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Optimism leads to optimality: Ambiguity in network formation*

Péter Bayer[†] Ani Guerdjikova[‡]

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Abstract

We analyze a model of endogenous two-sided network formation where players are affected by uncertainty about their opponents' decisions. We model this uncertainty using the notion of equilibrium under ambiguity as in Eichberger and Kelsey (2014). Unlike the set of Nash equilibria, the set of equilibria under ambiguity does not always include underconnected and thus inefficient networks such as the empty network. On the other hand, it may include networks with unreciprocated, one-way links, which comes with an efficiency loss as linking efforts are costly. We characterize equilibria under ambiguity and provide conditions under which increased player optimism comes with an increase in connectivity and realized benefits in equilibrium. Next, we analyze network realignment under a myopic updating process with optimistic shocks and derive a global stability condition of efficient networks in the sense of Kandori et al. (1993). Under this condition, a subset of the Pareto optimal equilibrium networks is reached, specifically, networks that maximize the players' total benefits of connections.

Keywords: Ambiguity, network formation, optimism.

JEL classification: C72, D81, D85.

1 Introduction

The question of how social networks form and which structures are both stable and efficient is of primary interest in economic literature. In this paper, we study the effect of perceived strategic uncertainty (ambiguity) and optimism on the outcomes of a network formation game. This question is of interest for two reasons. First, as explained below, empirical literature from social psychology as well as experimental studies on entrepreneurs, suggest that optimism is a relevant factor in network formation, which can improve both individual outcomes and social welfare. Using the concept of equilibrium under ambiguity by Eichberger and Kelsey (2014), our model provides a theoretical underpinning for these findings. Second, models of network formation exhibit large indeterminacy, allowing for both Pareto-dominated (and in particular, the empty network) and Pareto-dominant equilibrium networks and thus raising the question of equilibrium selection. We address this by introducing a best-response dynamic similar to Kandori et al. (1993), (henceforth KMR) and Young (1993) but with shocks affecting players'

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optimism. We derive conditions for this dynamic to reach a subset of the equilibrium networks which are not Pareto-dominated by other equilibria.

We model endogenous network formation, whereby a link between two players is treated as a value-generating opportunity (recording a duet, publishing a research paper, a productivity-increasing R&D partnership between two companies). Link values are pair-specific, (as, e.g., in Bloch and Dutta (2009) and Ding (2019)), exogenous and not subject to network externalities (the quality of a recording or of a published paper does not depend on the set of other recordings or papers; the increase in productivity does not affect the productivity of firms outside of the partnership). Linking is costly and the value of the link is realized if and only if both parties incur the cost (meeting and rehearsing for the duet, deriving the relevant results, adjusting and employing the available resources towards the partnership). Notably, an investing player bears the cost even if her intended partner does not reciprocate, thus generating strategic complementarities and a coordination problem on the level of link formation. Positive externalities arise because the efforts to a link are complements, while negative network externalities arise, because players' costs are assumed convex in the number of maintained links. A player is thus negatively affected by any two of the other players linking to each other, as this increases her potential partners' costs of linking to her and decreases her prospects of obtaining beneficial links.

Our first result extends the result of Gilles and Sarangi (2010) by characterizing the set of equilibrium networks for this game. Nash equilibria are undirected networks, such that the value of each link established by a player in equilibrium exceeds the marginal cost of establishing the link. Not surprisingly, this game has multiple Nash equilibria, some of which are Pareto dominated such as, notably, the empty network.

The multiplicity of equilibria naturally give rise to strategic uncertainty, which cannot be resolved using the refinement concepts introduced in the literature (see our discussion of the literature at the end of this section). We follow Eichberger and Kelsey (2014) and introduce ambiguity about the strategies chosen by others combined with an attitude towards such ambiguity (optimism/pessimism). Such ambiguity can be attributed to three distinct, but interconnected, sets of factors. First, the perception of ambiguity depends on institutional characteristics (the existence of formal or informal rules and the establishment of trust). As Koppenjan and Klijn (2004) puts it: "Networks can be distinguished from each other in terms of the degree to which actors more or less share certain perceptions. The more similar that actors' perceptions are [...] the easier it will be to reach consensus in complex games". Geographical, social or cultural proximity between players may reduce ambiguity. Second, ambiguity is related to the perceived complexity of the underlying interaction, for instance, multiplicity of equilibria, ease of describing and predicting the optimal behavior of others. Finally, on a social psychological level, previous experience of players with the game or with the other players, as well as their perceived competence level may determine perceived ambiguity. Ambiguity may thus be small within a local rural community, where rules and norms are well-established, people know each other and the number of possible links is small; it may be larger for a class of freshmen starting college in a different state or country and thus being exposed to different cultures and backgrounds; and even larger for newly recruited members of a criminal gang in which complexity is large, while norms and trust are difficult to establish. Within a network, more experienced players, or "insiders", who have better understanding of the rules, may perceive less ambiguity: companies with insider knowledge of the market or experienced professionals may face less ambiguity when establishing business partnerships.

When strategic uncertainty is present, behavior crucially depends on attitudes towards such uncertainty, optimism and pessimism. Empirical research shows that such attitudes are both individual- and context-specific: they may be inherited, learned through socialization, from experience, or even consciously acquired (from e.g., psychological therapy). Optimism also varies across cultures on the country level (Hostede, 2001). Thus, agents may exhibit different levels of optimism depending on their country

of origin, family background, training, experience, or mood. Self-selection or market selection may lead to certain attitudes becoming predominant in specific environments (Guerdjikova and Sciubba, 2015; Eichberger and Guerdjikova, 2018). Research in economics and social psychology points out the relevance of optimism in the process of social network formation, optimistic individuals being more likely to initiate and sustain beneficial social relationships, form alliances and partnerships and become successful entrepreneurs. The “tendency to expect favorable outcomes” (Srivastava and Angelo, 2009) means that optimists attach more weight to their partners reciprocating their efforts towards establishing a relationship. Such expectations may become self-fulfilling as partners reciprocate even without being optimistic.

We examine the impact of ambiguity and optimism on equilibrium networks. We characterize and prove existence of equilibrium (Proposition 3.3) by showing that, starting from any network, a sequence of unilateral best-responses (under ambiguity) leads to an equilibrium in a finite number of steps. This result provides an algorithm for identifying equilibria under ambiguity in a network formation game, and shows that such equilibria are stable under simple learning dynamics. We provide a characterization of the equilibrium networks under ambiguity.

We next discuss the effect of increased optimism in equilibrium networks under ambiguity. Optimism increases the weight players place on others reciprocating a link and thus increases the maximal number of links a player initiates as a best-response (Proposition 3.5). However, the effect of optimism on the number and total value of realized links is ambiguous: a player who becomes more optimistic might sever existing links in favor of more valuable but unreciprocated ones, resulting in a less connected network. This is precluded by a condition called *monotonicity*. A network is called *monotonic* (in linking decisions) if for each player, any initiated link is at least as valuable to her in expectation as any of the uninitiated links. If an equilibrium network is monotonic then any increase of the players’ optimism parameters results in a weakly more connected equilibrium (Proposition 3.7).

Ambiguity does not resolve the problem of equilibrium multiplicity. Our second set of results thus concerns the impact of ambiguity on network realignment. While equilibrium analysis offers interesting insights, social networks are, by their nature, changeable: relationships and partnerships continuously form and break apart with much of this variation unaccounted for by changes in economic fundamentals. In our model, such variations can be attributed to changes in the players’ subjective ambiguity perception and optimism. We investigate which conditions ensure that such network realignment is efficiency-improving.

The economic and psychological literature has found that both ambiguity perception, as well as optimism are subject to time variation based on the psychological state of the individual, changes in mood, quality of daily life and life-time events, see our literature section for a review. We thus consider the network’s evolution under best-response dynamics with random changes in the individuals’ ambiguity and optimism parameters. Similarly to KMR, we assume that players have realistic, utility-maximizing preferences by default and play the game accordingly, unaffected by ambiguity. This changes in random periods when they receive a shock to their ambiguity perception. Differently from KMR, the deviation in behavior related to the shock is not random, but optimal by their current ambiguity perception and attitude. Shocks are temporary and may be followed by the player becoming again realistic in the next period. Shocks that induce high levels of optimism lead to players more actively pursuing high-value links, possibly by discarding already existing ones. Highly pessimistic players discard existing links without forming new ones. Thus, to capture more interesting cases of network realignment, we focus on shocks that amount to surges of optimism.

This dynamic leads to a subset of Pareto-undominated equilibrium networks independently of the starting network or of the probability of optimistic surges, provided that a property we call the ‘aligned preferences’ is met (Propositions 4.4 and 4.5). In particular, our dynamic selects the networks that maximize the players’ total benefits of connections. Aligned preferences means that all networks that

maximize the players’ benefits from network connections in the set of Nash equilibrium networks (a social consideration) are monotonic (an individual consideration). Thus, social considerations ‘align’ with individual ones. This condition is always satisfied if the values of all links are identical, even if link costs may differ across players. Therefore, surges of individual optimism lead to the formation of an optimal equilibrium network if efficiency is measured by the size of the network, a property that even previously considered pairwise notions fail to satisfy in this relatively simple case. If the condition is not met, we show that a minimal network of “mutually favorite” links forms almost surely and never breaks up. However, a minimal counterexample of three players shows that these dynamics fail to produce an efficient network as the efficient network itself is unstable to surges of optimism.

The rest of the paper is organized as follows: The remainder of this section reviews the related literature. Section 2 introduces a game of network formation and adapts the notion of equilibrium under ambiguity to it. Section 3 contains our results for the static network formation game under ambiguity; we show existence of equilibria, we provide a characterization, and study the comparative statics with respect to the players’ attitude towards ambiguity. In Section 4, we turn towards a dynamic analysis of network realignment. We analyze a best-response dynamic with optimistic shocks and analyze its long-term behavior. For aligned preferences, the dynamic almost surely reaches the set of efficient networks in finite time. Section 5 concludes.

1.1 Related literature

Our model descends from the link formation game informally defined by Myerson (1991), where a link forms between pairs of individuals if and only if both parties consent to it. This idea has been developed to create a large literature of non-cooperative network formation models, starting with Bala and Goyal (2000). In our model, linking benefits are the sum of values of formed links and the cost of linking is a convex function of the number of individuals a player intends to link to. The convex cost assumption is a more general and realistic than the linear cost case studied in most of the literature¹. The convex cost function imposes natural limits on the size of both the equilibrium and the efficient networks (contrary to, e.g., Goyal and Joshi (2003)’s “small costs” case and König et al. (2014), whereby the complete network is efficient). Similarly to Gilles and Sarangi (2010)’s equilibrium characterization of such ‘mutual consent games’, we show that the Nash equilibria are undirected networks, such that the value of each link established by a player in equilibrium exceeds the marginal cost of establishing the link. This latter condition corresponds to the link-deletion-proof property (Jackson and Wolinsky, 1996) and to the strong deletion-proof property by Gilles et al. (2006). The resulting multiplicity of equilibria, some of which are Pareto dominated such as, notably, the empty network is commonly observed in the literature, (Brueckner, 2006; Westbrook, 2010; Griffith, 2020; Baumann, 2019). Pairwise stability (Jackson and Wolinsky, 1996) can be used as a refinement on this set, but as argued also by Gilles and Sarangi (2010), this does not solve the problem of multiplicity. More restrictive concepts such as the strong equilibrium (Jackson and van den Nouweland, 2005), or the bilateral equilibrium (Goyal and Vega-Redondo, 2007), can achieve efficiency.

Another strand of related literature is that of dynamic network formation games. The failure of static refinements to deliver efficiency has given rise to dynamic approaches based on best-response dynamics. Best-response dynamics as in KMR have been studied both in network formation games (Watts, 2001) as well as in network games with simultaneous choice of links and actions (Ely, 2002; Jackson and Watts, 2002a,b; Goyal and Vega-Redondo, 2005; Hojman and Szeidl, 2006; Galeotti and Goyal, 2010; Dawid and Hellman, 2014), see Hellman and Staudigl (2014) for a review. In general, neither uniqueness of the stochastically stable state, nor its efficiency within the set of Nash equilibria are guaranteed. Uniqueness and efficiency arise in cases in which network formation is costless and all

¹Bloch (1995) and Goyal and Joshi (2003) work with decreasing marginal cost, which seems realistic when the cost of links can be distributed across all members of an association but not when links are bilateral and costs are player-specific

link values are positive, as in König et al. (2014) and Staudigl (2011), and thus the complete network (which is both efficient and a Nash equilibrium) emerges in the limit.

The final broader topic related to our paper is optimism under strategic uncertainty. Experimental studies in economics have shown that ambiguity in the other players' strategies is indeed present, leading to deviations from Nash equilibrium behavior (Pulford and Colman, 2007; Di Mauro and Castro, 2008; Eichberger et al., 2008; Kelsey and Le Roux, 2015, 2017). Attitudes towards such ambiguity are heterogeneous and show non-negligible rates of both optimism and pessimism (Camerer and Karjalainen, 1994; Ivanov, 2011; Eichberger et al., 2008; Kocher et al., 2018; Li et al., 2019). Optimism emerges as a characteristic feature of successful entrepreneurs, which positively correlates with measures of performance, (Bengtsson et al., 2012; Bogliacino and Gallo, 2015; Persson and Seiler, 2022). Notably, Holm et al. (2013) identifies optimism viewed as willingness to accept ambiguity in social interactions and to bet on the trustworthiness of a partner, to be a key characteristic of entrepreneurs. Dollinger (1995) confirms that tolerance towards ambiguity is positively related to an entrepreneur's willingness to engage in alliances.

Research in social psychology confirms these findings on the level of social link formation. Optimistic individuals are found to be more likely to form and sustain beneficial links in situations of strategic uncertainty (freshmen forming friendships during their first months on campus) (Andersson, 2012; Brissette et al., 2002). This research suggests that ambiguity attitudes, and in particular, optimism, are also empirically relevant for the process of network formation. Notably, social psychologists identify optimism as a pervasive individual feature (Peterson, 2000), which is positive both for individuals and the society as a whole (Freud, 1928; Tiger, 1979). Both ambiguity perception and ambiguity attitudes have been found to exhibit significant intrapersonal variation. Indeed, high-arousal emotional states (joy, fear, anger) are known to increase ambiguity perception, while low-arousal states (sadness, contentment) decrease it (Baillon et al., 2013).

From a theoretical point of view, optimism has been shown to have interesting and relevant implications in supermodular games. Existence of equilibria under ambiguity, as well as comparative static results for such games can be established using well-known results, see Topkis (1979), as shown by Schipper (2021), and Eichberger and Kelsey (2014). In particular, optimism acts as an equilibrium selection device resulting in Pareto dominant equilibria in coordination games (Eichberger and Kelsey, 2002; Eichberger et al., 2008) and leads to an increase in payoffs in the maximal equilibrium in supermodular games (Schipper, 2021). Since the game we study is not supermodular due to the convexity of the cost functions, these results do not apply. Our paper thus extends results on existence and comparative statics in games with ambiguity to the class of network formation games studied here.

2 The model

2.1 A game of network formation

We now define a game of network formation with additive benefits gained from direct interactions. Let $n > 1$ and let $I = \{1, \dots, n\}$ be the finite set of players. Let W be an $n \times n$ non-negative, symmetric, real matrix with the convention $w_{ii} = 0$ for $i \in I$, called the matrix of benefits. Let $c_i: \mathbb{N} \rightarrow \mathbb{R}_+$ be a strictly increasing, weakly convex function, such that $c_i(0) = 0$, called player i 's cost function. We write $c = (c_i)_{i \in I}$ for the vector of cost functions. Our network formation game is parametrized by the pair (W, c) .

A strategy of player i is a binary n -vector $x_i = (x_{ij})_{j \in I}$ with $x_{ij} = 1$ representing the choice by i to link to j , and $x_{ij} = 0$ representing the choice not to link. By convention we take $x_{ii} = 0$ for all $i \in I$. Let X_i denote player i 's set of strategies and $\prod_{i \in I} X_i = X$ the set of strategy profiles. The aggregate of the players' decisions $(x_i)_{i \in I} = x \in X$ characterizes the *directed graph of linking intentions*. We let $\deg_i^+(x) = \sum_{j \in I \setminus \{i\}} x_{ij}$ denote the *out-degree* of player i , the number of players to whom she intends to

link in profile x ; and $\deg_i(x) = \sum_{j \in I \setminus \{i\}} x_{ij}x_{ji}$ denotes the number of reciprocated linking intentions. The players to whom i intends to link in strategy profile x are called her *out-neighbors*, denoted by the set $N_i^+(x)$, while the ones reciprocating the intention are called *neighbors*, denoted by the set $N_i(x)$.

Link formation is two-sided. Once linking intentions are announced, an undirected link forms between players i and j if and only if the linking intention is reciprocated: $x_{ij} = x_{ji} = 1$. In all other cases, no link ends up forming between the two. Let G denote the set of undirected networks (simple graphs) over set of nodes I and let the map $g: X \rightarrow G$ be defined such that for every $x \in X$, $\{i, j\} \in g(x)$ if and only if $x_{ij} = x_{ji} = 1$. For $X' \subseteq X$ we let $g(X')$ denote the image of X' under g . For $G' \subseteq G$ we let $g^{-1}(G')$ denote the pre-image of G' , that is, the set of strategy profiles that form any member of the set of networks G' . Notably, for $X' \subseteq X$, the set $g^{-1}(g(X'))$ contains all profiles that produce the same network as any member of X' .

The payoff of player $i \in I$ is then given as follows:

$$u_i(x) = \sum_{j: \{i,j\} \in g(x)} w_{ij} - c_i(\deg_i^+(x)). \quad (1)$$

In words, the players in game (1) may form costly links with other players to receive the benefit associated with that link as given by the weight matrix. Benefits of links are only realized by the participants if both decide to link to one another. Crucially, players incur costs for all outgoing links even ones that are not reciprocated. Our game therefore is similar to the models which require mutual consent for link formation, see Myerson (1991) and Gilles and Sarangi (2010).

We continue with a useful notation for marginal benefits and marginal costs.

Definition 2.1 (Marginal benefits, marginal costs). Let a strategy profile $x \in X$ be given. Then, player i 's *benefit from her marginal link* is $w'_i(x) = \min_{\{j: x_{ij}=1\}} w_{ij}$. For $r \in \mathbb{N}$, player i 's *marginal cost function* is given by $c'_i(r) = c_i(r) - c_i(r-1)$ if r is positive and $c'_i(0) = 0$.

By convention we have $\min \emptyset = \infty$. Note that player i 's benefit from her marginal link takes into account all her linking intentions, not just the reciprocated ones that end up forming. In the special case of linear cost functions, we denote constant marginal cost of player i by c_i .

We assume for simplicity that marginal costs are never equal to marginal benefits, that is, for every natural integer r and disjoint pair of players $i, j \in I$ we have $w_{ij} \neq c'_i(r)$. We make this assumption to simplify characterizations by avoiding indifference, but it may also be interpreted as the players always preferring not to link in case they are indifferent. This property holds for almost all game parametrizations.

We now introduce the equilibrium concepts of the game without ambiguity to serve as benchmarks. For a player $i \in I$, let $X_{-i} = \prod_{j \neq i} X_j$. The set of best response strategies of player i to the strategy combination chosen by the other players is denoted $B_i(x_{-i})$. As usual in the literature, we write $B_i(x)$. A strategy profile x is a Nash equilibrium of the game of network formation if $x_i \in B_i(x)$ for all $i \in I$.² The following characterization follows in a straightforward way.

