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Sparse Obstructions for Minor-Covering Parameters*

Dimitris Chatzidimitriou[†] Dimitrios M. Thilikos^{‡§} Dimitris Zoros[†]

Abstract

Given a finite set of graphs \mathcal{H} and a non-negative integer k , we define $\mathcal{A}_k(\mathcal{H})$ as the set containing every graph G that has k vertices whose removal provides a graph without any of the graphs in \mathcal{H} as a minor. It is known that if \mathcal{H} contains at least one planar graph then each obstruction in $\mathcal{A}_k(\mathcal{H})$ has at most $k^{c_{\mathcal{H}}}$ vertices, for some $c_{\mathcal{H}}$ depending only on the choice of \mathcal{H} . In this paper, we investigate the size of the graphs in $\mathcal{A}_k(\mathcal{H})$ that belong to certain classes of sparse graphs. In particular, we prove that for every graph F , if \mathcal{H} contains at least one planar graph and only connected graphs, all graphs in $\mathcal{A}_k(\mathcal{H})$ that are F -topological minor-free have at most $c_{F,\mathcal{H}} \cdot k$ vertices, where $c_{F,\mathcal{H}}$ depends exclusively on the choice of \mathcal{H} and F . Our result is a consequence of two more general conditions on graph parameters, namely the Finite Integer Index Property and Protrusion Decomposability, that can serve as a general framework for proving linear bounds for obstructions.

keywords Obstruction sets, graph minors, apex-extensions, protrusion decompositions, tree decompositions, finite integer index.

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1 Introduction

All graphs in this paper are simple, finite, and undirected. We use the term *graph collection* for finite sets of graphs, while for infinite sets of graphs we use the term *graph class*. A graph collection \mathcal{H} is *connected* if all its graphs are connected, while we say that \mathcal{H} is *planar* if at least one of its graphs is planar. A graph H is a *minor* of G if H can be obtained from some subgraph of G after a series of edge contractions. Also, H is a *topological minor* of G if G contains some subdivision of H . Given a collection of graphs \mathcal{H} , we define $\text{exc}(\mathcal{H})$ as the class of all graphs that do not contain any of the graphs in \mathcal{H} as a minor.

Given a graph class \mathcal{G} we say that \mathcal{G} is *minor-closed* if every minor of a graph in \mathcal{G} belongs to \mathcal{G} . Given a graph class \mathcal{G} , we define $\text{obs}(\mathcal{G})$ as the set of all minor-minimal graphs that do not belong to \mathcal{G} . Because of the *Robertson and Seymour theorem* [52], $\text{obs}(\mathcal{G})$ is *finite*. This implies that every minor-closed graph class can be *finitely characterised* by its obstruction set: indeed, for such a class \mathcal{G} it holds that $\mathcal{G} = \text{exc}(\text{obs}(\mathcal{G}))$, therefore a graph H belongs to \mathcal{G} iff none of the (finitely many) graphs in $\text{obs}(\mathcal{G})$ is a minor of H . To identify the obstruction set of a minor-closed graph class can be a difficult task and there is a long line of research on the characterization (partial or complete) of the obstruction sets for diverse minor-closed graph classes (see [2–5, 7, 9, 12, 14, 22, 24, 25, 27, 33, 37, 40, 42, 44–46, 50, 53–56]).

Apex extensions. In this paper we study the obstructions of the general family of minor-closed graph classes that are produced from simpler ones using the *k-apex extension operation*. More specifically, given a minor-closed graph class \mathcal{G} and a non-negative integer k , we define the *k-apex extension of \mathcal{G}* as the graph class containing every graph G that, by removing at

most k vertices, becomes a graph in \mathcal{G} . Given a collection of graphs \mathcal{H} , we denote by $\mathcal{A}_k(\mathcal{H})$ the k -apex extension of $\text{exc}(\mathcal{H})$. Notice that $\mathcal{A}_k(\mathcal{H})$ is quite general as it can express several known (parameterized) graph classes. For instance, $\mathcal{A}_k(\{K_2\})$ is the class of graphs with a vertex cover at most k , $\mathcal{A}_k(\{K_3\})$ is the class of graphs with a feedback vertex set at most k , $\mathcal{A}_k(\{K_4, K_{2,3}\})$ is the class of k -apex outerplanar graphs [20], and $\mathcal{A}_k(\{K_5, K_{3,3}\})$ is the class of k -apex planar graphs (see [44]).

In the general case, it has been proved in [3] that $\mathcal{A}_k(\mathcal{H})$ can be effectively computed, given \mathcal{H} and k . In practice, the identification of $\mathcal{A}_k(\mathcal{H})$ is a hard problem even for simple choices of \mathcal{H} and it becomes really hard for non-trivial values of k (see e.g. [21]).

Particular cases where $\mathcal{A}_k(\mathcal{H})$ has been completely identified are $\mathcal{A}_k(\{K_2\})$, for $k \leq 7$ [12, 24, 25], $\mathcal{A}_k(\{K_3\})$, for $k \leq 2$ [22], and $\mathcal{A}_1(\{K_4, K_{2,3}\})$ [20] – see also [44] for some interesting discussion on $\mathcal{A}_1(\{K_5, K_{3,3}\})$. Another direction is to identify graphs in $\mathcal{A}_k(\mathcal{H})$ with some particular property (e.g., [53] identified all the outerplanar graphs in $\mathcal{A}_k(\{K_3\})$ or to just identify members of $\mathcal{A}_k(\mathcal{H})$ (e.g., [44] proved that $|\mathcal{A}_1(\{K_5, K_{3,3}\})| \geq 36$). Another direction is to upper bound the sizes of the graphs in $\mathcal{A}_k(\mathcal{H})$ by some function $f_{\mathcal{H}}$ on k . Such a function has been found for $\mathcal{H} = \{K_2\}$ where $f_{\mathcal{H}}(k) = 2k + 1$ [23]. It was also proved in [49] that if \mathcal{H} is connected and contains the graph $K_{1,t}$, then $f_{\mathcal{H}}(k) = O(tk^7 + t^7k^2)$. Without doubt, the most general result in this direction was proved in [30] and says that if \mathcal{H} is planar, then all obstructions in $\mathcal{A}_k(\mathcal{H})$ have size bounded by a polynomial function of k , i.e. $f_{\mathcal{H}}(k) = k^{c_{\mathcal{H}}}$, where $c_{\mathcal{H}}$ is a constant that depends only on the choice of \mathcal{H} . A challenging open question is whether this bound can become uniform in the sense of improving it to $f_{\mathcal{H}}(k) = c_{\mathcal{H}} \cdot k^c$, where c is some universal constant not depending on \mathcal{H} . This question has been the main motivation of this paper (see also the Conclusion section in the end of the paper for some related discussion).

Our results. In this paper we consider some connected and planar collection of graphs \mathcal{H} . We prove a linear, in k , bound on the size of the graphs in $\text{obs}(\mathcal{A}_k(\mathcal{H}))$ that do not contain some fixed graph F as a topological minor. To be more precise, we define \mathcal{T}_F as the class of all graphs excluding F as a topological minor and we prove the following:

Theorem 1. *Let \mathcal{H} be a planar and connected graph class and let F be a graph. Then for every non-negative integer k , every graph in $\mathcal{T}_F \cap \text{obs}(\mathcal{A}_k(\mathcal{H}))$ has at most $c \cdot k$ vertices, for some c depending on \mathcal{H} and F .*

The above theorem has several implications. For instance, planar obstructions for k -apex extensions to \mathcal{H} have linear, in k , size. Indeed, just consider $F = K_6$ and observe that \mathcal{T}_F contains all planar graphs. Moreover, for every t , bounded (by t) degree graphs in $\text{obs}(\mathcal{A}_k(\mathcal{H}))$ have also size $O(k)$ (just take $F = K_{1,t}$).

Organization of the paper. The proof of Theorem 1 follows as a special case of a much more general result. This result is Theorem 2 and its presentation demands some additional concepts that are given in Section 2. Such concepts are the FI property (Subsection 2.1) and the notion of protrusion-decomposability (Subsection 2.2). Theorem 2 is presented in Subsection 2.3. The proof of the fact that Theorem 1 follows from Theorem 2 is given in Subsection 2.4. As the proof of Theorem 2 is lengthy, we dedicate the last subsection of Section 2 to an outline of it.

In Section 3 we present the main definitions required for the proof and some preliminary results. This includes notation on sets and functions in Subsection 3.1 and on graphs and trees in Subsection 3.2. We also give several types of graph-decompositions in Subsection 3.3. In Subsection 3.4, we provide extensions of those decompositions to bounded graphs along with some related preliminary results.

The bulk of the proof of Theorem 2 is given in Section 4. The structure of this section is presented in the outline of the proof in Subsection 2.5. Finally, in Section 5, we give some open problems and conjectures related to the size of the obstructions of the k -apex extensions of minor-closed graph classes.

2 A more general result

Given a graph G , we denote by $V(G)$ and $E(G)$ the vertex and the edge set of G respectively. We also set $|G| = |V(G)|$. Let $u \in V(G)$ and $e \in E(G)$. We denote by $G \setminus S$, for some $S \subseteq V(G)$, the graph obtained from G by removing the vertices in S . We also use $G \setminus u$ to denote $G \setminus \{u\}$.

2.1 The Fill property

Boundaried graphs. A *labeling* of a graph G is any injective function $\lambda : V(G) \rightarrow \mathbb{N}$. Let $t \in \mathbb{N}$. A t -*boundaried graph* is a triple $\mathbf{G} = (G, X, \lambda)$ where G is a graph, $X \subseteq V(G)$, $|X| = t$, and λ is a labelling of G . We call X the *boundary* of \mathbf{G} and we call the vertices of X the *boundary vertices* of \mathbf{G} . We also call G the *underlying graph* of \mathbf{G} and the integer $t = |X|$ *boundary size* of \mathbf{G} . When we do not want to specify what is the value t , we simply call \mathbf{G} *boundaried graph* instead of t -boundaried graph. We denote by $\mathcal{B}^{(t)}$ the class of all t -boundaried graphs and we set $\mathcal{B}^{(\leq t)} = \bigcup_{i \in \{0, \dots, t\}} \mathcal{B}^{(i)}$. Given a t -boundaried graph $\mathbf{G} = (G, X, \lambda)$, we define the *label normalising function* of \mathbf{G} $\psi_{\mathbf{G}} : X \rightarrow [t]$ such that for each $v \in X$, $\psi_{\mathbf{G}}(v) = |\{u \in X \mid \lambda(u) \leq \lambda(v)\}|$. Note that, as λ is an injective function, $\psi_{\mathbf{G}}$ is a bijection and, given a boundary vertex v of \mathbf{G} , we call $\psi_{\mathbf{G}}(v)$ the *index* of v . We also define the *frontier graph* of \mathbf{G} as

$$H_{\mathbf{G}} = (\{1, \dots, t\}, \{\{\psi_{\mathbf{G}}(x), \psi_{\mathbf{G}}(y)\} \mid \{x, y\} \in E(G[X])\}).$$

Two t -boundaried graphs \mathbf{G}_1 and \mathbf{G}_2 are *compatible* if $H_{\mathbf{G}_1} = H_{\mathbf{G}_2}$ (not just isomorphic).

Let $\mathbf{G}_1 = (G_1, X_1, \lambda_1)$ and $\mathbf{G}_2 = (G_2, X_2, \lambda_2)$ be two t -boundaried graphs. We define the *gluing operation* \oplus such that $(G_1, X_1, \lambda_1) \oplus (G_2, X_2, \lambda_2)$ is the graph G obtained by taking the disjoint union of G_1 and G_2 and then, for each $i \in [t]$, identifying the vertex $\psi_{\mathbf{G}_1}^{-1}(i)$ and the vertex $\psi_{\mathbf{G}_2}^{-1}(i)$ (i.e., we identify boundary vertices of the same index). Keep in mind that $\mathbf{G}_1 \oplus \mathbf{G}_2$ is a graph and not a boundaried graph. Moreover, the operation \oplus requires both boundaried graphs to have boundaries of the same size.

Finite Integer Index. We call every partial function \mathbf{p} mapping graphs to non-negative integers a *graph parameter*. We denote by $\text{dom}(\mathbf{p})$ (called the *domain* of \mathbf{p}) the set containing every graph G for which $\mathbf{p}(G)$ is defined.

Let \mathbf{p} be a graph parameter and $t \in \mathbb{N}$. Let $\mathbf{G}_i = (G_i, X_i, \lambda_i), i \in [2]$ be boundaried graphs of boundary size at most t . We say that $\mathbf{G}_1 \equiv_{\mathbf{p}, t} \mathbf{G}_2$ if \mathbf{G}_1 and \mathbf{G}_2 are both t' -boundaried graphs for some $t' \in \{0, \dots, t\}$, they are compatible, and

$$\exists c_{\mathbf{G}_1, \mathbf{G}_2} \in \mathbb{Z} \quad \forall \mathbf{F} \in \mathcal{B}^{(t')}, \quad \mathbf{p}(\mathbf{G}_1 \oplus \mathbf{F}) = \mathbf{p}(\mathbf{G}_2 \oplus \mathbf{F}) + c_{\mathbf{G}_1, \mathbf{G}_2} \quad (1)$$

If in the above definition, for some $i \in [2]$, the graph $\mathbf{G}_i \oplus \mathbf{F} \notin \text{dom}(\mathbf{p})$, we assume that $\mathbf{p}(\mathbf{G}_i \oplus \mathbf{F}) = \infty$. It is easy to observe that $\equiv_{\mathbf{p}, t}$ is an equivalence relation on the set $\mathcal{B}^{(\leq t)}$. Moreover, one of the equivalence classes of this relation is the set, denoted $\text{null}(\mathbf{p})$, of all boundaried graphs whose underlying graph does not belong to $\text{dom}(\mathbf{p})$.

