the normalization  $\int_{\mathbb{R}^d} \mathcal{M}_\beta(v') \phi_\eta(v') dv' = 1$ . Observe first that Hypothesis 2 implies

$$\|\phi_\eta - 1\|_{-\beta}^2 \leq \mu(\eta) \|\phi_\eta\|_{-\beta}^2$$

and thus, for  $\eta$  small enough

$$\|\phi_\eta\|_{-\beta}^2 \leq \frac{\lambda}{\lambda - \mu(\eta)}$$

is uniformly bounded as  $\eta \rightarrow 0$ . Observe second that

$$|L^{*,+}(\phi_\eta)(v)| \leq \|b(\cdot, v)\|_\beta \|\phi_\eta\|_{-\beta} \lesssim [v]^{-\beta}$$

which yields, for  $\eta$  small enough,

$$|\phi_\eta(v)| \lesssim \frac{[v]^{-\beta}}{\left[ (\nu(v) - \mu(\eta) [v]^{-\beta})^2 + \eta^2 (v \cdot \sigma)^2 \right]^{\frac{1}{2}}} \lesssim \frac{1}{[v]^\beta \nu(v) - \mu(\eta)} \lesssim 1,$$

i.e.  $\phi_\eta$  is uniformly bounded in  $L^\infty(\mathbb{R}^d)$  as  $\eta \rightarrow 0$ , and Hypothesis 4-(i) when  $\alpha > 2 + \beta$  follows.

The rescaled eigenvector  $\Phi_\eta$  satisfies

$$\Phi_\eta(u) := \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} u \right) = \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+} \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} u \right)}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - i(u \cdot \sigma) - \mu(\eta) |u|_\eta^{-\beta}}.$$

We turn to the case  $\alpha \leq 2 + \beta$ . Estimate (1.11) in Hypothesis 4-(ii) follows from  $\Phi_\eta$  being uniformly bounded and for  $\eta$  small and  $|u| \leq 1$  (using  $|L^{*,+}(\phi_\eta)(v)| \lesssim [v]^{-\beta}$ ),

$$|\operatorname{Im} \Phi_\eta(u)| = \left| \frac{(u \cdot \sigma) \left[ \eta^{\frac{\beta}{1+\beta}} L^{*,+} \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} u \right) \right]}{\left( \eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - \mu(\eta) |u|_\eta^{-\beta} \right)^2 + (u \cdot \sigma)^2} \right| \lesssim \frac{|u \cdot \sigma|}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right)} \lesssim |u|_\eta^{1+\beta}.$$

When  $\alpha \leq \beta$ , the integral moment bound (1.12) in Hypothesis 4-(ii)-(b) follows from (for small  $\eta$  and large  $u$  and using again  $|L^{*,+}(\phi_\eta)(v)| \lesssim [v]^{-\beta}$ )

$$|\Phi_\eta(u)| \lesssim \frac{1}{1 + |u|^\beta |u \cdot \sigma|}$$

which implies

$$\|\Phi_\eta\|_\beta^2 \lesssim \int_0^\pi \int_1^{+\infty} \frac{r^{-1-\alpha+\beta}}{1+r^{2+2\beta}\cos\theta^2} dr d\theta < +\infty.$$

To prove the remaining points we use  $L^*1 = 0$  to write

$$\begin{aligned} \Phi_\eta(u) - \frac{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right)}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - i(u \cdot \sigma) - \mu(\eta) |u|_\eta^{-\beta}} &= \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+} \phi_\eta \left( \eta^{-\frac{1}{1+\beta}} u \right) - \eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right)}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - i(u \cdot \sigma) - \mu(\eta) |u|_\eta^{-\beta}} \\ &= \frac{\eta^{\frac{\beta}{1+\beta}} L^{*,+} (\phi_\eta - 1) \left( \eta^{-\frac{1}{1+\beta}} u \right)}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - i(u \cdot \sigma) - \mu(\eta) |u|_\eta^{-\beta}}. \end{aligned}$$

Since then

$$\left| \eta^{\frac{\beta}{1+\beta}} L^{*,+} (\phi_\eta - 1) \left( \eta^{-\frac{1}{1+\beta}} u \right) \right| \lesssim \eta^{\frac{\beta}{1+\beta}} \left[ \eta^{-\frac{1}{1+\beta}} u \right]^{-\beta} \|\phi_\eta - 1\|_{-\beta} \lesssim \sqrt{\mu(\eta)} \eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right),$$

we deduce

$$\left\| \frac{\Phi_\eta}{\Phi_{\eta,0}} - 1 \right\|_{L^\infty(\mathbb{R}^d)} \lesssim \sqrt{\mu(\eta)} \xrightarrow{\eta \rightarrow 0} 0$$

with the simpler function

$$\Phi_{\eta,0}(u) := \frac{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right)}{\eta^{\frac{\beta}{1+\beta}} \nu \left( \eta^{-\frac{1}{1+\beta}} u \right) - i(u \cdot \sigma) - \mu(\eta) |u|_\eta^{-\beta}}.$$

To prove the convergence of  $\Phi_\eta$  it is thus enough to check the convergence of  $\Phi_{\eta,0}$ :

$$\lim_{\eta \rightarrow 0} \Phi_\eta(u) = \lim_{\eta \rightarrow 0} \Phi_{\eta,0}(u) = \frac{\nu_0}{\nu_0 - i|u|^\beta(u \cdot \sigma)} =: \Phi(u)$$

and in the case  $\alpha = 2 + \beta$  we also have

$$\Omega(u) = \lim_{\lambda \rightarrow 0, \lambda \neq 0} \lambda^{-(1+\beta)} \frac{\nu_0 |\lambda u|^\beta (\lambda u \cdot \sigma)}{\nu_0^2 + |\lambda u|^{2\beta} (\lambda u \cdot \sigma)^2} = \nu_0^{-1} |u|^\beta (u \cdot \sigma),$$

and the corresponding diffusion coefficients are given in the statement of Corollary 1.5.

Moreover, since

$$\text{Im } \Phi_{\eta,0}(u) := \frac{|u|_\eta^\beta \nu_\eta(u) |u|_\eta^\beta (u \cdot \sigma)}{\left( |u|_\eta^\beta \nu_\eta(u) - \mu(\eta) \right)^2 + |u|_\eta^{2\beta} (u \cdot \sigma)^2},$$

and

$$\text{Im } \Phi(u) := \frac{\nu_0 |u|^\beta (u \cdot \sigma)}{\nu_0^2 + |u|^{2\beta} (u \cdot \sigma)^2},$$

we deduce

$$\frac{\text{Im } \Phi(u)}{\text{Im } \Phi_{\eta,0}(u)} = \frac{\nu_0}{|u|_\eta^\beta \nu_\eta(u) |u|_\eta^\beta} \frac{\left( |u|_\eta^\beta \nu_\eta(u) - \mu(\eta) \right)^2 + |u|_\eta^{2\beta} (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta} (u \cdot \sigma)^2} = (1 + o(1)) \frac{|u|^\beta}{|u|_\eta^\beta}.$$

Since  $|\text{Im } \Phi_\eta - \text{Im } \Phi_{\eta,0}| \leq \sqrt{\mu(\eta)} |\text{Im } \Phi_{\eta,0}|$ ,

$$\frac{\text{Im } \Phi(u)}{\text{Im } \Phi_\eta(u)} = \frac{\text{Im } \Phi(u)}{\text{Im } \Phi_{\eta,0}(u)} \frac{\text{Im } \Phi_{\eta,0}(u)}{\text{Im } \Phi(u)} = (1 + o(1)) \frac{|u|^\beta}{|u|_\eta^\beta}$$

which completes the proof of Hypothesis 4-(ii)-(a).

**Remark 1.** Note that when  $\alpha < 0$  (assuming as always  $\alpha + \beta > 0$ ) the rescaled mass is positive:

$$\int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} du = \int_{\mathbb{R}^d} \operatorname{Re} \Phi(u) |u|^{-d-\alpha} du = \int_{\mathbb{R}^d} \frac{\nu_0^2}{\nu_0^2 + |u|^{2\beta} (u \cdot \sigma)^2} \frac{1}{|u|^{d+\alpha}} du > 0$$

by inspection and we also prove that it is finite:

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} du &= \int_{r \geq 0} \left( \int_{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \frac{\nu_0^2 |\sin^{d-2} \theta|}{\nu_0^2 + r^{2(1+\beta)} \sin^2 \theta} d\theta \right) \frac{1}{r^{1+\alpha}} dr \\ &= \int_{r=0}^1 \cdots + \int_{r \geq 1} \int_{|\theta| \in (\frac{\pi}{4}, \frac{\pi}{2})} \cdots + \int_{r \geq 1} \int_{|\theta| \leq \frac{\pi}{4}} \cdots =: I_1 + I_2 + I_3. \end{aligned}$$

Then  $I_1 \lesssim \int_{r \sim 0} r^{-1-\alpha} dr < +\infty$  and  $I_2 \lesssim \int_{r \geq 1} r^{-3-\alpha-2\beta} dr < +\infty$  and finally

$$\begin{aligned} I_3 &\lesssim \int_{r \geq 1} \left( \int_{|z| \leq \frac{1}{\sqrt{2}}} \frac{\nu_0^2}{\nu_0^2 + r^{2(1+\beta)} z^2} dz \right) \frac{1}{r^{1+\alpha}} dr \\ &\lesssim \int_{r \geq 1} \left( \int_{|w| \leq \frac{r^{1+\beta}}{\sqrt{2}}} \frac{\nu_0^2}{\nu_0^2 + w^2} dw \right) \frac{1}{r^{2+\alpha+\beta}} dr < +\infty. \end{aligned}$$

This implies the existence of the limit defining the diffusion coefficient in Lemma 1.3. However for such  $\alpha$ , no initial data seem to allow for a fractional diffusive limit, see also Subsection 9.5.

## 7. PROOF OF THE HYPOTHESES FOR KINETIC FOKKER-PLANCK EQUATIONS

Let us consider  $\beta = 2$ ,  $\alpha \geq 0$ ,  $\mathcal{M}$  given by Hypothesis 1, and let us consider the operators

$$\mathcal{L}(f) := \nabla_v \cdot \left( \mathcal{M} \nabla_v \left( \frac{f}{\mathcal{M}} \right) \right) \quad \text{and} \quad Lh := \mathcal{M}^{-1} \nabla_v \cdot (\mathcal{M} \nabla_v h),$$

which are self-adjoint in, respectively,  $L_v^2(\mathcal{M}^{-1})$  and  $L_v^2(\mathcal{M})$ .

**7.1. Proof of Hypothesis 2.** This hypothesis is:

$$\int_{\mathbb{R}^d} |\nabla_v h|^2 \mathcal{M}(v) dv \geq \lambda \|h - \mathcal{P}h\|_{-2} \quad \text{with} \quad \mathcal{P}h := \int_{\mathbb{R}^d} h(v') |v'|^{-2} \mathcal{M}(v') dv'$$

for some  $\lambda > 0$  (recall that  $\int [\cdot]^{-2} \mathcal{M} = 1$  as per Hypothesis 1). It is a form of the so-called *Hardy-Poincaré inequality*, see for instance [8] where references are collected for proving it for  $d \geq 3$  and  $\alpha > -2$ , [23, Corollary 1] and [9, Appendix A] where it is proved in all dimensions  $d \geq 1$  under the condition  $d + \alpha > 0$  (for instance the “ $\alpha$ ” in [23, Corollary 1] corresponds to our “ $-(d + \alpha)$ ”). Note that the case when  $d \geq 3$  and  $\alpha \in (-d, -2)$  would correspond in [23, 9] to situations where the Hardy-Poincaré inequality holds without the need of the zero-average condition. These cases are however excluded by our assumption  $\alpha \geq 0$ .