Proposition 2.2 (Nash equilibrium characterization). *A strategy profile x^* is a Nash equilibrium if and only if the following two conditions are satisfied*

1. $x_{ij}^* = 1 \Leftrightarrow x_{ji}^* = 1$ (*reciprocity*),
2. $w'_i(x^*) > c'_i(\deg_i(x^*))$ for all $i \in I$ (*marginal benefits higher than marginal costs*).

²Throughout the paper, we focus on equilibria in pure strategies.

Since linking to a player without reciprocation is costly without giving any immediate benefits, every Nash equilibrium consists of reciprocated linking intentions. Furthermore, each player's least profitable neighbor must bring more benefit than the marginal cost of the player's links. We denote the set of Nash equilibria by X^* . The set of networks $g(X^*)$ are called *Nash stable*. As we consider a game of two-sided link formation, we also adapt the notion of pairwise Nash stability (Bloch and Jackson, 2006).

Definition 2.3 (Pairwise Nash stability). A strategy profile x is *pairwise Nash stable* if

- x is a Nash equilibrium,
- for any pair $\{i, j\} \notin g(x)$ we have either $u_i(x) > u_i(x + ij)$ or $u_j(x) > u_j(x + ij)$,

where the profile $x + ij$ is defined as $(x + ij)_{ij} = (x + ij)_{ji} = 1$ and $(x + ij)_{k\ell} = x_{k\ell}$ for any other pair $\{k, \ell\} \neq \{i, j\}$.

Pairwise Nash stable profiles have two properties: no player should be better off by unilaterally severing any number of her existing links, and no two unlinked players should be better off by forming a new reciprocated link. We note that the notion of pairwise stability (Jackson and Wolinsky, 1996), which prescribes stability against bilateral formation of a new link or unilateral severance of any single present link, coincides with pairwise Nash stability in our game: as Proposition 2.2 indicates, being in a Nash equilibrium entails an unwillingness to sever a single link for each player; her least profitable one. The characterization of pairwise Nash stable profiles in our game follows in a straightforward way.

Lemma 2.4 (Pairwise Nash stability characterization). *A Nash equilibrium x^* is pairwise Nash stable if and only if for every $i, j \in I$ with $x_{ij}^* = x_{ji}^* = 0$ we have $w_{ij} < \max\{c'_i(\deg_i(x^*) + 1), c'_j(\deg_j(x^*) + 1)\}$.*

In a pairwise Nash stable profile, every mutually profitable link must form and thus any link that does not end up forming must be unprofitable for at least one participant. A network formed by a pairwise Nash stable profile is itself called pairwise Nash stable. We note that pairwise Nash stable networks coincide with the set of maximal Nash stable networks.³

In this paper we are interested in the conditions that allow the players' optimism to increase efficiency in network formation games. We first define Pareto optimal equilibria:

Definition 2.5 (Pareto optimal equilibrium). A Nash equilibrium $x \in X^*$ is *Pareto optimal* if there does not exist another Nash equilibrium $x' \in X^*$ such that $u_i(x') \geq u_i(x)$ for all $i \in I$ with at least one strict inequality.

Clearly, a Pareto optimal equilibrium has to be pairwise Nash stable, but not every pairwise Nash stable profile is Pareto optimal; for instance, pairwise Nash stable networks are not stable against mutually beneficial changes by coalitions of more than two players. Note that even Pareto optimal equilibria may be Pareto dominated by other profiles that are not themselves equilibria; our restriction to the set of Nash equilibria guarantees that individual participation constraints are satisfied, a crucial consideration in non-cooperative models of network formation.

As Pareto optimality is a relatively weak notion, we will use another criterion, measuring the players' total benefits from realized network connections, as well. For a profile $x \in X$, let $v(x) = \sum_{i < j: \{i, j\} \in g(x)} w_{ij}$ denote the total sum of realized benefits in the formed network $g(x)$.

Definition 2.6 (Maximum-benefit equilibria). The profile $x^* \in X^*$ is called a maximum-benefit equilibrium (MBE) if $x^* \in \operatorname{argmax}_{x \in X^*} v(x)$.

³A Nash stable network g is called maximal if for every $i, j \in I$ with $\{i, j\} \notin g$, the network we get by adding the link $\{i, j\}$, $g \cup \{i, j\}$, is not Nash stable.

Let the set of MBE profiles be denoted by V^* . Networks in the set $g(V^*)$ are called maximum-benefit networks. MBE profiles maximize total benefits from connections in the society, while ensuring that for each player i , the value she derives from her marginal link exceeds her cost, and i chooses her connections optimally given the behavior of the other players. Thus, they satisfy a participation constraint and an incentive constraint for each individual and can be, in principle, decentralized. While in general, maximum-benefit equilibria are not necessarily Pareto optimal (see Example A.1), our theorems will necessitate a condition (“aligned preferences”), under which they are (which we show formally in Lemma 4.3). As such, we use the maximal-benefit criterion for selection.

If all players’ cost functions are linear, that is, $c_i(r) = c_i \cdot r$ for constant values c_i , the notions of pairwise stability, Pareto optimality, and MBE networks coincide in a unique profile defined by $\{i, j\} \in x \Leftrightarrow w_{ij} > \max\{c_i, c_j\}$.

2.2 Ambiguity attitudes and equilibrium under ambiguity

The concept of Nash equilibrium presupposes that players have correct beliefs about their opponents’ behavior and best-respond to these beliefs. In games with multiple Nash equilibria, such an assumption seems rather strong. While players might indeed entertain beliefs about the behavior of the other players in the game, they might have limited confidence in such beliefs. We refer to such uncertainty as strategic ambiguity.

The game of network formation studied here presents several sources of strategic ambiguity. First, a coordination problem exists at the level of each mutually beneficial link. Second, if a player has more beneficial links than she can establish, the other players face ambiguity regarding her choice of partners. Finally, the multiplicity of Pareto-optimal equilibria leads to a coordination problem on the aggregate level. The implied complexity of the game renders the task of predicting the behavior of others highly non-trivial.

The theory of ambiguity allows us to model uncertainty about the other players’ strategies by the use of non-additive probabilities called capacities, see Schmeidler (1989).

Let \mathcal{X}_{-i} denote the set of all subsets of X_{-i} .

Definition 2.7 (Schmeidler (1989)). A *capacity* on $(X_{-i}, \mathcal{X}_{-i})$ is a function $\nu_i: \mathcal{X}_{-i} \rightarrow [0, 1]$ with $\nu_i(\emptyset) = 0$, $\nu_i(X_{-i}) = 1$ and $\nu_i(A) \leq \nu_i(A')$, whenever $A \subseteq A'$.

A capacity is thus a non-additive measure, which is normalized between 0 and 1 and satisfies monotonicity with respect to set inclusion. A capacity in general violates additivity: the measures assigned to an event $A \subseteq X_{-i}$, $\nu_i(A)$, and its complement $X_{-i} \setminus A$, $\nu_i(X_{-i} \setminus A)$ need not sum up to 1.

The notion of a capacity is very general. In this paper, we use a special class of capacities, called NEO-additive capacities, characterized by two parameters per player, $\alpha_i \in [0, 1]$, and $\delta_i \in [0, 1]$, as well as a probability distribution π_i on X_{-i} (which will be endogenously determined in equilibrium).

Definition 2.8 (Chateauneuf et al. (2007)). For a player i , given her ambiguity parameters α_i, δ_i , and a probability measure π_i on $(X_{-i}, \mathcal{X}_{-i})$, the *NEO-additive capacity* $\nu_i(\cdot | \alpha_i, \delta_i, \pi_i)$ on $(X_{-i}, \mathcal{X}_{-i})$ is defined as follows:

$$\nu_i(A | \alpha_i, \delta_i, \pi_i) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A = X_{-i}, \\ \delta_i \alpha_i + (1 - \delta_i) \pi_i(A) & \text{otherwise.} \end{cases}$$

A NEO-additive capacity can be interpreted as follows: the probability distribution π_i is player i ’s best estimate about the behavior of other players. The degree of confidence of i in this best estimate is given by $(1 - \delta_i)$, while δ_i represents the perceived ambiguity about the behavior of other players. For a given π_i , δ_i identifies a set of probability distributions given by the convex combination of π_i with the set of all possible probability measures on X_{-i} , $\Delta(X_{-i})$,

$$\Pi_i = (1 - \delta_i) \{\pi_i\} + \delta_i \Delta(X_{-i})$$

that the player takes into account when deciding on her strategy. In particular, if $\delta_i = 0$, the degree of confidence is 1, i is not affected by ambiguity, and her subjective beliefs are represented by π_i itself. When the degree of confidence $(1 - \delta_i) < 1$, the player perceives ambiguity δ_i with respect to her best estimate π_i , and Π_i contains multiple probability distributions.

The effect of such ambiguity on i 's beliefs about an event A depends on the player's degree of optimism, α_i . When $\alpha_i = 0$, the player is a complete pessimist and her beliefs scale down the probability of any event A (except for the whole event X_{-i}) by her degree of confidence, $(1 - \delta_i)$. For a fully optimistic player with $\alpha_i = 1$, the probabilities of all events (except the impossible one, \emptyset) are overweighted.

NEO-additive capacities provide a parsimonious way to capture both ambiguity and the players' potentially heterogeneous response towards it, while remaining analytically tractable and empirically measurable, (Baillon et al., 2018). In contrast to max-min expected utility (Gilboa and Schmeidler, 1989), they allow for both ambiguity-loving and ambiguity-averse behavior and have therefore the advantages of the general Choquet expected utility by Schmeidler (1989) and the α -max-min expected utility by Ghirardato et al. (2004). They are thus capable of capturing the role of optimism in network formation as outlined in the Introduction. Furthermore, the NEO-additive capacities allow for a differentiation between the "objective" best estimate captured by the probability distribution π_i , which, in equilibrium, will reflect the actually chosen strategy of the other players, and the subjective perception of and attitude towards strategic uncertainty, as captured by the parameters δ_i and α_i . In equilibrium, players' best estimates π_i are self-confirming in that they coincide with the actually played equilibrium strategies, while for $\delta_i > 0$, there is still room for strategic uncertainty which impacts the equilibrium in a non-trivial way (Eichberger and Kelsey, 2014).

The parameters of ambiguity and ambiguity attitude, δ_i and α_i are naturally allowed to vary across players. Perceived ambiguity may depend on the (perceived) complexity of the interaction, the number of equilibria and the ease of computing them, the availability of natural focal points, or, as pointed out by Koppenjan and Klijn (2004), the existence of rules and social conventions which guide behavior. More experienced players, as well as players who perceive themselves as insiders may be subject to less ambiguity than novices unfamiliar with the (formal or informal) rules governing the specific interaction. Ambiguity attitude is in general player- and context-specific and may reflect (similarly to risk-aversion) individual dispositions, or be attributed to previous experience acquired consciously or unconsciously as explained in the introduction.

In this paper, we will restrict attention to pure strategies and thus, to NEO-additive capacities for which π_i is a Dirac measure on some strategy combination of the other players, $x_{-i} \in X_{-i}$, $\pi_i(x_{-i}) = 1$. Slightly abusing notation, we will write $\nu_i(\cdot | \alpha_i, \delta_i, x_{-i})$ to denote such a capacity.

Given i 's subjective beliefs expressed by a NEO-additive capacity $\nu_i(\cdot | \alpha_i, \delta_i, x_{-i})$, her Choquet expected payoff from choosing a strategy x_i in the network formation game above is given by the Choquet integral of the payoff function u_i with respect to ν_i :

Definition 2.9 (Chateauneuf et al. (2007)). Given ambiguity parameters α_i, δ_i , and a probability measure π_i on $(X_{-i}, \mathcal{X}_{-i})$ such that $\pi_i(x_{-i}) = 1$ for some $x_{-i} \in X_{-i}$, the Choquet expected payoff with respect to the NEO-additive capacity ν_i is given by:

$$\begin{aligned} U_i(x_i, \nu_i(\cdot | \alpha_i, \delta_i, x_{-i})) &= \int u_i(x_i, \tilde{x}_{-i}) d\nu_i \\ &= \alpha_i \max_{\pi \in \Pi_i} E_\pi [u_i(x_i, \tilde{x}_{-i})] + (1 - \alpha_i) \min_{\pi \in \Pi_i} E_\pi [u_i(x_i, \tilde{x}_{-i})] \\ &= \delta_i \left[\alpha_i \max_{\tilde{x}_{-i} \in X_{-i}} u_i(x_i, \tilde{x}_{-i}) + (1 - \alpha_i) \min_{\tilde{x}_{-i} \in X_{-i}} u_i(x_i, \tilde{x}_{-i}) \right] + (1 - \delta_i) u_i(x_i, x_{-i}) \end{aligned} \quad (2)$$

The payoff of strategy x_i given belief ν_i is the sum of the expected payoff obtained under the best estimate x_i with weight $1 - \delta_i$, the highest possible payoff attainable by x_i with weight $\delta_i \alpha_i$, and the

lowest payoff attainable by x_i , with weight $\delta_i(1 - \alpha_i)$. In the network formation game, this simplifies to:

$$U_i(x_i, \nu_i(\cdot|\alpha_i, \delta_i, x_{-i})) = \sum_{j \neq i} x_{ij} w_{ij} (\delta_i \alpha_i + (1 - \delta_i) x_{ji}) - c_i(\deg_i^+(x)) \quad (3)$$

which obtains by noting that $u_i(x_i, x_{-i})$ is always maximized at $x_{ji} = 1$ for every $j \neq i$, whereas its minimum obtains at $x_{ji} = 0$ for every $j \neq i$.

There are three extreme cases: for a realistic player i ($\delta_i = 0$), the Choquet expected payoff of x_i given ν_i coincides with the payoff of x_i given x_{-i} , for an optimistic/pessimistic player i ($\delta_i = \alpha_i = 1$ and $\delta_i = 1, \alpha_i = 0$ respectively), the Choquet expected payoff of x_i ignores the best estimate π_i and equals the largest/smallest payoff available through x_i .

In the following, we will treat α_i and δ_i as parameters and we will write $\alpha = (\alpha_i)_{i \in I}$ and $\delta = (\delta_i)_{i \in I}$ for the vectors of ambiguity attitudes and ambiguity perceptions.

The best response correspondence of player i given that her beliefs are represented by a NEO-additive capacity ν_i with parameters α_i and δ_i is defined as usual by

$$B_i(\nu_i(\cdot|\alpha_i, \delta_i, x_{-i})) = \arg \max_{x_i \in X_i} U_i(x_i, \nu_i(\cdot|\alpha_i, \delta_i, x_{-i})).$$

As most definitions of equilibrium, the equilibrium under ambiguity imposes consistency between equilibrium beliefs and equilibrium behavior. Differently from the standard Nash equilibrium concept, in our framework, players perceive ambiguity about the behavior of their opponents. Hence, their beliefs will in general assign positive weight to strategies which are not used in equilibrium. The consistency notion is thus imposed on a subset of strategies assigned positive weights under ν_i , namely on those that are assigned a strictly positive probability for all probability distributions in the set Π_i . This defines the support of the capacity ν_i , and, in the case, in which π_i is a Dirac measure on some x_{-i} , is given by x_{-i} , the best estimate about the strategy combination played by the other players.

We now adapt the equilibrium under ambiguity concept of Eichberger and Kelsey (2014) to our framework as follows:

Definition 2.10 (Eichberger and Kelsey (2014)). For given vectors of parameters of optimism and ambiguity perception (α, δ) , a network x^* is an *equilibrium under ambiguity* (EUA) if for every $i \in I$, $x_i^* \in B_i(\nu_i^*(\cdot|\alpha_i, \delta_i, x_{-i}^*))$.

EUA imposes constraints both on players' strategies and on players' beliefs. In particular, whenever player i 's best estimate is that strategy combination x_{-i}^* will be played by the other players, her equilibrium strategy, x_i^* is a best response according to her respective equilibrium beliefs given by $\nu_i^*(\cdot|\alpha_i, \delta_i, x_{-i}^*)$. It is easily seen that this condition is equivalent to the definition of EUA in Eichberger and Kelsey (2014). It also illustrates the similarity of the concept to the Nash equilibrium as a mutual best-response.

Two features distinguish the EUA from the Nash equilibrium. First, beliefs are given by a capacity, and players take into account and best-respond to a set of strategies (Π_i) played by their opponents. In particular, their best estimate x_{-i}^* even though correct at the equilibrium, is still deemed ambiguous: the player is not certain that other players will indeed choose x_{-i}^* . As explained in Eichberger and Kelsey (2014), this may be due to a limited experience with the game or with the players involved, to the imperfection of the model of the game that captures the strategic situation at hand, or (most relevant in our setting) to the complexity of the game including, but not limited to the fact that it allows for multiple equilibria. As a result, each player allows for the possibility of a deviation by δ_i from the best estimate x_{-i}^* reflected by the set Π_i and chooses a strategy that maximizes her α -max-min

expected utility on this set. For $\delta_i = 0$, ambiguity is irrelevant and we recover probabilistic beliefs and the standard notion of best response.

The second difference consists in the consistency imposed between equilibrium beliefs and equilibrium strategies. In a Nash equilibrium, every strategy x_{-i} assigned a strictly positive probability by i (and thus, in the support of i 's probabilistic beliefs) is a best response of the other players. Similarly, in an EUA, the consistency is imposed on the support of the capacity, i.e., exactly on those strategies which are assigned a strictly positive probability *for all distributions* in Π_i . In the case of pure strategy equilibria, the only such strategy is x_{-i}^* . In this sense, the "best estimate" is a very cautious prediction about the others' play restricted to those profiles which are assigned a strictly positive probability by all priors consistent with the capacity. In contrast, different elements of Π_i assign strictly positive probabilities to strategy profiles which are not a best-response by the opponents, but are considered in i 's optimization, because though unlikely, their play would have substantial impact on i 's payoff.

This approach is in fact common to all notions of equilibrium with ambiguity. In fact, as pointed out by Eichberger and Kelsey (2014), different solution concepts, such as Marinacci (2000), Dow and Werlang (1994), and Ryan (1997), who consider general convex capacities, (and thus, exclude optimism), differ exactly in the definition of the set of strategies on which consistency is imposed. However, as shown by Eichberger and Kelsey (2014), for the case of NEO-additive capacities, all these definitions coincide.⁴

We denote by $X^*(\alpha, \delta)$ the set of equilibria under ambiguity given (α, δ) and continue using X^* for the set of equilibria of the game without ambiguity, that is, $\delta_i = 0$ for all $i \in I$.

3 Equilibria under ambiguity of the network formation game

We begin by a characterization of a player's best response under parameters α_i and δ_i . To simplify the analysis and avoid the cases of indifference we will discuss ambiguity parameters $\delta_i, \alpha_i, i \in I$ such that for every $i, j \in I$ and every $r \in \mathbb{N}$ we have $w_{ij}\delta_i\alpha_i \neq c'_i(r)$ and $w_{ij}(\delta_i\alpha_i + 1 - \delta_i) \neq c'_i(r)$, which is a generic setting of parameters. We call this the assumption of 'no indifference'.

The behavior of players maximizing (3) is showcased by the following characterization.

