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# Design of an acoustic energy distributor using thin resonant slits

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**Abstract.** We consider the propagation of time harmonic acoustic waves in a device with three channels. The wave number is chosen such that only the piston mode can propagate. The main goal of this work is to present a geometry which can serve as an energy distributor. More precisely, the geometry is first designed so that for an incident wave coming from one channel, the energy is almost completely transmitted in the two other channels. Additionally, tuning a bit two geometrical parameters, we can control the ratio of energy transmitted in the two channels. The approach is based on asymptotic analysis for thin slits around resonance lengths. We also provide numerical results to illustrate the theory.

**Key words.** Acoustic waveguide, energy distributor, asymptotic analysis, thin slit, scattering coefficients, complex resonance.

## 1 Introduction

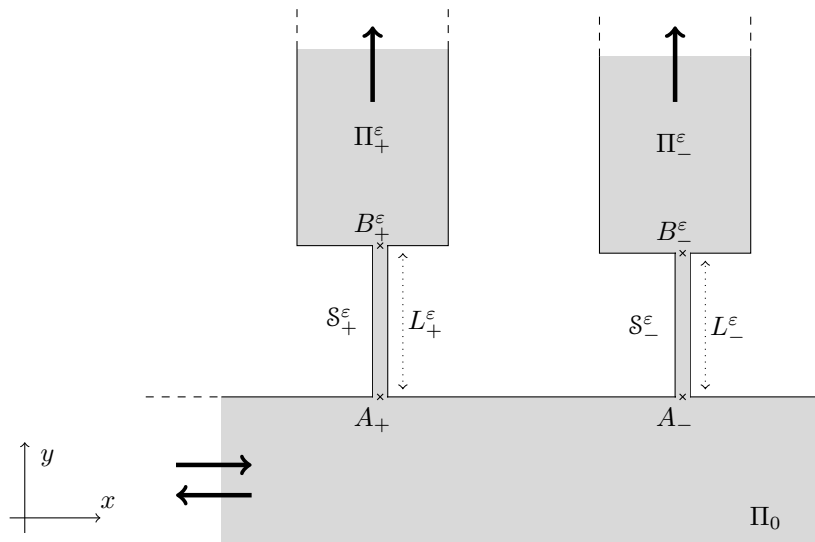


Figure 1: Geometry of the waveguide  $\Omega^\epsilon$ .

In this article, we are interested in the design of an acoustic energy distributor. More precisely, we study the propagation of time harmonic waves at a given wave number in a structure with one input/output and two outputs (see Figure 1). Our goal is to find a geometry where the energy of an incoming wave is almost completely transmitted and additionally where we can control the ratio

of energy transmitted in the two other output channels. The main difficulty of this problem lies in the fact that the dependence of the acoustic field with respect to the geometry is non linear and implicit. A device closed to the one that we wish to create is the acoustic power divider [18, 1, 28, 8]. The difference is that in our case, we want to be able to control the ratio of energy transmitted in the two output channels. We mention that such devices are very interesting for applications not only in acoustics but also for example in optics [29, 9] or in radio electronics [7] (see the literature concerning radio diplexers).

To exhibit such particular waveguides, contrary to for example [8], we will not use zero-index material or similar metamaterials which are, from our understanding, still hard to handle in practice. Instead we will work with thin slits as illustrated in Figure 1. In general, due to the geometrical features, almost no energy passes through the slits and it may seem a bit paradoxical to use them to have almost complete transmission. However working around the resonance lengths, it has been shown that we can observe this phenomenon. This has been studied for example in [17, 5, 23, 24, 22] in the context of the scattering of an incident wave by a periodic array of subwavelength slits. The approach that we will consider is based on matched asymptotic expansions. For related techniques, we refer the reader to [3, 10, 16, 26, 11, 27, 14, 2, 4]. We emphasize that an important feature of our work distinguishing it from the previous references is that the lengths, and not only the widths, of the slits depend on  $\varepsilon$  (see (1)). This way of considering the problem is an essential ingredient of the analysis.

Let us mention that techniques of optimization (see e.g. [20, 21, 19]) have been applied to exhibit energy acoustic distributors. However they involve non convex functionals and unsatisfying local minima exist. Moreover, they offer no control on the obtained shape compare to the approach we propose here. In particular, with our geometry, a small change of the geometry allows us to transmit the energy in one channel instead of the other. In the context of propagation of acoustic waves, another device which is interesting in practice is the modal converter [12, 6, 15]. At higher frequency, when several modes can propagate, the aim is to have a structure where the energy of an incident mode is transferred onto another mode. We do not know if thin slits can be useful to obtain such effect.

The outline is as follows. In the next section, we present the geometry and the notation. Then in Section 3, we introduce two auxiliary problems which will be used in the analysis. The Section 4 constitutes the heart of the article: here we compute an asymptotic expansion of the acoustic field and of the scattering coefficients with respect to  $\varepsilon$ , the width of the thin slits. Then we exploit the results in Section 5 to exhibit situations where the device acts as an energy distributor. We illustrate the theory in Section 6 with numerical experiments before proving two technical lemmas needed in the study in a short appendix.

## 2 Setting

First, we describe in detail the geometry (see Figure 1). Set  $\Pi_0 := \{z = (x, y) \in \mathbb{R}^2 \mid (x, y) \in (-\infty; 0) \times (0; 1)\}$ . Pick two different points  $A_{\pm} = (p_{\pm}, 1) \in (-\infty; 0) \times \{1\} \subset \partial\Pi_0$ . For  $\varepsilon > 0$ , define the lengths

$$L_{\pm}^{\varepsilon} := L_{\pm} + \varepsilon L'_{\pm} \tag{1}$$

where the values  $L_{\pm} > 0$ ,  $L'_{\pm} > 0$  will be fixed later on to observe interesting phenomena. Define the thin strips

$$\mathcal{S}_{\pm}^{\varepsilon} := (p_{\pm} - \varepsilon/2; p_{\pm} + \varepsilon/2) \times [1; 1 + L_{\pm}^{\varepsilon}].$$

Define the points  $B_{\pm}^{\varepsilon}$  such that  $B_{\pm}^{\varepsilon} = (p_{\pm}, 1 + L_{\pm}^{\varepsilon})$ . Set

$$\Pi_{\pm}^{\varepsilon} := (p_{\pm} - 1/2; p_{\pm} + 1/2) \times (1 + L_{\pm}^{\varepsilon}; +\infty).$$

And finally, we define the geometry

$$\Omega^\varepsilon := \Pi_0 \cup \mathcal{S}_+^\varepsilon \cup \mathcal{S}_-^\varepsilon \cup \Pi_+^\varepsilon \cup \Pi_-^\varepsilon.$$

Interpreting the domain  $\Omega^\varepsilon$  as an acoustic waveguide, we are led to consider the following problem with Neumann boundary condition

$$\begin{cases} \Delta u^\varepsilon + \omega^2 u^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ \partial_\nu u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases} \quad (2)$$