**7.2. Proof of Hypothesis 3.** We perform the following computation (the case of  $\tilde{\chi}_R$  is similar)

$$\begin{aligned} \|L(\chi_R)\|_2^2 &= \int_{\mathbb{R}^d} |\nabla \cdot (\mathcal{M} \nabla_v \chi_R)|^2 [\cdot]^2 \frac{dv}{\mathcal{M}} \\ &= \int_{\mathbb{R}^d} \left| \Delta \chi_R + \frac{\nabla_v \mathcal{M}}{\mathcal{M}} \cdot \nabla \chi_R \right|^2 [\cdot]^2 \mathcal{M} dv \\ &= \int_{B_{2R} \setminus B_R} \left| \Delta \chi_R + \frac{\nabla_v \mathcal{M}}{\mathcal{M}} \cdot \nabla \chi_R \right|^2 [\cdot]^2 \mathcal{M} dv \\ &= \int_{B_{2R} \setminus B_R} [\cdot]^{-2} \mathcal{M} dv \lesssim_X R^{-(2+\alpha)} = R^{-(\beta+\alpha)}. \end{aligned}$$

**7.3. Proof of Hypothesis 4.** The equation satisfied by  $\Phi_\eta$  is

$$(7.1) \quad -|u|_\eta^2 \Delta_u \Phi_\eta + (d + \alpha) u \cdot \nabla_u \Phi_\eta - i(u \cdot \sigma) |u|_\eta^2 \Phi_\eta = \mu(\eta) \Phi_\eta.$$

We now prove (1.11) in Hypothesis 4, but estimate first the non-rescaled eigenfunction.

**Lemma 7.1.** *The unique solution to*

$$-L\phi_\eta - i\eta(v \cdot \sigma)\phi_\eta = \mu(\eta)[v]^{-2}\phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) [v]^{-2} \mathcal{M}(v) dv = 1$$

satisfies for any  $R \geq 1$

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1 \quad \text{and} \quad \|\text{Im} \phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta, \mu(\eta))$$

with constants depending only on  $R$  but uniform in  $\eta \rightarrow 0$ .

*Proof of Lemma 7.1.* As for the scattering equation, Hypothesis 2 implies, for  $\eta$  small enough

$$\lambda \|\phi_\eta - 1\|_{-2}^2 \leq \mu(\eta) \|\phi_\eta\|_{-2} \quad \Rightarrow \quad \|\phi_\eta\|_{-2}^2 \leq \frac{\lambda}{\lambda - \mu(\eta)} \lesssim 1.$$

The elliptic regularity of the operator  $L = \Delta - (d + \alpha)\langle v \rangle^{-2} v \cdot \nabla_v$ , with uniform ellipticity constant, then classically implies that

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1.$$

Since  $\mathcal{P}\phi_\eta = 1$  in the decomposition  $\phi_\eta = \mathcal{P}\phi_\eta + \mathcal{P}^\perp\phi_\eta$ , one deduces

$$\|\text{Im} \phi_\eta\|_{-2} \leq \|\mathcal{P}^\perp\phi_\eta\|_{-2} \lesssim \mu(\eta)$$

and the imaginary part satisfies the equation

$$-L(\text{Im} \phi_\eta) - \mu(\eta)[v]^{-2} \text{Im} \phi_\eta = \eta(v \cdot \sigma) \text{Re} \phi_\eta.$$

Therefore the elliptic regularity combined with the integral bound on  $\text{Im} \phi_\eta$  and the bound

$$\|\eta(v \cdot \sigma) \text{Re} \phi_\eta\|_{L^2(B(0,R))} \lesssim \eta$$

on the right hand side implies that

$$\|\text{Im} \phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta, \mu(\eta))$$

which concludes the proof.  $\square$

The following lemma proves (1.11).

**Lemma 7.2.** *There are  $\eta_1 \in (0, \eta_0)$  small enough and  $\delta \in (0, \min(4 - \alpha, 3))$  and  $C$  large enough so that*

$$\forall \eta \in (0, \eta_1), \forall u \in \mathbb{R}^d, \quad |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)} \quad \text{and} \quad |\text{Im} \Phi_\eta(u)| \lesssim |u|_\eta^{\min(2+\alpha, 3)-\delta}.$$

*Proof of Lemma 7.2.* Multiply (7.1) by  $\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}$  and take the real part:

$$-|u|_\eta^2 \text{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \Delta_u \Phi_\eta \right) + (d + \alpha) u \cdot \text{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \nabla_u \Phi_\eta \right) = \mu(\eta) |\Phi_\eta|.$$

Since

$$\nabla_u |\Phi_\eta| = \text{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \nabla_u \Phi_\eta \right), \quad \Delta_u |\Phi_\eta| \geq \text{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \Delta_u \Phi_\eta \right),$$

one gets

$$-|u|_\eta^2 \Delta_u |\Phi_\eta| + (d + \alpha) u \cdot \nabla_u |\Phi_\eta| - \mu(\eta) |\Phi_\eta| \leq 0.$$

Then observe that the real function  $F(u) = |u|_\eta^{C\mu(\eta)}$  satisfies for  $|u| \geq A\eta^{\frac{1}{3}}$  with  $A$  large:

$$\begin{aligned} & -|u|_\eta^2 \Delta_u F + (d + \alpha)u \cdot \nabla_u F - \mu(\eta)F \\ & \geq \mu(\eta)F \left[ -Cd \frac{|u|_\eta^2}{|u|^2} - C(C\mu(\eta) - 2) \frac{|u|_\eta^2}{|u|^2} + C(d + \alpha) - 1 \right] \\ & \geq \mu(\eta)F [-Cd(1 + \varepsilon) - C(C\mu(\eta) - 2)(1 + \varepsilon) + C(d + \alpha) - 1] \\ & \geq \mu(\eta)F [C(2 + \alpha) - 1 - \varepsilon C(d - 2) - C^2\mu(\eta)(1 + \varepsilon)] \end{aligned}$$

where we have used that  $\frac{|u|_\eta^2}{|u|^2} \leq 1 + \varepsilon$  with  $\varepsilon$  small for  $|u| \geq A\eta^{\frac{1}{3}}$  when  $A$  large enough. The right hand side is thus positive for  $C$  large enough and  $\varepsilon$  and  $\eta$  small enough, since  $2 + \alpha > 0$ :

$$\forall |u| \geq A\eta^{\frac{1}{3}}, \quad -|u|_\eta^2 \Delta_u F + (d + \alpha)u \cdot \nabla_u F - \mu(\eta)F \geq 0$$

i.e.  $F$  is a super-solution in this region. Moreover, Lemma 7.1 shows that

$$\sup_{|u| \leq A\eta^{\frac{1}{3}}} |\Phi_\eta(u)| \leq \|\phi_\eta\|_{L^\infty(B(0,A))} \lesssim_A 1$$

and we can therefore compare  $\Phi_\eta$  and  $F$  on the ball  $|u| \leq A\eta^{\frac{1}{3}}$  with a bound uniform in  $\eta$ . The maximum principle thus implies that  $|\Phi_\eta| \lesssim |u|_\eta^{C\mu(\eta)}$  for all  $|u| \geq A\eta^{\frac{1}{3}}$  with a bound uniform in  $\eta$ . Finally, since  $\eta^{C\mu(\eta)} \sim 1$  as  $\eta \rightarrow 0$ , this bound extends to any  $u \in \mathbb{R}^d$  up to enlarging the comparison constant (independently of  $\eta \rightarrow 0$ ).

Take then the imaginary part of (7.1)

$$-|u|_\eta^2 \Delta_u \operatorname{Im} \Phi_\eta + (d + \alpha)u \cdot \nabla_u \operatorname{Im} \Phi_\eta - \mu(\eta) \operatorname{Im} \Phi_\eta = (u \cdot \sigma) |u|_\eta^2 \operatorname{Re} \Phi_\eta,$$

multiply by  $\frac{\operatorname{Im} \Phi_\eta}{|\operatorname{Im} \Phi_\eta|}$  and use the estimate  $|\Phi_\eta(u)| \lesssim |u|^{C\mu(\eta)}$  when  $|u| \geq A\eta^{\frac{1}{3}}$  to get for  $|u| \geq A\eta^{\frac{1}{3}}$

$$-|u|_\eta^2 \Delta_u |\operatorname{Im} \Phi_\eta| + (d + \alpha)u \cdot \nabla_u |\operatorname{Im} \Phi_\eta| - \mu(\eta) |\operatorname{Im} \Phi_\eta| \leq |u|_\eta^{3+C\mu(\eta)}.$$

Define  $G(u) := |u|_\eta^{\mathbf{e}}$  with  $\mathbf{e} := 2 + \min(\alpha, 1) - \delta$  and compute for  $|u| \in [A\eta^{\frac{1}{3}}, 1]$ :

$$-|u|_\eta^2 \Delta_u G + (d + \alpha)u \cdot \nabla_u G - \mu(\eta)G \geq G [\mathbf{e}\delta - \mu(\eta) - \mathcal{O}(A^{-2})] \gtrsim G \gtrsim |u|_\eta^{3+C\mu(\eta)}$$

for  $A$  large enough and  $\eta$  small enough. The maximum principle then shows that  $|\operatorname{Im} \Phi_\eta| \lesssim |u|_\eta^{\mathbf{e}}$  on  $|u| \in [A\eta^{\frac{1}{3}}, 1]$  by comparing  $\operatorname{Im} \Phi_\eta$  and  $G$  on  $|u| = A\eta^{\frac{1}{3}}$  thanks to the second inequality in Lemma 7.1. Again the bound extends to any  $|u| \leq A\eta^{\frac{1}{3}}$  using the second inequality in Lemma 7.1, since  $\max(\eta, \mu(\eta)) \lesssim \eta^{\frac{\mathbf{e}}{3}}$  uniformly as  $\eta \rightarrow 0$  (examining separately the cases  $\alpha \in [0, 1]$  and  $\alpha \in (1, 4)$ ).  $\square$

The next lemma allows us to prove the integral moment estimate (1.12) in Hypothesis 4-(ii)-(b).

**Lemma 7.3.** *There is  $\mathbf{g} > 0$  such that for any  $q \geq -2$  and  $G, H \in L^2([u]^{q-d-\alpha})$  such that*

$$(7.2) \quad -|u|_\eta^2 \Delta_u H + (d + \alpha)u \cdot \nabla_u H - i(u \cdot \sigma) |u|_\eta^2 H = G,$$

*the following gain of decay at infinity holds*

$$\int_{\mathbb{R}^d} |H(u)|^2 [u]^{q+\mathbf{g}-d-\alpha} du \lesssim_{q,\zeta} \int_{\mathbb{R}^d} |G(u)|^2 [u]^{q-d-\alpha} du + \int_{\mathbb{R}^d} |H(u)|^2 [u]^{q-d-\alpha} du.$$

*Proof of Lemma 7.3.* Consider a real-valued smooth function  $\chi_0(u)$  that is zero on  $|u| \leq \frac{1}{2}$  and equal to 1 on  $|u| \geq 1$ , and integrate (7.2) against  $\overline{H}\chi_0^2|u|_\eta^{q-d-\alpha}$  and take the real part:

$$\begin{aligned} & \int_{|u| \geq 1} |u|_\eta^2 |\nabla H(u)|^2 |u|_\eta^{q-d-\alpha} du \\ & \lesssim \int_{\mathbb{R}^d} \left( \chi_0^2 |H|^2 + |\Delta(\chi_0^2)| |u|_\eta^2 |H|^2 + |\nabla(\chi_0)| |u|_\eta |H|^2 + \chi_0^2 |G|^2 \right) |u|_\eta^{q-d-\alpha} du \\ & \lesssim \int_{|u| \geq \frac{1}{2}} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du. \end{aligned}$$

Integrate then (7.2) against  $\overline{H}(u \cdot \sigma)\chi_1^2|u|_\eta^{q-d-\alpha-1}$  where  $\chi_1$  is a real-valued smooth function that is zero on  $|u| \leq 1$  and equal to 1 on  $|u| \geq 2$ , and take the imaginary part:

$$\begin{aligned} & \int_{\mathbb{R}^d} (u \cdot \sigma)^2 \chi_1(u)^2 |H(u)|^2 |u|_\eta^{1+q-d-\alpha} du \\ & \lesssim \int_{|u| \geq 1} |u|_\eta^2 |\nabla H(u)|^2 |u|_\eta^{q-d-\alpha} du + \int_{|u| \geq 1} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du \\ & \lesssim \int_{|u| \geq \frac{1}{2}} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du \end{aligned}$$

where we have used the real part estimate in the last line. This yields

$$\int_{\mathbb{R}^d} (u \cdot \sigma)^2 |u| |H(u)|^2 |u|^{q-d-\alpha} du \lesssim \int_{\mathbb{R}^d} \left( |H|^2 + |G|^2 \right) |u|^{q-d-\alpha} du.$$