We say that \mathbf{p} has *Finite Integer Index* (FII) if $\equiv_{\mathbf{p},t}$ has a finite number of equivalence classes for every t , i.e., the number of its equivalence classes of $\equiv_{\mathbf{p},t}$, depends on \mathbf{p} and t only.

Given that \mathbf{p} has FII, we define $\text{card}_{\mathbf{p}}(t)$ as the number of equivalence classes of $\equiv_{\mathbf{p},t}$, for each $t \in \mathbb{N}$. We define the function $\text{transp}_{\mathbf{p}}$ that receives as inputs pairs $(\mathbf{G}_1, \mathbf{G}_2)$ where $\mathbf{G}_1 \equiv_{\mathbf{p},t} \mathbf{G}_2$ and $\mathbf{G}_i \notin \text{null}(\mathbf{p}), i \in [2]$ and outputs the number $c_{\mathbf{G}_1, \mathbf{G}_2}$ as in (1). Notice that demanding that $\mathbf{G}_i \notin \text{null}(\mathbf{p}), i \in [2]$ guaranties that $\text{transp}_{\mathbf{p}}(\mathbf{G}_1, \mathbf{G}_2)$ is an integer.

Notice that the above definition slightly deviates from the standard one, given in [8], as we additionally demand that the two boundary graphs \mathbf{G}_1 and \mathbf{G}_2 are compatible. We adopt this definition of FII as it suits better to the presentation our proofs. Moreover, it is not hard to prove that the two definitions (ours and the one in [8]) are equivalent. In fact, $\equiv_{\mathbf{p},t}$ (because of the compatibility demand) is a refinement of the equivalence relation in [8]. The notion of FII first appeared in the works of [10,18] and is similar to the notion of *finite state* [1,11,16]. Results and examples of parameters having the FII property can be found in [8,30,31,38].

2.2 Graph decompositions

Tree-decompositions. Let G be a graph. A *tree-decomposition* of G is a pair $D = (T, \chi)$, where T is a tree and $\chi : V(T) \rightarrow 2^{V(G)}$ such that:

1. $\bigcup_{q \in V(T)} \chi(q) = V(G)$,
2. for every edge $\{u, v\} \in E$, there is a $q \in V(T)$ such that $\{u, v\} \subseteq \chi(q)$, and
3. for each $v \in V(G)$ the set $\{t \mid v \in X_t\}$ induces connected subgraph of T .

We call the vertices of T *nodes* of D and the images of χ *bags* of D . The *width* of a tree-decomposition $D = (T, \chi)$ is $\max\{|\chi(q)| \mid q \in V(T)\} - 1$. The treewidth of a G is the minimum width over all tree-decompositions of G .

Protrusion decompositions. We denote by $\partial_G(S)$ the vertices of S that have neighbours outside S and by $N_G(S)$ the neighbours of vertices in S that do not belong to S . Let G be a graph and let $R \subseteq V(G)$. We say that R is a β -*protrusion* of G if $\max\{|\partial_G(R)|, \text{tw}(G[R])\} \leq \beta$. An (α, β) -*protrusion decomposition* of a graph G (see Figure 2) is a partition $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ of $V(G)$ such that

1. $\max\{\ell, |X_0|\} \leq \alpha$,
2. for every $i \in [\ell]$, the set $R_i = N_G[X_i]$ is a β -protrusion of G and
3. for every $i \in \{1, \dots, \ell\}$, $N_G(X_i) \subseteq X_0$.

Intuitively, an (α, β) -protrusion decomposition of a graph can be seen as a partition into at most $\alpha + 1$ sets where one of them, the central one, has size at most α and each of the rest has treewidth at most β and has at most β neighbors, all contained in the central set.

Protrusion decompositions have been introduced in [8] in the context of kernelization algorithms (see also [31,32]).

2.3 The main theorem

We say that a parameter \mathbf{p} is *protrusion decomposable*, if there exists some $c > 0$ such that

$$\forall G \in \text{dom}(\mathbf{p}), G \text{ has a } (c \cdot \mathbf{p}(G), c)\text{-protrusion decomposition.} \quad (2)$$

If \mathbf{p} is protrusion decomposable, we denote by $\text{dec}(\mathbf{p})$ the minimum c for which (2) is true and we call it *protrusion decomposability constant* of \mathbf{p} . We use notation $H \leq_m G$ to denote that H is a minor of G and we say that \mathbf{p} is *minor-closed* if $H \leq_m G \Rightarrow \mathbf{p}(H) \leq \mathbf{p}(G)$. The main result of this paper is a consequence of the following.

Theorem 2. *Let \mathbf{p} be a graph parameter that has Fll, is protrusion decomposable, and is minor-closed. There is a constant $c_{\mathbf{p}}$ such that for every graph $G \in \text{dom}(\mathbf{p})$, there exists a graph G' such that*

1. $G' \leq_m G$,
2. $\mathbf{p}(G') = \mathbf{p}(G)$, and
3. $|G'| \leq c_{\mathbf{p}} \cdot \mathbf{p}(G)$.

The constant $c_{\mathbf{p}}$ emerges from the proof of correctness of the above theorem, that is presented in Section 3 and in Section 4. The precise value of $c_{\mathbf{p}}$ is given in Subsection 4.1. We postpone the proof of Theorem 2, to the next sections. All results in the rest of this section are assuming this result.

2.4 Proof of Theorem 1

We define the graph class $\mathcal{G}_{\mathbf{p}}^k$ containing every graph G where $\mathbf{p}(G) \leq k$. We say that \mathbf{p} is *c-normal* for some operation, that can be *edge deletion*, *vertex deletion* or *edge contraction*, if for every graph $G \in \text{dom}(\mathbf{p})$ the application of the corresponding operation does not decrease the value of \mathbf{p} more than c . Furthermore, we say that \mathbf{p} is *normal* if there exists a constant c such that \mathbf{p} is *c-normal* for some of the above operations.

Theorem 2 implies that when \mathbf{p} is normal, there is a linear, in k , bound on the size of the graphs in $\text{obs}(\mathcal{G}_{\mathbf{p}}^k)$.

Lemma 1. *Let \mathbf{p} be a graph parameter that has Fll, is normal, protrusion-decomposable, and minor-closed. Then, for every $k \in \mathbb{N}$, all graphs in $\text{obs}(\mathcal{G}_{\mathbf{p}}^k)$ have at most $O(k)$ vertices.*

Proof. Let $G \in \text{obs}(\mathcal{G}_{\mathbf{p}}^k)$ and suppose towards a contradiction that $|G| > c_{\mathbf{p}} \cdot \mathbf{p}(G)$. From Theorem 2 there exists a graph G' such that $\mathbf{p}(G') = \mathbf{p}(G) > k$, $G' \leq_m G$, and $|G'| \leq c_{\mathbf{p}} \cdot \mathbf{p}(G)$, which implies that $|G'| < |G|$, a contradiction to the fact that $G \in \text{obs}(\mathcal{G}_{\mathbf{p}}^k)$.

We assume that \mathbf{p} is *c-normal* for vertex deletions (the same argument holds for edge deletions or edge contractions). Let $u \in V(G)$. It holds that $\mathbf{p}(G) - c \leq \mathbf{p}(G \setminus u)$ and, as G is a minor minimal graph with \mathbf{p} greater than k and $G \setminus u \leq_m G$, $\mathbf{p}(G \setminus u) \leq k$. This implies that $\mathbf{p}(G) \leq k + c$. Hence $|G| \leq c_{\mathbf{p}} \cdot (k + c)$. \square

There are no relevant parameters that are protrusion decomposable when defined in general graphs. However, there are many parameters that when we restrict their domain in (topological) minor free graph classes, then they become protrusion decomposable. See [8, 31, 38, 39] for results on general families of parameters and graph classes where this holds. In the rest of this subsection we give such a general framework based on the notion of treewidth-modulability.

We say that a graph parameter \mathbf{p} is *treewidth-modulable* if there are constants c_1, c_2 such that for every graph G and every non-negative integer k , $\mathbf{p}(G) \leq k$ implies that there is some $S \subseteq V(G)$ such that $|S| \leq c_1 \cdot k$ and $\text{tw}(G \setminus S) \leq c_2$. The following is proved in [38].

Proposition 1. *Let \mathbf{p} be a graph parameter and let F be a graph. If \mathbf{p} is treewidth-modulable and every graph in $\text{dom}(\mathbf{p})$ excludes F as topological minor, then \mathbf{p} is protrusion-decomposable.*

We now proceed with the proof of Theorem 1.

Proof of Theorem 1. Given a graph G we define the parameter $\mathbf{p}^{\mathcal{H},F}$ as a partial function mapping graphs in \mathcal{T}_F to numbers as follows:

$$\mathbf{p}^{\mathcal{H},F}(G) = \min\{k \in \mathbb{N} \mid \exists S \subseteq V(G) \text{ such that } |S| \leq k \text{ and } G \setminus S \in \text{exc}(\mathcal{H})\}.$$

Notice that the parameter $\mathbf{p}^{\mathcal{H},F}$ depends on both \mathcal{H} and F and that $\text{dom}(\mathbf{p}^{\mathcal{H},F}) = \mathcal{T}_F$. Observe also that $\mathbf{p}^{\mathcal{H},F}$ has been defined so to ensure that

$$\mathcal{T}_F \cap \mathcal{A}_k(\mathcal{H}) = \mathcal{G}_{\mathbf{p}^{\mathcal{H},F}}^k.$$

Because of Lemma 1, it suffices to prove that $\mathbf{p}^{\mathcal{H},F}$ has FII, is normal, protrusion-decomposable, and minor-closed.

The fact that $\mathbf{p}^{\mathcal{H},F}$ has FII has been proved in [8, Lemma 8.4]. We now claim that $\mathbf{p}^{\mathcal{H},F}$ is 1-normal for vertex deletions. Indeed, suppose in contrary that for some graph G and some $x \in V(G)$ $\mathbf{p}^{\mathcal{H},F}(G) \geq k$ while $\mathbf{p}^{\mathcal{H},F}(G \setminus x) \leq k - 2$. The last relation implies that there exists an $S \subseteq V(G \setminus x)$ such that $|S| \leq k - 2$ and $(G \setminus x) \setminus S \in \text{exc}(\mathcal{H})$. This implies that $G \setminus (S \cup \{x\}) \in \text{exc}(\mathcal{H})$ and as $|S \cup \{x\}| \leq k - 1$ we have that $\mathbf{p}^{\mathcal{H},F}(G) < k$, a contradiction.

Next we prove that $\mathbf{p}^{\mathcal{H},F}$ is protrusion-decomposable. As \mathcal{H} contains some planar graph H , we know that all graphs in $\text{exc}(\mathcal{H})$ have treewidth at most c_H , for some constant c_H that depends only on the choice \mathcal{H} [13, 15, 19, 51]. Therefore all graph in $\text{exc}(\mathcal{H})$ have treewidth at most c_H . This implies that $\mathbf{p}^{\mathcal{H},F}$ is treewidth-modulable, therefore, from Proposition 1 $\mathbf{p}^{\mathcal{H},F}$ is protrusion-decomposable.

Finally, it is easy to verify that $\mathbf{p}^{\mathcal{H},F}$ is minor-closed. This completes the proof. \square

2.5 Outline of the proof of Theorem 2

We now outline the main ideas of the proof of Theorem 2 with references to the concepts, definitions, and proofs that are given in Section 3 and in Section 4.

Rooted tree decompositions. For our proofs it is convenient to consider tree decompositions of boundaried graphs (see Subsection 3.4). A rooted tree decomposition of a boundaried graph $\mathbf{G} = (G, X, \lambda)$ is a triple (T, χ, r) where (T, χ) is a tree decomposition, the tree T is rooted on r and $\chi(r) = X$. For each $x \in V(T)$, we denote by $\mathbf{G}_x = (G_x, \chi(x), \lambda|_{V(G_x)})$ the subgraph of \mathbf{G} induced by the vertices in the bags of (T, χ) that are descendants of x , boundaried on $\chi(x)$. We also consider rooted tree decompositions, called *binary tree decompositions*, where T is a binary tree and for each two vertices i, j of T , where j is a child of i , $|\chi(i)| = |\chi(j)|$ implies that $|\mathbf{G}_j| < |\mathbf{G}_i|$ (see Subsection 3.4). Next, we define a special type of rooted tree decomposition, namely, the (α, β) -rooted tree-decomposition, whose root node contains at most α vertices, has at most α children and each of the rooted tree decompositions corresponding to its children is a binary tree decomposition of width at most β , where its root consists of the boundary of the vertices of its nodes and this boundary is different for each child of the root (see Subsection 3.4). (α, β) -rooted tree-decompositions are much more convenient to work with and we prove that if \mathbf{p} is protrusion decomposable, then every graph in $\text{dom}(\mathbf{p})$ admits an $(c \cdot \mathbf{p}(G), 2 \cdot c)$ -rooted tree-decomposition (see Subsection 3.4), where $c = \text{dec}(\mathbf{p})$ is the protrusion-decomposability constant of \mathbf{p} . Given such a decomposition (T, χ, r) of \mathbf{G} we denote by $r^{(1)}, \dots, r^{(s)}$ the children of r , for $i \in \{1, \dots, s\}$.