Here,  $\Delta$  is the Laplace operator while  $\partial_\nu$  corresponds to the derivative along the exterior normal. Furthermore,  $u^\varepsilon$  is the acoustic pressure of the medium while  $\omega > 0$  is the wave number of the piston modes  $w_h^\pm(x, y) = e^{\pm i\omega x}$  (resp.  $w_v^\pm(x, y) = e^{\pm i\omega y}$ ) propagating in  $\Pi_0$  (resp.  $\Pi_\pm^\varepsilon$ ). We fix  $\omega \in (0; \pi)$  so that no other mode can propagate. We are interested in the solution to the diffraction problem (2) generated by the incoming wave  $w_h^+$  in the trunk  $\Pi_0$ . This solution admits the decomposition

$$u^\varepsilon(x, y) = \begin{cases} w_h^+(x) + R^\varepsilon w_h^-(x) + \dots & \text{in } \Pi_0 \\ T_+^\varepsilon w_v^+(y - L^\varepsilon) + \dots & \text{in } \Pi_+^\varepsilon \\ T_-^\varepsilon w_v^+(y - L^\varepsilon) + \dots & \text{in } \Pi_-^\varepsilon \end{cases} \quad (3)$$

where  $R^\varepsilon \in \mathbb{C}$  is a reflection coefficient and  $T_\pm^\varepsilon \in \mathbb{C}$  are transmission coefficients. In this decomposition, the ellipsis stand for a remainder which decays at infinity with the rate  $e^{-(\pi^2 - \omega^2)^{1/2}|x|}$  in  $\Pi_0$  and  $e^{-(\pi^2 - \omega^2)^{1/2}|y|}$  in  $\Pi_\pm^\varepsilon$ . Due to conservation of energy, one has

$$|R^\varepsilon|^2 + |T_+^\varepsilon|^2 + |T_-^\varepsilon|^2 = 1.$$

In general, almost no energy of the incident wave  $w_h^+$  passes through the thin strips and one observes almost complete reflection. More precisely, one finds that there holds

$$R^\varepsilon = 1 + \tilde{R}^\varepsilon, \quad T_\pm^\varepsilon = \tilde{T}_\pm^\varepsilon, \quad (4)$$

where  $\tilde{R}^\varepsilon, \tilde{T}_\pm^\varepsilon$  tend to zero as  $\varepsilon$  goes to zero. The main goal of this work is to show that choosing carefully the lengths  $L_\pm^\varepsilon$  of the thin strips  $\mathcal{S}_\pm^\varepsilon$  as well as their positions, the energy of the wave  $w_h^+$  can be almost completely transmitted. Moreover we can control the energy transmitted respectively in  $T_+^\varepsilon$  and  $T_-^\varepsilon$ . More precisely, we will prove that choosing carefully  $L_\pm^\varepsilon$ , as  $\varepsilon$  tends to zero we can have

$$R^\varepsilon = \tilde{R}^\varepsilon, \quad T_\pm^\varepsilon = T_\pm^0 + \tilde{T}^\varepsilon,$$

where  $\tilde{R}^\varepsilon, \tilde{T}_\pm^\varepsilon$  tend to zero as  $\varepsilon$  goes to zero,  $|T_+^0|^2 + |T_-^0|^2 = 1$  and  $|T_+^0|/|T_-^0|$  can be any number in  $(0; +\infty)$  (see formulas (31) below). Thus we can select the energy ratio transmitted in  $\Pi_\pm^\varepsilon$  and the device acts as an energy distributor.

### 3 Auxiliary objects

In this section, we discuss a couple of boundary value problems whose solutions will appear in the construction of the asymptotic expansions of the acoustic field  $u^\varepsilon$ .

Considering the limit  $\varepsilon \rightarrow 0^+$  in the equation (2) restricted to the strips  $\mathcal{S}_\pm^\varepsilon$ , we are led to study the one-dimensional Helmholtz equations

$$\partial_y^2 v + \omega^2 v = 0 \quad \text{in } \ell_\pm := (1; 1 + L_\pm) \quad (5)$$

supplied with the artificially imposed Dirichlet conditions

$$v(1) = v(1 + L_\pm) = 0. \quad (6)$$

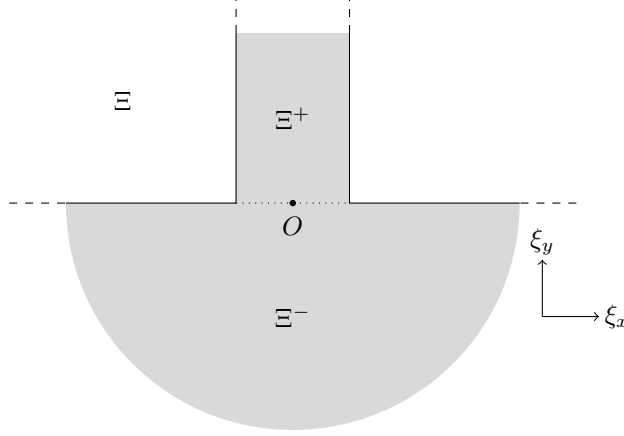


Figure 2: Geometry of the frozen domain  $\Xi$ .

Eigenvalues and eigenfunctions (up to a multiplicative constant) of the boundary value problem (5)–(6) are given by

$$\mu_m := (\pi m/L_{\pm})^2, \quad v_m(y) = \sin(\pi m(y-1)/L_{\pm}),$$

with  $m \in \mathbb{N}^* := \{1, 2, 3, \dots\}$ .

Now we present a second problem which is involved in the construction of asymptotics and which will be used to describe the boundary layer phenomenon near the points  $A_{\pm}, B_{\pm}^{\varepsilon}$ . To capture rapid variations of the field for example in the vicinity of  $A_{\pm}$ , we introduce the stretched coordinates  $\xi^{\pm} = (\xi_x^{\pm}, \xi_y^{\pm}) = \varepsilon^{-1}(z - A_{\pm}) = (\varepsilon^{-1}(x - p_{\pm}), \varepsilon^{-1}(y - 1))$ . Observing that

$$(\Delta_z + \omega^2)u^{\varepsilon}(\varepsilon^{-1}(z - A_{\pm})) = \varepsilon^{-2}\Delta_{\xi^{\pm}}u^{\varepsilon}(\xi^{\pm}) + \dots, \quad (7)$$

we are led to consider the Neumann problem

$$-\Delta_{\xi}Y = 0 \quad \text{in } \Xi, \quad \partial_{\nu}Y = 0 \quad \text{on } \partial\Xi, \quad (8)$$

where  $\Xi := \Xi^{-} \cup \Xi^{+} \subset \mathbb{R}^2$  (see Figure 2) is the union of the half-plane  $\Xi^{-}$  and the semi-strip  $\Xi^{+}$  such that

$$\Xi^{-} := \mathbb{R}_-^2 = \{\xi = (\xi_x, \xi_y) : \xi_y < 0\}, \quad \Xi^{+} := \{\xi : \xi_y \geq 0, |\xi_x| < 1/2\}.$$

In the method of matched asymptotic expansions (see the monographs [30, 13], [25, Chpt. 2] and others) that we will use, we will work with solutions of (8) which are bounded or which have polynomial growth in the semi-strip as  $\xi_y \rightarrow +\infty$ . One of such solutions is evident and is given by  $Y^0 = 1$ . Another solution, which is linearly independent with  $Y^0$ , is the unique function satisfying (8) and which has the representation

$$Y^1(\xi) = \begin{cases} \xi_y + C_{\Xi} + O(e^{-\pi\xi_y}) & \text{as } \xi_y \rightarrow +\infty, \quad \xi \in \Xi^{+} \\ \frac{1}{\pi} \ln \frac{1}{|\xi|} + O\left(\frac{1}{|\xi|}\right) & \text{as } |\xi| \rightarrow +\infty, \quad \xi \in \Xi^{-}. \end{cases} \quad (9)$$

Here,  $C_{\Xi}$  is a universal constant. Note that the coefficients in front of the growing terms in (9) are related due to the fact that a harmonic function has zero total flux at infinity.