This first estimate improves the decay at infinity outside a cone around  $u \perp \sigma$ . We now use the ellipticity of the equation to control this region. The operator is written  $\mathbb{L}_\eta = -|u|_\eta^{d+\alpha} \nabla_u \cdot [|u|_\eta^{-d-\alpha} \nabla_u]$  and we deduce by simple commutator estimates that

$$\int_{|u| \geq 2} \left| \nabla_u \left( H(u) |u|_\eta^{\frac{q-d-\alpha+2}{2}} \right) \right|^2 du \lesssim \int_{|u| \geq 1} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du.$$

Consider first the case  $d > 2$ . The Caffarelli-Kohn-Nirenberg inequality yields

$$\left\| H(\cdot) \cdot | \cdot |_\eta^{\frac{q-d-\alpha+2}{2}} 1_{|\cdot| \geq 2} \right\|_{L^{\frac{2d}{d-2}}}^2 \lesssim \int_{|u| \geq 1} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du.$$

Consider the cone  $\mathcal{C} := \{ |\frac{u}{|u|} \cdot \sigma| \leq |u|^{-\frac{\delta}{2}}, |u| \geq 2 \}$  for some  $\delta > 0$ , and a gain of weight  $|u|^\mathfrak{g}$  for some  $\mathfrak{g} > 0$  to be chosen later. The Hölder inequality then yields

$$\int_{\mathcal{C}} |H(u)|^2 |u|^{q-d-\alpha+\mathfrak{g}} du \leq \left\| H(u) |u|_\eta^{\frac{q-d-\alpha+2}{2}} 1_{|u| \geq 2} \right\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \left( \int_{\mathcal{C}} |u|_\eta^{\frac{d(\mathfrak{g}-2)}{2}} du \right)^{\frac{2}{d}}.$$

The extra volume integral may be estimated using spherical coordinates as

$$\int_{\mathcal{C}} |u|_\eta^{\frac{d(\mathfrak{g}-2)}{2}} du \lesssim \int_1^{+\infty} r^{d-1} \langle r \rangle^{\frac{d(\mathfrak{g}-2)}{2}} \langle r \rangle^{-\frac{\delta}{2}} dr \lesssim \int_1^{+\infty} r^{-1+\frac{d\mathfrak{g}}{2}-\frac{\delta}{2}} dr$$

which is finite as soon as  $\mathfrak{g} < \frac{\delta}{d}$  (which defines and restricts  $\delta$ ). Outside the cone we use the first estimate:

$$\int_{\mathcal{C}^c \cap \{|\cdot| \geq 2\}} |H(u)|^2 |u|^{q-d-\alpha+\mathfrak{g}} du \lesssim \int_{\mathbb{R}^d} |H(u)|^2 (u \cdot \sigma)^2 |u|^{-2+\delta+q-d-\alpha+\mathfrak{g}} du$$

which is controlled as soon as  $\mathfrak{g} \leq 3 - \delta$ . The constraints are compatible for  $\mathfrak{g} \in (0, \frac{3}{d+1})$ .

In the case  $d = 1$ , the gain of decay is immediate from the first estimate alone. In the case  $d = 2$ , we follow a similar argument but replace the Caffarelli-Kohn-Nirenberg inequality with the Onofri inequality:

$$\left\| H(u) |u|_\eta^{\frac{q-d-\alpha+2}{2}} 1_{|u| \geq 2} \right\|_{L^{2p}}^2 \lesssim_p \int_{|u| \geq 1} \left( |H|^2 + |G|^2 \right) |u|_\eta^{q-d-\alpha} du$$

for any  $p < \infty$ . The Hölder inequality then gives

$$\int_{\mathcal{C}} |H(u)|^2 |u|^{q-d-\alpha+\mathfrak{g}} du \leq \left\| |H(u)| |u|^{\frac{q-d-\alpha+2}{2}} 1_{|u| \geq 2} \right\|_{L^{2p}(\mathbb{R}^d)}^2 \left( \int_{\mathcal{C}} |u|^{q(\mathfrak{g}-2)} du \right)^{\frac{1}{q}}$$

where  $q = \frac{1}{1-1/p}$  is the exponent conjugate to  $p$ . The conclusion follows as before by taking  $p$  large enough.  $\square$

Let us deduce the moment bound (1.12) from Lemma 7.3. Observe first that the pointwise bound  $|\Phi_\eta(u)| \lesssim |u|^{C\mu(\eta)}$  proved in Lemma 7.2 implies that

$$\int_{|u| \geq 1} |\Phi_\eta(u)|^2 |u|^{q-d-\alpha} du < +\infty$$

for  $q = -2$  and  $\eta$  small enough with bound uniform in  $\eta$ , since  $\alpha + 2 > 0$ . We then repeatedly apply Lemma 7.3 with  $H = \Phi_\eta$  and  $G = \mu(\eta)\Phi_\eta$  to obtain that  $\Phi_\eta$  decays faster than any polynomial at infinity, with constants uniform in  $\eta$  (note that  $\mathfrak{g}$  is independent of  $q$  in the lemma).

To prove the asymptotic convergence we consider the solution  $\Phi : \mathbb{R}^d \rightarrow \mathbb{C}$  to

$$(7.3) \quad -|u|^2 \Delta \Phi + (d + \alpha)u \cdot \nabla \Phi + i(u \cdot \sigma)\Phi = 0, \quad \Phi(0) = 1, \quad \Phi \in C^0, \quad \Phi \in L^2(\langle u \rangle^\infty).$$

This solution exists by weak limit  $\Phi_\eta \rightarrow \Phi$ . The regularity of  $\Phi$  follows from the equation outside of zero, and the fact that  $\Phi$  goes to 1 as  $u$  goes to 0 comes from the fact that the normalisation of  $\phi_\eta$  yields  $\int_{\mathbb{R}^d} |\Phi(u) - 1|^2 |u|^{-d-2-\alpha} du < \infty$  (a different conclusion near zero would make this integral infinite). The solution  $\Phi$  is unique since integrating the equation for the difference of two solutions  $\Phi := \Phi_1 - \Phi_2$  against  $\overline{\Phi}|u|^{-d}$  yields  $\int_{\mathbb{R}^d} |\nabla \Phi|^2 |u|^{2-d} du = 0$  with all boundary terms near zero behaving like  $|\Phi|^2(u)$ ,  $u \sim 0$ . Note that this solution can be computed semi-explicitly as an entire series  $\Phi(u) = \sum_{k,l \geq 0} a_{k,l} |u|^{2k} (u \cdot \sigma)^l$  by solving the two-parameters induction (see the discussion later and Figure 3).

With the solution  $\Phi$  above at hand, the convergence  $\Phi_\eta \rightarrow \Phi$  in  $L^2_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$  follows easily from the bounds established and the convergence of the coefficients of the equation satisfied by  $\Phi_\eta$ : one can prove that  $\eta \mapsto \Phi_\eta$  is Cauchy in  $L^2$  on any such compact set as  $\eta \rightarrow 0$ , and such convergence has a polynomial rate and is uniform on any compact set in  $\mathbb{R}^d \setminus \{0\}$ .

We now prove Hypothesis 4-(ii)-(a). The equation for  $W_\eta := \Phi_\eta - \Phi$  is

$$-|u|_\eta^2 \Delta_u W_\eta + (d + \alpha)u \cdot \nabla_u W_\eta - i(u \cdot \sigma)|u|_\eta^2 W_\eta - \mu(\eta) W_\eta = \eta^{\frac{2}{3}}(d + \alpha) \frac{u}{|u|^2} \cdot \nabla_u \Phi + \mu(\eta)\Phi.$$

We derive a bound on  $\nabla_u \Phi$ . For this, differentiate the limit equation for  $\Phi$ ,

$$-|u|^2 \Delta(\nabla \Phi) + (d + \alpha)u \cdot \nabla(\nabla \Phi) + (d + \alpha)\nabla \Phi - 2(\Delta \Phi)u = i(u \cdot \sigma)|u|^2 \nabla \Phi + i \nabla((u \cdot \sigma)|u|^2) \Phi.$$

Test against  $\frac{\overline{\nabla \Phi}}{|\nabla \Phi|}$ , use

$$\begin{aligned} \nabla |\nabla \Phi| &= \operatorname{Re} \left( \frac{\overline{\nabla \Phi}}{|\nabla \Phi|} \nabla \nabla \Phi \right), & \operatorname{Re} \left( \frac{\overline{\nabla \Phi}}{|\nabla \Phi|} \Delta(\nabla \Phi) \right) &\leq \Delta(|\nabla \Phi|), \\ u \cdot \operatorname{Re} \left( \Delta \Phi \frac{\overline{\nabla \Phi}}{|\nabla \Phi|} \right) &= u \cdot \nabla(|\nabla \Phi|), & |\nabla((u \cdot \sigma)|u|^2)| &\leq 3|u|^2, \end{aligned}$$

and take the real part to get

$$-|u|^2 \Delta(|\nabla \Phi|) + (d + \alpha - 2)u \cdot \nabla(|\nabla \Phi|) + (d + \alpha)|\nabla \Phi| \leq 3|u|^2 \|\Phi\|_\infty.$$

The comparison principle (with the same sort of computations as for Lemma 7.2) then yields that  $|\nabla \Phi| \lesssim |u|^2$  since we know from the equation that  $\nabla \Phi(0) = 0$ . As a consequence,

$$-|u|_\eta^2 \Delta_u |W_\eta| + (d + \alpha)u \cdot \nabla_u |W_\eta| - \mu(\eta)|W_\eta| \lesssim \eta^{\frac{2}{3}}|u| + \mu(\eta) \lesssim \eta^{\frac{2}{3}}|u|_\eta,$$

since  $\Phi$  is uniformly bounded. From this, one deduces  $|W_\eta| \lesssim \eta^{\frac{2}{3}}|u|_\eta$  (since  $W_\eta$  is of order  $\eta$  near zero). This implies the hypothesis since then, recalling  $\alpha = 4$  and  $\beta = 2$ ,

$$\begin{aligned} & \left| \int_{1 \geq |u| \geq \eta^{\frac{1}{3}}} (u \cdot \sigma) \left[ \operatorname{Im} \Phi_\eta(u) - \operatorname{Im} \Phi(u) \right] |u|_\eta^{-d-4} du \right| \\ & \leq \int_{1 \geq |u| \geq \eta^{\frac{1}{3}}} \eta^{\frac{2}{3}} |u|_\eta^{-d-2} du = \int_{\eta^{-\frac{1}{3}} \geq |u| \geq 1} \frac{\eta^{\frac{d+2}{3}} du}{\eta^{\frac{d+2}{3}} |1 + |u|^2|^{\frac{d+2}{2}}} \lesssim 1. \end{aligned}$$

We now prove the second part of Hypothesis 4-(ii)-(a), regarding the existence of a scaling limit of  $\Phi$ . The rescaled limit  $\Phi$  satisfies  $\Phi(0) = 1$  and the Schrödinger equation

$$-|u|^2 \Delta_u \Phi + (d + \alpha) u \cdot \nabla_u \Phi - i(u \cdot \sigma) |u|^2 \Phi = 0.$$

Therefore,  $\Omega_\lambda(u) := \frac{\operatorname{Im} \Phi(\lambda u)}{\lambda^3}$  satisfies (using that  $\Phi$  is continuous at  $\Phi(0) = 1$ )

$$-|u|^2 \Delta_u \Omega_\lambda(u) + (d + \alpha) u \cdot \nabla_u \Omega_\lambda(u) = (u \cdot \sigma) |u|^2 \operatorname{Re} \Phi(\lambda u) \xrightarrow{\lambda \rightarrow 0} (u \cdot \sigma) |u|^2.$$

Since the limit equation  $-|u|^2 \Delta \Omega + (d + \alpha) u \cdot \nabla \Omega = (u \cdot \sigma) |u|^2$  has unique continuous solution satisfying  $\Omega(0) = 0$  given by  $\Omega(u) = \frac{(u \cdot \sigma) |u|^2}{d+8}$ , elliptic estimates imply  $\Omega_\lambda \rightarrow \Omega$  in  $L^1(\mathbb{S}^{d-1})$ .