Lemma 3.1 (Best response under ambiguity). *For $x \in X$ and $i \in I$ we have $x_i \in B_i(\nu_i(\cdot|\alpha_i, \delta_i, x_{-i}))$ if and only if for every disjoint pair $j, k \in I \setminus \{i\}$, with $x_{ij} = 1$ and $x_{ik} = 0$ we have*

1. $w_{ij}(\alpha_i\delta_i + (1 - \delta_i)x_{ji}) > c'_i(\deg_i(x))$,
2. $w_{ik}(\alpha_i\delta_i + (1 - \delta_i)x_{ki}) < c'_i(\deg_i(x))$,
3. $w_{ij}(\alpha_i\delta_i + (1 - \delta_i)x_{ji}) \geq w_{ik}(\alpha_i\delta_i + (1 - \delta_i)x_{ki})$.

Under ambiguity, players best respond to their beliefs on the opponents' actions x_{-i} but do not place full confidence on these beliefs. Instead, they rank their opponents based on a distorted value $w_{ij}(\alpha_i + (1 - \delta_i))x_{ji}$. The higher their perception of ambiguity, the less weight they place on their best estimate, and the more optimistic they are, the stronger they distort towards the positive outcome that any opponent will link to them. Highly optimistic players thus focus only on the available benefit of a link, w_{ij} , while highly pessimistic ones make their decisions in response to a distorted belief that no opponent will link to them. In best response, all out-links of i must have higher perceived benefits

⁴Other models of ambiguity in games include Hanany et al. (2020) who concentrate on games with incomplete information, where the ambiguity is about the types of other players, as well as the approach of Riedel and Sass (2014) in which players use ambiguous randomization devices to decide on their strategies. These motivations are quite different from the strategic uncertainty we seek to model here. Dominiak and Eichberger (2021) generalize the equilibrium under ambiguity with NEO-additive capacities to belief functions, which however necessitates a significantly larger number of parameters.

than player i 's marginal costs and must satisfy the consistency requirement of always linking to the opponents with the highest perceived value. The characterization of EUA networks follows:

Corollary 3.2 (EUA characterization). *For given parameters (α, δ) , the profile x^* is an EUA if and only if for every trio $i, j, k \in I$ such that $x_{ij}^* = 1$ and $x_{ik}^* = 0$ we have*

$$w_{ij}(\delta_i \alpha_i + x_{ji}^*(1 - \alpha_i)) \geq w_{ik}(\delta_i \alpha_i + x_{ki}^*(1 - \alpha_i))$$

and for every pair $i, j \in I$, x_{ij}^* and x_{ji}^* follow the rules presented by Table 1.

w_{ij}	$> \frac{c'_j(x^*)}{\delta_j \alpha_j}$	$\in \left(\frac{c'_j(x^*)}{\delta_j \alpha_j + 1 - \delta_j}, \frac{c'_j(x^*)}{\delta_j \alpha_j} \right)$	$< \frac{c'_j(x^*)}{\delta_j \alpha_j + 1 - \delta_j}$
$> \frac{c'_i(x^*)}{\delta_i \alpha_i}$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = 1, x_{ji}^* = 0$
$\in \left(\frac{c'_i(x^*)}{\delta_i \alpha_i + 1 - \delta_i}, \frac{c'_i(x^*)}{\delta_i \alpha_i} \right)$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 0$ or 1	$x_{ij}^* = x_{ji}^* = 0$
$< \frac{c'_i(x^*)}{\delta_i \alpha_i + 1 - \delta_i}$	$x_{ij}^* = 0, x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 0$	$x_{ij}^* = x_{ji}^* = 0$

Table 1: EUA characterization. Links which are perceived as high-value (top row and left column) by a given player are always extended, links with perceived low value (bottom row and right column) are never extended. In-between-value links (middle row and column) are extended only if reciprocated, either because the partner to the link perceives them as high-value, or as in-between-value. If, for both partners, the link is perceived as an 'in-between'-value, there may exist some equilibria where a reciprocated link is formed and some where neither direction of the link forms. Note: we use the convention $1/0 = \infty$.

As Table 1 shows, each player views each opponent one of three ways, resulting in a total of nine qualitative types of pairwise relations in equilibrium. A player may have links that she will initiate even if the opponent does not reciprocate. This can only happen if the player is sufficiently optimistic. If such a high-value opponent likewise considers the initiating player valuable enough, a reciprocated link forms between them. If the initiating player is low-value for the opponent, a one-way link forms, sustained in equilibrium by the initiator's optimism.

A player may have low-value opponents to whom she will not link even if that opponent initiates a link towards her. This may either be due to the fact that the player is highly pessimistic, or has links to other opponents, driving up the player's marginal cost. In this case the player may have incoming one-way links from low-value opponents who consider her high-value.

Every other opponent is 'in-between'-value; the player will only form such a link if she believes it will be reciprocated. If an 'in-between'-value opponent considers her as high-value, the opponent will initiate and the player will reciprocate. If she is considered low-value, no links will form. Two players who consider each other 'in-between'-value may link to each other but no directed link can exist between them.

We highlight two important differences between Nash equilibrium and EUA profiles. First, the profile in which no player links to anyone else may not be an EUA as high-value directions will always form in equilibrium. As a result, existence becomes a non-trivial issue (note that whenever the cost function is strictly convex, the game is not supermodular in the players' own strategies and thus, the standard existence result of Topkis (1979) cannot be used).⁵

⁵To see that the game is not supermodular in the players' own strategies under convex costs, let $n = 3$ and $w_{ij} = w$ for all i and $j \in \{1, 2, 3\}$, $i \neq j$. Take four strategy profiles which differ only in regard to x_1 . In particular, let for $x^1, x^2, x^3, x^4 \in X$, $x_{21}^k = x_{31}^k = 1$ and $x_{23}^k = x_{32}^k = 0$ for $k \in \{1, \dots, 4\}$. The four strategy profiles differ with respect to the strategy of player 1; we set $(x_{12}^1 = 0, x_{13}^1 = 0)$, $(x_{12}^2 = 1, x_{13}^2 = 0)$, $(x_{12}^3 = 0, x_{13}^3 = 1)$, $(x_{12}^4 = 1, x_{13}^4 = 1)$. We thus have: $x^1 = x^2 \wedge x^3$, $x^4 = x^2 \vee x^3$ and supermodularity of u_1 in 1's own strategy would require:

$$u_1(x_1^4, x_{-1}^4) - u_1(x_1^3, x_{-1}^3) = w - c'_1(1) \geq w - c'_1(0) = u_1(x_1^2, x_{-1}^2) - u_1(x_1^1, x_{-1}^1)$$

But for c_1 strictly convex, we have $c'_1(1) > c'_1(0)$, in violation of supermodularity.

Second, an EUA profiles may contain unreciprocated linking intentions. To obtain a direct comparison with the set of Nash equilibria, one can take $\delta_i = 0$ for all i , in which case the threshold for high-value opponents becomes infinity, effectively eliminating the top row and left column of Table 1, and leading to the characterization in Lemma 2.2.

We now state a general existence result of EUA.

Proposition 3.3 (Existence of EUA). *For any network formation game (W, c) and any set of ambiguity parameters (α, δ) such that the assumption of ‘no indifference’ is satisfied, an EUA profile exists.*

We obtain the general existence result of Proposition 3.3 by considering strict best-response dynamics: starting in an arbitrary profile, we show that no strict best-response improvement path can revisit a profile it had visited before. Thus, any such path must reach a profile where no player can improve; that network is an EUA. Furthermore, the set of EUA profiles is thus given by the absorbing sets of best-response dynamics under ambiguity.

We now examine the effects of increasing players’ optimism, starting with its effects on the number of out-neighbors. While we will not obtain a direct result, we are able to derive monotonicity conditions on player optimism and the minimal and maximum number of out-neighbors. We begin with a formal definition of these two concepts.

Definition 3.4 (Minimal and maximal out-degree). For player $i \in I$ her *minimal* and *maximal out-degree*, denoted by $\underline{d}_i(\alpha_i, \delta_i)$ and $\bar{d}_i(\alpha_i, \delta_i)$ are defined as

$$\begin{aligned}\underline{d}_i(\alpha_i, \delta_i) &= \deg_i^+(x) \text{ where } x_i \in B_i \left(\nu_i \left(\cdot | \alpha_i, \delta_i, (x_{ji} = 0)_{j \neq i} \right) \right), \\ \bar{d}_i(\alpha_i, \delta_i) &= \deg_i^+(x) \text{ where } x_i \in B_i \left(\nu_i \left(\cdot | \alpha_i, \delta_i, (x_{ji} = 1)_{j \neq i} \right) \right),\end{aligned}$$

respectively.

For fixed ambiguity parameters we write \underline{d}_i and \bar{d}_i for the number of links player i offers in a best response to the belief that no players link to i and all players link to i , respectively. Since indifference of linking and not linking is ruled out, these values are well defined, with \underline{d}_i being the largest degree r such that $r \geq |\{j \in I : \alpha_i \delta_i w_{ij} > c'_i(r)\}|$ holds and \bar{d}_i being the largest degree r such that $r \geq |\{j \in I : (\alpha_i \delta_i + (1 - \delta_i)) w_{ij} > c'_i(r)\}|$ holds. Thus, \underline{d}_i and \bar{d}_i are also lower and upper bounds, respectively, of the number of out-links that player i can have in any EUA network.

For the case of no ambiguity, $\delta = 0$, the values $\underline{d}_i(\alpha_i, 0)$ and $\bar{d}_i(\alpha_i, 0)$ are independent of α_i . In this case, $\underline{d}_i(\alpha_i, 0) = 0$ (absent ambiguity, a player does not link to anyone if no player links to her) and we denote $d_i = \bar{d}_i(\alpha_i, 0)$ as the number of links initiated by a player unaffected by ambiguity when she believes that all other players offer a link to her. It is the maximum number of neighbors a player can have in a best response.

Note that $d_i = \underline{d}_i(1, 1) = \bar{d}_i(1, 1)$, as a player with $\delta_i = \alpha_i = 1$ always best responds to the belief that every opponent links to her. Thus, optimists will always have d_i out-links in a best response. In contrast, $\underline{d}_i(0, 0) = \bar{d}_i(0, 0) = 0$, i.e. pessimistic players do not link to anyone.

We now show how changes in player’s optimism and confidence impact the lower- and upper bounds.

Proposition 3.5 (Minimal and maximal out-degree comparative statics). *Let α_i, δ_i be given. For every $i \in I$, every $\delta'_i > \delta_i$, and every $\alpha'_i > \alpha_i$, we have*

1. $\underline{d}_i(\alpha'_i, \delta_i) \geq \underline{d}_i(\alpha_i, \delta_i)$,
2. $\bar{d}_i(\alpha'_i, \delta_i) \geq \bar{d}_i(\alpha_i, \delta_i)$,
3. $\underline{d}_i(\alpha_i, \delta'_i) \geq \underline{d}_i(\alpha_i, \delta_i)$,

$$4. \bar{d}_i(\alpha_i, \delta'_i) \leq \bar{d}_i(\alpha_i, \delta_i).$$

Since the perceived value of all potential links increases with the degree of optimism, both \underline{d}_i and \bar{d}_i are increasing in α_i (points 1 and 2). An increase in the perception of ambiguity reduces the confidence the player assigns to the events that no one links to her and everyone links to her, respectively. This leads to an increase of \underline{d}_i (point 3) and a decrease of \bar{d}_i . A higher δ_i thus decreases the difference $\bar{d}_i - \underline{d}_i$ which becomes 0 at $\delta_i = 1$, as in this case the player's best response is independent of her best estimate of x_{-i} . If $\delta_i = 1$ for all i , the players' best responses are unique up to indifference between linking to equal-valued neighbors. Without indifference, for instance, in the case of $w_{ij} \neq w_{ik}$ for any triple i, j, k , the best responses are unique and there is a unique EUA profile.

Of interest is the relationship between the ambiguity parameters and the size of the realized network in equilibrium. Although points 1 and 2 show that the bounds of the players' out-links increase with respect to their optimism, the realized network may not be more connected than the old one if the players' optimism increases. For instance, a more optimistic player may sever existing links in an attempt to link to higher-value players but the newly initiated links may not be reciprocated. For this reason, the comparative statics between player optimism and the size of the equilibrium network is not straightforward. We identify a sufficient condition, called the monotonicity of networks, that guarantees that optimism increases the connectivity of equilibrium networks.

Definition 3.6 (Monotonic linking). For a player $i \in I$, a strategy $x_i \in X_i$ is called *monotonic* if for every $j, k \in I$ such that $w_{ij} \geq w_{ik}$ we have $x_{ij} \geq x_{ik}$. A profile of linking intentions $x \in X$ is monotonic if for every $i \in I$, x_i is monotonic.

If x is monotonic, then all players i link to some selection of their top $\deg_i^+(x)$ choices of opponents. In a monotonic profile, a player can only increase the value of her links by forming more links, not by replacing her existing links with better ones. Monotonic profiles always exist, for instance, every player linking to her top choice constitutes a monotonic profile, but there is no guarantee that monotonic EUA profiles exist.

We show that if an equilibrium is monotonic, then upon an increase in the players' optimism there will always exist an EUA under the new parameters which forms a strictly larger network than the old one.

Proposition 3.7 (Monotonic equilibrium comparative statics). *Let $x \in X^*(\alpha, \delta)$ be monotonic and consider an $\alpha' \geq \alpha$. Then, there exists an EUA network, $x'(\alpha', \delta) \in X^*(\alpha', \delta)$ such that $g(x) \subseteq g(x')$.*

We demonstrate Proposition 3.7 with an example.

Example 3.8 (Monotonic equilibrium with increased optimism). Consider a game between four players with $w_{12} = w_{23} = H$, $w_{14} = w_{23} = L$, and $w_{13} = w_{24} = 0$ with $H > L > 0$. Suppose that for all players we have $c'_i(2) < L$: all players can afford two links of value at least L . In the unique Pareto optimal equilibrium profile, which happens to be the pairwise Nash stable profile as well, all links worth H and L form, creating the network $g(x^*) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$ (Figure 1 left), while any subset of $g(x^*)$ is Nash stable. Consider the monotonic profile x where each player links to her high-value partner only, forming the network $g(x) = \{\{A, B\}, \{C, D\}\}$ (Figure 1 right). Suppose that parameters δ_i and α_i are such that $((\alpha_i \delta_i) + (1 - \delta_i))H > c'_i(1)$, $\delta_i L > c'_i(2)$, and $\alpha_i \delta_i L < c'_i(2)$. Then, players are too pessimistic to form the lower-value but still profitable links $\{1, 4\}$ and $\{2, 3\}$, making x the only EUA. Suppose that for all i α_i increases to α'_i in a way that $\alpha'_i \delta_i L > c'_i(2)$, such that players find it optimal to link to low-value opponents too, even without expecting reciprocation. Under the listed assumptions such values of α'_i are guaranteed to exist and x^* becomes the only EUA.

The next example demonstrates that, without monotonicity, increased optimism may lead to a smaller, less efficient equilibrium.

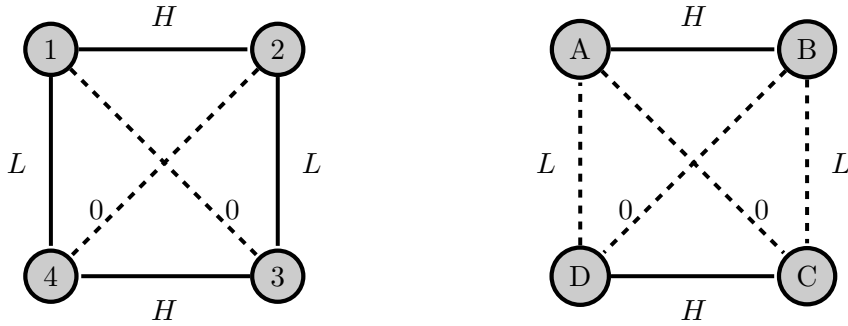


Figure 1: The weight network of Example 3.8. Reciprocated links are shown as continuous edges, no linking is indicated by dashed ones. If $H > L > 0$, and if $c'_i(2) < L$ then network $\{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$ (left) forms in the most efficient profile. If the α_i values are low, players' pessimism prevents the formation of lower-value but mutually profitable links, leading to network $\{\{1, 2\}, \{3, 4\}\}$ (right) forming in the unique EUA. As all players are linked to their highest-valued opponent, this equilibrium is monotonic. Increasing the optimism parameters to overcome can lead to the formation of the efficient network.

Example 3.9. Consider a game between three players with $w_{12} = H$, $w_{13} = M$, and $w_{23} = L$ with $H > M > L > 0$. Suppose that $c'_1(1) < H$ and $c'_1(2) > M$, as well as $c'_3(1) < L$ and $c'_3(2) > M$ meaning that players 1 and 3 can only link profitably to one opponent, while $c'_2(2) < L$, thus it is profitable for player 2 link to any of her two opponents. Let α_i and δ_i be such that (1) $(\delta_1\alpha_1 + (1 - \delta_1))H > c'_1(1)$, (2) $(\delta_2\alpha_2 + (1 - \delta_2))H > c'_2(2)$, (3) $(\delta_3\alpha_3 + (1 - \delta_3))L > c'_3(1)$. Then, the profile forming network $\{\{1, 2\}, \{2, 3\}\}$ with no linking attempt between players 2 and 3 is an EUA, which is not monotonic, as player 3's has one link with player 2 who is not her best choice (Figure 2 left).

Suppose that player 3 becomes more optimistic: $\alpha'_3 > \alpha_3$ such that $\delta_3\alpha'_3M > (\delta_3\alpha_3 + (1 - \delta_3))L$. Then, player 3's perceived benefit of linking to 1 is higher than to 2; in the new EUA, she initiates a link to player 1 without linking to 2 (Figure 2 right). The new EUA has fewer links and is less efficient than the original.

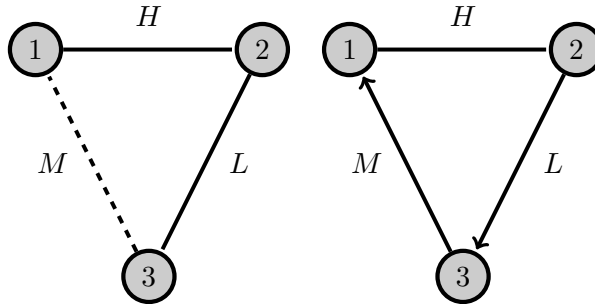


Figure 2: The networks of Example 3.9. Player 2 can accommodate two neighbors, players 1 and 3 only one. The unique EUA (left) is not monotonic as player 3 is not linked to the opponent that would give her the highest value. If 3 becomes more optimistic, who may sever her link with 2 and make an unreciprocated offer to 1 instead (right), leading to a loss of efficiency in the new EUA.

In the special case when all cost functions are linear in the number of neighbors, the game is supermodular and increased optimism raises equilibrium connectivity regardless of whether the initial equilibrium is monotonic (Schipper, 2021).

4 Network realignment through optimism

In this section we study the realignment of networks. By network realignment we refer to sequences of link formations and severances that lead from one equilibrium to another. This type of realignment

may happen for several reasons. Equilibria that are stable against individual or pairwise changes may break apart due to a coordinated improvement by larger coalitions. Farsighted players may experiment in order to trigger a sequence of changes that increase their individual payoffs. Deviations may also occur randomly. Instead, here we consider exogenous shocks to players' perception of and attitude towards uncertainty as a driver of network realignment. As demonstrated in Examples 3.8 and 3.9, such shocks modify players' subjective evaluations of their current linking decisions and thus induce changes in individual linking choices.