## 4 Asymptotic analysis

In this section, we compute the asymptotic expansion of the field  $u^{\varepsilon}$  in (3) as  $\varepsilon$  tends to zero. The final results are summarized in (26). We assume that the limit lengths  $L_{\pm}$  of the thin strips (see (1)) are such that

$$L_{\pm} = \frac{\pi m_{\pm}}{\omega} \quad \text{for some } m_{\pm} \in \mathbb{N}^*. \quad (10)$$

In other words, we assume that  $\omega^2$  is an eigenvalue of the problems (5)–(6). We emphasize that these problems are posed in the fixed lines  $\ell_\pm$  but the true lengths  $L_\pm^\varepsilon = L_\pm + \varepsilon L'_\pm$  of the strips  $\mathcal{S}_\pm^\varepsilon$  depend on the parameter  $\varepsilon$ . In the channels, we work with the ansatz

$$u^\varepsilon = u_0^0 + \varepsilon u'_0 + \dots \quad \text{in } \Pi_0, \quad u^\varepsilon = u_\pm^0 + \varepsilon u'_\pm + \dots \quad \text{in } \Pi_\pm^\varepsilon, \quad (11)$$

while in the thin strips, we deal with the expansion

$$u^\varepsilon(x, y) = \varepsilon^{-1} v_\pm^{-1}(y) + v_\pm^0(y) + \dots \quad \text{in } \mathcal{S}_\pm^\varepsilon.$$

Taking the formal limit  $\varepsilon \rightarrow 0^+$ , we find that  $v_\pm^{-1}$  must solve the homogeneous problem (5)–(6). Under the assumption (10) for the lengths  $L_\pm$ , non zero solutions exist for this problem and we look for  $v_\pm^{-1}$  in the form

$$v_\pm^{-1}(y) = a_\pm \mathbf{v}_\pm(y) \quad \text{with} \quad a_\pm \in \mathbb{C}, \quad \mathbf{v}_\pm(y) = \sin(\omega(y-1)).$$

Let us stress that the values of  $a_\pm$  are unknown and will be fixed during the construction of the asymptotics of  $u^\varepsilon$ . At  $A_\pm$ , the Taylor formula gives

$$\varepsilon^{-1} v_\pm^{-1}(y) + v_\pm^0(y) = 0 + (C_\pm^A \xi_y^\pm + v_\pm^0(1)) + \dots \quad \text{with} \quad C_\pm^A := a_\pm \partial_y \mathbf{v}(1) = a_\pm \omega. \quad (12)$$

Here  $\xi_y^\pm = \varepsilon^{-1}(y-1)$  is the stretched variable introduced just before (7). At  $B_\pm^\varepsilon$ , we have

$$\varepsilon^{-1} v_\pm^{-1}(y) + v_\pm^0(y) = 0 + (C_\pm^B \zeta_y^\pm + v_\pm^0(1+L_\pm)) + \dots \quad (13)$$

with

$$C_\pm^B := -a_\pm \partial_y \mathbf{v}(1+L_\pm) = -a_\pm \omega \cos(\omega L_\pm) = (-1)^{1+m_\pm} a_\pm \omega. \quad (14)$$

Here, we use the stretched coordinates  $\zeta^\pm = (\zeta_x^\pm, \zeta_y^\pm) = (\varepsilon^{-1}(x-p_\pm), \varepsilon^{-1}(1+L_\pm-y))$  (mind the sign of  $\zeta_y^\pm$ ).

We look for an inner expansion of  $u^\varepsilon$  in the vicinity of  $A_\pm$  of the form

$$u^\varepsilon(x) = C_\pm^A Y^1(\xi^\pm) + c_\pm^A + \dots$$

where  $Y^1$  is introduced in (9),  $C_\pm^A$  are defined in (12) and  $c_\pm^A$  are constants to determine. In a vicinity of  $B_\pm^\varepsilon$ , we look for an inner expansion of  $u^\varepsilon$  of the form

$$u^\varepsilon(x) = C_\pm^B Y^1(\zeta_x^\pm, \zeta_y^\pm + L'_\pm) + c_\pm^B + \dots$$

where  $C_\pm^B$  are defined in (14) and  $c_\pm^B$  are constants to determine.

Let us continue the matching procedure. Taking the limit  $\varepsilon \rightarrow 0^+$ , we find that the main term  $u_0^0$  in (11) must solve the problem

$$\Delta u_0^0 + \omega^2 u_0^0 = 0 \quad \text{in } \Pi_0, \quad \partial_\nu u_0^0 = 0 \quad \text{on } \partial\Pi_0 \setminus \{A_+, A_-\},$$

with the expansion

$$u_0^0 = w_h^+ + R^0 w_h^- + \tilde{u}_0^0.$$

Here  $R^0 \in \mathbb{C}$  and  $\tilde{u}_0^0$  decay exponentially at infinity. Moreover, we find that the term  $u_\pm^0$  in (11) must solve the problem

$$\Delta u_\pm^0 + \omega^2 u_\pm^0 = 0 \quad \text{in } \Pi_\pm^\varepsilon, \quad \partial_\nu u_\pm^0 = 0 \quad \text{on } \partial\Pi_\pm^\varepsilon \setminus B_\pm^\varepsilon,$$

with the expansion

$$u_\pm^0(x, y) = T_\pm^0 w_v^+(y - L^\varepsilon) + \tilde{u}_\pm^0(x, y),$$

Here  $T_{\pm}^0 \in \mathbb{C}$  and  $\tilde{u}_{\pm}^0$  decay exponentially at infinity. The coefficients  $R^0, T_{\pm}^0 \in \mathbb{C}$  will provide the first terms in the asymptotics of  $R^{\varepsilon}, T_{\pm}^{\varepsilon}$ :

$$R^{\varepsilon} = R^0 + \dots \quad \text{and} \quad T_{\pm}^{\varepsilon} = T_{\pm}^0 + \dots$$

Matching the behaviours of the inner and outer expansions of  $u^{\varepsilon}$  in  $\Pi_0$ , we find that at the points  $A^{\pm}$ , the function  $u_0^0$  must expand as

$$u_0^0(x, y) = C_{\pm}^A \frac{1}{\pi} \ln \frac{1}{r_{\pm}^A} + U_{0\pm}^0 + O(r_{\pm}^A) \quad \text{as } r_{\pm}^A := ((x - p_{\pm})^2 + (y - 1)^2)^{1/2} \rightarrow 0^+,$$

where  $U_0^0$  is a constant. Observe that  $u_0^0$  is singular both at  $A^+$  and  $A^-$ . Integrating by parts in

$$0 = \int_{\Pi_{0,\rho}} (\Delta u_0^0 + \omega^2 u_0^0)(e^{+i\omega x} + e^{-i\omega x}) - u_0^0 (\Delta(e^{+i\omega x} + e^{-i\omega x}) + \omega^2(e^{+i\omega x} + e^{-i\omega x})) dx dy,$$