Rich protrusions. The second concept that we need is the one of *rich protrusion*. Intuitively a rich protrusion can be seen as a β -protrusion X whose size is big enough (as a function of the size of its boundary) so to enable the replacement in G of some part of X by a smaller one so that the value of \mathbf{p} on the resulting graph is the same as on G (see Section 4.5 for the formal definitions).

Graphs without rich protrusions. Our next step is to prove that if G does not have rich protrusions, then $|G| = O(\mathbf{p}(G))$. For this, we already know that G has a $(c \cdot \mathbf{p}(G), 2c)$ -rooted tree-decomposition and, based on this, we prove that $\sum_{i \in [s]} \mathbf{p}(G_{r^{(i)}}) \leq \mathbf{p}(G) + O(1)$ (see Subsection 4.5). Notice that as G does not contain any rich protrusion and as each $\mathbf{G}_{r^{(i)}}$ can be seen as such a protrusion, we have that $|G_{r^{(i)}}| = O(\mathbf{p}(G_{r^{(i)}}))$. This, together with the previous inequality and the fact that $|G| = \sum_{i \in [s]} |G_{r^{(i)}}| + O(\mathbf{p}(G))$ imply that $|G| = O(\mathbf{p}(G))$ (see Lemma 19). We conclude that if G does not have rich protrusions, its size is already linear in the value of the parameter \mathbf{p} on G .

What remains now is to deal with the case where G contains a rich protrusion R . Notice that R is the vertex set of some boundaried graph $\mathbf{H} = (H, Y, \lambda)$ that has a binary rooted tree decomposition $D = (T, \chi, r)$ of width at most $2c$ (i.e., $R = V(H)$). We stress here that one might be tempted to apply directly the approach of [8] and replace in G the boundaried graph \mathbf{H} by a smaller equivalent protrusion \mathbf{H}' . This replacement creates a graph G' where $\mathbf{p}(G) = \mathbf{p}(G') + \text{transp}_{\mathbf{p}}(\mathbf{H}, \mathbf{H}')$. However, we do not have any guarantee that G' will be a minor of G and moreover that $\text{transp}_{\mathbf{p}}(\mathbf{H}, \mathbf{H}') = 0$ (in order to have that $\mathbf{p}(G) = \mathbf{p}(G')$). To enforce these additional properties, we follow an alternative approach that intuitively consists of “locally compressing” the protrusion rather than “replacing” it with something smaller (see also [43] for a similar approach). This is done in Lemma 23.

Pair collections. We now make a short break in order to give some additional definitions. Let T be the tree of the decomposition D that is rooted on r . We say that a pair $(a, b) \in V(T) \times V(T)$ is a *vertical pair* if b is a descendant of a in T , distinct from a (see Subsection 3.2 for the formal definitions). Two vertical pairs (a, b) and (a', b') of (T, r) are *non-interfering* if either a and a' are non-comparable in T (i.e., none is a descendant of the other), or b' is a descendant of a , or a' is a descendant of b . The *inner territory* of a pair (a, b) consists of the descendants of a that are not proper descendants of b . A *pair collection* of (T, r) is a set \mathcal{C} of pairwise non-interfering vertical pairs of T . (See Figure 1 for a visualization of these concepts.) The *capacity* of a pair (a, b) is the size of its inner territory. The *capacity* of \mathcal{C} is the minimum capacity of a vertical pair in \mathcal{C} . Finally, the *potential* of \mathcal{C} is $\sum_{(a,b) \in \mathcal{C}} \text{transp}_{\mathbf{p}}(\mathbf{H}_a, \mathbf{H}_b)$.

Transition pairs. We now go back to the decomposition $D = (T, \chi, r)$ of the protrusion \mathbf{H} . We say that a vertical pair (a, b) of T is a *transition pair* if \mathbf{H}_a and \mathbf{H}_b are compatible, $\mathbf{H}_a \equiv_{\mathbf{p}, t} \mathbf{H}_b$, and \mathbf{H}_b is a “rooted”-minor of \mathbf{H}_a in the sense that \mathbf{H}_b can be obtained from \mathbf{H}_a by operations that preserve the boundaried vertices and the index correspondence between them (see Subsection 4.3). We also say that a transition pair (a, b) is a *null-transition pair* if $\text{transp}_{\mathbf{p}}(\mathbf{H}_a, \mathbf{H}_b) = 0$ (see Subsection 4.6). Given a transition pair (a, b) of T , we define the *(a, b) -compression of \mathbf{H}* as the boundaried graph obtained from \mathbf{H} if we replace \mathbf{H}_a by \mathbf{H}_b (see Subsection 4.2).

As an important step of our proof we show that if a vertical pair (a, b) has “big enough” capacity, then we can revise the part of D that corresponds to the inner territory of (a, b) so that it now contains some transition pair (a', b') (see Subsection 4.3). The proof of this result makes extensive use of some suitable variants of the concepts of lean and linked tree decompositions introduced in [6, 57].

We next prove a combinatorial lemma asserting that if $|T| > c \cdot \mathbf{p}(G)$ for some constant $c \geq 1$, then T contains a pair collection of size bigger than $\mathbf{p}(G)$ and capacity “big enough” so as to guarantee the existence of a pair collection of more than $\mathbf{p}(G)$ transition pairs (see 4.6).

Null-transitions. The next step is to prove that \mathbf{H} has a null-transition pair (we postpone the argumentation on this to the next paragraph). We can prove that if (a, b) is a such a pair and \mathbf{H}' is the (a, b) -compression of \mathbf{H} , it holds that, for every boundaried graph \mathbf{F} that is compatible to \mathbf{H} , $\mathbf{p}(\mathbf{F} \oplus \mathbf{H}) = \mathbf{p}(\mathbf{F} \oplus \mathbf{H}')$. Given that there is some \mathbf{F} such that $\mathbf{F} \oplus \mathbf{H} = G$, we

obtain that $\mathbf{p}(G) = \mathbf{p}(G')$ where $G' = \mathbf{F} \oplus \mathbf{H}'$. As \mathbf{H}_b is a rooted minor of \mathbf{H}_a it also follows that G' is a minor of G and as $a \neq b$, $|G'| < |G|$. This means that, as long as we can find a null-transition pair, we can always “compress” G to a minor G' of it where $\mathbf{p}(G) = \mathbf{p}(G')$.

Existence of null-transition pairs. We conclude this exposition by explaining why a null-transition pair exists in \mathbf{H} . Since \mathbf{H} is a rich protrusion, the tree T in its tree decomposition is big enough to guarantee the existence of a pair collection \mathcal{C} of more than $\mathbf{p}(G)$ transition pairs. For each such pair the value $\text{transp}_{\mathbf{p}}(\mathbf{H}_a, \mathbf{H}_b)$ expresses how much the value of the parameter will change after a replacement. Using the fact that \mathbf{p} is a minor closed parameter, we prove that this value is never negative (see Subsection 4.6). Therefore the potential of \mathcal{C} is also non-negative (this is the only point in the proof that we use the minor-closedness of \mathbf{p}).

We next prove that if we apply all (a, b) -compressions in \mathcal{C} (in any order) the *total reduction* of the parameter in the compressed graph will be at least the potential of \mathcal{C} (see 4.6 and the proof of Lemma 23). This proof is strongly based on the fact that the pairs in \mathcal{C} are mutually non-interfering. As this total reduction cannot be bigger than $\mathbf{p}(G)$ we conclude that at least one of the pair in \mathcal{C} should be a null-transition pair.

3 Basic concepts and results

In this section we introduce all combinatorial concepts and supporting results that are required for the proof of Theorem 2.

3.1 Sets and functions

We denote by \mathbb{Z} , \mathbb{N} , and \mathbb{N}^+ the set of all integers, non-negative integers, and positive integers respectively.

For two positive integers $n, m \in \mathbb{N}$, such that $n \leq m$, we write $[n, m] = \{n, n + 1, \dots, m\}$. We also write $[n] = \{1, \dots, n\}$.

Given a set S , we denote by 2^S the set of all subsets of S and by $\binom{S}{2}$ the set of all subsets of S with cardinality 2. Given a function $f : A \rightarrow B$ and a set S , we define $f|_S = \{(x, f(x)) \mid x \in S \cap A\}$ and $f \setminus S = \{(x, f(x)) \mid x \in A \setminus S\}$. Moreover, we always assume that a function $\sigma : A \rightarrow B$ is also defined on 2^A so that for $S \subseteq A$, $\sigma(S) = \{\sigma(x) \mid x \in S\}$.

3.2 Graphs

Given an $S \subseteq V(G)$, the *closed neighbourhood* of S in G is $N_G[S] = S \cup N_G(S)$.

A *subgraph* $H = (V_H, E_H)$ of a graph $G = (V, E)$ is a graph such that $V_H \subseteq V(G)$ and $E_H \subseteq E(G) \cap \binom{V(H)}{2}$. If H is a subgraph of G and $V(H) = V(G)$, then we say that H is a *spanning* subgraph of G . If $S \subseteq V(G)$, the subgraph of G *induced by* S , denoted $G[S]$, is the graph $(S, E(G) \cap \binom{S}{2})$. We also define $G \setminus S$ to be the subgraph of G induced by $V(G) \setminus S$. If $S \subseteq E(G)$, we denote by $G \setminus S$ the graph $(V(G), E(G) \setminus S)$. Given a $v \in V(G)$, we call $|N_G(\{v\})|$ the *degree* of v in G .

Given two graphs H and G we say that H is a *minor* of G , denoted by $H \leq_m G$, if H can be obtained by contracting some of the edges of some subgraph of G . A more formal (and general) definition of the minor relation will be given in Subsection 4.3.

3.2.1 Trees

We say that a subset S of $V(G)$ is *connected* if $G[S]$ is connected. A *tree* is a connected graph T , where T is the only connected spanning subgraph of T . The *leaves* of T are the

vertices of T that have degree at most 1 and are denoted as $\text{Leaf}(T)$. Given a graph G and $a, b \in V(G)$, an (a, b) -path in G is every connected subgraph of G that is a tree of maximum degree 2 where a and b are its vertices of degree at most 1. A *path* in G is any (a, b) -path P where $a, b \in V(G)$. We call a and b *endpoints* of P and if $E(P) \neq \emptyset$, then we say that P is a *non-trivial path*. The vertices of a path P that are not leaves are called *internal vertices* of P . Given a tree T and two distinct vertices a, b of $V(T)$ we denote by aTb the unique (a, b) -path in T . Given a graph G and two sets $X, Y \subseteq V(G)$, a *collection of t internally vertex-disjoint paths between X and Y* are t paths P_1, \dots, P_t where, for $h \in [t]$, one endpoint of P_i is in X and the other is in Y and for every $i, j \in [t], i \neq j$ there is no internal vertex of P_i that is a vertex of P_j .

3.2.2 Rooted trees

A *rooted tree* is a pair (T, r) where T is a tree and $r \in V(T)$. We define $\text{Leaf}(T, r)$ as the set of all leaves of T that are different than r . Given two vertices a, b of T , we write $a \leq_{T, r} b$ to denote that $a \in V(rTb)$ and, in this case, we say that b is a *descendant* of a in (T, r) . We write $a <_{T, r} b$ to denote that $a \neq b$ and $a \leq_{T, r} b$. We also write $a \not\leq_{T, r} b$ to denote that neither $a \leq_{T, r} b$ nor $b \leq_{T, r} a$ is true and, in this case, we say that a and b are *non-comparable* in (T, r) . Given some $q \in V(T)$, we denote the *set of descendants* of q in (T, r) as $\text{desc}_{T, r}(q)$. The *children* of a vertex $q \in T$, in (T, r) are the vertices in $\text{desc}_{T, r}(q)$ that are adjacent to q and are denoted as $\text{children}_{T, r}(q)$. A rooted tree (T, r) is *binary* if every vertex of T has at most two children. Let $v \in V(T)$. The *depth* of v in (T, r) is $|V(rTv)|$.