**Remark 2.** Note that interestingly, and in contrast with scattering operators, the rescaled mass vanishes for the Fokker-Planck operator when  $\alpha \in (-\beta, 0) = (-2, 0)$ :

$$\int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} du = \int_{\mathbb{R}^d} \operatorname{Re} \Phi(u) |u|^{-d-\alpha} du = 0$$

Indeed, it follows from constructing  $\mathfrak{G} : \mathbb{R}^d \rightarrow \mathbb{C}$  that is  $C^2$  so that  $\mathfrak{G}\Phi$  and  $\Phi\nabla\mathfrak{G}$  decay faster than any polynomials at infinity, that  $\mathfrak{G}(0)$  and  $\nabla\mathfrak{G}(0) = 0$ , and

$$(7.4) \quad -|u|^{d+\alpha} \nabla \cdot \left[ |u|^{-(d+\alpha)} \nabla (|u|^2 \mathfrak{G}) \right] - i(u \cdot \sigma) |u|^2 \mathfrak{G} = 1.$$

Then integration by parts are justified near zero and infinity and imply

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha} du &= \int_{\mathbb{R}^d} \Phi(u) \left\{ -|u|^{d+\alpha} \nabla \cdot \left[ |u|^{-(d+\alpha)} \nabla (|u|^2 \mathfrak{G}) \right] - i(u \cdot \sigma) |u|^2 \mathfrak{G} \right\} |u|^{-d-\alpha} du \\ &= \int_{\mathbb{R}^d} \left\{ -|u|^2 \Delta_u \Phi + (d + \alpha) u \cdot \nabla_u \Phi - i(u \cdot \sigma) |u|^2 \Phi \right\} \mathfrak{G}(u) |u|^{-d-\alpha} du = 0. \end{aligned}$$

To construct such  $\mathfrak{G}$ , plug the ansatz  $\mathfrak{G}(u) = \sum_{k,l \geq 0} a_{k,l} B_{k,l}(u)$  with  $B_{k,l}(u) := |u|^{2k} (u \cdot \sigma)^l$  and  $a_{0,l} = 0$  for all  $l \in \mathbb{N}$ ,  $a_{k \neq 1, 0} = 0$  and  $a_{1,0} = 1$ , into (7.4) to get the sufficient condition

$$[2(k+1)(\alpha - 2l - 2k) + (d + \alpha)l] a_{k,l} = i a_{k-1, l-1} + l(l-1) a_{k-1, l+2}.$$

This relation forms a triangle in the  $(l, k)$  quadrant with horizontal base, and given the conditions above one can solve by induction on  $l$  for each  $k$  and then on  $k$ , see Figure 3. Moreover one can prove by induction (only large  $k, l$  matter) that  $|a_{k,l}| \lesssim \frac{1}{k!l!}$  so that  $\mathfrak{G}$  is well-defined. Finally  $\mathfrak{H}(u) := |u|^2 \mathfrak{G}(u)$  satisfies, arguing as before,  $-|u|^{d+\alpha} \nabla \cdot [|u|^{-(d+\alpha)} \nabla (|\mathfrak{H}|)] \leq 1$ . Then  $|\cdot|^{2+\alpha+0}$  provides a super-solution at infinity, and  $\mathfrak{G}$  bounded for  $u$  near zero, so this proves that  $\mathfrak{G}\Phi$  decays faster than any polynomial (see the discussion following Lemma 7.3 giving the decay of  $\Phi$ ). Similar arguments can be performed on  $|\nabla\mathfrak{H}|$  and  $(\nabla\mathfrak{G})\Phi$  by differentiating (7.4).

## 8. PROOF OF THE HYPOTHESES FOR KINETIC LÉVY-FOKKER-PLANCK EQUATIONS

Consider  $s \in (\frac{1}{2}, 1)$ ,  $\alpha > s$  and  $\mathcal{M}$  is given by Hypothesis 1, and the operator

$$\mathcal{L}(f) := \Delta_v^s f + \nabla_v \cdot (U f).$$

The fractional Laplacian is defined as in (1.5) but we use the equivalent definition (see for example [20] and the references therein)

$$\Delta_v^s f(v) := -C_{d,s} \int_{\mathbb{R}^d} \frac{[f(v) - f(v')]}{|v - v'|^{d+2s}} dv' \quad \text{with} \quad C_{d,s} := \frac{4^s \Gamma(\frac{d}{2} + s)}{\pi^{\frac{d}{2}} |\Gamma(-s)|}.$$

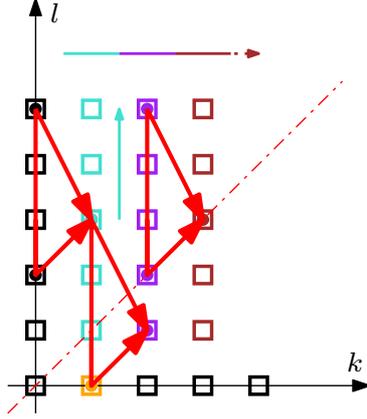


FIGURE 3. The black squares are zero values given to start the induction. The orange square has value 1. The black values for  $k = 0$  allow to compute the turquoise blue squares (that are all zero then). The turquoise blue squares together with the (non-zero) orange square give the values held by the purple squares, etc. On each column, values are computed from bottom to top, sliding the triangular red planer on the previous column.

The drift force  $U$  solves  $\Delta_v^s \mathcal{M} + \nabla_v \cdot (U \mathcal{M}) = 0$ . [13, Proposition 7] shows that the explicit radial solution  $U$  to this equation satisfies  $U(v) = \mathbb{U}(v)[v]^{-\beta} v$  with  $\beta := 2s - \alpha$  and  $\mathbb{U}$  a uniformly positive function bounded from above. The operator  $L$  is

$$Lh := \mathcal{M}^{-1} \Delta_v^s (\mathcal{M}h) + \mathcal{M}^{-1} \nabla_v \cdot (U \mathcal{M}h) = \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}h) - (\Delta_v^s \mathcal{M}) h \right] + U \cdot \nabla_v h.$$

**8.1. Proof of Hypothesis 2.** This hypothesis is implied by the fractional Hardy-Poincaré inequalities proved in [13] and earlier in [51]:

**Proposition 8.1** ([13, 51]). *Let  $d \geq 1$ ,  $s \in (0, 1)$ ,  $\alpha > s$  and  $\beta := 2s - \alpha$ . Then there is  $\lambda > 0$  (depending on  $s$ ) such that*

$$-\operatorname{Re} \langle Lh, h \rangle \geq \lambda \|h - \mathcal{P}h\|_{-\beta}^2.$$

*Proof.* Compute

$$\begin{aligned} -\operatorname{Re} \langle Lh, h \rangle &= -\operatorname{Re} \int_{\mathbb{R}^d} \left[ \Delta_v^s (\mathcal{M}h) + \nabla_v \cdot (U \mathcal{M}h) \right] \bar{h} \, dv' \\ &= -\operatorname{Re} \int_{\mathbb{R}^d} (\Delta_v^s \bar{h}) h \mathcal{M} \, dv + \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} U \cdot \nabla_v (|h|^2) \mathcal{M} \, dv \\ &= -\operatorname{Re} \int_{\mathbb{R}^d} (\Delta_v^s \bar{h}) h \mathcal{M} \, dv - \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} \nabla_v \cdot (U \mathcal{M}) |h|^2 \, dv \\ &= -\operatorname{Re} \int_{\mathbb{R}^d} (\Delta_v^s \bar{h}) h \mathcal{M} \, dv + \operatorname{Re} \int_{\mathbb{R}^d} \frac{1}{2} (\Delta_v^s \mathcal{M}) |h|^2 \, dv \\ &= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|h - h'|^2}{|v - v'|^{d+2s}} \mathcal{M} \, dv \, dv' \end{aligned}$$

and thus

$$-\operatorname{Re} \langle Lh, h \rangle = \frac{C_{d,s}}{4} \int_{\mathbb{R}^d} \frac{|h - h'|^2}{|v - v'|^{d+2s}} (\mathcal{M} + \mathcal{M}') \, dv \, dv'.$$

Note that there is  $\kappa > 0$  such that

$$\forall (v, v') \in \mathbb{R}^d \times \mathbb{R}^d, \quad [v]^{-\beta} [v']^{-\beta} \mathcal{M} \mathcal{M}' \leq \kappa \frac{\mathcal{M} + \mathcal{M}'}{|v - v'|^{d+2s}},$$

by matching the asymptotics at large  $v$  and  $v'$ . Hence we get that

$$-\operatorname{Re} \langle Lh, h \rangle \gtrsim \int_{\mathbb{R}^d} |h - h'|^2 [v]^{-\beta} [v']^{-\beta} \mathcal{M} \mathcal{M}' \, dv \, dv' \gtrsim \|h - \mathcal{P}h\|_{-\beta}^2,$$

where we used in the last line the classical coercivity for scattering operators discussed above.  $\square$

Note that in the coercivity inequality of Proposition 8.1,  $\lambda(s) \rightarrow 0$  as  $s \rightarrow 1$  since  $C_{d,s} \rightarrow 0$  as  $s \rightarrow 1$ . This explains why the coercivity weight  $\beta = 2$  of the Fokker-Planck operator differs from the coercivity weight  $\beta = 2 - \alpha$  of the Lévy-Fokker-Planck operator when  $s \rightarrow 1$ . In fact when  $\alpha = 2s$  and  $s \rightarrow 1$  the correct formal limit is the Fokker-Planck operator with Gaussian equilibrium, in view of the general theory of Lévy processes, for which  $\beta = 0$  is indeed the limit of  $\beta = 2 - \alpha = 2 - 2s$  as  $s \rightarrow 1$ .

**8.2. Proof of Hypothesis 3.** We estimate (the case of  $\tilde{\chi}_R$  is similar)

$$\|L(\chi_R)\|_\beta = \left\| \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right] + U \cdot \nabla_v \chi_R \right\|_\beta$$

in several steps. Write first

$$\begin{aligned} \|U \cdot \nabla_v \chi_R\|_\beta^2 &= \int_{\mathbb{R}^d} |U \cdot \nabla_v \chi_R|^2 [v]^\beta \mathcal{M}(v) \, dv = \int_{\mathbb{R}^d} \left| \mathbb{U}(v) [v]^{-\beta} v \cdot \nabla_v \chi_R \right|^2 [v]^\beta \mathcal{M}(v) \, dv \\ &\leq \| \mathbb{U} \|_\infty \int_{\mathbb{R}^d} |v \cdot \nabla_v \chi_R|^2 \mathcal{M}_\beta(v) \, dv \lesssim R^{-\alpha-\beta}. \end{aligned}$$

Then split the other term into

$$\begin{aligned} &\left\| \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right] \right\|_\beta^2 \\ &= \left\| \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right] \mathbf{1}_{|v| \leq R} \right\|_\beta^2 + \left\| \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right] \mathbf{1}_{|v| \geq R} \right\|_\beta^2. \end{aligned}$$

When  $|v| \leq R$ , write  $v = R w$  with  $|w| \leq 1$  and observe that  $\chi(w) = \chi(w')$  when  $|w'| \leq 1$  to get,

$$\begin{aligned} |[\Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R](Rw)| &= C_{d,s} \left| \int_{|w'| \geq 1} \frac{\chi(w) - \chi(w')}{R^{d+a} |w - w'|^{d+2s}} \mathcal{M}(Rw') R^d \, dw' \right| \\ &\lesssim \int_{|w'| \geq 1} \frac{|\chi(w) - \chi(w')|}{R^{d+a+\alpha} |w - w'|^{d+2s}} \frac{dw'}{|w'|^{d+\alpha}} \lesssim R^{-(d+2s+\alpha)}, \end{aligned}$$

which yields (using  $\beta = 2s - \alpha$  and  $\alpha > 0$ )

$$\left\| \mathcal{M}^{-1} \left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right] \mathbf{1}_{|v| \leq R} \right\|_\beta^2 \lesssim R^{-4s+\beta-\alpha} \lesssim R^{-\alpha-\beta}.$$

When  $|v| \geq R$ , we write

$$\begin{aligned} &\left[ \Delta_v^s (\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M}) \chi_R \right](v) \\ &= \int_{\mathbb{R}^d} \frac{[\chi_R(v) - \chi_R(v')]}{|v - v'|^{d+2s}} \mathcal{M}(v') \, dv' \\ &= \int_{|v-v'| \leq \frac{|v|}{2}} \frac{[\chi_R(v) - \chi_R(v')]}{|v - v'|^{d+2s}} \mathcal{M}(v') \, dv' + \int_{|v-v'| \geq \frac{|v|}{2}} \frac{[\chi_R(v) - \chi_R(v')]}{|v - v'|^{d+2s}} \mathcal{M}(v') \, dv'. \end{aligned}$$