with  $\Pi_{0,\rho} := \{(x, y) \in \Pi_0, x > -\rho \text{ and } r_{\pm}^A > 1/\rho\}$ , and taking the limit  $\rho \rightarrow +\infty$ , we get  $2i\omega(R^0 - 1) + 2C_+^A \cos(\omega p_+) + 2C_-^A \cos(\omega p_-) = 0$ . From the expressions of  $C_{\pm}^A$  (see (12)), this gives

$$R^0 = 1 + i(a_+ \cos(\omega p_+) + a_- \cos(\omega p_-)). \quad (15)$$

Then matching the behaviours of the inner and outer expansions of  $u^{\varepsilon}$  in  $\Pi_{\pm}^{\varepsilon}$ , we find that at the points  $B_{\pm}^{\varepsilon}$ , the function  $u_{\pm}^0$  must expand as

$$u_{\pm}^0(x, y) = C_{\pm}^B \frac{1}{\pi} \ln \frac{1}{r_{\pm}^B} + U_{\pm}^0 + O(r_{\pm}^B) \quad \text{as } r_{\pm}^B := ((x - p_{\pm})^2 + (y - 1 - L^{\varepsilon})^2)^{1/2} \rightarrow 0^+,$$

where  $U_{\pm}^0$  are constants. Note that  $u_{\pm}^0$  is singular at  $B_{\pm}^{\varepsilon}$ . Integrating by parts in

$$\begin{aligned} 0 &= \int_{\Pi_{\pm,\rho}^{\varepsilon}} (\Delta u_{\pm}^0 + \omega^2 u_{\pm}^0)(e^{+i\omega(y-1-L^{\varepsilon})} + e^{-i\omega(y-1-L^{\varepsilon})}) dx dy \\ &\quad - \int_{\Pi_{\pm,\rho}^{\varepsilon}} u_{\pm}^0 (\Delta(e^{+i\omega(y-1-L^{\varepsilon})} + e^{-i\omega(y-1-L^{\varepsilon})}) + \omega^2(e^{+i\omega(y-1-L^{\varepsilon})} + e^{-i\omega(y-1-L^{\varepsilon})})) dx dy, \end{aligned}$$

with  $\Pi_{\pm,\rho}^{\varepsilon} := \{(x, y) \in \Pi_{\pm}^{\varepsilon}, y < \rho \text{ and } r_{\pm}^B > 1/\rho\}$ , and taking the limit  $\rho \rightarrow +\infty$ , we get  $2i\omega T_{\pm}^0 + 2C_{\pm}^B \cos(\omega p_{\pm}) = 0$ . From the expressions of  $C_{\pm}^B$  (see (14)), this gives

$$T_{\pm}^0 = i(-1)^{1+m_{\pm}} a_{\pm} \cos(\omega p_{\pm}). \quad (16)$$

Matching the constant behaviour inside  $\Pi_0$ , we get

$$U_{0\pm}^0 = C_{\pm}^A \pi^{-1} \ln \varepsilon + c_{\pm}^A = -C_{\pm}^A \pi^{-1} |\ln \varepsilon| + c_{\pm}^A.$$

This sets the value of  $c_{\pm}^A$ . However  $U_{0\pm}^0$  depends on  $a_{\pm}$  and we have to explicit this dependence. For  $u_0^0$ , we have the decomposition

$$u_0^0 = w_h^+ + w_h^- + C_+^A \gamma_+ + C_-^A \gamma_- \quad (17)$$

where  $\gamma_{\pm}$  are the outgoing functions such that

$$\begin{cases} \Delta \gamma_{\pm} + \omega^2 \gamma_{\pm} = 0 & \text{in } \Pi_0 \\ \partial_{\nu} \gamma_{\pm} = \delta_{A_{\pm}} & \text{on } \partial \Pi_0. \end{cases} \quad (18)$$

Here  $\delta_{A_{\pm}}$  stands for the Dirac delta function at  $A_{\pm}$ . Denote by  $\Gamma_{\pm}$  the constant behaviour of  $\gamma_{\pm}$  at  $A_{\pm}$ , that is the constant such that  $\gamma_{\pm}$  behaves as

$$\gamma_{\pm}(x, y) = \frac{1}{\pi} \ln \frac{1}{r_{\pm}^A} + \Gamma_{\pm} + O(r_{\pm}^A) \quad \text{when } r_{\pm}^A = ((x - p_{\pm})^2 + (y - 1)^2)^{1/2} \rightarrow 0^+.$$

In Lemma 6.2 below, we will prove that the constant behaviours of  $\gamma_{\pm}$  at  $A_{\mp}$  are equal. We denote by  $\tilde{\Gamma} = \gamma_+(A_-) = \gamma_-(A_+)$  the value of this coupling constant. Then from (17), we derive

$$U_{0\pm}^0 = 2 \cos(\omega p_{\pm}) + \omega(a_{\pm}\Gamma_{\pm} + a_{\mp}\tilde{\Gamma}).$$

Matching the constant behaviour at  $A_{\pm}$  inside the thin strips, we obtain

$$\begin{aligned} v_{\pm}^0(1) &= C_{\pm}^A C_{\Xi} + c_{\pm}^A = U_{0\pm}^0 + C_{\pm}^A (\pi^{-1} |\ln \varepsilon| + C_{\Xi}) \\ &= 2 \cos(\omega p_{\pm}) + a_{\mp}\omega\tilde{\Gamma} + a_{\pm}\omega (\pi^{-1} |\ln \varepsilon| + C_{\Xi} + \Gamma_{\pm}). \end{aligned} \quad (19)$$

Now, matching the constant behaviour inside  $\Pi_{\pm}^{\varepsilon}$ , we get

$$U_{\pm}^0 = C_{\pm}^B \pi^{-1} \ln \varepsilon + c_{\pm}^B = -C_{\pm}^B \pi^{-1} |\ln \varepsilon| + c_{\pm}^B.$$

This sets the value of  $c_{\pm}^B$ . However  $U_{\pm}^0$  depends on  $a_{\pm}$  and we have to explicit this dependence. For  $u_{\pm}^0$ , we have the decomposition

$$u_{\pm}^0(x, y) = C_{\pm}^B g(x - p_{\pm}, y + 1 + L^{\varepsilon})$$

where  $g$  is the outgoing function such that

$$\begin{cases} \Delta g + \omega^2 g = 0 & \text{in } \Pi := (-1/2; 1/2) \times (0; +\infty) \\ \partial_{\nu} g = \delta_O & \text{on } \partial\Pi. \end{cases} \quad (20)$$

Here  $\delta_O$  stands for the Dirac delta function at  $O$ . Denote by  $G$  the constant behaviour of  $g$  at  $O$ , that is the constant such that  $g$  behaves as

$$g(x, y) = \frac{1}{\pi} \ln \frac{1}{r} + G + O(r) \quad \text{when } r = (x^2 + y^2)^{1/2} \rightarrow 0^+.$$

Then we have

$$U_{\pm}^0 = C_{\pm}^B G = (-1)^{1+m_{\pm}} a_{\pm}\omega G.$$

Matching the constant behaviour at  $B_{\pm}^{\varepsilon}$  inside the thin strips, we obtain

$$\begin{aligned} v_{\pm}^0(1 + L_{\pm}) &= C_{\pm}^B (L'_{\pm} + C_{\Xi}) + c_{\pm}^B = U_{\pm}^0 + C_{\pm}^B (\pi^{-1} |\ln \varepsilon| + L'_{\pm} + C_{\Xi}) \\ &= (-1)^{1+m_{\pm}} a_{\pm}\omega (\pi^{-1} |\ln \varepsilon| + L'_{\pm} + C_{\Xi} + G). \end{aligned} \quad (21)$$