3.2.3 Pair collectons

Consider a rooted tree (T, r) . We say that a pair (a, b) of $V(T) \times V(T)$ is a *vertical pair* of (T, r) if $a <_{T, r} b$ (notice that in a vertical pair (a, b) , a and b should be different vertices). We call a (resp. b) upper (resp. lower) vertex of (a, b) . If $\{a, b\}$ is an edge of T we call (a, b) an *edge-pair*. The *inner part* of a vertical pair (a, b) is $\text{inner}_{T, r}(a, b) = \{b\} \cup (\text{desc}_{T, r}(a)) \setminus \text{desc}_{T, r}(b)$. Notice that in the tree $T[\text{inner}_{T, r}(a, b)]$ the vertex b is a leaf. The *outer part* of (a, b) is defined as $\text{outer}_{T, r}(a, b) = \{a, b\} \cup (V(T) \setminus \text{inner}_{T, r}(a, b))$. Notice that $\text{outer}_{T, r}(a, b) \cap \text{inner}_{T, r}(a, b) = \{a, b\}$ (see Figure 1). The *capacity* of (a, b) is defined as $\text{capacity}_{T, r}(a, b) = |\text{inner}_{T, r}(a, b)|$.

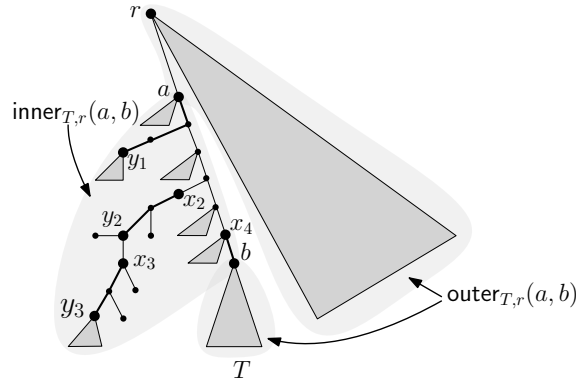


Figure 1: The vertical pairs (x_2, y_2) , (x_3, y_3) and (x_4, b) are pairwise non-interfering. Notice that a and y_1 are not (a, b) -aligned.

Two vertical pairs (a, b) and (a', b') of (T, r) are *non-interfering* if $a \not\leq_{T, r} a'$ or $b <_{T, r} a'$, or $b' <_{T, r} a$ (see Figure 1). A *pair collection* of (T, r) is a set \mathcal{C} of pairwise non-interfering

vertical pairs of T . The *minimum* (resp. *maximum*) *capacity* of \mathcal{C} is the minimum (resp. maximum) capacity of a vertical pair in \mathcal{C} .

If (a, b) is a vertical pair of (T, r) , the (a, b) -*compression* of (T, r) is the rooted tree (T', r) where T' is obtained from T if we remove all vertices in $\text{inner}_{T,r}(a, b) \setminus \{a, b\}$, identify a and b , and call this vertex a again. We denote this new rooted tree by $(T, r) \setminus (a, b)$. Notice that if (T, r) is binary, then $(T, r) \setminus (a, b)$ is also a binary tree.

Let (a, b) be a vertical pair of (T, r) and $x, y \in \text{inner}_{T,r}(a, b)$ (x and y are not necessarily distinct). We say that x and y are (a, b) -*aligned* if either $V(xTy) \subseteq V(aTb)$ or $V(xTy) \cap V(aTb) \subseteq \{x, y\}$ (see Figure 1). Notice that x and y are not (a, b) -aligned if the path joining x and y contains edges from the path joining a and b and edges outside this path. Certainly, a and b are (a, b) -aligned.

Lemma 2. *Let (T, r) be a rooted binary tree and let (a, b) be a vertical pair of (T, r) . For every $d \geq 2$, if (a, b) has capacity $(d - 2) \cdot 2^{d-2} + 1$, then there are $x, y \in \text{inner}_{T,r}(a, b)$ such that x and y are (a, b) -aligned and $|xTy| \geq d$.*

Proof. We set $Y = T[\text{inner}_{T,r}(a, b)]$. Observe that (Y, a) is a rooted graph where $b \in \text{Leaf}(Y, a)$. If $|aYb| \geq d$ then a and b are the required vertices of $\text{inner}_{T,r}(a, b)$. Suppose then that $|aYb| \leq d - 1$. Let Y_1, \dots, Y_z be the connected components of $Y \setminus E(aYb)$. Each Y_i is a binary tree whose root is some vertex of aYb that has only one child. Clearly, as $b \in \text{Leaf}(Y, a)$ and $|aYb| \leq d - 1$, it holds that $z \leq d - 2$. Also the vertices of Y_1, \dots, Y_z form a partition of $V(Y)$. This means that one, say Y_i , of Y_1, \dots, Y_z has at least $1 + 2^{d-2}$ vertices. As Y_i is binary and its root has only one child, it follows that Y_i contains a path P on d vertices. Let x and y be its endpoints. We obtain that $|xYy| \geq d$. The lemma follows as the endpoints x and y are (a, b) -aligned. \square

3.3 Graph decompositions

Lean tree-decompositions. In this section we provide some variants of tree decompositions.

We say that (T, χ) is *lean* if for every $t \in \mathbb{N}$, every pair $u_1, u_2 \in V(T)$, and every $Z_i \subseteq \chi(u_i), i \in [2]$, where $|Z_1| = |Z_2|$, either there is an $e = \{w_1, w_2\} \in E(iTj)$ such that $\chi(w_1) \cap \chi(w_2) < t$ or there is a collection of t internally vertex-disjoint paths in G between Z_1 and Z_2 .

Proposition 2 ([6]). *Every graph G has a lean tree-decomposition of width $\text{tw}(G)$.*

Small tree-decompositions We say that a tree-decomposition (T, χ) is *small* if $\forall \{i, j\} \in E(T)$, $\chi(i) \setminus \chi(j) \neq \emptyset$ and $\chi(j) \setminus \chi(i) \neq \emptyset$. For the proof of the following, we copy [29, Lemma 11.9].

Proposition 3. *Let G be a graph and $D = (T, \chi)$ a (lean) tree-decomposition of G of width at most $\text{tw}(G)$. Then there exists a small (and lean) tree-decomposition $D' = (T', \chi')$ of G of width at most $\text{tw}(G)$, where $|T'| \leq |G|$.*

3.3.1 Protrusion decompositions

Let G be a graph and let $R \subseteq V(G)$. We say that R is *tight in G* if $\partial_G(R) = N_G(R \setminus \partial_G(R))$. If R is a β -protrusion of G and R is tight, then we say that R is a *tight β -protrusion* of G . Notice that if $X = R \setminus \partial_G(R)$, then R is tight iff $N_G(X) = \partial_G(N_G[X])$.

Let $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ be an (α, β) -protrusion decomposition of a graph G (for the definition, see Page 5). We call the sets $R_i, i \in [\ell]$, the *protrusions* of \mathcal{P} and the set X_0 the

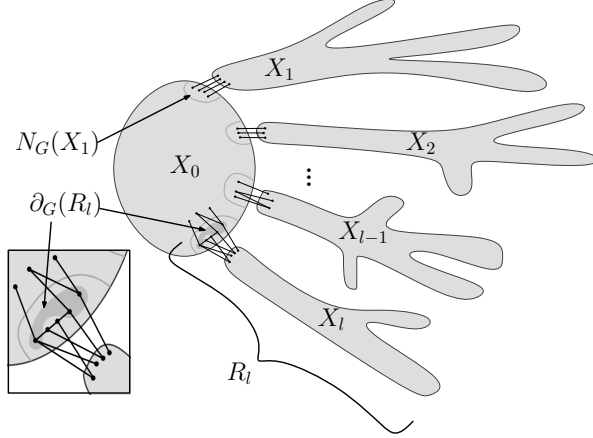


Figure 2: Notice that for every $i \in [\ell]$ the set $\partial_G(R_i)$ contains the vertices of R_i that are incident to vertices of X_0 and that $\partial_G(R_i) \subseteq N_G(X_i)$.

core of \mathcal{P} (see Figure 2). An (α, β) -protrusion decomposition $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ is *tight* if all its β -protrusions are tight.

Lemma 3. *If a graph G has an (α, β) -protrusion decomposition $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ then it also has a tight (α, β) -protrusion decomposition.*

Proof. Let $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ be an (α, β) -protrusion decomposition of G . Recall that $R_i = N_G[X_i], i \in [\ell]$. For every $i \in [\ell]$, we set $Z_i = N_G(X_i) \setminus \partial_G(R_i)$ and observe that the sets Z_1, \dots, Z_ℓ are pairwise disjoint. Indeed this follows by the fact that, for each $i \in [\ell]$, each vertex in Z_i is incident only with edges in $G[R_i]$. For the same reason, none of these vertices can be a vertex of some $\partial_G(R_j), j \in [\ell]$. We conclude that $\bigcup_{i \in [\ell]} (Z_i \cap \partial_G(R_i)) = \emptyset$. We define $X'_i = X_i \cup Z_i, i \in [\ell]$ and observe that

$$N_G[X'_i] = N_G[X_i] \quad \text{and} \quad N_G(X'_i) = \partial_G(R_i). \quad (3)$$

We also define $X'_0 = X_0 \setminus \bigcup_{i \in [\ell]} Z_i$. We claim that $\mathcal{P}' = \{X'_0, X'_1, \dots, X'_\ell\}$ is also an (α, β) -protrusion decomposition of G . As the sets in the collection $\{Z_1, \dots, Z_\ell, X_1, \dots, X_\ell\}$ are mutually disjoint we obtain that \mathcal{P}' is a partition of $V(G)$. As $X'_0 \subseteq X_0$, Condition 1 holds. (3) and the fact that $R_i = N_G(X_i)$ is a β -protrusion decomposition of G , imply Condition 2. To prove Condition 3, we recall first that $\partial_G(R_i) \subseteq N_G(X_i) \subseteq X_0$. Combining this with the fact that $\bigcup_{i \in [\ell]} (Z_i \cap \partial_G(R_i)) = \emptyset$ and the definition of X'_0 , we obtain that $\partial_G(R_i) \subseteq X'_0$. From (3), we deduce that $N_G(X'_i) \subseteq X'_0$ and the claim holds. From (3) we get the additional property that $\forall i \in [\ell], N_G(X'_i) = \partial_G(N_G[X'_i])$, therefore \mathcal{P}' is tight. \square

3.4 Tree-decompositions of boundaried graphs

We now extend several of the definitions of the previous sections to boundaried graphs.

3.4.1 Treewidth of boundaried graphs

Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph. A *tree-decomposition* of \mathbf{G} is a triple $D = (T, \chi, r)$ where (T, χ) is a tree-decomposition of G and r is a vertex of T such that $\chi(r) = X$. The *width* of a tree-decomposition $D = (T, \chi, r)$ is the width of the tree-decomposition (T, χ) . The

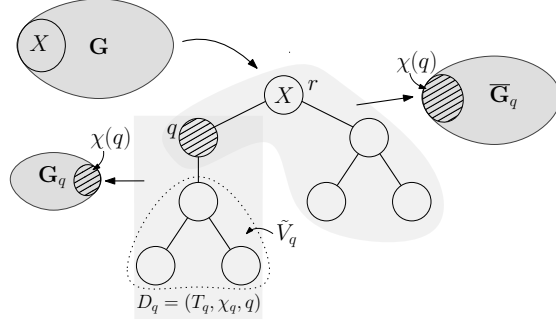


Figure 3: A boundaried graph $\mathbf{G} = (G, X, \lambda)$ and a tree-decomposition $D = (T, \chi, r)$ of it. Notice that \tilde{V}_q contains the vertices of the three bags inside the dotted curve.

treewidth of a boundaried graph \mathbf{G} is the minimum width over all its tree-decompositions and is denoted by $\mathbf{tw}(\mathbf{G})$.

Let $D = (T, \chi, r)$ be a tree-decomposition of a boundaried graph \mathbf{G} . For each $q \in V(T)$, we set $t_q = |\chi(q)|$ and we denote the corresponding frontier graph by $H_q = H_{\mathbf{G}_q}$. Also we set $T_q = T[\text{desc}_{T,r}(q)]$ and we denote by \mathbf{G}_q the t_q -boundaried graph $(G_q, \chi(q), \lambda_q)$ where

$$G_q = G \left[\bigcup_{q' \in V(T_q)} \chi(q') \right] \quad \text{and} \quad \lambda_q = \lambda|_{V(G_q)}.$$

Notice that if $a, b \in V(T)$ and $a \leq_{T,r} b$, then G_b is a subgraph of G_a . We also set $V_q = V(G_q)$, $\tilde{V}_q = V_q \setminus \chi(q)$, $\bar{G}_q = G \setminus \tilde{V}_q$ and we define the t_q -boundaried graph $\bar{\mathbf{G}}_q = (\bar{G}_q, \chi(q), \bar{\lambda}_q)$ where $\bar{\lambda}_q = \lambda|_{V(\bar{G}_q)}$, for $q \in V(T)$. Notice that \mathbf{G}_q and $\bar{\mathbf{G}}_q$ are compatible and that $\mathbf{G}_q \oplus \bar{\mathbf{G}}_q = \mathbf{G}$. Finally, for every $q \in V(T)$, we use the notation $\bar{V}_q = V(\bar{G}_q)$, $\chi_q = \chi|_{V_q}$, and $D_q = (T_q, \chi_q, q)$ and observe that D_q is a tree-decomposition of \mathbf{G}_q (see Figure 3).