Start with the first integral in the right hand side:

$$\begin{aligned} &\left| \int_{|v-v'| \leq \frac{|v|}{2}} \frac{[\chi_R(v) - \chi_R(v')]}{|v - v'|^{d+2s}} \mathcal{M}(v') \, dv' \right| \\ &\lesssim \int_{|v-v'| \leq \frac{|v|}{2}} \frac{\sup_{B(v, \frac{|v|}{2})} |\nabla_{v'}^2 [(\chi_R(v) - \chi_R(v')) \mathcal{M}(v')]|}{|v - v'|^{d+2s-2}} \, dv' \\ &\lesssim |v|^{2-2s} \sup_{B(v, \frac{|v|}{2})} |\nabla_{v'}^2 [(\chi_R(v) - \chi_R(v')) \mathcal{M}(v')]|. \end{aligned}$$

One has

$$\begin{aligned} & \sup_{B(v, \frac{|v|}{2})} |D_{v'}^2((\chi_R(v) - \chi_R(v'))\mathcal{M}(v'))| \\ & \lesssim \frac{|v|^{-d-\alpha}}{R^2} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi''\left(\frac{v'}{R}\right) \right| + \frac{|v|^{-d-\alpha-1}}{R} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi'\left(\frac{v'}{R}\right) \right| + |v|^{-d-\alpha-2}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \left| \int_{|v-v'| \leq \frac{|v|}{2}} \frac{[\chi_R(v) - \chi_R(v')]}{|v-v'|^{d+2s}} \mathcal{M}(v') dv' \right| \\ & \lesssim |v|^{2-2s} \left[ \frac{|v|^{-d-\alpha}}{R^2} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi''\left(\frac{v'}{R}\right) \right| + \frac{|v|^{-d-\alpha-1}}{R} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi'\left(\frac{v'}{R}\right) \right| + |v|^{-d-\alpha-2} \right] \\ & \lesssim |v|^{-d-\alpha-2s} \left[ \frac{|v|^2}{R^2} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi''\left(\frac{v'}{R}\right) \right| + \frac{|v|}{R} \sup_{v' \in B(v, \frac{|v|}{2})} \left| \chi'\left(\frac{v'}{R}\right) \right| + 1 \right] \lesssim |v|^{-d-\alpha-2s}, \end{aligned}$$

where we have used that  $\chi'$  and  $\chi''$  have compact support and  $|v| \leq 2|v'|$  in this region.

Focus now on the second integral (using  $\alpha > 0$ )

$$\left| \int_{|v-v'| \geq \frac{|v|}{2}} \frac{[\chi_R(v) - \chi_R(v')]}{|v-v'|^{d+2s}} \mathcal{M}(v') dv' \right| \leq \int_{|v-v'| \geq \frac{|v|}{2}} \frac{|\chi_R(v) - \chi_R(v')|}{|v-v'|^{d+2s}} \mathcal{M}(v') dv' \lesssim |v|^{-d-2s}.$$

As a conclusion,

$$\left\| \mathcal{M}^{-1} \left[ \Delta_v^s(\mathcal{M}\chi_R) - (\Delta_v^s \mathcal{M})\chi_R \right] \mathbf{1}_{|v| \geq R} \right\|_{\beta}^2 \lesssim \int_{|v'| > R} |v'|^{2\alpha-4s} |v'|^{\beta-d-\alpha} dv' \lesssim R^{-\alpha-\beta},$$

since  $\beta = 2s - \alpha$ . This concludes the proof.

**8.3. Proof of Hypothesis 4.** The adjoint of  $L$  is  $L^* = \Delta_v^s - U \cdot \nabla_v$  and following exactly the same arguments as in the proof of Lemma 7.1 for the Fokker-Planck operator yields

**Lemma 8.2.** *The unique solution to the eigenvalue equation*

$$-L^* \phi_\eta - i\eta(v \cdot \sigma)\phi_\eta = \mu(\eta) |v|^{-\beta} \phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) |v|^{-\beta} \mathcal{M}(v) dv = 1$$

satisfies for any  $R \geq 1$ ,

$$\|\phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R 1 \quad \text{and} \quad \|\text{Im} \phi_\eta\|_{L^\infty(B(0,R))} \lesssim_R \max(\eta, \mu(\eta)),$$

with constants depending only on  $R$  and uniform in  $\eta \rightarrow 0$ .

Note that local fractional ellipticity results are present in [47]. We now come to the pointwise estimates on the rescaled eigenvector. This is when  $\alpha \leq 2 + \beta$ , that is  $\alpha \leq 1 + s$ . Observe indeed that when  $\alpha > 1 + s$ , the scaling is diffusive, and the diffusion coefficient is obtained by solving

$$\Delta_v^s(\mathcal{M}F) + \nabla_v \cdot (U \mathcal{M}F) = -(v \cdot \sigma)\mathcal{M}(v),$$

with  $\int_{\mathbb{R}^d} F(v) \mathcal{M}_\beta(v) dv = 0$ .

**Lemma 8.3.** *Assume  $s \in (\frac{1}{2}, 1)$ . There are  $\eta_1 \in (0, \eta_0)$  small enough and  $A$  and  $C$  large enough so that*

$$\forall \eta \in (0, \eta_1), \forall u \in \mathbb{R}^d, \quad |\Phi_\eta(u)| \lesssim |u|_\eta^{C\mu(\eta)} \quad \text{and} \quad |\text{Im} \Phi_\eta(u)| \lesssim |u|_\eta^{\min(1,\alpha)+\beta-C\mu(\eta)}.$$

*Proof.* The rescaled equation for  $\Phi_\eta$  is, using  $U_\eta(u) := \eta^{\frac{1-\beta}{1+\beta}} U(u\eta^{-\frac{1}{1+\beta}})$  and  $2s - \beta = \alpha$ ,

$$-\eta^{\frac{\alpha}{1+\beta}} \Delta_u^s \Phi_\eta + U_\eta(u) \cdot \nabla_u \Phi_\eta - i(u \cdot \sigma) \Phi_\eta = \mu(\eta) |u|_\eta^{-\beta} \Phi_\eta.$$

Multiply this equation by  $\frac{\overline{\Phi_\eta}}{|\Phi_\eta|}$  and take the real part:

$$-\eta^{\frac{\alpha}{1+\beta}} \operatorname{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \Delta_u^s \Phi_\eta \right) + U_\eta(u) \cdot \operatorname{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \nabla_u \Phi_\eta \right) = \mu(\eta) |u|_\eta^{-\beta} |\Phi_\eta|.$$

Using the Kato inequality  $\Delta_u^s |\Phi_\eta| \geq \operatorname{Re} \left( \frac{\overline{\Phi_\eta}}{|\Phi_\eta|} \Delta_u^s \Phi_\eta \right)$  (see [14] for the Laplacian and [19] for the fractional Laplacian), one gets

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s |\Phi_\eta| + |u|_\eta^\beta U_\eta(u) \cdot \nabla_u |\Phi_\eta| - \mu(\eta) |\Phi_\eta| \leq 0.$$

Then observe that the real function  $F(u) = |u|_\eta^{C\mu(\eta)}$  satisfies for  $|u| \geq A\eta^{\frac{1}{3}}$ :

$$\begin{aligned} & -\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s F + |u|_\eta^\beta U_\eta(u) \cdot \nabla_u F - \mu(\eta) F \\ & = -\eta^{\frac{\alpha}{1+\beta}} |u|_\eta^\beta \Delta_u^s F + C\mu(\eta) |u|_\eta^{C\mu(\eta)-2} U_\eta(u) |u|^2 - \mu(\eta) |u|_\eta^{C\mu(\eta)} \end{aligned}$$

where we have used that  $U_\eta(u) = |u|_\eta^{-\beta} \mathbb{U}_\eta(u) u$  with some  $\mathbb{U}_\eta$  positive bounded from below (independently of  $\eta$ ). We now estimate  $\Delta_u^s \left( |\cdot|_\eta^{C\mu(\eta)} \right) (u)$ . By scaling:

$$\forall u \in \mathbb{R}^d, \quad \Delta_u^s \left( |\cdot|_\eta^{C\mu(\eta)} \right) (u) = \eta^{\frac{C\mu(\eta)-2s}{1+\beta}} \Delta_v^s \left( |\cdot|^{C\mu(\eta)} \right) \left( u\eta^{-\frac{1}{1+\beta}} \right).$$

We then estimate  $\Delta_v^s \left( |\cdot|^{C\mu(\eta)} \right)$  using

$$\begin{aligned} \Delta_v^s \left( |\cdot|^{C\mu(\eta)} \right) (v) & = C_{d,s} \int_{\mathbb{R}^d} \frac{|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)}}{|v' - v|^{d+2s}} dv', \\ & = C_{d,s} \int_{|v-v'| < \frac{|v|}{2}} \frac{|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)}}{|v' - v|^{d+2s}} dv' + C_{d,s} \int_{|v-v'| > \frac{|v|}{2}} \frac{|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)}}{|v' - v|^{d+2s}} dv'. \end{aligned}$$

To control the first term in the right hand side, use that

$$|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)} - \nabla_v \left( |\cdot|^{C\mu(\eta)} \right) \cdot (v' - v) \lesssim C\mu(\eta) |v|^{C\mu(\eta)-2} |v' - v|^2$$

to get

$$\begin{aligned} \int_{|v-v'| < \frac{|v|}{2}} \frac{|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)}}{|v' - v|^{d+2s}} dv' & \lesssim \int_{|v-v'| < \frac{|v|}{2}} \frac{C\mu(\eta) |v|^{C\mu(\eta)-2} |v' - v|^2}{|v' - v|^{d+2s}} dv' \\ & \lesssim C\mu(\eta) |v|^{C\mu(\eta)-2} \int_{|v-v'| < \frac{|v|}{2}} |v' - v|^{2-2s-d} dv' \lesssim C\mu(\eta) |v|^{C\mu(\eta)-2s}. \end{aligned}$$

To control the second term, use that

$$\left( |v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)} \right) \lesssim C\mu(\eta) |v|^{C\mu(\eta)-1} |v' - v|$$

to get (using here  $s > \frac{1}{2}$ )

$$\begin{aligned} \int_{|v-v'| > \frac{|v|}{2}} \frac{|v'|^{C\mu(\eta)} - |v|^{C\mu(\eta)}}{|v' - v|^{d+2s}} dv' & \lesssim \int_{|v-v'| > \frac{|v|}{2}} \frac{C\mu(\eta) |v|^{C\mu(\eta)-1} |v' - v|}{|v' - v|^{d+2s}} dv' \\ & \lesssim C\mu(\eta) |v|^{C\mu(\eta)-1} \int_{|v-v'| > \frac{|v|}{2}} \frac{dv'}{|v' - v|^{d+2s-1}} \\ & \lesssim C\mu(\eta) |v|^{C\mu(\eta)-2s}. \end{aligned}$$

We therefore have (using the scaling)

$$\Delta_v^s \left( |\cdot|^{C\mu(\eta)} \right) (v) \lesssim C\mu(\eta) |v|^{C\mu(\eta)-2s} \implies \Delta_u^s \left( |\cdot|_\eta^{C\mu(\eta)} \right) (u) \lesssim C\mu(\eta) |u|_\eta^{C\mu(\eta)-2s}.$$

This estimate implies, for some absolute constant  $C_0 > 0$ ,

$$\begin{aligned} \eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s F &\leq C_0 C\mu(\eta) \eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{C\mu(\eta)+\beta-2s} \\ &\leq C_0 C\mu(\eta) |u|_{\eta}^{C\mu(\eta)} \eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{-\alpha} \lesssim C_0 C\mu(\eta) |u|_{\eta}^{C\mu(\eta)} (1+A^2)^{-\frac{\alpha}{2}} \end{aligned}$$

in the region  $|u| \geq A\eta^{\frac{1}{1+\beta}}$ . As a consequence

$$\begin{aligned} &-\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s F + \mathbb{U}_{\eta}(u) u \cdot \nabla_u F - \mu(\eta) F \\ &= -\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s F + C\mu(\eta) |u|_{\eta}^{C\mu(\eta)-2} \mathbb{U}_{\eta}(u) |u|^2 - \mu(\eta) |u|_{\eta}^{C\mu(\eta)} \\ &\geq -C_0 C\mu(\eta) |u|_{\eta}^{C\mu(\eta)} (1+A^2)^{-\frac{\alpha}{2}} + C\mu(\eta) |u|_{\eta}^{C\mu(\eta)-2} (\inf \mathbb{U}_{\eta}) |u|^2 - \mu(\eta) |u|_{\eta}^{C\mu(\eta)} \\ &\geq C\mu(\eta) |u|_{\eta}^{C\mu(\eta)} \left[ -C_0 (1+A^2)^{-\frac{\alpha}{2}} + |u|_{\eta}^{-2} (\inf \mathbb{U}_{\eta}) |u|^2 - C^{-1} \right] \\ &\geq C\mu(\eta) |u|_{\eta}^{C\mu(\eta)} \left[ -C_0 (1+A^2)^{-\frac{\alpha}{2}} + (1+A^{-2})^{-1} (\inf \mathbb{U}_{\eta}) - C^{-1} \right] \geq 0 \end{aligned}$$

for  $A$  and  $C$  sufficiently large, and we deduce  $|\Phi_{\eta}| \lesssim F$  on  $|u| \geq A\eta^{\frac{1}{1+\beta}}$  and, for the same reasons as for the Fokker-Planck operator, the bound extends to any  $u \in \mathbb{R}^d$ .