Writing the compatibility condition so that the problem (5) supplemented with the boundary conditions (19)–(21) admits a non zero solution, we get

$$v_{\pm}^0 \partial_y \mathbf{v}|_1 - v_{\pm}^0 \partial_y \mathbf{v}|_{1+L_{\pm}} - (\mathbf{v} \partial_y v_{\pm}^0|_1 - \mathbf{v} \partial_y v_{\pm}^0|_{1+L_{\pm}}) = 0.$$

Since  $\mathbf{v}(1) = \mathbf{v}(1 + L_{\pm}) = 0$ , we obtain

$$v_{\pm}^0(1) + (-1)^{1+m_{\pm}} v_{\pm}^0(1 + L_{\pm}) = 0.$$

This gives the relations

$$2 \cos(\omega p_{\pm}) + a_{\mp}\omega\tilde{\Gamma} + a_{\pm}\omega (2\pi^{-1} |\ln \varepsilon| + L'_{\pm} + 2C_{\Xi} + \Gamma_{\pm} + G) = 0. \quad (22)$$

Below, we will prove that  $\Im m(\omega\Gamma_{\pm}) = |\cos(\omega p_{\pm})|^2$ ,  $\Im m(\omega G) = 1$  (Lemma 6.2) and  $C_{\Xi} \in \mathbb{R}$  (Lemma 6.1). Therefore (22) writes equivalently

$$a_{\pm}(\beta_{\pm} + i(1 + |\cos(\omega p_{\pm})|^2)) + a_{\mp}\omega\tilde{\Gamma} = -2 \cos(\omega p_{\pm}) \quad (23)$$

where  $\beta_{\pm}$  are the real valued quantities such that

$$\beta_{\pm} := \omega (2\pi^{-1} |\ln \varepsilon| + L'_{\pm} + 2C_{\Xi} + \Re e \Gamma_{\pm} + \Re e G). \quad (24)$$



Identities (23) form a system of two equations whose unknowns are  $a_{\pm}$ . Solving it, we get

$$a_{\pm} = \frac{2 \cos(\omega p_{\mp}) \omega \tilde{\Gamma} - 2(\beta_{\mp} + i(1 + |\cos(\omega p_{\mp})|^2)) \cos(\omega p_{\pm})}{(\beta_{+} + i(1 + |\cos(\omega p_{+})|^2))(\beta_{-} + i(1 + |\cos(\omega p_{-})|^2)) - \omega^2 \tilde{\Gamma}^2}. \quad (25)$$

And from (15), (16), we obtain explicit expressions for  $R^0$ ,  $T_{\pm}^0$ . This ends the asymptotic analysis. To sum up, when  $\varepsilon$  tends to zero, we have obtained the following expansions

$$\begin{aligned} u^{\varepsilon}(x, y) &= w_h^{+}(x, y) + w_h^{-}(x, y) + a_{+} \omega \gamma_{+}(x, y) + a_{-} \omega \gamma_{-}(x, y) + \dots \text{ in } \Pi_0, \\ u^{\varepsilon}(x, y) &= (-1)^{1+m_{\pm}} a_{\pm} \omega g(x - p_{\pm}, y + 1 + L^{\varepsilon}) + \dots \text{ in } \Pi_{\pm}^{\varepsilon}, \\ u^{\varepsilon}(x, y) &= \varepsilon^{-1} a_{\pm} \sin(\omega(y - 1)) + \dots \text{ in } \mathcal{S}_{\pm}^{\varepsilon}, \\ R^{\varepsilon} &= 1 + i(a_{+} \cos(\omega p_{+}) + a_{-} \cos(\omega p_{-})) + \dots, \quad T_{\pm}^{\varepsilon} = i(-1)^{1+m_{\pm}} a_{\pm} \cos(\omega p_{\pm}) + \dots, \end{aligned} \quad (26)$$

where  $a_{\pm}$  are given by (25). Here, the functions  $\gamma_{\pm}$ ,  $g$  are respectively introduced in (18), (20). Note in particular that when  $a_{\pm} \neq 0$ , the amplitude of the field blows up in the thin slits as  $\varepsilon$  tends to zero.

## 5 Analysis of the results

In this section, we explain how to use the asymptotic results (26) to exhibit settings where the waveguide  $\Omega^{\varepsilon}$  acts as an energy distributor. The degrees of freedom we can play with are  $L'_{\pm}$  and  $p_{\pm}$ , that is the lengths and the abscissa of the thin strips.

★ When  $p_{\pm}$  are chosen such that  $\cos(\omega p_{\pm}) = 0$ , from (25) we get  $a_{\pm} = 0$ . This implies  $R^0 = 1$  and  $T_{\pm}^0 = 0$ . In this case, the energy brought to the system is almost completely backscattered and the thin strips have almost no influence on the incident field. Definitely, this is not an acoustic distributor. Roughly speaking, in this situation what happens is that the resonant eigenfunctions associated with complex resonances existing due to the presence of the thin slits are not excited.

★ When  $p_{\pm}$  are chosen such that  $\cos(\omega p_{\pm}) = 1$ , we find

$$a_{\pm} = \frac{2\omega \tilde{\Gamma} - 2(\beta_{\mp} + 2i)}{(\beta_{+} + 2i)(\beta_{-} + 2i) - \omega^2 \tilde{\Gamma}^2}.$$

Then, we have

$$R^0 = 1 + i(a_{+} + a_{-}) = \frac{\beta_{+} \beta_{-} + 4 - \omega^2 \tilde{\Gamma}^2 + 4i\omega \tilde{\Gamma}}{(\beta_{+} + 2i)(\beta_{-} + 2i) - \omega^2 \tilde{\Gamma}^2}.$$

and

$$T_{\pm}^0 = i(-1)^{1+m_{\pm}} \frac{2\omega \tilde{\Gamma} - 2(\beta_{\mp} + 2i)}{(\beta_{+} + 2i)(\beta_{-} + 2i) - \omega^2 \tilde{\Gamma}^2}.$$

According to Lemma 6.2 below, we have  $\omega \tilde{\Gamma} = \eta + i$  for a certain  $\eta \in \mathbb{R}$  which characterizes the coupling between the two strips. With this notation, we find

$$R^0 = \frac{\beta_{+} \beta_{-} + 1 - \eta^2 + 2i\eta}{\beta_{+} \beta_{-} + 2i(\beta_{+} + \beta_{-}) - 3 - \eta^2 - 2i\eta}, \quad T_{\pm}^0 = \frac{2i(-1)^{m_{\pm}}(\beta_{\mp} + i - \eta)}{\beta_{+} \beta_{-} + 2i(\beta_{+} + \beta_{-}) - 3 - \eta^2 - 2i\eta}. \quad (27)$$

Notice that the denominator in the expressions for  $R^0$ ,  $T_{\pm}^0$  cannot vanish. Indeed it vanishes if and only if  $\beta_{+} + \beta_{-} = \eta$  and  $\beta_{+} \beta_{-} = 3 + \eta^2$ . One can verify that this cannot occur for  $(\beta_{+}, \beta_{-}) \in \mathbb{R}^2$ .