Our motivations for studying realignment is to select for equilibria that survive these changes, identify conditions under which such equilibria even exist, and investigate the efficiency properties of surviving equilibria. Since the seminal works of KMR and Young (1993), models of myopic adaptation with random mutations have been used as a selection device in games with multiple equilibria. In general, such models postulate a best-response type of adaptation. In each period, a player is chosen at random and revises her strategy by playing a best response to the current strategy profile of her opponents. With strictly positive probability, deviations (such as mistakes, experimentation) occur; instead of playing her best response, the player chooses a strategy at random.

In this section we discuss the deviations that, instead of affecting strategies directly, affect the players' ambiguity parameters, their perception of ambiguity and optimism. Players then behave accordingly, given their ambiguity parameters. We choose to focus on deviations that amount to increases in players' optimism parameters that we call 'optimistic surges'. As the static comparative analysis in Section 3 shows, players affected by increases in optimism seek out more and/or higher-value neighbors. We thus explore whether optimistic surges select for efficient equilibria in the long-run. In contrast, pessimistic surges cause links to break down without new ones forming. We comment on the possibility of introducing pessimistic shocks in our concluding Section 5.

Our optimism parameter α_i represents the tendency to overweigh good outcomes. The ambiguity parameter δ_i determines the extent to which the player's beliefs are concentrated on the actually observed outcomes or assign positive weight to other, unobserved outcomes. Formally, the combination of these two effects is reflected in the product $\alpha_i\delta_i$, which determines the impact of optimism on strategy choice. Optimism may be subject to exogenous stochastic shocks (such as mood, positive emotions and life-events, see literature review), which are uncorrelated with the network formation game. Our dynamic captures such rare idiosyncratic shocks, while allowing the players to act in a realistic way for most of the time.

We consider a dynamic process taking place in discrete time, $t \in \mathbb{N}$. The profile of linking intentions at time t will be denoted by x^t . Take an initial profile, x^{t-1} . In period t , a player i^t is chosen at random with uniform probability $1/n$ and receives an opportunity to revise her strategy, leading strategy x_{i^t} being chosen. All other players continue playing the strategies corresponding to $x_{-i^t}^{t-1}$, producing profile $x^t = (x_{i^t}, x_{-i^t}^{t-1})$.

Players alternate between acting "realistically" and "optimistically". The former is characterized by an ambiguity parameter $\delta_i = 0$ under which the value of α does not matter. The latter is calibrated as follows: for player $i \in I$ set $\bar{\delta}_i \in (0, 1)$ and $\bar{\alpha}_i \in [0, 1]$ such that for every $j \in I \setminus \{i\}$ and $r \in \{0, n-1\}$ we have $c'_i(r) < \bar{\alpha}_i\bar{\delta}_i w_{ij}$ if and only if $c'_i(r) < w_{ij}$, and for $j, k \in I \setminus \{i\}$ with $w_{ij} > w_{ik}$ we have $\bar{\delta}_i\bar{\alpha}_i w_{ij} > w_{ik}$. Such values are guaranteed to exist with both $\bar{\delta}_i$ and $\bar{\alpha}_i$ being close to 1, as the player set is finite. These parameters represent highly optimistic behavior but not complete ignorance of the player's beliefs (which occurs for $\delta_i = 1$). Thus, optimistic players' behavior will satisfy three properties: (1) if pressed to make a choice, a player will strictly prefer an unreciprocated link of a higher value to a reciprocated link of a lower value, (2) due to $\bar{\delta}_i < 1$ the player will strictly prefer a reciprocated link to an unreciprocated one of equal value, (3) the player always finds it optimal to have exactly d_i out-links to her top d_i choices, i.e. $\underline{d}_i(\bar{\alpha}_i, \bar{\delta}_i) = \bar{d}_i(\bar{\alpha}_i, \bar{\delta}_i) = d_i$.

The main parameter governing the dynamic process is the frequency of optimistic shocks, denoted by

$\beta \in (0, 1)$. If close to zero, our process resembles a best-response dynamic with regular, but infrequent optimistic shocks; if close to 1, the players almost always react optimistically. We imagine small values of β to be the most plausible but our analysis is not sensitive to the exact value as long as all players perform both optimistic and realistic actions with positive probability. The ambiguity parameters of player i^t at the time of revision are thus given by the following time-independent probability distribution:

$$\begin{aligned} \delta_{i^t}^t &= 0 \text{ and } \alpha_{i^t}^t \text{ arbitrary with probability } 1 - \beta, \\ \delta_{i^t}^t &= \bar{\delta}_i \text{ and } \alpha_{i^t}^t = \bar{\alpha}_i, \text{ with probability } \beta, \end{aligned}$$

Given her current ambiguity parameters, the revising player behaves myopically and chooses a best response to $x_{-i^t}^{t-1}$:

$$x_{i^t}^t \in B_{i^t}(\nu_{i^t}(\cdot | \alpha_{i^t}^t, \delta_{i^t}^t, x_{-i^t}^{t-1}))$$

Whenever the set of best responses $B_{i^t}(\cdot)$ is not a singleton, player i^t chooses a best response from the set $B_{i^t}(\cdot)$ uniformly at random. The resulting linking profile at time t , x^t , is then given by

$$x^t(x^{t-1}, i^t, \alpha_{i^t}^t, \delta_{i^t}^t) = (x_{i^t}^t, x_{-i^t}^{t-1}).$$

In words, the updating player is an optimist with uniform probability β and a best-responder with probability $1 - \beta$.

In what follows, we will shorten the notation of an optimistic player's best response to x to $B_i^o(x) = B_i(\nu_i(\cdot | \bar{\alpha}_i, \bar{\delta}_i, x_{-i}))$.

Our best-response dynamics with optimistic surges can be interpreted as a myopic response of a player to the current linking profile. As in the standard case, the player holds “static perceptions on the evolution of any variable of the process outside of her control”, see Vega-Redondo (2003). In this case, she is informed of the strategy profile $x_{-i^t}^{t-1}$ that prevailed during the preceding period. However, rather than assuming that the same strategy profile will be repeated at period t , she considers $x_{-i^t}^{t-1}$ as the best available estimate of the behavior of other players, assigning to it a (possibly limited) confidence of $1 - \delta_{i^t}^t$. In particular, when an optimistic surge occurs, a strictly positive weight is assigned to the strategy profile of the others which would maximize her payoff. This process thus generalizes the standard best-response dynamics to the case in which players assume that others will continue playing their preceding strategies, but perceive ambiguity with regards to such predictions.

The dynamic process describes a Markov chain on the set X . For $x, x' \in X$ and $\beta \in [0, 1]$ let $p_\beta(x, x')$ denote the one-step transition probability from x to x' , given that the probability of an optimistic shock is β . It is clear that for every $x, x' \in X$ with $p_\beta(x, x') > 0$ there exists $i \in I$ such that either $x'_i \in B_i(x)$ or $x'_i \in B_i^o(x)$ and for every $j \neq i$ we have $x'_{-i} = x_{-i}$. For $m \in \mathbb{N}$ let $p_\beta^m(x, x')$ denote the transition probability from x to x' in exactly m steps. For a set of networks $X' \subseteq X$ let $p_\beta^m(x, X') = \sum_{x' \in X'} p_\beta^m(x, x')$ denote the probability of reaching this set from x in exactly m steps.

Definition 4.1 (Successor). For $x, x' \in X$ and $m \in \mathbb{N}$ we say that x' is an m -successor of x if $p_\beta^m(x, x') > 0$.

Denote the set of m -successors of network x by $S^m(x)$ and let $\bigcup_{m \in \mathbb{N}} S^m(x) = S(x)$ denote the set of all successors of x . By convention we include $x \in S(x)$.

We introduce the key condition of our convergence results.

Definition 4.2 (Aligned preferences). The players exhibit *aligned preferences* if every MBE profile is monotonic.

Under aligned preferences, MBE profiles are not only optimal for a benefit-maximizing society, but also satisfy an individual optimality condition for the players: each player is linked to a selection of her

best opponents. This is an appealing property for a game to have, as without it we observe a conflict between efficiency in benefits and the players' incentives.

The notion of aligned preferences also partially resolves the conflict between the two notions of efficiency, MBE and Pareto optimal equilibria, discussed in Section 2. While in general, MBE profiles can be Pareto dominated by other equilibria (see Example A.1 in the appendix), our next Lemma shows that in the case of aligned preferences, MBE are Pareto-optimal in the set of Nash equilibria.

Lemma 4.3. *Under aligned preferences, MBE profiles are Pareto optimal in the set of Nash equilibria.*

Our first main result establishes the stability of networks formed by efficient profiles under aligned preferences.

Proposition 4.4 (Stability of networks formed by efficient profiles). *Suppose that the players exhibit aligned preferences. For every $x^* \in V^*$ and every $x' \in S(x^*)$ we have $g(x') \in g(V^*)$.*

By Proposition 4.4, under aligned preferences, the set of MBE profiles is stable under our dynamic of optimistic and best-responding players. More precisely, starting the dynamic process in an efficient profile, all successor profiles form networks identical to the ones formed by MBE profiles. Successor profiles amount to MBE profiles plus some additional one-way links. Formally, when only looking at the networks formed, the set $g(V^*)$ is absorbing under our dynamic process.

We now present our second main result. If preferences are aligned, then the dynamic described above always pushes the game towards networks that are identical to ones formed by MBE profiles. If there is a unique MBE profile, then the network it forms is stable under our dynamic, allowing for the creation and deletion of additional one-way links. If there is more than one maximum-benefit network, then they may communicate with each other under our dynamic but the total realized benefit of players never changes.

Proposition 4.5 (Maximum benefit networks form almost surely). *Suppose that the players exhibit aligned preferences. Then, for every $x \in X$ and every $\beta \in (0, 1)$ we have*

$$\lim_{m \rightarrow \infty} p_{\beta}^m(x, g^{-1}(g(V^*))) = 1.$$

Proposition 4.5 shows that, under aligned preferences, by the individual updates of players, a maximum-benefit network will form almost surely. Combined with 4.4, this means that networks formed by MBE profiles are globally stable.

The key intuition behind this result is that, from a sufficiently connected but inefficient equilibrium (from which, as we show, we can initiate the process) we can choose a link $\{i, j\}$ which is part of the efficient network but not part of the current network, and a player i who would initiate this link under an optimistic shock. If this is followed by j being given a best-responding revision opportunity, she reciprocates this link, possibly severing some others. There will then exist a series of best-responses through which we obtain a network which is strictly closer to the desired efficient network (notably, by the inclusion of the link $\{i, j\}$ and the exclusion of some unnecessary links).⁶ Since the set of all networks is finite, repeating this process allows the dynamic to reach V^* in finite time. Once a profile in V^* is reached, the value of the forming network in its successors cannot decrease (Proposition 4.4).

We note the difference between our approach and that of KMR. Similarly to their approach, our best-response dynamic with optimistic shocks defines a Markov process on the set of strategy combinations and thus, on the set of strategy profiles. In KMR, the random mutations imply that any strategy can be chosen with strictly positive probability and thus, the Markov process visits any of the Nash equilibria

⁶We interpret the distance between two networks $g, g' \in G$ as the number of links present in g but not in g' plus the number of links present in g' but not in g .

infinitely often. In contrast, in our model, the asymmetric nature of the optimistic shocks implies that the Markov process reaches the set of maximum-benefit networks, with probability 1, regardless of initial conditions. Furthermore, the set $g^{-1}(V^*)$ is absorbing.

We note that letting the probability of optimistic surges β go to 0 does not lead to a further refinement of the set of equilibria. In particular, as the following proposition shows, all MBE are stochastically stable.

Proposition 4.6. *There exists a set of probability distributions $(\hat{\mu}_\beta)_{\beta \in (0,1)}$ such that for each $\beta \in (0,1)$, $\hat{\mu}_\beta$ is an invariant probability distribution of the Markov chain $\left((X, p_\beta)_{\beta \in (0,1)}\right)$ and*

$$\text{supp} \left(\lim_{\beta \rightarrow 0} \hat{\mu}_\beta \right) = V^*.$$

The following example demonstrates network realignment through optimistic shocks under aligned preferences.

Example 4.7 (Network realignment). Consider the benefit weights given by Example 3.8, but with $c'_i(1) < L$ and $c'_i(2) > H$ for all i , that is, each player can have at most one neighbor in equilibrium. Then, the network formed by the unique MBE profile x^* is $g(x^*) = \{\{1, 2\}, \{3, 4\}\}$, but any network where each player has at most one neighbor is also supported by an equilibrium. Notably, x such that $g(x) = \{\{1, 4\}, \{2, 3\}\}$ is also pairwise stable, but not stable against optimistic surges.

The realignment process that leads from x to x^* is shown in Figure 3 and proceeds as follows. (1) under an optimistic shock, 1 severs her link to 4 and extends one to 2. (2) a best-responding 2 severs her link to 3 and reciprocates 1's link. (3) 3 best responds, cutting her unreciprocated link to 2. The network now has the link $\{1, 2\}$ established, which corresponds to a Nash equilibrium profile, and in network distance terms, closer to $g(x^*)$. (4) under another optimistic shock, 4 extends a link to 3. (5) a best-responding 3 reciprocates, leading to the profile x^* .

If preferences are not aligned, then MBE profiles are not always Pareto optimal, and they are also not stable against optimistic shocks; a player who is not linked to her best choice of neighbors, upon experiencing an optimistic shock, will invariably delete her links to her existing neighbors and make offers to her best choices. This can be demonstrated by a minimal counterexample of three players, very similar to Example 3.9.

Example 4.8. Consider a network of three players with $w_{12} = H$, $w_{13} = M$, and $w_{23} = L$ with $H > M > L > 0$. Suppose that $c'_1(1) < H$ and $c'_1(2) > M$, as well as $c'_3(1) < L$ and $c'_3(2) > M$ meaning that players 1 and 3 can entertain only one neighbor, while $c'_2(2) < L$, player 2 can entertain two neighbors. Then, the unique MBE is x^* with $g(x^*) = \{\{1, 2\}, \{2, 3\}\}$, which is not monotonic, thus preferences are not aligned. Starting from x^* if player 3 receives an optimistic shock she will sever her link running to 2 and offers one to 1 which will be unreciprocated (see Figure 2).

We conclude this section by showing convergence in the linear case, which is not covered by the case of aligned preferences.

Proposition 4.9. *Let the cost functions be linear. Then, for every $x \in X$ and every $\beta \in (0,1)$ and the unique $x^* \in V^*$ we have $\lim_{m \rightarrow \infty} p_\beta^m(x, g^{-1}(g(x^*))) = 1$.*

Proposition 4.9 extends the convergence result laid out in Proposition 4.5 to all games with linear costs, without the precondition of aligned preferences. This is due to the fact that under linear costs the unique MBE coincides with the unique pairwise Nash stable equilibrium, which is stable for both optimistic and best-responding deviations.

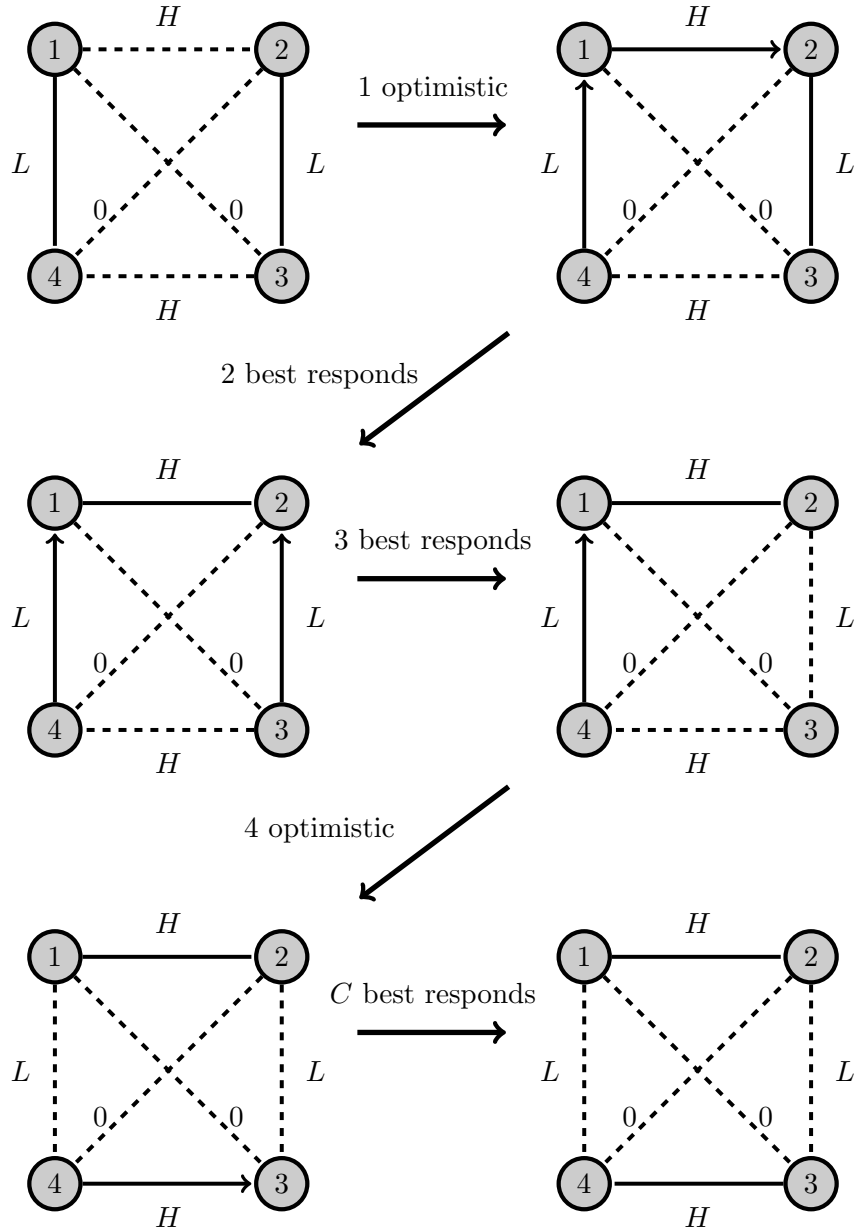


Figure 3: Network realignment from a Pareto dominated pairwise Nash stable equilibrium to an MBE profile. An optimistic shock compels 1 to seek out a new, better partner, severing her link to 4 and extending one to 2. A best-responding 2 reciprocates 1's link and severs the one to 3. A best-responding 3 deletes her unreciprocated link to 2. Finally, when 4 receives an optimistic shock herself, she will extend a link to 3 which a best-responding 3 accepts.