Taking now the imaginary part of the equation, one gets

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s |\operatorname{Im} \Phi_{\eta}| + \mathbb{U}_{\eta}(u) u \cdot \nabla_u |\operatorname{Im} \Phi_{\eta}| - \mu(\eta) |\operatorname{Im} \Phi_{\eta}| \lesssim |u|_{\eta}^{1+\beta+C\mu(\eta)}.$$

Define then  $\gamma := \min(\alpha, 1) + \beta - C\mu(\eta) \in (0, 2s)$  and the real function  $G(u) := |u|_{\eta}^{\gamma}$ . Note that  $\gamma \in (0, 2s)$  for  $\eta$  small enough, which implies that  $\Delta_u^s G$  makes sense. Write for  $|u| \geq A\eta^{\frac{1}{3}}$

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s G + |u|_{\eta}^{\beta} \mathbb{U}_{\eta}(u) \cdot \nabla_u G - \mu(\eta) G = -\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s G + \gamma |u|_{\eta}^{\gamma-2} \mathbb{U}_{\eta}(u) |u|^2 - \mu(\eta) G.$$

Let us estimate  $\Delta_u^s (|\cdot|_{\eta}^{\gamma})(u)$ . Note that by scaling

$$\forall u \in \mathbb{R}^d, \quad \Delta_u^s (|\cdot|_{\eta}^{\gamma})(u) = \eta^{\frac{\gamma-2s}{1+\beta}} \Delta_v^s (|\cdot|^{\gamma})(u\eta^{-\frac{1}{1+\beta}}).$$

One then estimates  $\Delta_v^s (|\cdot|^{\gamma})$  using

$$\begin{aligned} \Delta_v^s (|\cdot|^{\gamma})(v) &= C_{d,s} \int_{\mathbb{R}^d} \frac{(|v'|^{\gamma} - |v|^{\gamma})}{|v' - v|^{d+2s}} dv', \\ &= C_{d,s} \int_{|v-v'| < \frac{|v|}{2}} \frac{(|v'|^{\gamma} - |v|^{\gamma})}{|v' - v|^{d+2s}} dv' + C_{d,s} \int_{|v-v'| > \frac{|v|}{2}} \frac{(|v'|^{\gamma} - |v|^{\gamma})}{|v' - v|^{d+2s}} dv'. \end{aligned}$$

Small  $v$ 's are fine since  $\Delta_v^s (|\cdot|^{\gamma})$  is locally bounded. Continue with large  $v$ . In the first integral,

$$\begin{aligned} \int_{|v-v'| < \frac{|v|}{2}} \frac{(|v'|^{\gamma} - |v|^{\gamma})}{|v' - v|^{d+2s}} dv' &= \int_{|v-v'| < \frac{|v|}{2}} \frac{|v'|^{\gamma} - |v|^{\gamma} - \nabla_v (|\cdot|^{\gamma})(v)(v' - v)}{|v' - v|^{d+2s}} dv' \\ &\lesssim \int_{|v-v'| < \frac{|v|}{2}} \frac{\sup_{z \in B(v, \frac{|v|}{2})} |\nabla^2 (|\cdot|^{\gamma})(z)| |v' - v|^2}{|v' - v|^{d+2s}} dv' \lesssim |v|^{\gamma-2s}. \end{aligned}$$

The second integral may be estimated from above using that  $|v - v'| > \frac{|v|}{2}$  implies  $|v - v'| > \frac{|v'|}{3}$ ,

$$\int_{|v-v'| > \frac{|v|}{2}} \frac{|v'|^{\gamma} - |v|^{\gamma}}{|v' - v|^{d+2s}} dv' \lesssim \int_{|v-v'| > \frac{|v|}{2}} \frac{|v - v'|^{\gamma}}{|v' - v|^{d+2s}} dv' \lesssim |v|^{\gamma-2s}.$$

From this, we deduce  $\Delta_u^s (|\cdot|_{\eta}^{\gamma})(u) \lesssim \eta^{\frac{\gamma-2s}{1+\beta}} [u\eta^{-\frac{1}{1+\beta}}]^{\gamma-2s} = |u|_{\eta}^{\gamma-2s}$  which implies

$$\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s G \lesssim \eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\gamma+\beta-2s} \lesssim (1+A^2)^{-\frac{\alpha}{2}} |u|_{\eta}^{\gamma}$$

in the region  $|u| \geq A\eta^{\frac{1}{1+\beta}}$ . As a consequence, as previously,

$$-\eta^{\frac{\alpha}{1+\beta}} |u|_{\eta}^{\beta} \Delta_u^s G + |u|_{\eta}^{\beta} \mathbb{U}_{\eta}(u) \cdot \nabla_u G - \mu(\eta) G \gtrsim |u|_{\eta}^{\gamma}$$

for  $A$  sufficiently large and we deduce  $|\operatorname{Im} \Phi_{\eta}| \lesssim G$  on  $|u| \geq A\eta^{\frac{1}{1+\beta}}$  and, for the same reasons as for the Fokker-Planck operator, the bound extends to any  $u \in \mathbb{R}^d$ .  $\square$

**8.4. Rescaled drift force and limit equation.** We formally discuss the behaviour of the force  $U_\eta$  when  $\eta$  goes to 0: setting  $v = u\eta^{-\frac{1}{1+\beta}}$  gives the equation

$$\eta^{\frac{\alpha-\beta}{1+\beta}} \Delta_v^s \mathcal{M}_\eta + \nabla_v \cdot (U_\eta \mathcal{M}_\eta) = 0.$$

Observe that when  $u \neq 0$ ,

$$\begin{aligned} \eta^{\frac{\alpha}{1+\beta}} \Delta_v^s \mathcal{M}_\eta(u) &= -c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \frac{(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha})}{|u - u'|^{d+2s}} du' \\ &= -c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha})}{|u - u'|^{d+2s}} du' - c_{\alpha,\beta} C_{d,s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)^c} \frac{(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha})}{|u - u'|^{d+2s}} du'. \end{aligned}$$

The second term in the right hand side goes to zero as  $\eta \rightarrow 0$  since the singularity around zero has been removed from the integration domain. To deal with the first term, decompose

$$\eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{(|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha})}{|u - u'|^{d+2s}} du' = \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u|_\eta^{-d-\alpha}}{|u - u'|^{d+2s}} du' - \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u'|_\eta^{-d-\alpha}}{|u - u'|^{d+2s}} du'.$$

The first part goes to zero if  $\varepsilon < |u|$ . The second part is written as

$$\begin{aligned} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u'|_\eta^{-d-\alpha}}{|u - u'|^{d+2s}} du' &\sim_\varepsilon |u|^{-d-2s} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} |u'|_\eta^{-d-\alpha} du' \\ &= |u|^{-d-2s} \int_{B(0,\varepsilon\eta^{-\frac{\alpha}{1+\beta}})} (1 + |v'|^2)^{-d-\alpha} dv'. \end{aligned}$$

Taking  $\eta$  small then  $\varepsilon$  small yields

$$\lim_{\eta \rightarrow 0, \varepsilon \rightarrow 0} \left( -c_{\alpha,\beta} \eta^{\frac{\alpha}{1+\beta}} \int_{B(0,\varepsilon)} \frac{|u|_\eta^{-d-\alpha} - |u'|_\eta^{-d-\alpha}}{|u - u'|^{d+2s}} du' \right) = \frac{c_{\alpha,\beta}}{c_{\alpha,0}} \cdot \frac{1}{|u|^{d+2s}}.$$

Since  $\nabla_v (|v|^{-d-2s} v) = -2s|v|^{-d-2s}$ , we deduce that

$$\lim_{\eta \rightarrow 0} \eta^{\frac{1-\beta}{1+\beta}} U \left( u\eta^{-\frac{1}{1+\beta}} \right) = U_\infty(u) = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} \frac{u}{|u|^{d+2s}} |u|^{d+\alpha} = \frac{c_{\alpha,\beta}}{2sc_{\alpha,0}} |u|^{-\beta} u.$$

This proves the scaling limit of the drift force.

From the rescaled equation for  $\Phi_\eta$ , we deduce that  $\Phi_\eta$  goes to  $\Phi$ , where  $\Phi$  solves,

$$\frac{c_{\alpha,\beta}}{2asc_{\alpha,0}} \frac{u}{|u|^\beta} \cdot \nabla_u \Phi - i(u \cdot \sigma) \Phi = 0 \quad \text{with} \quad \Phi(0) = 1 \quad \implies \quad \Phi(u) := \exp \left( i \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u \cdot \sigma)}{1 + \beta} \right).$$

Thus, in the limit case  $\alpha = 1 + s = 2 + \beta$ ,  $\Omega(u) = \lim_{\lambda \rightarrow 0, \lambda \neq 0} \frac{\text{Im} \Phi(\lambda u)}{\lambda^{1+\beta}}$  satisfies,

$$\frac{c_{\alpha,\beta}}{2ac_{\alpha,0}} u \cdot \nabla_u \Omega = (u \cdot \sigma) |u|^\beta \quad \implies \quad \Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}} \frac{|u|^\beta (u \cdot \sigma)}{1 + \beta}.$$

**8.5. The particular case  $\alpha = 2s$ .** More explicit calculations are available when  $\alpha = 2s$ . In this case  $\beta = 0$ ,  $U(v) = c_0 v$  for some constant  $c_0 > 0$ , and the eigenproblem is

$$-\Delta_v^s \phi + c_0 v \cdot \nabla_v \phi - i\eta(v \cdot \sigma) \phi = \mu(\eta) \phi.$$

Taking the Fourier transform (in the dual of the Schwarz space) gives

$$-|\xi|^{2s} \hat{\phi} - c_0 \xi \cdot \nabla_\xi \hat{\phi} + \eta \sigma \cdot \nabla_\xi \hat{\phi} = (\mu(\eta) + c_0) \hat{\phi}$$

or equivalently

$$(\eta \sigma - c_0 \xi) \cdot \nabla_\xi \hat{\phi} = (\mu(\eta) + c_0 + |\xi|^{2s}) \hat{\phi}.$$

The solution to this equation is given by  $\hat{\phi} = \delta_{c_0^{-1} \eta \sigma}$  and  $\mu(\eta) = -|c_0^{-1} \eta \sigma|^{2s} = c_0^{-2s} \eta^{2s}$ , which yields by inverse Fourier transform  $\phi_\eta(v) := \exp(i c_0^{-1} \eta (v \cdot \sigma))$ . This agrees with the expression of  $\Phi$  given above, and allows us to compute  $c_0 = \frac{1}{2s}$ .

## 9. REMARKS AND EXTENSIONS

In Hypothesis 1, the equilibrium  $\mathcal{M}$  is an explicit power law, and in particular is centered and even. We discuss in this section the changes required for our proofs to deal with more general  $\mathcal{M}$  that are (i) characterised by *asymptotic* power-law estimates rather than exact formulae, and (ii) not necessarily even or centered. This means replacing Hypothesis 1 with:

**Hypothesis 1'** (Equilibria). *The equilibrium distribution satisfies*

$$(9.1) \quad \mathcal{M} = [\cdot]^{-(d+\alpha)} \mathcal{S}(v),$$

where  $\mathcal{S}$  is a slowly varying function, and the **generalised mass condition** (1.8).