If  $\eta = 0$  was zero, then we would have  $R^0 = \mathcal{R}^0$ ,  $T_{\pm}^0 = \mathcal{T}_{\pm}^0$  with

$$\mathcal{R}^0 = \frac{\beta_{+} \beta_{-} + 1}{\beta_{+} \beta_{-} + 2i(\beta_{+} + \beta_{-}) - 3}, \quad \mathcal{T}_{\pm}^0 = \frac{2i(-1)^{m_{\pm}}(\beta_{\mp} + i)}{\beta_{+} \beta_{-} + 2i(\beta_{+} + \beta_{-}) - 3}. \quad (28)$$

In particular, we would have  $R^0 = 0$  for all pairs  $(\beta_+, \beta_-) \in \mathbb{R}^2$  such that

$$\beta_+ \beta_- = -1, \quad (29)$$

and then  $|T_+^0(\beta_+, \beta_-)|^2 + |T_-^0(\beta_+, \beta_-)|^2 = 1$  (conservation of energy) as well as

$$\frac{|T_+^0(\beta_+, \beta_-)|}{|T_-^0(\beta_+, \beta_-)|} = \sqrt{\frac{\beta_-^2 + 1}{\beta_+^2 + 1}} = \sqrt{\frac{1/\beta_+^2 + 1}{\beta_+^2 + 1}}. \quad (30)$$

Observing in (24) that  $\beta_+$  varies in  $\mathbb{R}$  when  $L'_\pm$  runs in  $\mathbb{R}$ , we see that the ratio (30) could take any value in  $(0; +\infty)$ .

However, the coupling constant  $\eta$  cannot be chosen as we wish because we have already set  $p_\pm$  to impose  $\cos(\omega p_\pm) = 1$ . In Lemma 6.2 below, we will prove that the function  $e_\pm := \gamma_\pm - s_\pm w_h^-$ , with  $s_\pm = i \cos(\omega p_\pm)/\omega$ , is real and exponentially decaying at infinity. As a consequence, we infer that for  $p_- < p_+$  with  $|p_+ - p_-|$  large, we have

$$\omega \tilde{\Gamma} = \omega \gamma_+(p_-, 1) \approx \omega s_+ w_h^-(p_-) = i \cos(\omega p_+) e^{-i\omega p_-}.$$

With the above choice for  $p_\pm$  as well as the relation (29) for  $\beta_\pm$ , which translates into a condition relating  $L'_+$  to  $L'_-$ , we have  $\eta = \Re e(\omega \tilde{\Gamma}) \approx 0$ . From the previous analysis, we deduce that

$$R^0 \approx 0 \quad \text{together with} \quad \frac{|T_+^0(\beta_+, \beta_-)|}{|T_-^0(\beta_+, \beta_-)|} \approx \sqrt{\frac{\beta_-^2 + 1}{\beta_+^2 + 1}} = \sqrt{\frac{1/\beta_+^2 + 1}{\beta_+^2 + 1}}. \quad (31)$$

Thus, we can get almost complete transmission and varying  $L'_+$ , and so in practice, varying slightly  $L_+^\varepsilon$  around  $\pi m_+/\omega$ , we can control the ratio of energy transmitted respectively in  $\Pi_+^\varepsilon$  and in  $\Pi_-^\varepsilon$ . We emphasize that the tuning becomes more or more subtle as  $\varepsilon$  gets smaller. More precisely, for a fixed value of  $\beta_\pm$ , according to (24), the corresponding value of  $L'_\pm$  is equal to  $-2\pi^{-1} |\ln \varepsilon| + \beta_\pm/\omega - 2C_\Xi - \Re e \Gamma_\pm - \Re e G$  (note in particular that it is negative for  $\varepsilon$  small enough). As a consequence,

$$\begin{aligned} L_\pm^\varepsilon &= \pi m_\pm/\omega + \varepsilon L'_\pm \\ &= \pi m_\pm/\omega + \varepsilon(-2\pi^{-1} |\ln \varepsilon| + \beta_\pm/\omega - 2C_\Xi - \Re e \Gamma_\pm - \Re e G) \end{aligned}$$

converges to  $(\pi m_\pm/\omega)^-$  as  $\varepsilon$  tends to zero.

**Remark 5.1.** *In the method, we tune the lengths of the slits around the resonance lengths. One may imagine to play with other parameters. For example one could work with slits of variable width*

$$S_\pm^\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid y \in [1; 1 + L_\pm], |x - p_\pm| < \varepsilon H_\pm(y)\},$$

where  $H_\pm$  are smooth profile functions, and then perturb  $H_\pm$ . Or we could also perturb slightly the position of the slits, that is  $p_\pm$ .

**Remark 5.2.** *A priori the approach presented here can also be used to construct an acoustic energy distributor with one input/output channel and more than two output channels.*

**Remark 5.3.** *From this analysis, a natural idea is to try to work in the waveguide represented in Figure 3 with two slits at the end of the input channel. In this case, the asymptotic expansion is similar to the one above and we have  $\cos(\omega p_\pm) = 1$ . But the problem is that in this geometry we cannot reduce the coupling constant  $\eta$  (see (27)) as we wish. As a consequence, this device is not interesting for our purpose.*

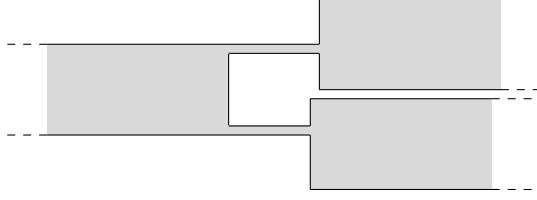


Figure 3: A device which fails to behave as an acoustic distributor.

## 6 Numerics

In this section, we illustrate the results we have obtained above. We work with

$$\omega = 0.8\pi, \quad p_- = -2.5, \quad p_+ = 0, \quad \varepsilon = 0.05.$$

Observe that in this case indeed we have  $\cos(\omega p_+) = \cos(\omega p_-) = 1$ . We compute numerically the scattering solution  $u^\varepsilon$  in (2). To proceed, we use a P2 finite element method in a truncated geometry. On the artificial boundary created by the truncation, a Dirichlet-to-Neumann operator with 15 terms serves as a transparent condition. Once we have computed  $u^\varepsilon$ , we get the scattering coefficients  $R^\varepsilon, T_\pm^\varepsilon$  in the representation (3).

In the first line of Figure 4, we represent  $R^\varepsilon, T_+^\varepsilon, T_-^\varepsilon$  as functions of  $(L_+^\varepsilon, L_-^\varepsilon)$ . In the second line of Figure 4, we display  $\mathcal{R}^0, \mathcal{T}_+^0, \mathcal{T}_-^0$  (see (28)) as functions of  $(\beta_+, \beta_-)$ . We observe that the true scattering coefficients and their asymptotic approximations coincide very well. From these results, we extract the curve  $(L_+^\varepsilon, L_-^\varepsilon)$  where  $|R^\varepsilon(L_+^\varepsilon, L_-^\varepsilon)|$  is minimum (see Figure 5). We note that indeed we can have almost no reflection. Additionally, we show the quantities  $|T_\pm^\varepsilon(L_+^\varepsilon, L_-^\varepsilon)|$  on this curve. As predicted, we notice that we can indeed control the ratio of energy transmitted in the channels  $\Pi_+^\varepsilon, \Pi_-^\varepsilon$ . Finally, in Figure 6, we represent the field  $u^\varepsilon$  in different geometries: one without particular tuning of the lengths of the slits, one where  $R^\varepsilon \approx 0, |T_+^\varepsilon| \approx 1$  and  $T_-^\varepsilon \approx 0$  and one where  $R^\varepsilon \approx 0, T_+^\varepsilon \approx 0$  and  $|T_-^\varepsilon| \approx 1$ . Our device indeed can act as an acoustic energy distributor.