We now consider the case in which preferences are not aligned. Denote by $D_i = \{j \mid w_{ij} > c'_i(d_i)\}$ the set of d_i -highest value ("favorite") partners for player i and by $D_i^* = \{j \in D_i \mid i \in D_j\}$ the subset of favorite players for whom i is also a favorite. Denote the network of "strict mutually favorite" links⁷

⁷In the special case in which $|D_i| = d_i$ for all i , this network indeed contains all mutually favorite links. When $|D_i| > d_i$, however, player i may not be able to establish all of his favorite links in an equilibrium. This is the case in which the set of minima $\arg \min_{k' \in D_i} \{w_{ik'}\}$ is not a singleton and i is indifferent among all such links when receiving an optimistic shock. Such links may thus form and break repeatedly under our best-response dynamic. In contrast (as shown below), g^f represents the set of links which will form and never break under a best-response dynamic with optimistic shocks.

as

$$g^f = \left\{ \begin{array}{l} \{i, j\} \mid i \in D_j, j \in D_i, \text{ and for each } k \in \{i, j\}, \\ \text{either } w_{ij} \neq \min_{k' \in D_k} \{w_{kk'}\} \text{ or } |D_k| = d_k \end{array} \right\}$$

Note that this network is a Nash equilibrium, $g^f \in g(X^*)$. Let $d_i^* = \deg_i(g^{-1}(g^f))$ be the degree of i in g^f . Write

$$D_i^{sb} = \left\{ j \mid i \in D_j, \{i, j\} \notin g^f \text{ s.t. } w_{ij} > c'_i(d_i^* + 1) \right\}$$

for the set of "second-best" partners of i : players for which i is a favorite partner, and to whom i is not connected in g^f , but may want to connect after having established all of her "strict mutually favorite" links in g^f . The set of networks G^{sb} is the set of all equilibrium networks in which each established link is either between strict mutually favorite players in g^f or between a favorite player and a second-best player:

$$G^{sb} = \left\{ \begin{array}{l} g \in g(X^*) \mid g \supseteq g^f \text{ and} \\ g \setminus g^f \subseteq \left\{ \begin{array}{l} \{i, j\} \mid \text{either } j \in D_i \text{ and } i \in D_j^{sb} \\ \text{or } i \in D_j \text{ and } j \in D_i^{sb} \end{array} \right\} \end{array} \right\}$$

We state the following result:

Proposition 4.10. *For every $x^0 \in X$ and every $\beta \in (0, 1)$, there exists almost surely a finite m such that $g(S^m(x^0)) \in G^{sb}$ and $g(S^{m'}(x_0)) \in G^{sb}$ for all $m' \geq m$. Furthermore,*

- (i) *if $|D_i| = d_i$ for all i , every $x \in g^{-1}(G^{sb})$ is recurrent;*
- (ii) *if for each $j \in D_i^{sb}$, $i \in D_j$, we have $w_{ij} > \min_{k \in D_j} w_{jk}$, then there exists almost surely a finite \bar{m} for which $g(S^{\bar{m}}(x^0)) \in G^{sb}$ is maximal (w.r.t. inclusion) in G^{sb} .*

Proposition 4.10 shows that in the general case, the dynamic with optimistic shocks reaches the set of networks in which all links are either strict mutually favorite or second-best links almost surely in finite time. In particular, all strict mutually favorite links in g^f are formed almost surely in finite time and once formed, are never severed. Once a network in G^{sb} is reached, all successive updates have the property that their undirected counterparts remain in the set: G^{sb} is thus an invariant set of the dynamic. Two special cases obtain: first, if the number of favorite links for each player exactly corresponds to their maximal degree, d_i , all networks in G^{sb} will be reached infinitely often. This recurrence may not hold in general for two reasons. First, G^{sb} may contain a network g for which a player i has more than $d_i - |j \in D_i \mid w_{ij} \neq \min_{k \in D_i} \{w_{ik}\}|$ connections with minimal value in D_i with players j such that i is not j 's favorite partner, $i \notin D_j$. In this case, i will not be able to extend all of these connections as part of an optimistic best-response. Since i is not these players' favorite partner, they will not initiate these links, either. Thus, some of these connections will be never established, even though they are mutually beneficial. In particular, the system may not reach a maximal network in G^{sb} .

Second, recurrence of the set G^{sb} may fail because the optimistic best-responses favor reciprocated links to unreciprocated ones when two links have equal value. In particular, for two players, i and j , such that the value of the link with i is minimal in D_j and vice-versa, $i \in D_j \cap D_j^{sb}$ and $j \in D_i \cap D_i^{sb}$, and such that the link $\{i, j\}$ is already formed in x^0 , this link may never be broken. The set of networks reached by the system is then path-dependent.

Both failure of recurrence and the failure to reach a maximal network in G^{sb} are avoided by requiring that $|D_i| = d_i$ for all i as in (i). On the other hand, stipulating, as in (ii), that whenever a link $\{i, j\}$ is second-best for i , and a favorite for j , it is a strict favorite for j , implies that a maximal network in G^{sb} is reached. Which maximal network is reached will in general depend on the initial condition x_0 .

The case of aligned preferences studied above represents a special case whereby the network g^f is monotonic and the set G^{sb} only contains mutually favorite links. Taken together, Propositions 4.5

and 4.10 then imply that the set of maximal benefit networks is a subset of the maximal elements of G^{sb} . If, furthermore, $|D_i| = d_i$ holds for all i , G^{sb} contains a single element, g^f , which is reached with probability 1.

We can use the examples studied earlier to understand the dynamic in absence of aligned preferences. In Example 4.8, $g^f = \{1, 2\}$, whereas the network $g(x^*) = \{\{1, 2\}, \{2, 3\}\} \in G^{sb}$ is the unique maximal network in G^{sb} . These two networks represent the set of recurrent states of the dynamic with optimistic shocks: the Pareto-optimal and maximal benefit network $g(x^*)$ is reached infinitely often. In Example A.1 in Appendix A.1, the network g^f corresponds to the Pareto-optimal equilibrium $\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$, whereas the MBE-network is not an element of G^{sb} . In this case, the Pareto-optimal equilibrium g^f is reached in finite time, and its links are never severed. The only long-term impact of the dynamic is for player 1 to extend a (never reciprocated) link to either player 2 or 3 and for 4 to extend, following an optimistic shock, a (never reciprocated) link to either player 5 or 6, which are withdrawn once the players become realistic again. These examples show that optimistic shocks have a non-trivial impact on network formation even when preferences are not aligned, allowing, at least in special cases, for the formation of Pareto-optimal equilibria.

5 Concluding remarks

In this paper we investigate a model of network formation with players facing ambiguity with respect to their opponents' linking decisions. We model this uncertainty using NEO-additive beliefs: players respond to mixtures of their rational beliefs as expressed by a probability measure of their opponents' actions, their optimistic beliefs, and their pessimistic ones.

To our knowledge, ours is the first project to consider ambiguity in strategies in a network formation game. This combination of frameworks is natural and fits well with behavioral considerations of network formation: optimistic players believe their linking offers are likely to be reciprocated, pessimistic players believe that opponents are unlikely to reciprocate. These considerations provide behavioral explanation to persistent one-way linking attempts such as stalking or refusing a profitable match in the hopes of attaining a better but highly implausible one.

By explicitly modeling player optimism we also allow the link-formation process to be governed proactively by the players. Under Nash equilibrium, link formation is governed by the players passively best-responding to each other's actions or their beliefs about their opponents. Under pairwise (Nash) stability, link-formation is governed by the logic of random pairwise interactions where, upon meeting a mutually beneficial opponent, the pair decides to form a link. Monadic and unilateral stability extend this idea to groups of players, but do not consider the possibility of links being severed to establish new and more beneficial ones. To put it differently, players are assumed to remain passive in creating these interactions and only act if the opportunity arises through events outside their control. Under our framework, optimistic players proactively seek out their best possible links, creating linking opportunities through a costly investment of initiating a link without immediate reciprocation.

Our investigation of equilibrium under ambiguity reveals that optimism can, as expected, lead to better equilibrium outcomes by selecting against underconnected equilibria. Too much optimism, on the other hand, makes players blind to their available prospects in favor of the best they could possibly achieve and thus there is no general monotonic relationship between equilibrium connectivity and optimism. Monotonic equilibria, however, do improve in terms of connectivity if the players become more optimistic, as they feature no tradeoff between the players' available links and their best possible links. In these cases, optimism is helpful for the players to seek out additional profitable partners without severing their links to existing ones.

In addition to equilibrium, we study network realignment under ambiguity. Under aligned preferences, best-response dynamics with optimistic shocks lead to all links of a Pareto optimal equilibrium

network forming; specifically, one that maximizes benefits from links. This insight sheds some light to the role of optimism in a society: individual optimism begets an optimal social network without relying on formal or informal multilateral coordination devices. It is noteworthy that both optimism and realism are found to be necessary attitudes to produce this result, with optimism being the driving force to seek out newer and better links, while realism compels players to act rationally. If the frequency of optimistic shocks is high, efficient networks are reached faster but players launch more unreciprocated linking attempts – even after all links of the Pareto optimal equilibrium network have been formed. If this frequency is lower, Pareto optimal equilibria are reached later but players spend more time at their objective best response – that is, in the actual equilibrium network.

Two practical exports emerge as a result. First, on an individual level, interventions such as psychological therapy or motivational courses meant to increase optimism may be beneficial. An increase in optimism (or decrease in pessimism) motivates individuals to extend links to new partners and sever less beneficial ones. Doing so may lead to an increase in both individual and social welfare. Combining optimism with realism is, however, essential for the individual to be able to discern situations of “aligned preferences” in which the new links will be reciprocated and those in which excessive optimism will lead to severing links without replacing them by more profitable ones. This is in-line with common wisdom which cites optimism as a decisive feature of entrepreneurs, writers and artists, who have to deal with multiple rejections before establishing fruitful partnerships which allow them to realize their ideas and goals.

Second, companies and organizations may be able to mimic the optimistic shock dynamics introduced in our paper by occasionally relying on young and less experienced (for high ambiguity) but optimistic employees to establish new partnerships. Such an internal dynamics within companies generates better connections and higher profits than restricting networking and alliance building only to established managers. Furthermore, under aligned preferences, this dynamics may provide a stimulus for growth of the industry by promoting connectedness and cooperation.

If preferences are not aligned, optimistic shocks are not enough to produce efficient networks. Instead, a Pareto-dominated network may emerge and persist. We thus cannot resolve the prevailing conflict between stability and efficiency in the general case. A possible way to relax the aligned preferences condition is by considering a higher degree of player sophistication. For instance, in Examples 3.9 and 4.8, if player 2 is informed of player 1’s payoff function, she may anticipate that player 1 will not sever her first-best link and is thus, unavailable as a partner. Alternatively, after several unsuccessful tries, she can also learn that player 1 will never reciprocate. Both would allow player 2 to be satisfied with her second-best outcome, which stabilizes the Pareto optimal equilibrium network. This type of sophisticated selection of neighbors is also left for future work.

Our model is without explicit network externalities. This choice simplifies our analysis as players evaluate their linking choices “link-wise”. Furthermore, this avoids the additional complexity of modeling the players’ beliefs of their neighbors’ links (and those of neighbors’ neighbors, etc.), as well as their responses to these beliefs. As the players’ reliable information of distant links can be scarce, a more general model of ambiguity in networks under explicit network externalities necessitates an epistemic foundation of players’ beliefs of distant links. Nevertheless, our base model has implicit network externalities due to the convexity of the cost function. For a player, any link that is formed between two opponents raises the opponents’ marginal costs, making it more difficult for these opponents to sustain links with the player.

Finally, we mention pessimism, which, in our model, does not seem to have any positive role in building the network or realigning efficiently. Pessimistic shocks in our model amount to (possibly temporary) severances of links without forming new ones. This is, however, due to the fact that the benefits to the interactions are certain. If links provide mutual insurance against an idiosyncratic risk, optimists may be less willing to initiate connections than pessimists, because they overweigh the

possibility of the good state occurring. Pessimism will also play a role when network interactions bring negative externalities, such as during a pandemic where being part of a connected social component exposes the individual to risk of infection. Investigating the role and mechanisms of pessimism is yet another interesting direction for future research.

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A Appendix

A.1 Aligned preferences conditions

In this section we discuss implications and conditions of the preferences being aligned. Aligned preferences represent a restriction on the players' preferences in a way that individual considerations to link to play monotonic strategies (i.e., link to one's top choice of opponents), are in agreement with social considerations that seek to maximize the players' sum of realized benefits.

If preferences are not aligned, maximum-benefit networks are not guaranteed to be Pareto optimal. This is illustrated by an example.

Example A.1. Consider a game played between six players. The values of linking between pairs are shown in Figure 4.

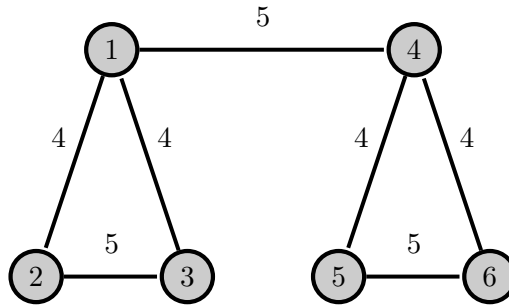


Figure 4: The weight network of example A.1. Links not shown in the figure are assumed to offer a benefit of 0.

Suppose that the cost structure is such that $d_1 = d_4 = 2$, while $d_2 = d_3 = d_5 = d_6 = 1$. Then, the only maximum-benefit network is $\{\{1, 2\}, \{1, 3\}, \{4, 5\}, \{4, 6\}\}$ for a total weight of 16. In this network, no player is linked to her top choice of opponent, hence preferences are not aligned.

Suppose that $c_1(1) = c_4(1) = 3.8$ and $c_1(2) = c_4(2) = 7.7$, which respects $d_1 = d_4 = 2$. Then, the network $\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$, has a weight of 15 and it Pareto dominates the maximum-benefit network.

Example A.1 also shows that the reverse of Lemma 4.3 is not true, that is, the existence of an equilibrium which is monotonic and Pareto optimal does not imply that preferences are aligned. In the example, the Pareto optimal equilibrium is monotonic but preferences are not aligned as it is not an MBE profile.

We highlight the special case where the benefits of all links are equal and players are only interested in the number of their neighbors. In this case, preferences are always aligned and maximum-benefit networks are the maximal Nash networks. Then, we can derive an even stronger statement, every Nash network that is not a maximum-benefit network is Pareto dominated by the maximum-benefit network closest to it.

Formally, the *distance* between two networks g, g' is defined as the the number of links present in exactly one of g or g' , formally: $\phi(g, g') = |(g \setminus g') \cup (g' \setminus g)|$.

Lemma A.2. *Let $w_{ij} = w$ for every disjoint pair $i, j \in I$ for some $w \in \mathbb{R}_+$. Then for every $x \in X^* \setminus V^*$, any profile in the set $\text{argmin}_{x^* \in V^*} \phi(g(x), g(x^*))$, Pareto dominates x .*

Proof: We proceed by contradiction. Let $x \in X^* \setminus V^*$, let x^* be an element of V^* whose induced network is closest to $g(x)$, and assume that x does not Pareto dominate x^* . Then, there exists $i \in I$ such that $\text{deg}_i(x) > \text{deg}_i(x^*)$. Hence, there exists a player $j \in N_i(x) \setminus N_i(x^*)$. If $\text{deg}_j(x^*) < \text{deg}_j(x)$, then either $\text{deg}_j(x) < d_j$ and hence, $\{i, j\}$ could be added to x , contradicting $x \in V^*$, or we have $\text{deg}_j(x) > d_j = \text{deg}_j(x^*)$, contradicting the fact that x is a Nash equilibrium.

If $\deg_j(x^*) > \deg_j(x)$, then there must exist $k \in N_j(x^*) \setminus N_j(x)$. Consider the profile x' which we get from x^* by removing the link $\{j, k\}$ and adding the link $\{i, j\}$. It is clear that $x' \in X^*$, since $\deg_i(x') \leq \deg_i(x) \leq d_i$, $\deg_j(x') = \deg_j(x) = d_j$, and $\deg_k(x') = \deg_k(x^*) - 1 < d_k$. Furthermore, $x' \in V^*$, since x^* has the same number of links as x' . However, we have $\phi(g(x), g(x')) = \phi(g(x), g(x^*)) - 2$, thus we contradict the choice of x^* as a closest element of V^* to x . Thus, x^* must Pareto dominate x . ■

There is no equivalent statement for games with aligned preferences in general due to the fact that a profile x which is player i 's first-best outcome is not necessarily an MBE thus it is possible for this player to be worse off in every MBE profile.

The condition of aligned preferences is a condition on the game's equilibrium structure and not the primitives. We conclude this section by deriving a set of sufficient conditions on the game's primitives (W, c) that implies that preferences are aligned. In fact, the sufficient condition is stronger, as it produces a unique MBE profile which is monotonic.

Lemma A.3. *Suppose that*

1. *for every $i \in I$, $|D_i| = d_i$, and*
2. *for every $i, j \in I$ we have $j \in D_i \Leftrightarrow i \in D_j$.*

Let x^ be given as follows: $x_{ij}^* = 1$ iff $i \in D_j$ and $j \in D_i$. Then, x^* is the unique MBE profile and it is monotonic.*

Proof: It is clear that x^* is monotonic and, due to the second assumption, it is undirected. Furthermore, by the first assumption, for every $i \in I$ we have $\deg_i(x^*) = d_i$. Thus, every player has the maximum number of neighbors and is linked to her best choice of d_i opponents. Thus, x^* is the only MBE. ■

Lemma A.3 lays out two sufficient conditions for preferences to be aligned: (1) each player can feasibly establish links to all of her 'favorite' opponents without violating her satiation constraint, and (2) reciprocity in being a 'favorite' opponent: all favorite links are mutually favorite. The first condition precludes, among other settings, the uniform benefits case; even though uniform benefits represent aligned preferences they do not generally imply uniqueness of the MBE profile. If indifferences are not allowed, as is often the case in matching where individuals have a strict ranking of opponents, this condition is automatically satisfied. The second condition ensures reciprocity of links if all players link to their set of favorite opponents and is the main restrictive condition in the lemma. These conditions are strong but, given the matrix of weights, easy to check.

A.2 Proofs for Section 2

Proposition 2.2

The 'only if' condition is straightforward: if any player has an unreciprocated link or the benefit of her marginal link is higher than the cost, the player can remove that link and be strictly better off. For 'if' direction, suppose that a strategy profile x satisfies both conditions but x_i is not a best response for player $i \in I$. Then, take $x'_i \in B_i(x)$. For all $j \neq i$ must have $x'_{ij} \leq x_{ij}$, otherwise i has an unreciprocated and thus unprofitable linking intention in x'_i and cannot be a best response. At the same time, we must have $x'_{ij} \leq x_{ij}$, otherwise i has severed a reciprocated and profitable link and x'_i cannot be a best response. Hence, x_i must have been a best response. ■

Proposition 2.4

The ‘only if’ direction is once again straightforward: if x^* is a Nash equilibrium and there exists an unformed link between i and j that would benefit both players, it cannot be pairwise Nash stable. To show the ‘if’ direction, suppose that x^* has the listed property but is not pairwise Nash stable. Since x^* has been chosen from the set of Nash equilibria, it is stable to individual deviations, so if it must be unstable against the pairwise formation of a single link. Then, a link $\{i, j\} \notin g(x^*)$ exists that would be profitable for both i and j . However, that necessarily means $w_{ij} > c'_i(\deg_i(x^*) + 1)$ and $w_{ij} > c'_j(\deg_j(x^*) + 1)$. ■

A.3 Proofs for Section 3

Lemma 3.1

The \Leftarrow direction is clear; if the link towards a player j appears profitable and there is no better alternative to link to, then player i 's best response is to link to j . The \Rightarrow direction consists of two simple parts: (1) linking to j cannot be a best response if it does not appear profitable, (2) linking to j but not linking to an apparently more profitable k cannot be a best response. ■

Proposition 3.3

We show this statement by the use of strict best-response paths. A strict best-response path is a sequence of profiles $(x^t)_{t \in \{0, \dots, T\}}$ such that for every $t \in \{1, \dots, T\}$ there exists a player i^t such that $x^t_{-i^t} = x^{t-1}_{-i^t}$, $x^{t-1}_{i^t} \notin B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t-1}))$, and $x^t_{i^t} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t-1}))$. In a strict best-response path every period has exactly one player move from a strategy that is not a best response to the current strategies of her opponents to a best response. We will show that no matter the starting profile x^0 , the order of updates, or the length of the sequence, there are no cycles in this process; there does not exist a non-trivial strict best-response path for which $x^0 = x^T$. Given this fact and due to the finiteness of the set of possible networks with n players, starting from any profile, any sequence of such strict improvements by players who are not at their best response will lead to an EUA.