We say that $D = (T, \chi, r)$ is a *lean tree-decomposition* if (T, χ) is lean. $D = (T, \chi, r)$ is a *binary tree-decomposition* if (T, r) is a binary tree and for every edge-pair (i, j) , if $|\chi(i)| = |\chi(j)|$ then G_j is a *proper* subgraph of G_i , i.e. $|G_j| < |G_i|$.

Lemma 4. *Let \mathbf{G} be a boundaried graph and let $D = (T, \chi, r)$ be a tree-decomposition of \mathbf{G} of width $\leq t - 1$. Then $|\mathbf{G}| \leq t \cdot |T|$.*

Proof. Notice that a set of $n = |\mathbf{G}|$ vertices is covered by $|T|$ sets, the bags of D , each of size at most t . As each bag of D covers at most t vertices, it follows that $n \leq t \cdot |T|$ as required. \square

Lemma 5. *Let \mathbf{G} be a boundaried graph and let $D = (T, \chi, r)$ be a binary tree-decomposition of \mathbf{G} . Then $|T| \leq 4 \cdot |\mathbf{G}|$.*

Proof. The proof is based on the fact that a binary tree-decomposition $D = (T, \chi, r)$ of \mathbf{G} can be transformed to a decomposition $D' = (T', \chi', r)$ with the following properties:

- A. T' is a binary tree
- B. if some vertex of t has two children t_1 and t_2 , then $\chi'(t) = \chi'(t_1) = \chi'(t_2)$.
- C. if some vertex of t has one child t' , then either $|\chi'(t) \setminus \chi'(t')| = 1$ or $|\chi'(t') \setminus \chi'(t)| = 1$.

A tree-decomposition as above is called *nice* and it is known (see e.g., [41]) that $|T'| \leq 4 \cdot |\mathbf{G}|$. It now remains to provide a way to transform (T, χ, r) to a nice tree-decomposition of \mathbf{G} of the same width where $|T| \leq |T'|$. For this we apply the following transformations as long as this is possible.

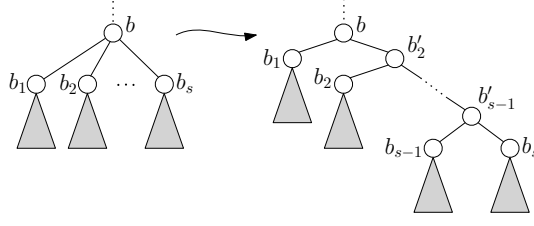


Figure 4: The transformation of Lemma 6.

1. If $t \in V(T)$, t has two children and for one, say t' , of them it holds that $\chi(t') \neq \chi(t)$, then subdivide the edge $\{t, t'\}$ and, if t_{new} is the subdivision vertex, set $\chi(v_{\text{new}}) = \chi(t)$.
2. If for some edge-pair (t, t') of T it holds that $|\chi(t) \cap \chi(t')| < \min\{|\chi(t)|, |\chi(t')|\}$, then subdivide the edge $\{t, t'\}$ and, if t_{new} is the subdivision vertex, set $\chi(v_{\text{new}}) = \chi(t) \cap \chi(t')$.
3. If for some edge $\{t, t'\}$ of T there are at least two vertices, a and a' in $\chi(t) \setminus \chi(t')$, then subdivide the edge $\{t, t'\}$ and, if t_{new} is the subdivision vertex, set $\chi(v_{\text{new}}) = \chi(t') \cup \{a\}$.

Let (T', χ', r) be the resulting tree-decomposition of \mathbf{G} . Clearly T' is binary, therefore property A holds. Property B follows by the fact that Transformation 1 cannot be applied any more and Property C follows because transformations 2 and 3 cannot be applied any more. Therefore (T', χ', r) is a nice tree-decomposition, as required. \square

Lemma 6. *Let \mathbf{G} be a graph and $D = (T, \chi, r)$ a (lean) tree-decomposition of \mathbf{G} of width at most $k \in \mathbb{N}$. Then there exists a (lean) binary tree-decomposition of \mathbf{G} of width at most k that has at most $2 \cdot |\mathbf{G}| - 1$ nodes.*

Proof. Because of Proposition 3, we can assume that (T, χ) is small and that T has at most $|\mathbf{G}|$ nodes. Clearly it holds that

$$\forall \{x, y\} \in E(T), \quad \chi(x) \setminus \chi(y) \neq \emptyset \text{ and } \chi(y) \setminus \chi(x) \neq \emptyset \quad (4)$$

Let B be the set of vertices of T that have more than 2 children in T . For each $b \in B$ we apply the following transformation on D . Let $\{b_1, \dots, b_s\}$ be the children of b in (T, r) . We set $D' := (T', \chi', r)$ where:

- T' is obtained by T after removing the edges $\{b, b_i\}, i \in [2, s]$, adding the new vertices $\{b'_2, \dots, b'_{s-1}\}$, the edge $\{b, b'_2\}$, the edges $\{b'_i, b'_{i+1}\}, i \in [2, s-2]$, the edges $\{b_i, b'_i\}, i \in [2, s-1]$ and the edge $\{b'_{s-1}, b_s\}$ (see Figure 4), and
- $\chi' = \chi \cup \{(b'_i, \chi(b)) \mid i \in [2, s-1]\}$.

We call $D' = (T', \chi', r)$ the tree-decomposition of \mathbf{G} that is obtained after applying the above transformation for every $b \in B$. Notice that T is a binary tree and that the width of D' is equal to the width of D . Also, it is easy to observe that the number of new nodes is upper bounded by the number of leaves of T , therefore the number of nodes of the new decomposition cannot increase more than twice. Moreover, it can be easily verified that if D is lean, then D' is lean as well.

Notice that D satisfied Condition (4) before the application of the above transformation. Moreover for every vertical pair (a, b) where Condition (4) is still true after the transformation, it holds that $|G_b| < |G_a|$. If Condition (4) is not any more true for some vertical pair (x, y) , created after the application of the above transformation, then $(x, y) \in \{(b'_i, b'_{i+1}) \mid i \in$

$[2, s - 2]$ for some $b \in B$ (here we denote $b'_2 := b$). In this case, $\chi(b'_i) = \chi(b)$, $i \in [2, s - 1]$. But then b'_i has another child, that is b_i , where both $\chi(b_i) \setminus \chi(b'_i)$ and $\chi(b'_i) \setminus \chi(b_i)$ are non-empty, because both $\chi(b_i) \setminus \chi(b)$ and $\chi(b) \setminus \chi(b_i)$ were initially non-empty. This implies that if $\chi(b'_i) = \chi(b'_{i+1})$, $V_{b'_{i+1}} \subsetneq V_{b'_i}$ then $|G_{b'_{i+1}}| < |G_{b'_i}|$ as required. Therefore D' is a binary tree-decomposition.

Notice now that the number of new nodes of D' is equal to $\sum_{v \in V_{\geq 3}} (\deg_T(v) - 2)$, where $V_{\geq 3}$ is the number of vertices in T with more than 2 children. It is easy to observe that $\sum_{v \in V_{\geq 3}} (\deg_T(v) - 2) \leq |\text{Leaf}(T, r)| \leq |T| - 1$. Therefore, $|T'| \leq 2 \cdot |T| - 1 \leq 2 \cdot |\mathbf{G}| - 1$. \square

Lemma 7. *Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph where $G[X]$ is a clique and $\text{tw}(G) \leq t - 1$. Then \mathbf{G} has a lean binary tree-decomposition $D = (T, \chi, r)$ with width at most $t - 1$ where $\chi(r) = X$ and $|T| \leq 2 \cdot |\mathbf{G}|$.*

Proof. From Proposition 2, G has lean tree-decomposition (T, χ) of width $\text{tw}(G)$. As $G[X]$ is a clique, all vertices of X will go in the same bag. Let $q \in V(T)$ such that $X \subseteq \chi(q)$ and set $\mathbf{G}' = (G, \chi(q), \lambda)$. Notice that (T, χ, q) is a lean tree-decomposition of \mathbf{G}' of width $\text{tw}(G)$. From Lemma 6, (T, χ, q) can be transformed to a lean binary tree-decomposition $D = (T', \chi', q)$ of width $\text{tw}(G)$, such that T' has at most $2 \cdot |\mathbf{G}'| - 1 = 2 \cdot |G| - 1$ nodes. Recall that $\chi'(q) = \chi(q)$. In the case where $\chi(q) = X$, D is the required lean binary tree decomposition of \mathbf{G} and we are done. In the case where $X \subsetneq \chi(q)$, rename q to q_{old} , add in T a new vertex q_{new} along with the edge $\{q, q_{\text{new}}\}$, set $\chi := \chi \cup (q_{\text{new}}, X)$ and rename q_{new} to q . Notice that, as $X \subsetneq \chi(q)$, after this modification D remains a lean binary tree-decomposition of G of width $\text{tw}(G)$. \square

Lemma 8. *Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph and $D = (T', \chi')$ a tree-decomposition of $G' = G \setminus X$ of width $\text{tw}(G')$. Then there exists a binary tree-decomposition of \mathbf{G} of width at most $\text{tw}(G') + |X|$.*

Proof. Pick any vertex \bar{r} of T' and let $S = \chi'(\bar{r})$. Clearly $D' = (T', \chi', \bar{r})$ is a rooted tree-decomposition of $\mathbf{G}' = (G', S, \lambda \setminus X)$. From Lemma 6, there exists a binary tree-decomposition $D'' = (T'', \chi'', \bar{r})$ of \mathbf{G}' of width $\text{tw}(G')$ and $2 \cdot |\mathbf{G}'| = 2 \cdot (|\mathbf{G}| - |X|)$ nodes. We update χ'' so that for every $q \in V(T'')$, $\chi''(q) := \chi''(q) \cup \{X\}$. Clearly, the new D'' is a binary rooted tree-decomposition of \mathbf{G} , where $\chi''(\bar{r}) = S \cup X$. Let $s = |S|$ and pick a collection A_1, \dots, A_s of subsets of $S \cup X$ such that $X = A_s \subsetneq A_{s-1} \subsetneq \dots \subsetneq A_1 \subsetneq S \cup X$. We use $D'' = (T'', \chi'', \bar{r})$ in order to construct a binary tree-decomposition $D^+ = (T^+, \chi^+, r^+)$ of \mathbf{G} of width $\text{tw}(G')$ as follows. Let T^+ be the disjoint union of T'' and a path on s vertices q_1, \dots, q_s . The construction of T^+ is completed by adding the edge $\{\bar{r}, q_1\}$. We also set $r^+ = q_s$. Finally we set $\chi^+ = \chi'' \cup \{(q_i, A_i) \mid i \in [s]\}$ and notice that $D^+ = (T^+, \chi^+, r^+)$ is a binary tree-decomposition of \mathbf{G} of width at most $\text{tw}(G') + |X|$ and with at most $2 \cdot (|\mathbf{G}| - |X|) + \text{tw}(G') + 1$ nodes, as $|S| \leq \text{tw}(G') + 1$. \square

3.4.2 (α, β) -rooted tree-decompositions

Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph and let $D = (T, \chi, r)$ be a tree-decomposition of \mathbf{G} . Given $\alpha, \beta \in \mathbb{N}$, we say that D is an (α, β) -tree-decomposition of \mathbf{G} if the following conditions hold (see Figure 5):

1. $|\chi(r)| \leq \alpha$. Also, if $\{r^{(1)}, \dots, r^{(s)}\}$ is the set of children of r in (T, r) , then $s \leq \alpha$,
2. $\forall h \in [s]$, then $D^{(h)} = (T_{r^{(h)}}, \chi_{r^{(h)}}, r^{(h)})$ is a binary tree-decomposition of $\mathbf{G}_{r^{(h)}}$ of width at most β ,

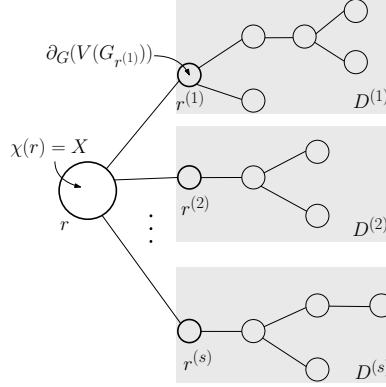


Figure 5: An (α, β) -tree-decomposition of a boundaried graph \mathbf{G} .

3. $\forall h \in [s]$, $V(G_{r^{(h)}})$ is tight in G ,
4. $\forall h \in [s]$, $\chi(r^{(h)}) = \partial_G(V(G_{r^{(h)}}))$, and
5. for each $\{i, j\} \in \binom{[s]}{2}$, $\chi(r^{(i)}) \neq \chi(r^{(j)})$.

We also define t_q , H_q , T_q , \mathbf{G}_q , V_q , \tilde{V}_q , $\overline{\mathbf{G}}_q$ and D_q , for every $q \in V(T)$, as we did in the case of tree-decompositions.