Slowly varying functions are non-vanishing measurable functions that satisfy  $\mathcal{S}(ax) \sim \mathcal{S}(x)$  as  $x$  goes to infinity, for any  $a > 0$ . Some examples of slowly varying functions are positive constants, functions that converge to positive constants, logarithms and iterated logarithms.

**9.1. Equilibria characterised only asymptotically.** If one considers a *centered* equilibrium  $\mathcal{M}$  that satisfies Hypothesis 1', the proof of Theorem 1.4 in Section 2 and the proof of Lemma 1.1 in Section 3 are essentially unchanged. The formulas for  $\mu_0$  and  $\kappa$  in Lemmas 1.2 and 1.3 are slightly modified, and rely on the existence of a scaling limit of  $\eta^{-\frac{d+\alpha}{1+\beta}} \mathcal{M}(u\eta^{-\frac{1}{1+\beta}})$  as  $\eta \rightarrow 0$ , which follows from Hypothesis 1'. Everything else remains unchanged and the structures of the proofs in Sections 4 and 5 are the same. Rates of convergence will depend on the form of  $\mathcal{S}$ .

**9.2. Non-centered equilibria.** When the microscopic equilibrium  $\mathcal{M}(v)$  is not centered, it results in a drift in the macroscopic equation. Our approach however allows us to tackle such a situation, with the following changes depending on whether this macroscopic drift is of higher, comparable or smaller order than the resulting (fractional) macroscopic diffusion. In view of Theorem 1.4 in the centered situation, we expect a macroscopic diffusion of order  $\zeta(\alpha, \beta) = \min\left(2, \frac{\alpha+\beta}{1+\beta}\right)$ , and therefore we expect the drift to be dominant when  $\alpha > 1$  and dominated when  $\alpha \in [0, 1)$ , with a borderline case at  $\alpha = 1$ . Observe that  $\alpha = 1$  is also the threshold for the absolute convergence of the integral  $\int_{\mathbb{R}^d} (v \cdot \sigma) \mathcal{M}(v) dv$  defining the macroscopic drift.

Consider a solution  $f$  in  $L^\infty([0, +\infty); L_{x,v}^2(\mathcal{M}^{-1}))$  to (1.1) and denote

$$f_\varepsilon(t, x, v) := f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon} + \frac{\bar{v}_\varepsilon t}{\theta(\varepsilon)}, v\right) \in L_t^\infty([0, +\infty); L_{x,v}^2(\mathcal{M}^{-1}))$$

where  $\varepsilon > 0$  and  $\theta(\varepsilon)$  is defined in (1.15), and where the *velocity corrector*  $\bar{v}_\varepsilon$  is defined by

$$(9.2) \quad \bar{v}_\varepsilon := \begin{cases} \frac{\int_{\mathbb{R}^d} v \mathcal{M}(v) dv}{\int_{\mathbb{R}^d} \mathcal{M}(v) dv} & \text{when } \alpha > 1, \\ \left( \lim_{R \rightarrow \infty} \frac{1}{\ln(R)} \frac{\int_{\mathbb{R}^d} v \chi_R(v) \mathcal{M}(v) dv}{\int_{\mathbb{R}^d} \chi_R(v) \mathcal{M}(v) dv} \right) \frac{|\ln(\varepsilon)|}{1+\beta} & \text{when } \alpha = 1, \\ 0 & \text{when } \alpha \in [0, 1). \end{cases}$$

The equation satisfied by  $f_\varepsilon$  is

$$(9.3) \quad \theta(\varepsilon) \partial_t f_\varepsilon + \varepsilon(v - \bar{v}_\varepsilon) \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon.$$

With this definition of  $f_\varepsilon$ , Theorem 1.4 holds and yields the (fractional) diffusive limit of  $f_\varepsilon$ . The changes in the proofs are as follows. The arguments presented in Section 2 are essentially unchanged with a few modifications to obtain the scaling of the eigenvalue resulting from (9.3).

We chose  $\bar{v}_\varepsilon$  in such a way that the dominant eigenmode has the scaling obtained in Lemmas 1.2 and 1.3. The new spectral problem to be considered in the modified Lemma 1.1 is

$$-L^* \phi_\eta - i\eta [(v - \bar{v}_\varepsilon) \cdot \sigma] \phi_\eta = \mu(\eta) [v]^{-\beta} \phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) \mathcal{M}_\beta(v) dv = 1.$$

Line-by-line technical modifications are needed in the proof of Lemmas 1.2 and 1.3 due to the additional drift but the procedure and method are preserved and we do not repeat the arguments. Let us just explain why we define the correction velocity  $\bar{v}_\varepsilon$  in this way. The spectral projector estimate of Section 3 follows the same procedure, with (3.20) replaced by

$$(9.4) \quad -L^* F - i\eta [(v - \bar{v}_\varepsilon) \cdot \sigma] F - z[v]^{-\beta} F = (v - \bar{v}_\varepsilon) \cdot \sigma.$$

The  $L^2$  estimate is unchanged and the crucial estimate (3.25) remains true as long as

$$q(R) := \int_{\mathbb{R}^d} [v - \bar{v}_\varepsilon] \chi_R(v) \mathcal{M}(v) dv \quad \text{at} \quad R := \eta^{-\frac{1}{1+\beta}}$$

is small compared with  $R_1 \eta^{-1} \Theta(\eta)$ , when  $R_1$  is large enough. This implies that the influence of the drift is smaller than the size of the fluid mode, which is of order  $\eta^{-1} \Theta(\eta)$ . Recall that

$$(9.5) \quad \frac{\Theta(\eta)}{\eta} := \begin{cases} \eta & \text{when } \alpha > 2 + \beta, \\ \eta |\ln(\eta)| & \text{when } \alpha = 2 + \beta, \\ \eta^{\frac{\alpha-1}{1+\beta}} & \text{when } 0 \leq \alpha < 2 + \beta. \end{cases}$$

One can then prove that for all  $\alpha \geq 0$ , one has  $q(\eta^{-\frac{1}{1+\beta}}) \lesssim \eta^{\frac{\alpha-1}{1+\beta}}$ , which proves that  $q(\eta^{-\frac{1}{1+\beta}})$  is small compared with  $R_1 \eta^{-1} \Theta(\eta)$  when  $R_1$  is large enough.

**9.3. More general velocity fields.** One could replace the transport operator  $v \cdot \nabla_x$  by a more general  $a(v) \cdot \nabla_x$ , where  $a$  is odd. All our results and proofs can be extended, even though the scalings found may be changed since the scaling of  $\ell(R)$  in (3.15) will be different. If  $a(v)$  scales like  $|v|^\delta$ , redoing the computations as in Section 2 and Section 3 then one would find

$$(9.6) \quad \Theta(\eta) := \begin{cases} \eta^2 & \text{when } \alpha > 2\delta + \beta, \\ \eta^2 |\ln(\eta)| & \text{when } \alpha = 2\delta + \beta, \\ \eta^{\frac{\alpha+\beta}{\delta+\beta}} & \text{when } 0 \leq \alpha < 2\delta + \beta. \end{cases}$$

An example is given by relativistic particles, for which  $a(v) := c \frac{v}{\sqrt{c^2 + v^2}}$ , where  $c$  is the speed of light. Such transport operators are relevant to special relativity, see for instance [49] in physics and [29] in mathematics. There,  $\Theta$  is given by (9.6) with  $\delta = 0$ .

**9.4. Kinetic Fokker-Planck equation with non gradient confining force.** All the results we obtain for the Fokker-Planck equation with gradient force can be extended to Fokker-Planck operators with non-gradient confining force at little expense. We chose not to present this more general setting in the core of the paper to stay consistent with the clean and simple Hypothesis 1 and to help with readability. It is however possible to consider

$$\mathcal{L}(f) = \Delta_v f + \nabla_v \cdot (U f) \quad \text{where } U \text{ satisfies} \quad \Delta_v \mathcal{M} + \nabla_v \cdot (U \mathcal{M}) = 0,$$

provided that quantitative bounds are available on  $U$  to ensure it is comparable to the drift in the Fokker-Planck operator. The analysis is then similar.

**9.5. The case  $\alpha < 0$ .** Assume in this subsection that  $\beta > 0$ . Observe that the natural condition for Hypothesis 2 (weighted coercivity inequality) to hold is  $\alpha + \beta > 0$  and include the cases of negative values of  $\alpha$ , and indeed the construction of the fluid mode in Lemmas 1.1 and 1.2 is valid for  $\alpha \in (-\beta, 0)$ . However, our main result, Theorem 1.4, assumes that  $\alpha \geq 0$  and this restriction comes from the convergence estimates in Section 2: in the case  $\alpha < 0$ , it is not possible to find initial conditions such that both error terms  $\partial_t E_1$  and  $E_2$  vanish in the limit.

This obstacle is in fact structural. Let us consider the simplest case of a scattering equation

$$\theta(\varepsilon)\partial_t h_\varepsilon + \varepsilon v \cdot \nabla_x h_\varepsilon = \nu(v) (r_\varepsilon - h_\varepsilon),$$

with  $\nu(v) = \nu_0 |v|^{-\beta}$ ,  $\theta(\varepsilon) = \varepsilon^{\frac{\beta}{1+\beta}}$  and  $h_\varepsilon(0, x, v)$  radially symmetric in  $v$  and satisfying

$$\hat{h}_\varepsilon(0, \xi, \varepsilon^{-\frac{1}{1+\beta}} u) \rightarrow \hat{H}(\xi, u).$$

In the spirit of [42] we compute the equation satisfied by the Laplace-Fourier transform  $\tilde{r}_\varepsilon(p, \xi)$  (Laplace in  $t$  and Fourier in  $x$ ):

$$\frac{1}{\theta(\varepsilon)} \left[ \int_{\mathbb{R}^d} \left( 1 - \frac{\nu(v)}{\theta(\varepsilon)p + i\varepsilon v \cdot \xi + \nu(v)} \right) \mathcal{M}(v) |v|^{-\beta} dv \right] \tilde{r}_\varepsilon(p, \xi) = \int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, v) \mathcal{M}(v) |v|^{-\beta}}{\theta(\varepsilon)p + i\varepsilon v \cdot \xi + \nu(v)} dv$$

Observe that (using that  $\mathcal{M}$  is even):

$$\begin{aligned} & \frac{1}{\theta(\varepsilon)} \int_{\mathbb{R}^d} \left( 1 - \frac{\nu(v)}{\theta(\varepsilon)p + i\varepsilon v \cdot \xi + \nu(v)} \right) \mathcal{M}(v) |v|^{-\beta} dv \\ &= \int_{\mathbb{R}^d} \left( \frac{p + i\varepsilon \theta(\varepsilon)^{-1} v \cdot \xi}{\theta(\varepsilon)p + i\varepsilon v \cdot \xi + \nu(v)} \right) \mathcal{M}(v) |v|^{-\beta} dv \\ &= \int_{\mathbb{R}^d} \left( \frac{p(\theta(\varepsilon)p + \nu(v)) + \varepsilon^2 \theta(\varepsilon)^{-1} (v \cdot \xi)^2}{(\theta(\varepsilon)p + \nu(v))^2 + (\varepsilon v \cdot \xi)^2} \right) \mathcal{M}(v) |v|^{-\beta} dv \end{aligned}$$

and (using that  $\hat{h}_\varepsilon(0, \xi, v)$  is radially symmetric in  $v$ )

$$\int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, v) \mathcal{M}(v) |v|^{-\beta}}{\theta(\varepsilon)p + i\varepsilon v \cdot \xi + \nu(v)} dv = \int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, v) \mathcal{M}(v) |v|^{-\beta} (\theta(\varepsilon)p + \nu(v))}{(\theta(\varepsilon)p + \nu(v))^2 + (\varepsilon v \cdot \xi)^2} dv.$$