For contradiction, suppose that such a cycle $(x^t)_{t \in \{0, \dots, T\}}$ exists with $x^0 = x^T$. We say that the link $\{i, j\}$ is *changing* in this sequence if there exist two time periods t and t' such that $x^t_{ij} \neq x^{t'}_{ij}$. Such links must exist in any strict best-response path.

Claim A.4. *In a strict best-response cycle there must exist a changing link $\{i, j\}$ and a time period t such that $x^t_{ij} = x^t_{ji} = 1$.*

To show this, suppose for contradiction that such a changing link does not exist. Take a player i and two distinct time periods $t < t'$ such that $i^t = i^{t'} = i$ and for every $t'' \in \{t+1, \dots, t'-1\}$ we have $i^{t''} \neq i$. For any cycle, such periods may be found by an appropriate time shift. Clearly, time periods t and t' cannot be consecutive in a strict best-response path and that there must exist a link $\{i, j\}$ such that $x^{t+1}_{ij} \neq x^{t'+1}_{ij}$ otherwise $x^t_{ij} = x^{t'+1}_{ij}$, contradicting that $x^t_{ij} \notin B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^t))$. Since we assumed that there are no changing reciprocal links, $\{i, j\} \in g(x^t) \cap g(x^{t+1}) \cap g(x^{t'}) \cap g(x^{t'+1})$. Furthermore, for every $k \in I$ with $x^t_{ik} = 0$ and $x^{t'}_{ki} = 1$ we have $x^{t+1}_{ik} = 0$, otherwise a changing reciprocal link would appear. Similarly, for every $k \in I$ with $x^{t'}_{ik} = 0$ and $x^{t'+1}_{ki} = 1$ we have $x^{t'+1}_{ik} = 0$.

It follows that $x^{t+1}_{ij} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t+1}))$ implies $x^{t'+1}_{ij} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t'+1}))$ as no player can form a link to i , nor delete a link to i that she would reciprocate forming or deleting in t' as there can

be no changing reciprocated links. Since $x_i^{t+1} = x_i^{t'}$ due to the fact that i hasn't changed her action between t and t' we have $x_i^{t'} \in B_i(\nu_i(\cdot|\alpha_i, \delta_i, x^{t'}))$, a contradiction by the fact that we assumed a strict best-response cycle. ■

Secondly, we show that a strict best-response cycle also cannot contain a reciprocated changing link. The two statements together imply that there are no strict best-response cycles.

Claim A.5. *In a strict best-response cycle there cannot be a changing link $\{i, j\}$ and a time period t such that $x_{ij}^t = x_{ji}^t = 1$.*

Once again, suppose for contradiction that such a link exists in a strict best-response cycle. Let i, j be a pair and t a time period such that $\{i, j\}$ is a reciprocated changing link with the largest value w_{ij} , $i^t = i$, $x_{ij}^t = x_{ji}^t = 1$, $x_{ij}^{t+1} = 0$, and $|\{k \in N_i(x^t) : w_{ik} = w_{ij}\}| \geq |\{k \in N_i(x^{t+1}) : w_{ik} = w_{ij}\}|$. In words, i, j is a best-value reciprocated changing link that was deleted by i in period t and she does not have more links of value w_{ij} in period $t+1$ than in period t . Such i, j and t are guaranteed to exist in a strict best-response cycle.

Let time period $t' < t$ be such that $i^{t'} = i$, $x_{ij}^{t'} = 0$, $x_{ij}^{t'+1} = 1$. As before, for any cycle, such periods may be found by an appropriate time shift.

Let $K_i = \{k : w_{ik}\alpha_i\delta_i > w_{ij}\alpha_i\delta_i + 1 - \delta_i\}$ denote the set of players whom i strictly prefers to j even if j reciprocates but they do not. If this set is empty, player i will never delete a link with j . It is clear that we must have $x_{ik}^{t'+1} = 1$ for every $k \in K_i$, otherwise we cannot have $x_{ij}^{t'+1} = 1$ as a best response to $x^{t'}$ by Lemma 3.1. This in turn implies that $x_{ik}^{t'} = 1$ for every $k \in K_i$, otherwise we would have a time period $t'' \in \{t' + 1, \dots, t - 1\}$ where i deletes her link to a $k \in K_i$ but keeps the link to j , which cannot happen in a best response.

So, at time t , when i deletes her link to j , every link that offers a better payoff was already formed, and since she cannot have more neighbors of equal value in $t+1$ by the choice of j , it is impossible that x_i^{t+1} is a best response to x^t provided that x_i^t was not a best response to x^t . This concludes the proof of Claim A.5 and thus of Proposition 3.3. ■

Corollary 3.5

To prove points 1 and 3, the statements on the lower bounds, we set $x_{ji} = 0$ for all $j \in I \setminus \{i\}$. We then have $w_{ij}\alpha'_i\delta_i \geq w_{ij}\alpha_i\delta_i$ and $w_{ij}\alpha_i\delta'_i \geq w_{ij}\alpha_i\delta_i$, meaning that for any r we have $\{j : w_{ij}\alpha'_i\delta_i > c'_i(r)\} \supseteq \{j : w_{ij}\alpha_i\delta_i > c'_i(r)\}$, implying point 1, as well as $\{j : w_{ij}\alpha_i\delta'_i > c'_i(r)\} \supseteq \{j : w_{ij}\alpha_i\delta_i > c'_i(r)\}$, implying point 3.

For points 2 and 4, the statements on the upper bounds, we set $x_{ji} = 1$ for all $j \in I \setminus \{i\}$. Then, $w_{ij}(\alpha'_i\delta_i + (1 - \delta_i)) \geq w_{ij}(\alpha_i\delta_i + (1 - \delta_i))$ and $w_{ij}(\alpha_i\delta'_i + (1 - \delta'_i)) \leq w_{ij}(\alpha_i\delta_i + (1 - \delta_i))$, meaning that for any r we have $\{j : w_{ij}(\alpha'_i\delta_i + (1 - \delta_i)) > c'_i(r)\} \supseteq \{j : w_{ij}(\alpha_i\delta_i + (1 - \delta_i)) > c'_i(r)\}$, implying point 2, and $\{j : w_{ij}(\alpha_i\delta'_i + (1 - \delta'_i)) > c'_i(r)\} \subseteq \{j : w_{ij}(\alpha_i\delta_i + (1 - \delta_i)) > c'_i(r)\}$, implying point 4. ■

Proposition 3.7

We decompose the network formation game with parameters (α', δ) into two parts: the mutual links that are formed in the equilibrium x of the game (α, δ) , $g(x)$ and the remaining links. In particular,

we will look at the network formation game with parameters (α', δ) such that

$$\begin{aligned}\tilde{w}_{ij} &= 0 \text{ for all } \{i, j\} \in g(x) \\ \tilde{w}_{ij} &= w_{ij} \text{ for all } \{i, j\} \notin g(x) \\ \tilde{c}_i(r) &= c_i(r + \deg_i(x)) \text{ for } r \in \mathbb{N}, i \in I\end{aligned}\tag{4}$$

Clearly, for every $i \in I$ the maximal number of links in this game is given by $\tilde{d}_i = \bar{d}_i - \deg_i(x)$. Furthermore, by Proposition 3.3, this game has an equilibrium – denote such an equilibrium by \tilde{x} . Note that since $\tilde{c}_i(r) > 0$ for all r , $\tilde{x}_{ij} = 0$ holds for all $\{i, j\} \in g(x)$.

For the original game with parameters (α', δ) , define the strategy combination x' by $x'_{ij} = 1$ for all $\{i, j\} \in g(x)$ and $x'_{ij} = \tilde{x}_{ij}$ for all $\{i, j\} \notin g(x)$. Note that for each i ,

$$\deg_i^+(x') = \deg_i^+(\tilde{x}) + \deg_i(x)\tag{5}$$

We now show that $x' \in X^*(\alpha', \delta)$ by showing that x'_i satisfies the two conditions in Lemma 3.1, i.e. it is a best response to x'_{-i} , starting with condition 1. Take an $\{i, j\}$ such that $x'_{ij} = 1$.

Case 1: If $\{i, j\} \in g(x)$, and there exists some k such that $x'_{ik} = 1$ and $\{i, k\} \notin g(x)$, then we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] \geq w_{ik} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ki}] > \tilde{c}'_i(\deg_i^+(\tilde{x})) = c'_i(\deg_i^+(x')),$$

where the first inequality follows from the fact that x is monotonic and thus $w_{ij} \geq w_{ik}$, the second inequality follows from the fact that $x'_{ki} = \tilde{x}_{ki}$ and \tilde{x} is an equilibrium of (4) and the equality is a consequence of the definition of \tilde{c} and (5).

Case 2: If $\{i, j\} \in g(x)$ and $N_i^+(x') = N_i(x)$ then we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i)] \geq w_{ij} [\alpha_i \delta_i + (1 - \delta_i)] > c'_i(\deg_i(x)) = c'_i(\deg_i^+(x')),$$

where the first inequality is due to $\alpha'_i \geq \alpha_i$, the second is due to the fact that $\{i, j\} \in g(x)$ and x was an EUA of the game with α , and the equality is by the definition of \tilde{c} and (5).

Case 3: If $\{i, j\} \notin g(x)$, then $x'_{ij} = \tilde{x}_{ij}$

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] > \tilde{c}'_i(\deg_i(\tilde{x})) = c'_i(\deg_i(x)),$$

where the inequality holds due to the fact that \tilde{x} is an equilibrium of (4) and the equality is a consequence of the definition of \tilde{c} and (5).

Thus, condition 1 of Lemma 3.1 is satisfied whenever $x'_{ij} = 1$.

Take next an $\{i, j\}$ such that $x'_{ij} = 0$. Clearly, $\{i, j\} \notin g(x)$ and thus $x'_{ij} = \tilde{x}_{ij}$. Since \tilde{x}_{ij} is an equilibrium of the game defined by (4), we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] < \tilde{c}'_i(\deg_i^+(\tilde{x})) = c'_i(\deg_i^+(x')).$$

Thus, condition 1 of Lemma 3.1 is satisfied whenever $x'_{ij} = 0$ as well.

Finally, we check condition 2. Suppose that there exist distinct players i, j and k such that $w_{ij}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}) > w_{ik}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ki})$, but $x'_{ik} = 1$ and $x'_{ij} = 0$. Clearly, $\{i, j\} \notin g(x)$. If $\{i, k\} \in g(x)$ then $x'_{ki} = 1$, and since $w_{ik} \geq w_{ij}$ by the fact that x is monotonic, this contradicts $w_{ij}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}) > w_{ik}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ki})$. Hence, $\{i, k\} \notin g(x)$. Since \tilde{x} is an EUA of the game (4), by condition 2 of Lemma 3.1 we have

$$\begin{aligned}w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) \tilde{x}_{ji}] &= w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] \leq \\ &w_{ik} (\alpha'_i \delta_i + (1 - \delta_i) x'_{ki}) = w_{ik} (\alpha'_i \delta_i + (1 - \delta_i) \tilde{x}_{ki}),\end{aligned}$$

a contradiction.

Therefore, all conditions of Lemma 3.1 hold. ■

A.4 Proofs for Section 4

Lemma 4.3

Proof: Take an $x^* \in V^*$. For contradiction, assume an $x \in X^*$ exists that Pareto dominates x^* . Suppose that for every $i \in I$ we have $\deg_i(x) \geq \deg_i(x^*)$. Then, for every $i \in I$ we have $c_i(x) \geq c_i(x^*)$, thus, for $u_i(x) \geq u_i(x^*)$ to hold we must have $\sum_{j \in N_i(x)} w_{ij} \geq \sum_{j \in N_i(x^*)} w_{ij}$, for each $i \in I$ and strict inequality for at least one player, contradicting $x^* \in V^*$. Thus, there exists a player i with $\deg_i(x) < \deg_i(x^*)$. However, since x^* is monotonic, $u_i(x) \geq u_i(x^*)$ requires $\deg_i(x) \geq \deg_i(x^*)$, a contradiction. ■

Proposition 4.4

We first prove a supporting statement.

Lemma A.6. *Suppose that the players exhibit aligned preferences. For a $T \in \mathbb{N}$ let $(x^t)_{t \in \{0, \dots, T\}}$ be a finite sequence of networks following the defined updating rule such that $x^0 \in V^*$ and $g(x^t) \in g(V^*)$ for $t \in \{0, \dots, T\}$. Then, for every $t \in \{0, \dots, T\}$ and every $i \in I$ we have $w'_i(x^t) > c'_i(d_i)$.*

Proof: Suppose that we have $w'_i(x^t) < c'_i(d_i)$. Let t denote the first such period, that is, $x^t_i \in B_i(x^t)$ or $x^t_i \in B_i^o(x^t)$. It is clear that the latter is impossible as optimistic players never initiate or accept such links, thus we must have $x^t_i \in B_i(x^t)$. Since we assumed $g(x^t) \in g(V^*)$ and since preferences are aligned, for every k such that $w'_i(x^t) < w_{ik}$ we have $x^t_{ik} = 1$. However, by the definition of d_i , this has to imply $\deg_i^+(x^t) > d_i$, which is also impossible as a best-responding player would not initiate more than d_i links. ■

Proof of Proposition 4.4: We first show that $g(x') \in g(X^*)$. For this, we show that every link formed in successors of x^* are profitable. To see this, take a sequence $(x^t)_{t \in \{0, \dots, T\}}$ obeying the defined updating rule with $x^0 \in V^*$. Since x^0 is a Nash equilibrium which has no unprofitable links, and since the updating rule has only optimistic and best-responding updates, for every x^t we have $w'_{it}(x^t) > c'_{it}(\deg_{it}(x^t))$. Hence, for every $i, j \in I$ with $x^t_{ij} = x^t_{ji} = 1$ we have $w_{ij} > c'_i(\deg_i^+(x^t)) \geq c'_i(\deg_i(x^t))$, so all links appearing in $g(x^t)$ are profitable.

Next, for contradiction, suppose that we have $x' \in S(x^*)$ and $g(x') \notin g(V^*)$. Then we must have $v(x') < v(x^*)$ as $v(x') = v(y(x')) > v(x^*)$ would contradict the choice of x^* as a maximum-benefit equilibrium, while $v(x') = v(x^*)$ would imply $g(x') \in g(V^*)$. Thus, the value of the network must decrease along the sequence between x^* and x' . Suppose that x' is the earliest element of the sequence with $v(x') < v(x^*)$, let x denote the element immediately preceding it and let i denote the player who updated between x and x' , that is, $x_{-i} = x'_{-i}$. Since $v(x) > v(x')$ there must exist $\{i, j\} \in g(x) \setminus g(x')$ and by Lemma A.6 it must satisfy $w_{ij} > c'_i(d_i)$.

Suppose that $x'_i \in B_i(x)$. Then, since i drops her link with j as a best response even though j reciprocates, we must have $w_{ij} < c'_i(\deg_i^+(x') + 1) = c'_i(\deg_i(x') + 1)$. Thus, $\deg_i(x') + 1 > d_i$, meaning that $\deg_i(x') = d_i$. Since i made a best response, $u_i(x) \leq u_i(x')$, and since the number of neighbors in x' is at maximum, and thus the weak increase in utility did not come through a decrease in costs, the total value of i 's links must have weakly increased from x to x' . Hence, we have

$$v(x) = \sum_{j \neq i} w_{ij} x_{ij} x_{ji} + \sum_{k \neq i} \sum_{k > j \neq i} w_{jk} x_{jk} x_{kj} \geq \sum_{j \neq i} w_{ij} x'_{ij} x_{ji} + \sum_{k \neq i} \sum_{k > j \neq i} w_{jk} x_{jk} x_{kj} = v(x'),$$

a contradiction to $v(x') < v(x)$.

As $x'_i \in B_i(x)$ leads to contradiction, we are left with the case where $x'_i \in B_i^o(x)$. Then we must have $\deg_i^+(x') = d_i$. Since $w_{ij} > c'_i(d_i)$ by Lemma A.6, and yet i chose not to reciprocate j 's link, there must exist $k \neq j$ with $w_{ij} = w_{ik}$ such that $x'_{jk} = x'_{ki} = 1$. Since x_i is monotonic, and $\{i, j\} \in x$ for every $\ell \in I$ with $w_{i\ell} > w_{ij}$ we have $\{i, \ell\} \in x$. Since $v(x) = v(x^*)$, we must have $g(x) \in g(V^*)$, and since preferences are aligned, for any ℓ with $w_{i\ell} > w_{ij}$ we must have $x_{i\ell} = x_{\ell i} = 1$. As x'_i is monotonic for i , $x'_{i\ell} = x'_{\ell i}$ must hold for the same set as well. Furthermore, for every $\ell \in I$ with $w_{ij} = w_{i\ell}$ and $x'_{i\ell} = 1$ we must have $x'_{\ell i} = x_{\ell i} = 1$ as well, otherwise deleting a link to an unreciprocated opponent of equal value is better for i than deleting her link to j . Thus, in x' , every linking intention of i is reciprocated: $\deg_i^+(x') = \deg_i(x') = d_i$, and she is linked to exactly d_i of her most favored opponents. It is therefore impossible that her update reduces total linking benefits: as before we have

$$v(x) = \sum_{j \neq i} w_{ij} x_{ij} x_{ji} + \sum_{k \neq i} \sum_{k > j \neq i} w_{jk} x_{jk} x_{kj} \geq \sum_{j \neq i} w_{ij} x'_{ij} x_{ji} + \sum_{k \neq i} \sum_{k > j \neq i} w_{jk} x_{jk} x_{kj} = v(x'),$$

a contradiction to $v(x') < v(x)$. ■

Proposition 4.5

Lemma A.7. *For every $x \in X$ there exists $x^* \in X^* \cap S(x)$ such that for all $i \in I$ we have $w'_i(x^*) > c'_i(d_i)$.*

Proof: First we show that there exists $x' \in X^* \cap S(x)$. Take an arbitrary $x \in X$. Then, suppose that in the next n periods, all players $\{1, \dots, n\}$ are given an ambiguity-neutral best-responding revision opportunity in increasing order. Call the resulting network x' . The probability of this happening is $((1 - \beta)/n)^n > 0$, so we must have $x' \in S(x)$. Then, for every $i \in I$ we have $w'_i(x') > c'_i(\deg_i^+(x'))$ and for every pair with $i > j$ we can have $x'_{ij} = 1$ only if $x_{ji} = 1$. Thus, repeating the same process in the reverse order will lead to an undirected network x'' with $w'_i(x'') > c'_i(\deg_i(x''))$ for all $i \in I$, satisfying the equilibrium conditions.