Lemma 9. *Let G be a graph that has an (α, β) -protrusion decomposition. Then there exists some boundaried graph \mathbf{G} whose underlying graph is G and such that \mathbf{G} has an $(\alpha, 2\beta)$ -tree-decomposition.*

Proof. Let $\mathcal{P} = \{X_0, X_1, \dots, X_\ell\}$ be an (α, β) -protrusion decomposition of G . From Lemma 3, we may assume that \mathcal{P} is tight, i.e.,

$$\forall i \in [\ell], \quad N_G(X_i) = \partial_G(N_G[X_i]). \quad (5)$$

Recall that $R_i = N_G[X_i]$, $|\partial_G(R_i)| \leq \beta$ and $\mathbf{tw}(G[R_i]) \leq \beta$, $i \in [\ell]$. We set $G_i = G[X_i]$, $i \in [\ell]$ and, as $X_i \subseteq R_i$, we obtain that $\mathbf{tw}(G_i) \leq \mathbf{tw}(G[R_i])$, $i \in [\ell]$. Notice that the vertex sets of the graphs in G_1, \dots, G_ℓ are pairwise disjoint, therefore

$$\forall I \subseteq [\ell], \quad \mathbf{tw}(\bigcup_{i \in I} G_i) \leq \beta \quad (6)$$

We say that $G_i \sim G_j$ iff $\partial_G(R_i) = \partial_G(R_j)$. Clearly, \sim defines an equivalence relation. Let $\{\mathcal{G}_1, \dots, \mathcal{G}_s\}$ be the partition of $\{G_1, \dots, G_\ell\}$ into the equivalence classes of \sim . For $h \in [s]$, we set $G^{(h)} = \bigcup \mathcal{G}_h$ and observe that, because of (6), $\mathbf{tw}(G^{(h)}) \leq \beta$. Also we set $\overline{G}^{(h)} = G[V(G^{(h)}) \cup Z^{(h)}]$ where $Z^{(h)}$ is the common open neighbourhood, in G , of all the vertex sets of the graphs in $\mathcal{G}^{(h)}$, $h \in [s]$. From (5), $Z^{(h)} = \partial_G(V(\overline{G}^{(h)}))$. As each $Z^{(h)}$ is some of the sets in $\{\partial_G(R_1), \dots, \partial_G(R_\ell)\}$, taking into account (5), we have that $|Z^{(h)}| \leq \beta$, $h \in [s]$.

Let λ be some labelling of G . We set, for every $h \in [s]$, $\mathbf{G}^{(h)} = (\overline{G}^{(h)}, Z^{(h)}, \lambda|_{V(\overline{G}^{(h)})})$, and $t^{(h)} = |Z^{(h)}|$. Recall that $t^{(h)} \leq \beta$, $h \in [s]$. From Lemma 8, each $\mathbf{G}^{(h)}$ has a binary tree-decomposition $D^{(h)} = (T^{(h)}, \chi^{(h)}, r^{(h)})$ of width at most $\mathbf{tw}(G^{(h)}) + t^{(h)} \leq 2 \cdot \beta$.

We now construct a tree-decomposition $D = (T, \chi, r)$ of \mathbf{G} as follows. To construct the tree T we take the disjoint union of $T^{(1)}, \dots, T^{(s)}$, then we add a new vertex r and we make r adjacent with $r^{(1)}, \dots, r^{(s)}$. We then define $\chi = \{(r, X_0)\} \cup \chi^{(1)} \cup \dots \cup \chi^{(s)}$. Notice that D is a tree-decomposition of G . We claim that D is an $(\alpha, 2 \cdot \beta)$ -decomposition. Moreover,

Condition 2 follows as $|\chi(r)| = |X_0| \leq \alpha$ and the fact that s is the number of equivalence classes of \sim that is at most $\ell \leq \alpha$. Condition 2 follows directly by the construction of D . Condition 3 follows from the tightness of \mathcal{P} . For Condition 4 it is enough to observe that for every $h \in [s]$, $\chi(r^{(h)}) = Z^{(h)} = \partial_G(V(\overline{G}^{(h)})) = \partial_G(V(G_{r^{(h)}}))$. As $\chi(r^{(h)}) = Z^{(h)}$, and, by their definition, all sets in $\{Z^{(1)}, \dots, Z^{(s)}\}$ are pairwise distinct, Condition 5 is satisfied. \square

4 The proof of Theorem 2

4.1 Some functions

Given a graph parameter \mathbf{p} that has Fill, we define the functions $\tau_{\mathbf{p}}, \theta_{\mathbf{p}}, \mu_{\mathbf{p}}, \delta_{\mathbf{p}}, \xi_{\mathbf{p}} : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\begin{aligned} \tau_{\mathbf{p}}(x) &= \max\{\mathbf{p}(G) \mid G \text{ is a graph in } \text{dom}(\mathbf{p}) \text{ where } |G| \leq x\}, \\ \theta_{\mathbf{p}}(x) &= (\text{card}_{\mathbf{p}}(x) \cdot x! + 1)^{x+1}, \\ \mu_{\mathbf{p}}(x) &= (\theta_{\mathbf{p}}(x) - 2) \cdot 2^{\theta_{\mathbf{p}}(x)-2} + 1, \\ \delta_{\mathbf{p}}(x) &= x((4x \cdot \mu_{\mathbf{p}}(x) - 1)^2 + 4x \cdot \mu_{\mathbf{p}}(x)), \text{ and} \\ \xi_{\mathbf{p}}(x) &= \tau_{\mathbf{p}}(x \cdot (2^{4x \cdot \mu_{\mathbf{p}}(x)-1} - 1)). \end{aligned}$$

We also define the constant $c_{\mathbf{p}} = (\delta_{\mathbf{p}}(2 \cdot \text{dec}(\mathbf{p})) \cdot (2 \cdot \text{dec}(\mathbf{p}) \cdot \xi_{\mathbf{p}}(2 \cdot \text{dec}(\mathbf{p})) + 1) + \text{dec}(\mathbf{p}))$.

4.2 A lemma on the compression of admissible pairs

4.2.1 Replacements

Let $\mathbf{G}_i = (G_i, X_i, \lambda_i), i \in [2]$ be two boundaried graphs. We say that \mathbf{G}_2 is a *part of* \mathbf{G}_1 if G_2 is a subgraph of G_1 such that $V(G_2)$ is the union of some of the connected components of $G_1 \setminus X_2$ that do not contain vertices of X_1 and $\lambda_1|_{V(G_2)} = \lambda_2$.

Let $\mathbf{G}'_2 = (G'_2, X'_2, \lambda'_2)$ be a boundaried graph. We say that \mathbf{G}'_2 and \mathbf{G}_2 are *strongly compatible* if they are compatible, and $\lambda'_2(V(G'_2)) \subseteq \lambda_2(V(G_2))$. Given that \mathbf{G}_2 is a part of \mathbf{G}_1 and that \mathbf{G}_2 and \mathbf{G}'_2 are strongly compatible, we define the *replacement of \mathbf{G}_2 by \mathbf{G}'_2 in \mathbf{G}_1* as the boundaried graph $\mathbf{G}'_1 = (G'_1, X'_1, \lambda'_1)$ where

- $G'_1 = (G_1 \setminus (V(G_2) \setminus X_2), X_2, \lambda \setminus (V(G_2) \setminus X_2)) \oplus G'_2$,
- $X'_1 = X_1$, and
- $\lambda'_1 = (\lambda_1 \setminus (\lambda_2 \setminus (V(G_2) \setminus X_2))) \cup \lambda'_2$.

See Figure 6 for a visualization of the replacement operation.

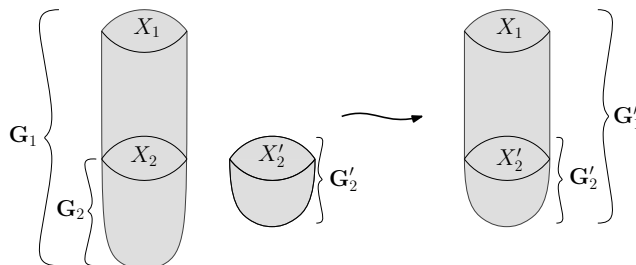


Figure 6: The replacement of \mathbf{G}_2 by \mathbf{G}'_2 in \mathbf{G}_1 .

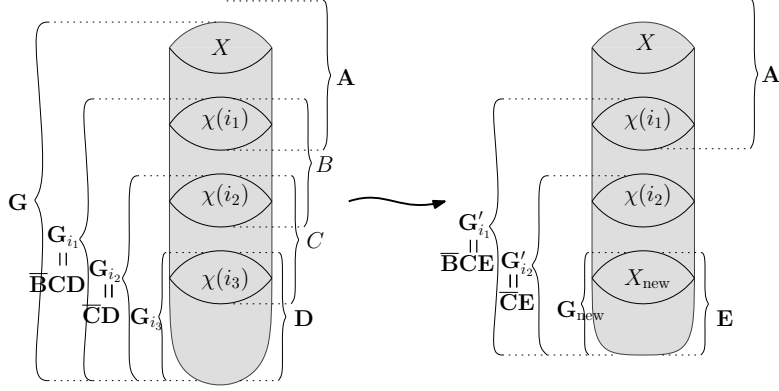


Figure 7: The boundaried graphs in Lemma 10.

4.2.2 Compressible and admissible pairs

Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph and let $D = (T, \chi, r)$ be a binary tree-decomposition of G of width at most $t-1$. We say that a vertical pair (a, b) of (T, r) is *compressible* for (\mathbf{G}, D) if \mathbf{G}_a and \mathbf{G}_b are compatible. Given a vertical pair (a, b) that is compressible for (\mathbf{G}, D) , we define the (a, b) -*compression of the pair* (\mathbf{G}, D) as the pair (\mathbf{G}', D') where $\mathbf{G}' = (G', X, \lambda')$ is the replacement of \mathbf{G}_a by \mathbf{G}_b in \mathbf{G} , $D' = (T', \chi', r)$, $T' = T \setminus (a, b)$, and $\chi' = \chi|_{V(T')}$. Notice that if D is a binary tree-decomposition of \mathbf{G} , then D' is again a binary tree-decomposition of \mathbf{G} .

Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph and let $D = (T, \chi, r)$ be a binary tree-decomposition of G of width at most $t-1$. Let $t \in \mathbb{N}$ and \mathbf{p} be a graph parameter that has **Fill**. We say that a vertical pair (a, b) of (T, r) is $\equiv_{\mathbf{p}, t}$ -*admissible* for (\mathbf{G}, D) if (a, b) is compressible for (\mathbf{G}, D) and $\mathbf{G}_a \equiv_{\mathbf{p}, t} \mathbf{G}_b$.

Lemma 10. *Let \mathbf{p} be a graph parameter that has **Fill** and let $t \in \mathbb{N}$. Let $\mathbf{G} = (G, X, \lambda)$ be a boundaried graph and let $D = (T, \chi, r)$ be a rooted tree-decomposition of \mathbf{G} of width at most $t-1$. Let (i_1, i_2) be an $\equiv_{\mathbf{p}, t}$ -admissible pair for (\mathbf{G}, D) where $\mathbf{G}_{i_1} \equiv_{\mathbf{p}, t_{i_1}} \mathbf{G}_{i_2}$ and let $i_3 \in V(T)$ such that $i_2 \leq_{T, r} i_3$. Let also $\mathbf{G}_{\text{new}} = (G_{\text{new}}, X_{\text{new}}, \lambda_{\text{new}})$ be a boundaried graph that is strongly compatible with \mathbf{G}_{i_3} and $\mathbf{G}_{i_3} \equiv_{\mathbf{p}, t_{i_3}} \mathbf{G}_{\text{new}}$ and let \mathbf{G}'_{i_i} be the replacement of \mathbf{G}_{i_3} by \mathbf{G}_{new} in \mathbf{G}_{i_i} , for $i \in [2]$. Then the following hold:*

- $\mathbf{G}'_{i_1} \equiv_{\mathbf{p}, t_{i_1}} \mathbf{G}'_{i_2}$, and $\text{transp}_{\mathbf{p}}(\mathbf{G}'_{i_1}, \mathbf{G}'_{i_2}) = \text{transp}_{\mathbf{p}}(\mathbf{G}_{i_1}, \mathbf{G}_{i_2})$.
- $\mathbf{G}_{i_2} \equiv_{\mathbf{p}, t} \mathbf{G}'_{i_2}$, and $\text{transp}_{\mathbf{p}}(\mathbf{G}_{i_2}, \mathbf{G}'_{i_2}) = \text{transp}_{\mathbf{p}}(\mathbf{G}_{i_3}, \mathbf{G}_{\text{new}})$.

Proof. We define (see Figure 7):

$$\begin{array}{lll}
 B = G_{i_1} \cap \overline{G}_{i_2} & C = G_{i_2} \cap \overline{G}_{i_3} & D = G_{i_3} \\
 \overline{BCE} = \mathbf{G}'_{i_1} & \overline{CE} = \mathbf{G}'_{i_2} & \overline{BCD} = \mathbf{G}_{i_1} \\
 \overline{CD} = \mathbf{G}_{i_2} & D = \mathbf{G}_{i_3} & E = \mathbf{G}_{\text{new}}.
 \end{array}$$

If $\mathbf{A} = (A, Y, \lambda)$ is some t_{i_1} -boundaried graph, we denote $\mathbf{A}\overline{BC} = (A \cup B \cup C, \chi(i_3), \lambda')$, where we insist that $V(\mathbf{A}) \cap V(\mathbf{G}_{i_1}) = Y = \chi(i_1)$ and we pick λ' as some vertex labelling of $A \cup B \cup C$ such that $\lambda|_{\chi(i_3)} \subseteq \lambda'$.