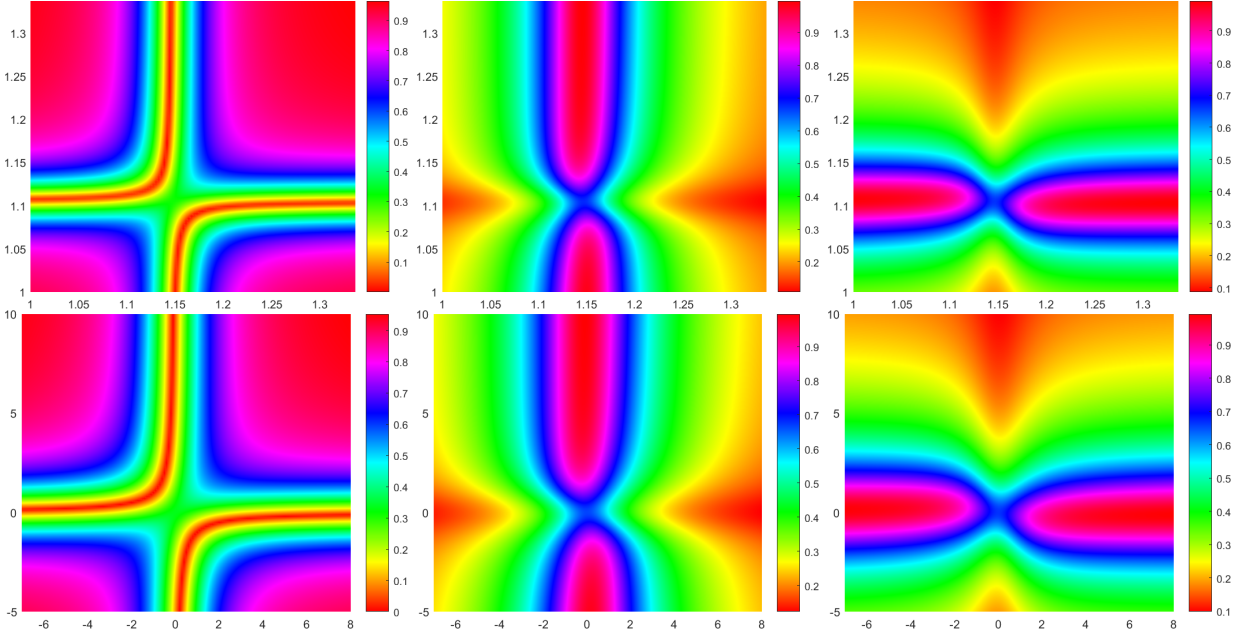


Figure 4: On the first line, from left to right, we represent  $R^\varepsilon, T_+^\varepsilon, T_-^\varepsilon$  as functions of  $(L_+^\varepsilon, L_-^\varepsilon)$ . On the second line, from left to right, we represent  $\mathcal{R}^0, \mathcal{T}_+^0, \mathcal{T}_-^0$  as functions of  $(\beta_+, \beta_-)$ . The true scattering coefficients and their principal asymptotic approximations coincide very well.

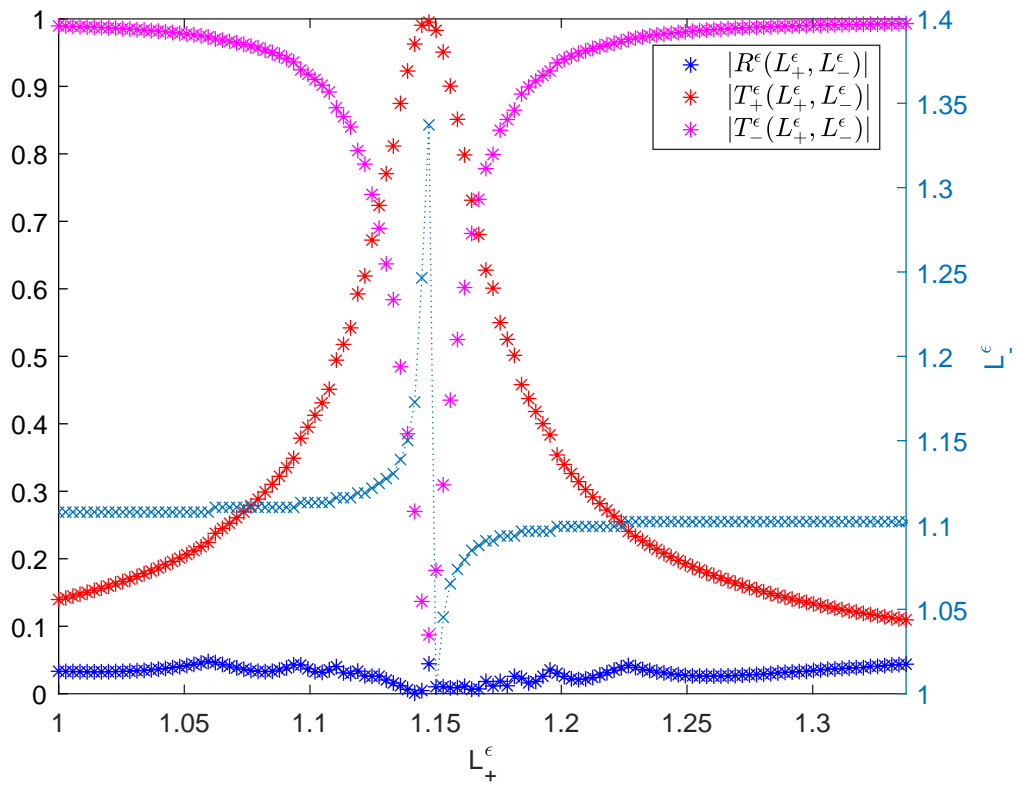


Figure 5: From the results of Figure 4, we extract the curve  $(L_+^\epsilon, L_-^\epsilon)$  where  $|R^\epsilon(L_+^\epsilon, L_-^\epsilon)|$  is minimum. Then we display  $|T_\pm^\epsilon(L_+^\epsilon, L_-^\epsilon)|$  on this curve.