This shows that we can take $x \in X^*$ at the start. Take a player i such that $w'_i(x) < c'_i(d_i)$. If none can be found, we are done. If such a player exists, suppose that she receives an optimistic revision opportunity. Then since for every $x_i^o \in B_i^o(x)$ we have $w'_i(x_i^o, x_{-i}) > c'_i(d_i)$, she will sever all links with players giving her less benefits than $c'_i(d_i)$ and possibly initiate some links to others. Then, allow all her neighbors with whom she severed her links to best-respond in an ambiguity-neutral way, their only best response being is to sever their unreciprocated links towards i and make no other changes. Finally, let player i best-respond to sever all unreciprocated links she had initiated as an optimist. Call the resulting network x' . Since $p_\beta^m(x, x') > 0$ for some $m \leq N_i(x) + 1$ we must have $x' \in S(x)$.

At x' , we clearly have $w'_i(x') > c'_i(d_i)$ since all players $j \neq i$ with $w_{ij} < c'_i(d_i)$ now have no outgoing links towards i while $N_i(x) \supseteq N_i(x')$. Thus, we have $x'_i \in B_i(x')$, meaning that $x' \in X^*$. Furthermore, $w'_i(x') > c'_i(d_i)$, as well as $g(x') \subset g(x)$, since player i lost links and no player gained any. If there exists $j \neq i$ with $w'_j(x') < c'_j(d_j)$, we can repeat this process. If the process does not terminate sooner with a desired network, it will terminate by reaching the empty network which trivially satisfies the conditions of the statement of the lemma. ■

The next lemma shows that the learning process enters the set of maximum-benefit equilibria that are monotonic, V^* , for any starting state with a positive probability.

Lemma A.8. *Suppose that the players exhibit aligned preferences. For every $x \in X$ there exists an $x^* \in V^* \cap S(x)$.*

Proof: By Lemma A.7 we may take $x \in X^*$ such that $w'_i(x) > c'_i(d_i)$ for all $i \in I$. Let $x^* \in V^*$ be the closest element of V^* to x . If $x \geq x^*$, then we must have $x = x^*$ otherwise we contradict the choice of x^* as an element of V^* , in which case we are done. If $x \not\geq x^*$, we have $g(x^*) \setminus g(x) \neq \emptyset$.

We will show that there exists an $x'' \in X^* \cap S(x)$ which also satisfies $w'_i(x'') > c'_i(d_i)$ for every $i \in I$ and for which $\phi(g(x''), g(x^*)) < \phi(g(x), g(x^*))$. Through a repeated application of this argument the distance eventually decreases to 0, meaning that we reach x^* . We distinguish two cases.

Case 1. There exists a player i and a link $\{i, j\} \in g(x^*) \setminus g(x)$ such that $\deg_i(x) < d_i$. Let $j \in \operatorname{argmax}_{j' \in N_i(x^*) \setminus N_i(x)} w_{ij'}$ be the most valuable such link for player i . Then let x'_i be a strategy which is monotonic for i , $\deg_i^+(x'_i, x_{-i}) = d_i$, and $N_i^+(x'_i, x_{-i}) \supseteq (N_i(x) \cap N_i(x^*)) \cup \{j\}$. By Lemma 3.1, and due to x satisfying $w'_i(x) > c'_i(d_i)$, we must have $x'_i \in B_i^o(x)$. Let player i receive an optimistic revision opportunity and suppose that she revises to x'_i .

Case 1a. If $\deg_j(x) < d_j$, then we must have $w_{ij} \geq w'_j(x^*) > c'_j(d_j) \geq c'_j(\deg_j(x) + 1)$. Therefore, $B_j(x'_i, x_{-i})$ is a singleton with its only element, which we denote by x'_j , satisfying $x'_{jk} = 1$ if and only if $k \in N_j(x) \cup \{i\}$. Suppose that player j receives a best-responding revision opportunity, and thus, selects the strategy x'_j . For each $k \in N_i(x) \setminus N_i^+(x'_i, x_{-i})$ let k receive a best-responding revision opportunity, their only best response being is to keep all of their existing links except those with i who had severed her link to each k .

Finally, let player i also receive best-responding revision opportunity. Her only best response, which we denote by x''_i , is to delete all unreciprocated links she initiated when she moved to x'_i , that is, $x''_{ik} = 1$ if and only if $k \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$. Let the resulting network be denoted by x'' . It is clear that x'' is an equilibrium network, $w'_\ell(x'') \geq c'_\ell(d)$ for all $\ell \in I$, and $\phi(g(x''), g(x^*)) < \phi(g(x), g(x^*))$. Since the above process leads to x'' from x , we have $x'' \in S(x)$ as well.

Case 1b. If $\deg_j(x) = d_j \geq \deg_j(x^*)$, then there must exist a link $\{j, k\} \in g(x) \setminus g(x^*)$. Since x^* is monotonic we must also have $w_{ij} \geq w_{jk}$. Thus, there must exist an $x'_j \in B_j(x'_i, x_{-i})$ such that $x'_{j\ell} = 1$ if and only if $\ell \in (N_j(x) \setminus \{k\}) \cup \{i\}$. As before let j receive a best-responding revision opportunity and suppose that she selects x'_j . Let this be followed by player k best-responding; her only best response being to cut her now unreciprocated link towards j and keeping all her other neighbors.

Then, as before, for each $k' \in N_i(x) \setminus N_i^+(x'_i, x_{-i})$ let k' receive a best-responding revision opportunity, their only best response being to keep all of their existing links except those with i , who had severed her link to each k' . Finally, let i best-respond, her only best response being to delete all unreciprocated links she initiated when she moved to x'_i , that is, $x''_{ik} = 1$ if and only if $k \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$. Let the resulting network be denoted by x'' . As before, $x'' \in X^*$, satisfies $w'_\ell(x'') > c'_\ell(d)$ for all $\ell \in I$, and $\phi(g(x''), g(x^*)) < \phi(g(x), g(x^*))$, as well as $x'' \in S(x)$, completing Case 1.

Case 2. For every link $\{i, j\} \in g(x^*) \setminus g(x)$ we have $\deg_i(x) = d_i$ and $\deg_j(x) = d_j$. We will show that there must exist such a link $\{i, j\}$ for which there exists $\{i, k\} \in g(x) \setminus g(x^*)$ such that $w_{ij} > w_{ik}$.

Suppose this is not true, i.e. for every i, j, k with $\{i, j\} \in g(x^*) \setminus g(x)$ and $\{i, k\} \in g(x) \setminus g(x^*)$ we have $w_{ij} \leq w_{ik}$. Since $\deg_i(x) = d_i$ for all i with $N_i(x) \neq N_i(x^*)$ we must have $\deg_i(x) \geq \deg_i(x^*)$ for all i . These two facts imply that $v(x) \geq v(x^*)$ which can only hold with equality as x^* is a maximum-benefit equilibrium network. If $\deg_i(x) > \deg_i(x^*)$ for some $i \in I$ and $\deg_j(x) \geq \deg_j(x^*)$ for all other $j \neq i$, then $v(x) = v(x^*)$ can hold only if we have such a trio i, j, k with $\{i, j\} \in g(x^*) \setminus g(x)$, $\{i, k\} \in g(x) \setminus g(x^*)$, and we have $w_{ij} > w_{ik}$, a contradiction. So suppose that $\deg_i(x) = \deg_i(x^*)$ for all i .

Now we will show that under these conditions x is monotonic, a contradiction since that would mean $x \in V^*$. Take a player i and suppose that there exist $j, k \neq i$ such that $\{i, j\} \in g(x)$, $\{i, k\} \notin g(x)$ and $w_{ij} < w_{ik}$. If no such player can be found, x is monotonic. We must have $\{i, k\} \notin g(x^*)$ by assumption thus $\{i, k\} \in g(x) \setminus g(x^*)$. However, due to $\deg_i(x) = \deg_i(x^*)$ there must exist an $\ell \in N_i(x^*) \setminus \{j\}$ and by assumption we must have $w_{ik} > w_{ij} \geq w_{i\ell}$. Thus, since $\{i, k\} \notin g(x^*)$, this is a contradiction as x^* is monotonic.

Therefore if for every link $\{i, j\} \in g(x^*) \setminus g(x)$ we have $\deg_i(x) = d_i$ and $\deg_j(x) = d_j$, then there must exist $\{i, j\} \in g(x^*) \setminus g(x)$ and a $\{i, k\} \in g(x) \setminus g(x^*)$ such that $w_{ij} > w_{ik}$. In particular, let $j \in \operatorname{argmax}_{j' \in N_i(x^*) \setminus N_i(x)} w_{ij'}$ and let $k \in \operatorname{argmin}_{j' \in N_i(x) \setminus N_i(x^*)} w_{ij'}$.

Construct a strategy x'_i such that it is monotonic and for every $\ell \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$ we have $x'_{i\ell} = 1$. By the fact that $w_{ij} > w_{ik}$, we must have such a strategy with $\deg_i^+(x'_i, x_{-i}) = d_i$ as player i can include j to her set of neighbors and exclude k and still maintain an out-degree of d_i . By Lemma 3.1, we have $x'_i \in B_i^o(x)$. As in Case 1, let i receive an optimistic revision opportunity and suppose she picks the optimistic best response x'_i . From here by the same process as in Case 1b we can reach $x'' \in X^* \cap S(x)$ with $w'_\ell(x'') > c'_\ell(d)$ for all $\ell \in I$, and $\phi(g(x''), g(x^*)) < \phi(g(x), g(x^*))$. ■

Finally, given that we reach V^* , the proof of the Proposition itself follows simply from Proposition 4.4.

Proof of Proposition 4.5. By Lemma A.8 for every $x \in X$ we have a $x^* \in V^*$ with $x^* \in S(x)$. Since there are finitely many networks, with probability one, the process enters V^* and thus $g^{-1}(g(V^*))$. By Proposition 4.4, the game never leaves $g^{-1}(g(V^*))$. ■

Proposition 4.6

As shown in Propositions 4.4 and 4.5, for every $\beta > 0$, the set $g^{-1}(V^*)$, is an absorbing set of the Markov chain (X, p_β) , while all states outside of $g^{-1}(V^*)$ are transient. We thus restrict attention to the set $g^{-1}(V^*)$. Since the Markov process $(g^{-1}(V^*), p_\beta)$ need not be irreducible, we partition the set $g^{-1}(V^*)$ into $K \geq 1$ communicating classes: each class consists of a non-empty set of MBE, $V_k \subseteq V^*$ and the corresponding set $g^{-1}(V_k)$ and $(V_k)_{k=1}^K$ is a partition of V^* . This partition is independent of β as long as $\beta \in (0, 1)$.

Fix $k \in \{1, \dots, K\}$ and consider the set of Markov chains $(g^{-1}(V_k), p_\beta)_{\beta \in (0, 1)}$. Note that each such chain is aperiodic, irreducible and defined on a finite state space and thus has a unique invariant distribution $\hat{\mu}_{k, \beta}$. The set of stochastically stable states of such a chain is defined as

$$\operatorname{supp}(\hat{\mu}_k) = \operatorname{supp} \left(\lim_{\beta \rightarrow 0} \hat{\mu}_{k, \beta} \right) \quad (6)$$

provided that the limit exists.

To show existence and characterize the limit, we follow Vega-Redondo (2003, p. 480-481). For $x, x' \in X$, denote by $p_0^\infty(x, x')$ the transition probability from x to x' after an infinite number of applications of the best-response dynamics without optimistic surges, $p_0^\infty = \lim_{m \rightarrow \infty} p_0^m$. For a given $\beta \in (0, 1)$ and $k \in \{1, \dots, K\}$, we define a Markov chain on $(V_k, Q_{k, \beta})$ as follows: for $x, x' \in V_k$, the transition probability $Q_{k, \beta}(x, x')$ is the concatenation of the one-step-ahead best-response dynamics with optimistic surges, p_β , with p_0^∞ :

$$Q_{k, \beta}(x, x') = \sum_{x'' \in g^{-1}(V_k)} p_\beta(x, x'') p_0^\infty(x'', x')$$

As we know from the proof of Proposition 3.3 and Proposition 4.5, the best-response dynamics without optimistic surges with a starting point in $g^{-1}(V_k)$ reaches V_k in finite time. Thus, the process is well-defined and satisfies for every $\beta \in (0, 1)$:

1. For every $x \in V_k$, $Q_{k, \beta}(x, x) > 0$, and thus, $(V_k, Q_{k, \beta})$ is aperiodic, because the largest common denominator of the periods in which an element x is reached is 1.

2. For any $x, x' \in V_k$, $x \neq x'$, either $Q_{k,\beta}(x, x') = 0$ or $Q_{k,\beta}(x, x') = \beta\xi(x, x')$, where $\xi(x, x') \in (0, 1)$ is the probability that the best-response dynamics (without optimistic surges) reaches x' from x , conditional on an optimistic shock occurring, and is thus independent of β .
3. $Q_{k,\beta}(x, x') = 0$ if and only if $Q_{k,\beta'}(x, x') = 0$ for all $\beta' \in (0, 1)$.

By the definition of V_k , $(V_k, Q_{k,\beta})$ is irreducible for all $\beta \in (0, 1)$ and since it is defined on a finite state space and aperiodic, it has a unique probability distribution $\mu_{k,\beta}$. If $\mu_{k,\beta}$ converges to μ_k as $\beta \rightarrow 0$, then the limit on the right hand side of (6) exists and the support of μ_k is exactly the set of stochastically stable states of $(g^{-1}(V_k), p_\beta)$, see Young (1993), KMR and Vega-Redondo (2003, p. 481):

$$\text{supp}(\mu_k) = \text{supp}\left(\lim_{\beta \rightarrow 0} \mu_{k,\beta}\right) = \text{supp}(\hat{\mu}_k). \quad (7)$$

Lemma A.9. For all $k \in \{1, \dots, K\}$, $\text{supp}(\lim_{\beta \rightarrow 0} \mu_{k,\beta}) = V_k$.

Proof: Fix a $k \in \{1, \dots, K\}$. Clearly, if $|V_k| = 1$, then $\mu_{k,\beta}(x) = 1$ for the unique $x \in V_k$ and for all $\beta \in (0, 1)$ and the statement of the lemma is trivially satisfied.

Suppose thus that $|V_k| \geq 2$. The resistance function defined in Young (1993, p. 68),⁸ which we here denote by $e_k(x, x')$, satisfies:

$$0 < \frac{Q_{k,\beta}(x, x')}{\beta e_k(x, x')} < \infty.$$

By property 2, for $x \neq x'$ such that $Q_{k,\beta}(x, x') > 0$, we have

$$0 < \frac{\beta\xi(x, x')}{\beta e_k(x, x')} < \infty, \text{ or}$$

$$e_k(x, x') = 1 \text{ iff } Q_{k,\beta}(x, x') > 0,$$

and, by convention,

$$e_k(x, x') = \infty \text{ iff } Q_{k,\beta}(x, x') = 0.$$

Following Young (1993, p. 69), consider the set of undirected graphs on V_k such that every $x' \in V_k$ is the initial point of exactly one edge (x', x'') and for every x' there is a path leading to x . Denote the set of all such graphs by $\mathcal{Y}_{k,x}$. The resistance of such a graph $Y \in \mathcal{Y}_{k,x}$ is given by

$$E_{k,x}(Y) = \sum_{(x', x'') \in Y} e_k(x', x'').$$

Denote by $E_{k,x}(\mathcal{Y}_{k,x})$ the minimal resistance over graphs in $\mathcal{Y}_{k,x}$:

$$E_{k,x}(\mathcal{Y}_{k,x}) = \min_{Y \in \mathcal{Y}_{k,x}} \sum_{(x', x'') \in Y} e_k(x', x'').$$

We will call a graph $Y \in \mathcal{Y}_{k,x}$ feasible if $e_k(x', x'') < \infty$, for all $(x', x'') \in Y$. Since $(V_k, Q_{k,\beta})$ is irreducible, there exists a feasible graph for each x . If $|V_k| = 2$, this graph is unique, it is given by (x', x) and has a resistance of

$$E_{k,x}(\mathcal{Y}_{k,x}) = e_k(x', x) = 1 = |V_k| - 1.$$

⁸Intuitively, the resistance function in Young (1993) corresponds to the number of shocks necessary to induce a transition from x to x' .

If $|V_k| > 2$, the feasible graph need not be unique. Notably, the graph

$$Y = \left((x', x^1), (x^1, x^2), \dots, (x^l, x) \right),$$

where $l = |V_k| - 2$, $\{x', x^1, \dots, x^l\} = V_k \setminus \{x\}$ and $Q_{k,\beta}(x', x^1) > 0$, $Q_{k,\beta}(x^\lambda, x^{\lambda+1}) > 0$ for $\lambda \in \{1, \dots, l-1\}$ and $Q_{k,\beta}(x^l, x) > 0$ is feasible. The resistance of such a graph is given by the number of its edges:

$$E_{k,x}(Y) = |V_k| - 1.$$

Since each feasible graph in $\mathcal{Y}_{k,x}$ has the same number of edges, it follows that for all $x \in V_k$,

$$\min_{G \in \mathcal{Y}_{k,x}} E_{k,x}(G) = E_{k,x}(\mathcal{Y}_{k,x}) = |V_k| - 1 = \min_{x \in V_k} \min_{G \in \mathcal{Y}_{k,x}} \sum_{(x', x'') \in G} e(x', x'') = E_{k,x}(\mathcal{Y}_x).$$

Theorem 2 of Young (1993) implies that $\mu_k = \lim_{\beta \rightarrow 0} \mu_{k,\beta}$ exists and satisfies $\text{supp}(\mu_k) = V_k$, concluding the proof of the lemma. ■

Let $\kappa = (\kappa_1, \dots, \kappa_K)$ be such that $\sum_{k=1}^K \kappa_k = 1$ and $\kappa_k > 0$ for all $k \in \{1, \dots, K\}$ and consider the probability distribution

$$\hat{\mu}_\beta(\kappa) = \sum_{k=1}^K \kappa_k \hat{\mu}_{k,\beta}.$$

For each $\beta \in (0, 1)$, $\hat{\mu}_\beta(\kappa)$ is an invariant (though not unique) probability distribution of the Markov chain $(g^{-1}(V), p_\beta)$, and since all states outside of $g^{-1}(V)$ are transient, also of (X, p_β) . Furthermore, for a given κ with $\kappa_k > 0$ for all $k \in \{1, \dots, K\}$, (7), together with Lemma A.9, implies:

$$\text{supp} \left(\lim_{\beta \rightarrow 0} \hat{\mu}_\beta(\kappa) \right) = \text{supp} \left(\lim_{\beta \rightarrow 0} \sum_{k=1}^K \kappa_k \hat{\mu}_{k,\beta} \right) = \text{supp} \left(\sum_{k=1}^K \kappa_k \mu_k \right) = \cup_{k=1}^K V_k = V^*,$$

concluding the proof of the proposition. ■

Proposition 4.9

As before, we first show that the dynamic system will take the game into V^* , which is a unique profile x^* in the linear case.