Similarly, if $\mathbf{A} = (A, Y, \lambda)$ is some t_{i_1} -boundaried graph, we denote $\mathbf{A}\overline{C} = (A \cup C, \chi(i_3), \lambda')$, where we insist that $V(\mathbf{A}) \cap V(\mathbf{G}_{i_2}) = Y = \chi(i_2)$ and we pick λ' as some vertex labelling of $A \cup B \cup C$ such that $\lambda|_{\chi(i_3)} \subseteq \lambda'$.

We also set $ABCD = \mathbf{A} \oplus \overline{\mathbf{BCD}}$, $ACD = \mathbf{A} \oplus \overline{\mathbf{CD}}$, $ABCE = \mathbf{A} \oplus \overline{\mathbf{BCE}}$, and $ACE = \mathbf{A} \oplus \overline{\mathbf{CE}}$. Clearly,

$$ABCD = \mathbf{ABC} \oplus \mathbf{D} \quad (7)$$

$$ACD = \mathbf{AC} \oplus \mathbf{D} \quad (8)$$

$$ABCE = \mathbf{ABC} \oplus \mathbf{E} \quad (9)$$

$$ACE = \mathbf{AC} \oplus \mathbf{E} \quad (10)$$

Finally we set $t = t_{i_1} = t_{i_2}$, $t' = t_{i_3} = |X_{\text{new}}|$, $c_{1,2} = \text{transp}_{\mathbf{p}}(\mathbf{G}_{i_1}, \mathbf{G}_{i_2})$, and $c_{3,4} = \text{transp}_{\mathbf{p}}(\mathbf{G}_{i_3}, \mathbf{G}_{\text{new}})$. The proof is based on the following observations:

$\mathbf{G}_{i_1} \equiv_{\mathbf{p},t} \mathbf{G}_{i_2}$ means that

$$\begin{aligned} \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(\mathbf{A} \oplus \mathbf{G}_{i_1}) = \mathbf{p}(\mathbf{A} \oplus \mathbf{G}_{i_2}) + c_{1,2} \iff \\ \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(ABCD) = \mathbf{p}(ACD) + c_{1,2} \end{aligned} \quad (11)$$

$\mathbf{G}_{i_3} \equiv_{\mathbf{p},t'} \mathbf{G}_{\text{new}}$ means that

$$\begin{aligned} \forall \mathbf{S} \in \mathcal{B}^{(t')}, \quad & \mathbf{p}(\mathbf{S} \oplus \mathbf{G}_{i_3}) = \mathbf{p}(\mathbf{S} \oplus \mathbf{G}_{\text{new}}) + c_{3,4} \iff \\ \forall \mathbf{S} \in \mathcal{B}^{(t')}, \quad & \mathbf{p}(\mathbf{S} \oplus \mathbf{D}) = \mathbf{p}(\mathbf{S} \oplus \mathbf{E}) + c_{3,4} \end{aligned} \quad (12)$$

We apply (12) for all $\mathbf{S} \in \{\mathbf{ABC} \mid \mathbf{A} \in \mathcal{B}^{(t')}\}$,

$$\begin{aligned} \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(\mathbf{ABC} \oplus \mathbf{D}) = \mathbf{p}(\mathbf{ABC} \oplus \mathbf{E}) + c_{3,4} \stackrel{(7),(9)}{\iff} \\ \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(ABCD) = \mathbf{p}(ABCE) + c_{3,4} \end{aligned} \quad (13)$$

From (11) and (13), we obtain that

$$\forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad \mathbf{p}(ABCE) = \mathbf{p}(ACD) + c_{1,2} - c_{3,4} \quad (14)$$

We apply (12) for all $\mathbf{S} \in \{\mathbf{AC} \mid \mathbf{A} \in \mathcal{B}^{(t')}\}$,

$$\begin{aligned} \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(\mathbf{AC} \oplus \mathbf{D}) = \mathbf{p}(\mathbf{AC} \oplus \mathbf{E}) + c_{3,4} \stackrel{(8),(10)}{\iff} \\ \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(ACD) = \mathbf{p}(ACE) + c_{3,4} \end{aligned} \quad (15)$$

From (14) and (15) we conclude that

$$\begin{aligned} \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(ABCE) = \mathbf{p}(ACE) + c_{1,2} \stackrel{(9),(10)}{\iff} \\ \forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad & \mathbf{p}(\mathbf{A} \oplus \overline{\mathbf{BCE}}) = \mathbf{p}(\mathbf{A} \oplus \overline{\mathbf{CE}}) + c_{1,2} \end{aligned}$$

which implies that $\overline{\mathbf{BCE}} \equiv_{\mathbf{p},t'} \overline{\mathbf{CE}}$ and $\text{transp}_{\mathbf{p}}(\overline{\mathbf{BCE}}, \overline{\mathbf{CE}}) = c_{1,2}$, as required.

Notice now that (15) can be rewritten

$$\forall \mathbf{A} \in \mathcal{B}^{(t)}, \quad \mathbf{p}(\mathbf{A} \oplus \overline{\mathbf{CD}}) = \mathbf{p}(\mathbf{A} \oplus \overline{\mathbf{CE}}) + c_{3,4} \quad (16)$$

From (16) we obtain that $\overline{\mathbf{CD}} \equiv_{\mathbf{p},t} \overline{\mathbf{CE}}$ and $\text{transp}_{\mathbf{p}}(\overline{\mathbf{CD}}, \overline{\mathbf{CE}}) = c_{3,4}$. Also, (12) implies that $\text{transp}_{\mathbf{p}}(\mathbf{D}, \mathbf{E}) = c_{3,4}$. Therefore $\text{transp}_{\mathbf{p}}(\overline{\mathbf{CD}}, \overline{\mathbf{CE}}) = \text{transp}_{\mathbf{p}}(\mathbf{D}, \mathbf{E})$, thus $\mathbf{G}_{i_2} \equiv_{\mathbf{p},t} \mathbf{G}'_{i_2}$ and $\text{transp}_{\mathbf{p}}(\mathbf{G}_{i_2}, \mathbf{G}'_{i_2}) = \text{transp}_{\mathbf{p}}(\mathbf{G}_{i_3}, \mathbf{G}_{\text{new}})$. \square

4.3 Compressing transition pairs

4.3.1 The minor relation for boundaried graphs

Let $\mathbf{G}_1 = (G_1, X_1, \lambda_1)$ and $\mathbf{G}_2 = (G_2, X_2, \lambda_2)$ be two t -boundaried graphs. We say that \mathbf{G}_1 is a *minor* of \mathbf{G}_2 , denoted by $\mathbf{G}_1 \leq_m \mathbf{G}_2$, if there is a function $\sigma : V(G_1) \rightarrow 2^{V(G_2)}$ where

1. $\forall x, y \in V(G_1), x \neq y \Rightarrow \sigma(x) \cap \sigma(y) = \emptyset$,
2. $\forall x \in V(G_1), G_2[\sigma(x)]$ is connected.
3. $\forall \{x, y\} \in E(G_1), G_2[\sigma(x) \cup \sigma(y)]$ is connected.
4. $\forall i \in [t], \psi_{\mathbf{G}_2}^{-1}(i) \in \sigma(\psi_{\mathbf{G}_1}^{-1}(i))$

We also say that the graph H is a *minor* of G if $(H, \emptyset, \emptyset) \leq_m (G, \emptyset, \emptyset)$ and we denote it by $H \leq_m G$. Notice that if $\mathbf{G}_1 \leq_m \mathbf{G}_2$, then $H_{\mathbf{G}_2}$ is a spanning subgraph of $H_{\mathbf{G}_1}$.

The following observation indicates how the minor relation on boundaried graphs is generating the minor relation under the gluing operation.

Observation 1. *Let $\mathbf{G} = (G, X, \lambda)$, $\mathbf{G}_1 = (G_1, X_1, \lambda_1)$ and $\mathbf{G}_2 = (G_2, X_2, \lambda_2)$ be three t -boundaried graphs such that $\mathbf{G}_1 \leq_m \mathbf{G}_2$. Then the graph $\mathbf{G} \oplus \mathbf{G}_1$ is a minor of the graph $\mathbf{G} \oplus \mathbf{G}_2$.*

4.3.2 Transition pairs

Let \mathbf{p} be a graph parameter that has FII. Let also \mathbf{G} be a boundaried graph and let $D = (T, \chi, r)$ be an (α, β) -rooted tree-decomposition of \mathbf{G} . We say that a vertical pair (a, b) of (T, r) is a *transition pair* of (\mathbf{G}, D) if (a, b) is $\equiv_{\mathbf{p}, t}$ -admissible (\mathbf{G}, D) and $\mathbf{G}_b \leq_m \mathbf{G}_a$.

Lemma 11. *Let \mathbf{p} be a graph parameter that has FII and let $t \in \mathbb{N}$. Let also \mathbf{G} be a boundaried graph, $D = (T, \chi, r)$ be a binary tree-decomposition of \mathbf{G} of width at most $t - 1$, and (a, b) be a transition pair of (\mathbf{G}, D) . If (\mathbf{G}', D') is the (a, b) -compression of the pair (\mathbf{G}, D) then $\mathbf{G}' \leq_m \mathbf{G}$, $|\mathbf{G}'| < |\mathbf{G}|$, $\mathbf{G}' \equiv_{\mathbf{p}, t} \mathbf{G}$ and $\text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b)$.*

Proof. $\mathbf{G}' \leq_m \mathbf{G}$ follows from the fact that \mathbf{G}_a and \mathbf{G}_b are compatible, thus the corresponding frontier graphs H_a and H_b are equal, and the fact that $\mathbf{G}_b \leq_m \mathbf{G}_a$. Moreover, $|\mathbf{G}'| < |\mathbf{G}|$ follows from the definition of a binary tree-decomposition. To prove that $\mathbf{G}' \equiv_{\mathbf{p}, t} \mathbf{G}$ and $\text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b)$, it is enough to apply Lemma 10 for \mathbf{G} and D , by setting $i_1 := r, i_2 := r, i_3 := a$, and $\mathbf{G}_{\text{new}} := \mathbf{G}_b$. \square

4.3.3 Weak leanness and linkedness

Let $D = (T, \chi, r)$ be a tree-decomposition of a boundaried graph $\mathbf{G} = (G, X, \lambda)$. Given a vertical pair $a, b \in V(T)$, we say that D is *weakly (a, b) -lean* for \mathbf{G} if for every $x, y \in \text{inner}_{T, r}(a, b)$, such that x and y are (a, b) -aligned, and every $s \in \mathbb{N}$, either there are s vertex disjoint paths between $\chi(x)$ and $\chi(y)$ in G or there is an edge $\{q, q'\} \in E(xTy)$ such that $|\chi(q) \cap \chi(q')| < s$. We also say that D is *(a, b) -linked* if for every $x, y \in \text{inner}_{T, r}(a, b)$, such that x and y are (a, b) -aligned, and every $s \in \mathbb{N}$, either there are s vertex disjoint paths between $\chi(x)$ and $\chi(y)$ in G or there is a vertex $w \in V(xTy)$ such that $|\chi(w)| < s$.

Lemma 12. *Let $\mathbf{G} = (G, A, \lambda)$ be a boundaried graph where $G[A]$ is a clique, let $B \subseteq V(G)$ where $G[B]$ is a clique, and let $D = (T, \chi, r)$ be a lean binary tree-decomposition of G such that T has a leaf l where $\chi(l) = B$. Let also \mathbf{G}' be a graph obtained by removing from G edges that belong to $E(G[A])$ or in $E(G[B])$. Then D is weakly (r, l) -lean for $\mathbf{G}' = (G', A, \lambda)$.*

Proof. Let $P = rTl$ and let T_1, \dots, T_z be the connected components of $T \setminus E(P)$. Recall that if x and y are (r, l) -aligned, then they both belong to one of P, T_1, \dots, T_z . In any case, we have to prove that if $\chi(x)$ and $\chi(y)$ are joined by a collection $\mathcal{Q} = \{Q_1, \dots, Q_z\}$ of pairwise vertex disjoint paths of G , then they are also joined by the same collection of paths in G' . We choose \mathcal{Q} , so that $V(\mathbf{U}\mathcal{Q})$ is minimized. This minimization forces each internal vertex of a path in \mathcal{Q} to belong to $\chi(h)$ for some $h \in V(xTy)$ but not in $\chi(x) \cup \chi(y)$.