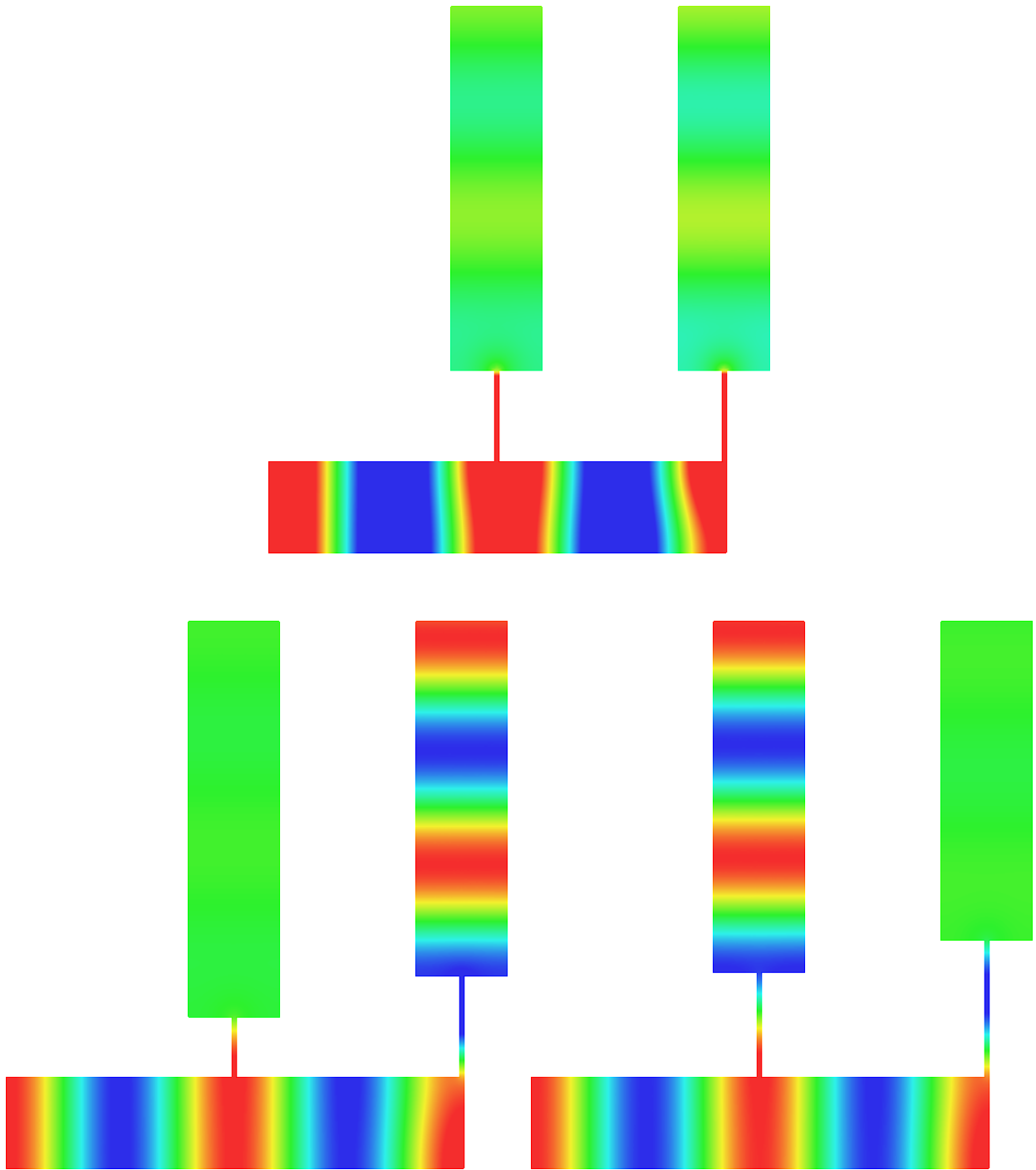


Figure 6: First line: without particular tuning of the lengths of the slits, the energy of the incoming wave is almost completely backscattered. Second line, left: setting where  $R^\varepsilon \approx 0$ ,  $T_+^\varepsilon \approx 0$  and  $|T_-^\varepsilon| \approx 1$ . Second line, right: setting where  $R^\varepsilon \approx 0$ ,  $|T_+^\varepsilon| \approx 1$  and  $T_-^\varepsilon \approx 0$ .

## Appendix: auxiliary results

**Lemma 6.1.** *The constant  $C_{\Xi}$  is real.*

*Proof.* Let  $Y^1$  be the function introduced in (9). For all  $\rho > 0$ , we have

$$0 = \int_{\Xi_\rho} (Y^1 - \overline{Y^1}) \Delta Y^1 - \Delta(Y^1 - \overline{Y^1}) Y^1 d\xi_x d\xi_y$$

with  $\Xi_\rho := \{(\xi_x, \xi_y) \in \Xi, \xi_y < 0 \text{ and } |\xi| < \rho\} \cup \{(\xi_x, \xi_y) \in (-1/2; 1/2) \times [0; \rho]\}$ . Integrating by parts and taking the limit  $\rho \rightarrow +\infty$ , we get  $C_{\Xi} - \overline{C_{\Xi}} = 0$ . This shows that  $C_{\Xi}$  is real.  $\square$

**Lemma 6.2.** *The constant behaviours of  $\gamma_{\pm}$  at  $A_{\mp}$  are equal. We denote by  $\tilde{\Gamma} \in \mathbb{C}$  this constant. We have  $\Im m(\omega \tilde{\Gamma}) = \cos(\omega p_+) \cos(\omega p_-)$ . The constants  $\Gamma_{\pm}$  are such that  $\Im m(\omega \Gamma_{\pm}) = |\cos(\omega p_{\pm})|^2$ .*

*Proof.* Since the functions  $\gamma_{\pm}$  are outgoing at infinity, we have the expansions

$$\gamma_{\pm} = s_{\pm} w_h^- + \tilde{\gamma}_{\pm}$$

where  $s_{\pm} \in \mathbb{C}$  and  $\tilde{\gamma}_{\pm}$  are exponentially decaying at infinity. We have

$$0 = \int_{\Pi_{0,\rho}} (\Delta \gamma_+ + \omega^2 \gamma_+) \gamma_- - \gamma_+ (\Delta \gamma_- + \omega^2 \gamma_-) dx dy,$$

with  $\Pi_{0,\rho} = \{(x, y) \in \Pi_0, x > -\rho \text{ and } r_{\pm}^A > 1/\rho\}$ . Integrating by parts and taking the limit  $\rho \rightarrow +\infty$ , we find that the constant behaviours of  $\gamma_{\pm}$  at  $A_{\mp}$  are equal. On the other hand, integrating by parts in

$$0 = \int_{\Pi_{0,\rho}} (\Delta \gamma_{\pm} + \omega^2 \gamma_{\pm})(e^{+i\omega x} + e^{-i\omega x}) - \gamma_{\pm} (\Delta(e^{+i\omega x} + e^{-i\omega x}) + \omega^2(e^{+i\omega x} + e^{-i\omega x})) dx dy,$$

and taking the limit  $\rho \rightarrow +\infty$ , we obtain

$$s_{\pm} = i \cos(\omega p_{\pm}) / \omega. \tag{32}$$

Then one can verify that the function  $e_{\pm} := \gamma_{\pm} - s_{\pm} w_h^-$  is real. Indeed  $e_{\pm} - \overline{e_{\pm}}$  is exponentially decaying and solves the homogeneous problem. We deduce that  $\Im m(\omega \tilde{\Gamma}) = \cos(\omega p_+) \cos(\omega p_-)$ . Finally, integrating by parts in

$$0 = \int_{\Pi_{0,\rho}} (\Delta \gamma_{\pm} + \omega^2 \gamma_{\pm}) \overline{\gamma_{\pm}} - \gamma_{\pm} (\Delta \overline{\gamma_{\pm}} + \omega^2 \overline{\gamma_{\pm}}) dx dy,$$

and taking again the limit  $\rho \rightarrow +\infty$ , we obtain  $2i\omega |s_{\pm}|^2 - 2\Im m \Gamma_{\pm} = 0$ . From (32), this yields the desired result.  $\square$

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