Lemma A.10. *Let the cost functions be linear. For every $x \in X$ and the unique $x^* \in V^*$ we have $x^* \in S(x)$.*

Proof: By Lemma A.7 we can take $x \in X^*$. We show that if $x = x^*$ there exists $x' \in S(x) \cap X^*$ such that $\phi(g(x'), g(x^*)) < \phi(g(x), g(x^*))$. Take a link $\{i, j\} \in g(x^*) \setminus g(x)$. Then, we must have $w_{ij} > \max\{c_i, c_j\}$. Take the following series of updates: i receives a revision opportunity as an optimist, j receives a revision opportunity as a best-responder, then i receives a revision opportunity as a best-responder. It is clear that an optimistic player i will link to j , as well as keep all of her links to her neighbors in x . A best-responder j will make no changes except reciprocate player i 's link. Finally, player i deletes all the unreciprocated links she made as an optimist. For the resulting network, $g(x') = g(x) \cup \{i, j\}$, we have $x' \in X^*$ as well as $\phi(g(x'), g(x^*)) = \phi(g(x), g(x^*)) - 1$.

■

Secondly, we show that once x^* is reached, the game never leaves the set $g^{-1}(g(x^*))$.

Lemma A.11. *Let the cost functions be linear. For the unique $x^* \in V^*$ we have $S(x^*) \subseteq g^{-1}(g(x^*))$.*

Proof: Suppose for contradiction that there exist a pair $x, x' \in S(x^*)$ such that $x \in g^{-1}(g(x^*))$, $x' \in S(x)$, and $x' \notin g^{-1}(g(x^*))$. Then, there must exist $\{i, j\} \in g(x') \setminus g(x)$, with $w_{ij} > \max\{c_i, c_j\}$. Clearly, one of i and j must have updated to get from x to x' so suppose it was i , that is, $x_{-i} = x'_{-i}$. If $x'_i \in B_i(x)$, then, since $x'_{ji} = x_{ji} = 1$, deleting the link to j cannot be a best response. If $x'_i \in B_i^o(x)$, then, since $w_{ij} > c_i$, deleting the link to j cannot be an optimistic best response, so we have arrived at a contradiction.

This completes the proof of the proposition.

■

A.4.1 Proposition 4.10

As before, we can take $x^0 \in X^*$ and in particular, $w_{ij} > c'_i(\deg_i(x^0))$ for all i and all j with $x_{ij}^0 = 1$. We prove the proposition by a sequence of Lemmata, which, taken together, imply the result.

Lemma A.12. *If for some $m' \geq 1$, $\{i, j\} \in g(S^{m'}(x^0))$ for some $\{i, j\} \in g^f$, then $\{i, j\} \in g(S^{m''}(x^0))$ for all $m'' \geq m'$.*

Proof: It suffices to show that if $x_{ij}^0 = x_{ji}^0 = 1$ for some $\{i, j\} \in g^f$, then $\{i, j\} \in g(x)$ for all $x \in S^{m''}(x^0)$. To see this, suppose that $x_{ji} = 1$ and consider two cases. First, if $\deg_i^+(x) < d_i$, we have that since $j \in D_i$, $w_{ij} > c'_i(\deg_i(x))$ and thus, $x_{ij} = 1$ obtains as a best-response whether or not i experiences an optimistic shock. Second, if $\deg_i^+(x) > d_i$, then there is a $k \neq j$ such that $x_{ik} = 1$ and

$$w_{ik} = \min_{\tilde{k} \neq i} \{w_{i\tilde{k}} \mid x_{i\tilde{k}} = 1\} \leq \min_{k' \in D_i} \{w_{ik'}\} \leq w_{ij}.$$

Since, by assumption, either $|D_i| = d_i$, and thus, $\tilde{k} \notin D_i$, or $w_{ij} > \min_{k' \in D_i} \{w_{ik'}\}$, we have that at least one of the two weak inequalities is strict, i.e., $w_{ik} < w_{ij}$. Thus, the best-response of i (whether optimistic or realistic) includes setting $x'_{ij} = 1$ and dropping the lower-value link with k , $x'_{ik} = 0$. We conclude that no matter which case obtains, sustaining the link with j , $x_{ij} = 1$, is always part of the best-response of i . Since a symmetric argument applies to j , we conclude that the link $\{i, j\}$ once established will never be severed.

■

Lemma A.13. *There exists almost surely a finite period m such that $g(S^m(x^0)) \in G^{sb}$ and $g^f \subseteq g(S^{m'}(x^0))$ for all $m \geq m'$. Furthermore, if $|D_i| = d_i$ for all i , there exists almost surely a finite period m such that $g(S^m(x^0)) = g^f$.*

Proof: Consider the following sequence of optimistic shocks and best-responses. Let player 1 receive an optimistic shock and extend links to d_1 of the players $j \in D_1$, including those for which $\{1, j\} \in g^f$, while severing all other of her existing links in x^0 . This is an optimistic best-response because either $|D_1| = d_1$, or $w_{1j} > \min_{k \in D_1} w_{1k} > c'_1(d_1)$ for all j s.t. $\{1, j\} \in g^f$. Next, let each of the agents j s.t. $\{1, j\} \in g^f$ (and thus, $1 \in D_j^*$) realistically best-respond to 1 by either reciprocating the link to 1, $x_{j1} = 1$, provided that $w_{1j} > c'_j(\deg_j(x^0) + 1)$, or, if $w_{1j} < c'_j(\deg_j(x^0) + 1)$, simultaneously

reciprocating 1 and deleting the worst one of their existing links in x^0 , $x_{j1} = 1$ and $x_{jk} = 0$ for $k = \min_{\tilde{k}} \{w_{j\tilde{k}} \mid x_{j\tilde{k}}^0 = 1\}$. To see that this is optimal, note that $\{j, k\} \notin g^f$ and consider two cases. If $|D_j| = d_j$, then for every $\{j, k\} \in g^f$, $k \in D_j$ and $w_{jk} > c'_j(d_j) \geq c'_j(\deg_j(x^0))$ whenever x^0 satisfies the assumption above. Alternatively, if $|D_j| > d_j$, $\{j, k\} \in g^f$ implies that

$$w_{jk} > \min_{k' \in D_j} \{w_{jk'}\} > c'_j(d_j) \geq c'_j(\deg_j(x^0))$$

We thus have that whenever a link with k is deleted, $w_{1j} > w_{kj} > c'_j(\deg_j(x^0))$. The rest of their strategies remains unchanged. Let all players k such that $x_{jk}^0 = 1$ and $x_{jk} = 0$ for some j s.t. $\{1, j\} \in g^f$ best-respond realistically by deleting their link to j while leaving all their other links unchanged. Finally, let player 1 best-respond realistically by deleting all links extended to j such that $j \in D_1$, $\{1, j\} \notin g^f$, except for at most $d_1 - d_1^*$ links to players $j \in \arg \min_{j \in D_1} w_{1j}$ such that $x_{1j}^0 = x_{j1}^0 = 1$. At the end of this first iteration, all links $\{1, j\} \in g^f$ such that $j \in D_1$ have been established and the resulting network satisfies $x^1 \in X^*$ s.t. $x_{1j}^1 = x_{j1}^1 = 1$ for all $j \in D_1^*$ s.t. $\{1, j\} \in g^f$. Furthermore, $w_{ij} > c'_i(\deg_i(x^1))$ for all i and all j with $x_{ij}^1 = 1$.

To define the iteration, let i be the last player to receive an optimistic shock and x^i be the corresponding equilibrium network satisfying $x_{kj}^i = x_{jk}^i = 1$ for all $k \leq i$ and all $j \in D_i^*$ s.t. $\{i, j\} \in g^f$, and $w_{kj} > c'_k(\deg_k(x^i))$ for all k and all j with $x_{kj} = 1$. Consider player $i + 1$ and suppose that she experiences an optimistic shock and thus extends links to exactly d_{i+1} of the players $j \in D_{i+1}$, including all those for which $\{i + 1, j\} \in g^f$ while severing all other of her existing links in x^i . This is an optimistic best-response because either $|D_{i+1}| = d_{i+1}$, or $w_{(i+1)j} > \min_{k \in D_{i+1}} w_{(i+1)k} > c'_{i+1}(d_{i+1})$ for all j s.t. $\{i + 1, j\} \in g^f$. Note that none of the existing links to $k \leq i$ s.t. $\{i + 1, k\} \in g^f$ are severed. Next, let each of the agents $j > i + 1$ s.t. $i + 1 \in D_j^*$, $\{i + 1, j\} \in g^f$ realistically best-respond to $i + 1$ by either reciprocating the link to $i + 1$, $x_{j(i+1)} = 1$, provided that $w_{(i+1)j} > c'_j(\deg_j(x^i) + 1)$, or, if $w_{(i+1)j} < c'_j(\deg_j(x^i) + 1)$, simultaneously reciprocating $(i + 1)$ and deleting the worst one of their existing links in x^i , $x_{j(i+1)} = 1$ and $x_{jk} = 0$ for $k = \min_{\tilde{k}} \{w_{j\tilde{k}} \mid x_{j\tilde{k}}^0 = 1\}$. As above, we have that if a link with k is deleted, then $\{j, k\} \notin g^f$. The rest of their strategies remains unchanged. Let all players k such that $x_{jk}^i = 1$ and $x_{jk} = 0$ for some j s.t. $i + 1 \in D_j^*$ best-respond realistically by deleting their link to j while leaving all their other links unchanged. Finally, let player $i + 1$ best-respond realistically by deleting all links extended to j such that $\{i + 1, j\} \notin g^f$, except for at most $d_i - d_i^*$ links to players $j \in \arg \min_{j \in D_i} w_{ij}$ such that $x_{ij}^0 = x_{ji}^0 = 1$, and either $i \in D_j$ or $j > i + 1$. At the end of this $(i + 1)$ st iteration, all links $\{k, j\}$ such that $k \in D_j$, $j \in D_k$, and $\{k, j\} \in g^f$ for $k \leq (i + 1)$ have been established and the resulting network satisfies $x^{i+1} \in X^*$ s.t. $x_{kj}^{i+1} = x_{jk}^{i+1} = 1$ for all $k \leq i + 1$, $j \in D_k^*$, $\{k, j\} \in g^f$. Furthermore, $w_{jk} > c'_j(\deg_j(x^{i+1}))$ for all j and all k with $x_{jk}^{i+1} = 1$.

Clearly, after the $(n - 1)$ -th iteration, which requires $(n - 1)$ optimistic shocks, all links in g^f are thus established. If $|D_i| = d_i = d_i^*$ for all i , only the links in g^f have been established. In contrast, if $|D_i| > d_i$, those links in x^0 such that $x_{ij}^0 = x_{ji}^0 = 1$ with $i \in D_j \cap D_j^{sb}$, $j \in D_i \cap D_i^{sb}$, and thus, $\{i, j\} \notin g^f$ will be preserved. The resulting network is then a superset of g^f . In both cases, the resulting network is in G^{sb} . Each iteration i requires at most $|j \in D_i^* \mid j > i| + n - 1$ updates. Since the so-defined sequence of updating steps is finite, it occurs with strictly positive probability. Therefore, almost surely, in a finite number of periods m , $g(S^m(x^0)) \in G^{sb}$ with $g(S^m(x^0)) = g^f$ if $|D_i| = d_i$ for all i . Furthermore, by Lemma A.12, $g^f \subseteq g(S^{m'}(x^0))$ for all $m' \geq m$. ■

Lemma A.14. *If $g(S^m(x^0)) \in G^{sb}$, then $g(S^{m'}(x^0)) \in G^{sb}$ for all $m \geq m'$.*

Proof: We have to show that the only reciprocal links $\{i, j\}$ that can form starting from $g \in G^{sb}$ are of the type $j \in D_i, i \in D_j^{sb}$. Indeed, let $x \in g^{-1}(G^{sb})$, let i be an arbitrary player and consider two cases. First, if i has the opportunity for a best-response, her best-response strategy will coincide with x_i . If, in contrast, i receives an optimistic best-response opportunity, her best-response will be given by $x'_{ij} = 1$ only if $j \in D_i$. In both cases, all links that i may extend are in the set $D_i \cup D_i^{sb}$. Consider now a player j and a strategy combination x' such that for each $i, x'_{ij} = 1$ only if $j \in D_i \cup D_i^{sb}$. The best-response of j if optimistic will be $x''_{jk} = 1$ only if $k \in D_j$, and if realistic, $x''_{jk} = 1$ only if $x'_{kj} = 1$ and $k \in D_j \cup D_j^{sb}$. Thus, no best-response starting from G^{sb} will induce a best-response for any player i which extends a link to a player $j \notin D_i \cup D_i^{sb}$, establishing that $S^{m'}(x^0) \in g^{-1}(G^{sb})$ for all $m \geq m'$. ■

Lemma A.15. *If $|D_i| = d_i$ for all i , then each $g^{sb} \in G^{sb}$ is recurrent.*

Proof: By Lemma A.13, g^f is reached almost surely in a finite number of steps from any $x^0 \in G^{sb}$, and in particular from g^f itself. It is thus, recurrent. So let the initial network be $g^f \in G^{sb}$ with $x^f = g^{-1}(g^f)$ and let $g^{sb} \in G^{sb} \setminus g^f$. Note in particular that $g^f \subset g^{sb}$ and that for $x^{sb} = g^{-1}(g^{sb})$, we have $w_{ij} > c'_i(\deg_i(x^{sb}))$ for all $\{i, j\} \in g^{sb}$.

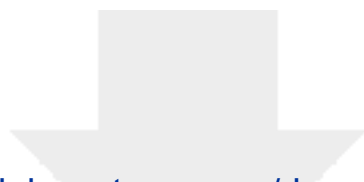
For two networks g and g' such that $g \subset g'$, let $J_i(g, g') = \{j \in D_i \mid \{i, j\} \in g' \setminus g\}$ be the set of links of i to her favorite partners, which are in g' , but not in g . Note that for each $j \in J_i, i \notin D_j$. Consider the following sequence of updating: starting at g^f , let player 1 receive an optimistic updating opportunity, extending links to all $j \in D_1$, including all $J_1(g^f, g^{sb}), x_{1j} = 1$ for all $j \in D_1$. Next, let each of the players $j \in J_1(g^f, g^{sb})$ receive a best-response opportunity. Since $\{i, j\} \in g^{sb} \setminus g^f$ and since $\deg_j(x^{sb}) \geq \deg_j(x^f) + 1 > \deg_j(x^f)$, we have that $w_{ij} > c'_j(\deg_j(x^f) + 1)$ and thus, it is optimal for each of the players $j \in J_1(g^f, g^{sb})$ to reciprocate the link to 1, setting $x_{j1}^1 = 1$, without severing any of her existing links. Next, let player 1 receive a realistic best-response opportunity, thereby deleting all extended links to $k \in D_1 \setminus (D_1^* \cup J_1(g^f, g^{sb}))$ and setting $x_{1j}^1 = 1$ iff $j \in D_1^* \cup J_1(g^f, g^{sb})$. The new network is $g^1 = g^f \cup_{j \in J_1(g^f, g^{sb})} \{1, j\}$. If $g^1 = g^{sb}$, we are done. If $g^1 \neq g^{sb}$, then it must be that $g^1 \subset g^{sb}$, in which case we continue by iteration as follows.

Proceeding iteratively, suppose that we have obtained the network $g^i = g^f \cup_{k \leq i} \left(\cup_{j \in J_k(g^k, g^{sb})} \{k, j\} \right)$ for some $i \geq 1$. Consider player $i+1$ and suppose that she receives an optimistic updating opportunity, thereby establishing all links with $j \in D_{i+1}$, including all players $j \in J_{i+1}(g^i, g^{sb}), x_{(i+1),j} = 1$ for all $j \in D_{i+1}$. Next, let each of the players $j \in J_{i+1}(g^i, g^{sb})$ receive a best-response opportunity. Since $\{i+1, j\} \in g^{sb} \setminus g^i$ and since $\deg_j(x^{sb}) \geq \deg_j(g^{-1}(g^i)) + 1 > \deg_j(g^{-1}(g^i))$, we have that $w_{(i+1),j} > c'_j(\deg_j(x^f) + 1)$ and thus, it is optimal for each of the players $j \in J_{i+1}(g^i, g^{sb})$ to reciprocate the link to $(i+1)$, setting $x_{j(i+1)}^{i+1} = 1$ without severing any of her existing links. Next, let player $(i+1)$ receive a realistic best-response opportunity. She will then keep all links to $k \in D_i$ such that either $i \in D_k$, or $k \in J_{i+1}(g^i, g^{sb})$, reciprocate all previously extended links by $k \leq i$ such that $\{i+1, k\} \in g^i \setminus g^f$ and delete all other extended but unreciprocated links: $x_{(i+1),j}^{i+1} = 1$ iff either $\{(i+1), j\} \in g^i$ or $j \in J_{i+1}(g^i, g^{sb})$. The new network is $g^{i+1} = g^i \cup_{j \in J_{i+1}(g^i, g^{sb})} \{i+1, j\}$. If $g^{i+1} = g^{sb}$, we are done. If $g^{i+1} \neq g^{sb}$, then it must be that $g^{i+1} \subset g^{sb}$ and we proceed iteratively.

The iteration finishes in no more than n steps. Since each step has a strictly positive probability, almost surely in finite time, the best-response dynamic with optimistic shocks reaches every element of G^{sb} . It follows that when $|D_i| = d_i$ for all i , each $g^{sb} \in G^{sb}$ is recurrent. ■

Lemma A.16. *If for each $j \in D_i^{sb}, i \in D_j$, we have $w_{ij} > \min_{k \in D_j} w_{jk}$, then there exists, almost surely a finite \bar{m} for which $g(S^{\bar{m}}(x^0)) \in G^{sb}$ is maximal (w.r.t. inclusion) in G^{sb} .*

By Lemma A.13, $g \supseteq g^f$ is reached almost surely in finite time. If g is maximal in G^{sb} , then we are done. If not, let $g^{\max} \in G^{sb}$ with $x^{\max} = g^{-1}(g^{\max})$ be a maximal network in G^{sb} which includes g , $g^{\max} \supset g$ with $x = g^{-1}(g)$. Let $\{i, j\} \in g^{\max} \setminus g$. Since $g \supseteq g^f$, we have that either $j \in D_i^{sb}$ or $i \in D_j^{sb}$. Let, w.l.o.g., $i \in D_j^{sb}$, and, therefore, by assumption, $j \in D_i$ such that $w_{ij} > \min_{k \in D_i} w_{ik}$. Consider the following sequence of updates. Let i receive an optimistic updating opportunity and thus extend a link to d_i of her favorite partners, including j . Note that since $w_{ij} > \min_{k \in D_j} w_{jk}$, the link to j will be part of any optimistic best-response, regardless of the already existing connections of player i in g . Next, let j realistically best-respond by reciprocating i 's link and leaving all of her other links unchanged. This is a best-response, because $g^{\max} \supset g \in G^{sb}$ implies that $w_{ij} > c_j(\deg_j(x^{\max})) \geq c_j(\deg_j(x) + 1) > c_j(\deg_j(x))$, and thus, j finds it optimal to establish the connection with i . Finally, let i receive a realistic best-response opportunity, severing all unreciprocated links, keeping the newly established link to j and reestablishing all of her original links in g . We obtain $g' = g \cup \{i, j\}$. Proceeding by iteration, g^{\max} is reached within a finite number of updates, \bar{m} . Since each such sequence has a strictly positive probability, the result of the Lemma obtains. ■



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