We then change variable  $v = \varepsilon^{-\frac{1}{1+\beta}} u$  in these two integrals. Since then  $\nu(v) = \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta}$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \left( \frac{p(\theta(\varepsilon)p + \nu(v)) + \varepsilon^2 \theta(\varepsilon)^{-1} (v \cdot \xi)^2}{(\theta(\varepsilon)p + \nu(v))^2 + (\varepsilon v \cdot \xi)^2} \right) \mathcal{M}(v) |v|^{-\beta} dv \\ &= \int_{\mathbb{R}^d} \left( \frac{p(\theta(\varepsilon)p + \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta}) + \varepsilon^2 \theta(\varepsilon)^{-1} (\varepsilon^{-\frac{1}{1+\beta}} u \cdot \xi)^2}{(\theta(\varepsilon)p + \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta})^2 + (\varepsilon \varepsilon^{-\frac{1}{1+\beta}} u \cdot \xi)^2} \right) \mathcal{M}(\varepsilon^{-\frac{1}{1+\beta}} u) \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta} \varepsilon^{-\frac{d}{1+\beta}} du \\ &= c_{\alpha, \beta} \varepsilon^{\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \left( \frac{p(p + |u|^{-\beta}) + (u \cdot \xi)^2}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} \right) |u|^{-d-\alpha} |u|^{-\beta} du \\ &\underset{\varepsilon \rightarrow 0}{\sim} c_{\alpha, \beta} \varepsilon^{\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \left( \frac{p(p + |u|^{-\beta}) + (u \cdot \xi)^2}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} \right) \frac{du}{|u|^{d+\alpha+\beta}} \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, v) \mathcal{M}(v) |v|^{-\beta} (\theta(\varepsilon)p + \nu(v))}{(\theta(\varepsilon)p + \nu(v))^2 + (\varepsilon v \cdot \xi)^2} dv \\ &= \int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, \varepsilon^{-\frac{1}{1+\beta}} u) \mathcal{M}(\varepsilon^{-\frac{1}{1+\beta}} u) \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta} (\theta(\varepsilon)p + \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta})}{(\theta(\varepsilon)p + \varepsilon^{\frac{\beta}{1+\beta}} |u|^{-\beta})^2 + (\varepsilon^{\frac{\beta}{1+\beta}} u \cdot \xi)^2} \varepsilon^{-\frac{d}{1+\beta}} du \\ &= \int_{\mathbb{R}^d} \frac{\hat{h}_\varepsilon(0, \xi, \varepsilon^{-\frac{1}{1+\beta}} u) \varepsilon^{-\frac{d}{1+\beta}} \mathcal{M}(\varepsilon^{-\frac{1}{1+\beta}} u) |u|^{-\beta} (p + |u|^{-\beta})}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} du \\ &\underset{\varepsilon \rightarrow 0}{\sim} c_{\alpha, \beta} \varepsilon^{\frac{\alpha}{1+\beta}} \int_{\mathbb{R}^d} \hat{H}(\xi, u) \frac{|u|^{-\beta} (p + |u|^{-\beta})}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} \frac{du}{|u|^{d+\alpha}}. \end{aligned}$$

All in all, we deduce that  $\tilde{r}_\varepsilon(p, \xi)$  converges towards  $\tilde{r}(p, \xi)$  defined by

$$\left[ \int_{\mathbb{R}^d} \left( \frac{p(p + |u|^{-\beta}) + (u \cdot \xi)^2}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} \right) \frac{du}{|u|^{d+\alpha+\beta}} \right] \tilde{r}(p, \xi) = \int_{\mathbb{R}^d} \hat{H}(\xi, u) \frac{(p + |u|^{-\beta})}{(p + |u|^{-\beta})^2 + (u \cdot \xi)^2} \frac{du}{|u|^{d+\alpha+\beta}}.$$

This is, written in Laplace-Fourier variables, an equation with fractional derivative in space but which is non local in time, instead of a fractional diffusion equation.

## REFERENCES

- [1] P. Aceves-Sanchez and L. Cesbron. Fractional diffusion limit for a fractional Vlasov-Fokker-Planck equation. *SIAM J. Math. Anal.*, 51(1):469–488, 2019.
- [2] P. Aceves-Sanchez and A. Mellet. Anomalous diffusion limit for a linear Boltzmann equation with external force field. *Math. Models Methods Appl. Sci.*, 27(5):845–878, 2017.
- [3] G. Ariel, A. Rabani, S. Benisty, J. D. Partridge, R. M. Harshey, and A. Be’er. Swarming bacteria migrate by Lévy Walk. *Nature Communications*, 6(1):8396, Sept. 2015.
- [4] C. Bardos, F. Golse, and I. Moyano. Linear Boltzmann equation and fractional diffusion. *Kinet. Relat. Models*, 11(4):1011–1036, 2018.
- [5] E. Barkai, E. Aghion, and D. A. Kessler. From the area under the Bessel excursion to anomalous diffusion of cold atoms. *Phys. Rev. X*, 4:021036, May 2014.
- [6] N. Ben Abdallah, A. Mellet, and M. Puel. Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach. *Kinet. Relat. Models*, 4(4):873–900, 2011.
- [7] G. Birkhoff and E. P. Wigner, editors. *Nuclear Reactor Theory*, volume 11 of *Proceedings of Symposia in Applied Mathematics*. American Mathematical Society, 1961.
- [8] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. *C. R. Math. Acad. Sci. Paris*, 344(7):431–436, 2007.
- [9] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Asymptotics of the Fast Diffusion Equation via Entropy Estimates. *Archive for Rational Mechanics and Analysis*, 191(2):347–385, Feb. 2009.
- [10] L. Boltzmann. Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungsberichte Akad. Wiss., Vienna*, 66(275), 1872. English translation: L. Boltzmann, Further studies on the thermal equilibrium of gas molecules, in S. G. Brush *Kinetic Theory Vol. 2*, pp.88-175, Oxford 1966.
- [11] C. Börgers, C. Greengard, and E. Thomann. The diffusion limit of free molecular flow in thin plane channels. *SIAM Journal on Applied Mathematics*, 52(4):1057–1075, 1992.
- [12] F. Bouchut, F. Golse, and M. Pulvirenti. *Kinetic equations and asymptotic theory*, volume 4 of *Series in Applied Mathematics (Paris)*. Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 2000. Edited and with a foreword by Benoît Perthame and Laurent Desvillettes.
- [13] E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Fractional Hypocoercivity. ArXiv:1911.11020, 2019.
- [14] H. Brezis and A. C. Ponce. Kato’s inequality when  $\Delta u$  is a measure. *C. R. Math. Acad. Sci. Paris*, 338(8):599–604, 2004.
- [15] P. Cattiaux, E. Nasreddine, and M. Puel. Diffusion limit for kinetic Fokker-Planck equation with heavy tails equilibria: the critical case. *Kinet. Relat. Models*, 12(4):727–748, 2019.
- [16] L. Cesbron. Anomalous diffusion limit of kinetic equations in spatially bounded domains. *Comm. Math. Phys.*, 364(1):233–286, 2018.
- [17] L. Cesbron, A. Mellet, and M. Puel. Fractional diffusion limit of a kinetic equation with diffusive boundary conditions in the upper-half space. *Arch. Ration. Mech. Anal.*, 235(2):1245–1288, 2020.
- [18] L. Cesbron, A. Mellet, and K. Trivisa. Anomalous transport of particles in plasma physics. *Applied Mathematics Letters*, 25(12):2344–2348, 2012.
- [19] H. Chen and L. Véron. Semilinear fractional elliptic equations with gradient nonlinearity involving measures. *J. Funct. Anal.*, 266(8):5467–5492, 2014.
- [20] M. Daoud and E. H. Laamri. Fractional laplacians: A short survey. *Discrete & Continuous Dynamical Systems - S*, 15(1):95–116, 2022.
- [21] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3):1175–1198, 2000.
- [22] C. Dogbé. Anomalous diffusion limit induced on a kinetic equation. *J. Statist. Phys.*, 100(3-4):603–632, 2000.
- [23] J. Dolbeault and G. Toscani. Fast diffusion equations: Matching large time asymptotics by relative entropy methods. *Kinetic & Related Models*, 4(3):701–716, 2011.
- [24] R. S. Ellis and M. A. Pinsky. The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pures Appl. (9)*, 54:125–156, 1975.
- [25] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [26] N. Fournier and C. Tardif. Anomalous diffusion for multi-dimensional critical kinetic Fokker-Planck equations. *Ann. Probab.*, 48(5):2359–2403, 2020.

- [27] N. Fournier and C. Tardif. One dimensional critical kinetic Fokker-Planck equations, Bessel and stable processes. *Comm. Math. Phys.*, 381(1):143–173, 2021.
- [28] U. Frisch and H. Frisch. Non-LTE transfer—III. Asymptotic expansion for small  $\varepsilon$ . *Monthly Notices of the Royal Astronomical Society*, 181(2):273–280, 1977.
- [29] R. T. Glassey. *The Cauchy Problem in Kinetic Theory*. Society for Industrial and Applied Mathematics, 1996.
- [30] F. Golse. Anomalous diffusion limit for the Knudsen gas. *Asymptot. Anal.*, 17(1):1–12, 1998.
- [31] S. Hittmeir and S. Merino-Aceituno. Kinetic derivation of fractional Stokes and Stokes-Fourier systems. *Kinet. Relat. Models*, 9(1):105–129, 2016.
- [32] M. Jara, T. Komorowski, and S. Olla. Limit theorems for additive functionals of a Markov chain. *Ann. Appl. Probab.*, 19(6):2270–2300, 2009.
- [33] M. Jara, T. Komorowski, and S. Olla. Superdiffusion of energy in a chain of harmonic oscillators with noise. *Comm. Math. Phys.*, 339(2):407–453, 2015.
- [34] T. Komorowski and S. Olla. Ballistic and superdiffusive scales in the macroscopic evolution of a chain of oscillators. *Nonlinearity*, 29(3):962–999, 2016.
- [35] T. Komorowski, S. Olla, and L. Ryzhik. Fractional diffusion limit for a kinetic equation with an interface. *Ann. Probab.*, 48(5):2290–2322, 2020.
- [36] G. Lebeau and M. Puel. Diffusion approximation for Fokker Planck with heavy tail equilibria: a spectral method in dimension 1. *Comm. Math. Phys.*, 366(2):709–735, 2019.
- [37] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor equations*. Springer-Verlag, Vienna, 1990.
- [38] S. Marksteiner, K. Ellinger, and P. Zoller. Anomalous diffusion and Lévy walks in optical lattices. *Phys. Rev. A*, 53:3409–3430, May 1996.
- [39] J. C. Maxwell. On the process of diffusion of two or more kinds of moving particles among one another. *Philosophical Magazine*, 20:21–38, 1860.
- [40] J. C. Maxwell. On the dynamical theory of gases. *Philosophical transactions of the Royal Society of London*, 157:49–88, 1867.
- [41] A. Mellet. Fractional diffusion limit for collisional kinetic equations: A moments method. *Indiana Univ. Math. J.*, 59:1333–1360, 2010.
- [42] A. Mellet, S. Mischler, and C. Mouhot. Fractional diffusion limit for collisional kinetic equations. *Archive for Rational Mechanics and Analysis*, 199(2):493–525, 2011.
- [43] E. Nasreddine and M. Puel. Diffusion limit of Fokker-Planck equation with heavy tail equilibria. *ESAIM Math. Model. Numer. Anal.*, 49(1):1–17, 2015.
- [44] B. Nicolaenko. Dispersion Laws for plane wave propagation. In A. Grünbaum, editor, *The Boltzmann Equation*. Courant Institute, 1971.
- [45] B. Perthame, W. Sun, and M. Tang. The fractional diffusion limit of a kinetic model with biochemical pathway. *Zeitschrift für angewandte Mathematik und Physik*, 69(3):67, May 2018.
- [46] G. C. Pomraning. *The Equations of Radiation Hydrodynamics*. Pergamon Press, 1973.
- [47] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. *Journal of Differential Equations*, 260(12):8675–8715, 2016.
- [48] Y. Sagi, M. Brook, I. Almog, and N. Davidson. Observation of anomalous diffusion and fractional self-similarity in one dimension. *Phys. Rev. Lett.*, 108:093002, Mar 2012.
- [49] J. M. Stewart. *Non-equilibrium relativistic kinetic theory [by] John M. Stewart*. Springer-Verlag Berlin, New York, 1971.
- [50] Y. Tu and G. Grinstein. How white noise generates power-law switching in bacterial flagellar motors. *Phys. Rev. Lett.*, 94:208101, May 2005.
- [51] F.-Y. Wang and J. Wang. Functional inequalities for stable-like Dirichlet forms. *J. Theoret. Probab.*, 28(2):423–448, 2015.
- [52] A. M. Weinberg and E. P. Wigner. *The physical theory of neutron chain reactors*. The University of Chicago Press, Chicago, Ill., 1958.
- [53] E. P. Wigner. Mathematical problems of nuclear reactor theory. In *Proc. Sympos. Appl. Math., Vol. XI*, pages 89–104. American Mathematical Society, Providence, R.I., 1961.

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