Suppose that some edge $e = \{i, j\}$ of some path $Q \in \{Q_1, \dots, Q_z\}$ does not exist in G' . Clearly, the endpoints of e belong to one, say A , of A and B . Also, w.l.o.g., we assume that i is the vertex of $\{i, j\}$ that is closer to r in T and, again w.l.o.g., we assume that i is an endpoint of $\{i, j\}$ that does not belong to $\chi(x)$ (by the choice of \mathcal{Q} it is not possible that both i and j belong to $\chi(x)$ or to $\chi(y)$). We now claim that $i \in \chi(h)$ for some $h \in V(xTy) \setminus \{x\}$. Indeed, as we already mentioned, this holds when i is an internal vertex of Q while if i is not an internal vertex, then i should belong to $\chi(y)$. We conclude that i belongs both to $\chi(r)$ and to $\chi(y)$ and as $x \in V(rTy), i \in \chi(x)$, a contradiction to the choice of i . Therefore all edges of the paths in \mathcal{Q} are also edges of G' . \square

Lemma 13. *Let \mathbf{G} be a boundaried graph, $D = (T, \chi, r)$ a binary tree-decomposition of \mathbf{G} of width at most $t - 1$, $t \in \mathbb{N}$, and (a, b) a vertical pair of (T, r) . Then there exists a binary tree-decomposition $D' = (T', \chi', r)$ of \mathbf{G} where: (a, b) is a vertical pair of (T', r) , $\text{outer}_{T', r}(a, b) = \text{outer}_{(T, r)}(a, b)$, $\chi|_{\text{outer}_{T', r}(a, b)} = \chi|_{\text{outer}_{(T, r)}(a, b)}$, D' is (a, b) -linked, and $\text{capacity}_{(T', r)}(a, b) \geq \frac{1}{4t} \cdot \text{capacity}_{(T, r)}(a, b)$.*

Proof. Let $I = \text{inner}_{T, r}(a, b)$ and keep in mind that $\text{capacity}_{T, r}(a, b) = |I|$. Let $A = \chi(a)$ and $B = \chi(b)$. Let $J = G[\bigcup_{x \in I} \chi(x)]$ and let \bar{J} be the graph obtained from J if we make all pairs of vertices in A adjacent and all pairs of vertices in B adjacent, i.e., both A and B induce cliques in \bar{J} . Let $\bar{\mathbf{J}} = (\bar{J}, A, \lambda|_I)$. Let also $D^* = (T^*, \chi^*, a)$, where $T^* = T[I]$ and $\chi^* = \chi|_I$. Observe that D^* is a binary tree-decomposition of $\bar{\mathbf{J}}$ of width $\leq t - 1$. From Lemma 4, $|J| = |\bar{J}| \leq t \cdot |I| \leq t \cdot c$, where $c = \text{capacity}_{T, r}(a, b)$. Finally, from Lemma 5 it holds that

$$|T^*| \leq 4 \cdot |\bar{\mathbf{J}}| \Rightarrow |\bar{\mathbf{J}}| \geq |T^*|/4 \quad (17)$$

From Lemma 7 there exists a lean binary tree-decompositions $D^\bullet = (T^\bullet, \chi^\bullet, a)$ of $\bar{\mathbf{J}}$, with width at most $t - 1$, such that $\chi^\bullet(a) = A$ and $|T^\bullet| \leq 2 \cdot |\bar{\mathbf{J}}| \leq 2 \cdot t \cdot c$.

We name b the vertex of T^\bullet that has the biggest possible depth and $B \subseteq \chi(b)$. Our target is to modify D^\bullet so that b will be a leaf of T^\bullet and $\chi^\bullet(b) = B$. To this aim we assume that b is not a leaf of T^\bullet and we apply modifications according to the following case analysis:

Case 1: $B \subsetneq \chi(b)$ and b has at most one child in T^\bullet , rename b to b_{old} in D^\bullet and add in T^\bullet a new vertex b_{new} along with the edge $\{b_{\text{old}}, b_{\text{new}}\}$ (see Figure 8, Case 1). Finally, set $\chi^\bullet := \chi^\bullet \cup (b_{\text{new}}, B)$ and rename b_{new} to b . Notice that, as $B \subsetneq \chi^\bullet(b_{\text{old}})$, this modification creates again a lean binary tree-decomposition of $\bar{\mathbf{J}}$.

Case 2: If $B = \chi(b)$ and b has one child d in T^\bullet , we copy the transformation of Case 1 (see Figure 8, Case 2). Notice that as $\chi(d)$ cannot be a subset of $\chi(b)$, the new decomposition D^\bullet is again a lean binary decomposition of $\bar{\mathbf{J}}$.

Case 3: b has two children d and d' in T^\bullet . We remove the edges $\{b, d\}$ and $\{b, d'\}$ rename b to b_{old} in D^\bullet and add two new vertices b_{new} and b'_{new} , and the edges $\{b_{\text{old}}, b_{\text{new}}\}$, $\{b'_{\text{new}}, d'\}$, $\{b_{\text{old}}, b'_{\text{new}}\}$, and $\{b'_{\text{new}}, d\}$ (see Figure 8, Case 3). We set $\chi^\bullet := \chi^\bullet \cup \{(b_{\text{new}}, B), (b'_{\text{new}}, \chi^\bullet(b))\}$ and then we rename b_{new} to b . Observe that the fact that $\chi(d)$ is not a subset of $\chi(b)$ and $\chi(d')$ is not a subset of $\chi(b)$ in the original D^\bullet , the “new” D^\bullet is again a lean binary decomposition of $\bar{\mathbf{J}}$.

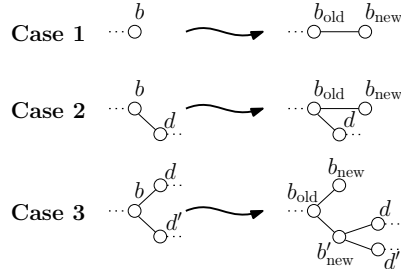


Figure 8: The transformations of Lemma 13.

After the above modifications, we have that b is indeed a leaf of T^\bullet . Also, these modifications may add at most 2 more nodes in D^\bullet . Now we are in position to apply Lemma 12 on $\bar{\mathbf{J}}$, B , D^\bullet , b , and J and obtain that D^\bullet is a binary tree-decomposition that is weakly (a, b) -lean for $\mathbf{J} = (J, A, \lambda|_J)$.

We now construct $D' = (T', \chi', r)$ by setting $T' = (T \setminus I) \cup T^\bullet$, and $\chi' = \chi|_{V(G) \setminus I} \cup \chi^\bullet$. As b is a leaf of T^\bullet , D' is a binary tree-decomposition of \mathbf{G} with width at most the width of D . Moreover, D' is a binary tree-decomposition that is weakly (a, b) -lean for \mathbf{G} . We now transform D' to a tree-decomposition that is (a, b) -linked for \mathbf{G} by applying the following transformation for every edge-pair (x, y) of (T^\bullet, a) : if $|\chi'(x) \cap \chi'(y)| < \min\{|\chi'(x)|, |\chi'(y)|\}$, then remove $\{x, y\}$ from T' , add a new vertex $v_{x,y}$ and the edges $\{x, v_{x,y}\}$ and $\{v_{x,y}, y\}$ and set $\chi' := \chi' \cup \{(v_{x,y}, \chi'(x) \cap \chi'(y))\}$. This transformation makes D' (a, b) -linked while it does not harm the status of D' of being a binary tree-decomposition.

To prove the last property, first observe that $\text{capacity}_{T,r}(a, b) = |T^*|$ and then use (17). Now notice that $\text{capacity}_{(T',r)}(a, b) \geq |T^\bullet|$, as we may add some vertices to T' when transforming T^\bullet to an (a, b) -linked tree-decomposition. From Lemma 4, it holds that $|T^\bullet| \geq |\bar{\mathbf{J}}|/t$. Combining this with (17) we conclude that $|T^\bullet| \geq |T^*|/(4 \cdot t)$. \square

4.3.4 Looking for transition pairs

We need the following proposition (for a proof, see [35]).

Proposition 4. *Let t, y be positive integers, and let $w \in \Sigma^*$, where Σ is an alphabet whose symbols are the numbers in $[t]$. If $|w| \geq y^t$, then there is a number $t' \in [t]$ and a subword u of w such that all letters in u are at least t' and u contains the number t' at least y times.*

Lemma 14. *Let \mathbf{p} be a graph parameter that has FII. Let also \mathbf{G} be a boundaried graph, $D = (T, \chi, r)$ a binary tree-decomposition of \mathbf{G} , and (a, b) a vertical pair of (T, r) such that D has width at most $t - 1$, $t \in \mathbb{N}$, $\text{capacity}_{T,r}(a, b) \geq \mu_{\mathbf{p}}(t)$, and D is (a, b) -linked. Then there exists a transition pair (a', b') of (\mathbf{G}, D) where $a', b' \in \text{inner}_{T,r}(a, b)$.*

Proof. We set $\theta = \theta_{\mathbf{p}}(t)$. From Lemma 2 and the definition of $\mu_{\mathbf{p}}$, there are $x, y \in \text{inner}_{T,r}(a, b)$ such that x and y are (a, b) -aligned and $|xTy| \geq \theta = (\text{card}_{\mathbf{p}}(t) \cdot t! + 1)^{t+1}$. Let $P = xTy$ and keep in mind that all vertices of P belong to $\text{inner}_{T,r}(a, b)$.

Let $\langle q_1, \dots, q_\theta \rangle$ be the vertices of P ordered in the way they appear in P starting from $q_1 = x$. We see the sequence $w = \langle |\chi(q_1)|, |\chi(q_1)|, \dots, |\chi(q_\theta)| \rangle$ as a word on the alphabet $\Sigma = [0, t]$. As $\theta = (\text{card}_{\mathbf{p}}(t) \cdot t! + 1)^{t+1}$, from Proposition 4, there is some $t' \in [0, t]$, a pair $i', j' \in [\theta]$ and a set $I \subseteq [i', j']$ such that

$$\mathbf{A.} \quad |\chi(q_{i'})|, |\chi(q_{i'+1})|, \dots, |\chi(q_{j'})| \geq t'$$

B. $|I| \geq \text{card}_{\mathbf{p}}(t) \cdot t! + 1$ and

C. for all $f \in I$, $|\chi(q_f)| = t'$.

Let $\sigma : \{\mathbf{G}_{q_h} \mid h \in I\} \rightarrow \mathcal{R}$ such that $\sigma(\mathbf{G}_{q_h}) = \text{rep}_{\mathbf{p}}(\mathbf{G}_{q_h})$. As $|I| \geq \text{card}_{\mathbf{p}}(t) \cdot t! + 1$ and $t \geq t'$, there is a set $I' \subseteq I$ where $|I'| = t! + 1$ and such that all graphs in $\{\mathbf{G}_{q_h} \mid h \in I'\}$ are equivalent with respect to $\equiv_{\mathbf{p},t}$. Let $\mathcal{W} = \langle p_1, \dots, p_{|I'|} \rangle$ be the vertices of I' ordered in the way they appear in P .

As D is (a,b) -linked, and because of **A.** and **C.**, for every $i, j \in [t! + 1]$, the vertices of $\chi(p_i)$ and the vertices of $\chi(p_j)$ are connected in \mathbf{G}_{p_i} by a collection $\mathcal{Q}_{i,j}$ of t' internally vertex-disjoint paths. For every $i, j \in [t! + 1], i \leq j$, we define the bijection $\kappa_{i,j} : [t'] \rightarrow [t']$ such that for each $m \in [t]$, it holds that the vertices $\psi_{\mathbf{G}_{p_i}}^{-1}(m)$ and $\psi_{\mathbf{G}_{p_j}}^{-1}(m)$ are the two endpoints of some path in $\mathcal{Q}_{i,j}$ (recall that $\psi_{\mathbf{G}_{p_i}}$ and $\psi_{\mathbf{G}_{p_j}}$ are the label normalising functions of \mathbf{G}_{p_i} and \mathbf{G}_{p_j} respectively). Notice that if q_i, q_j , and q_h are vertices in \mathcal{W} where $i \leq j \leq h$, then, $\kappa_{i,h} = \kappa_{i,j} \circ \kappa_{j,h}$. As $[t']$ has $t'!$ different permutations, there exist $i, j \in [t! + 1], i < j$ such that $\kappa_{i,j}$ is the identity mapping. This means that $\mathcal{Q}_{i,j}$ contains t' vertex-disjoint paths $P_1, \dots, P_{t'}$ in \mathbf{G}_{p_i} , from $\chi(p_i)$ to $\chi(p_j)$, such that the endpoints of each path are equally indexed. Let $a' = p_i$ and $b' = p_j$ and we assume that for $m \in [t']$, P_m is the path with endpoints $\psi_{\mathbf{G}_{a'}}^{-1}(m)$ and $\psi_{\mathbf{G}_{b'}}^{-1}(m)$. It remains to show that $\mathbf{G}_{b'} \leq_m \mathbf{G}_{a'}$. For this, we define the injection $\mu : V(\mathbf{G}_{b'}) \rightarrow V(\mathbf{G}_{a'})$ where:

$$\mu(v) = \begin{cases} V(P_{\psi_{\mathbf{G}_{b'}}(v)}) & , \text{ if } v \in \chi(b') \\ v & , \text{ if } v \in \tilde{V}_{b'} = V_{b'} \setminus \chi(b') \end{cases}$$

Intuitively, $\mathbf{G}_{b'}$ can be obtained from $\mathbf{G}_{a'}$ if we remove all vertices in $V_{a'} \setminus V_{b'}$ that do not belong to some of the paths in $\mathcal{Q}_{i,j} = \{P_1, \dots, P_{t'}\}$ and then, for each $h \in [t']$ contract the path P_h to its endpoint in $\chi(w)$, i.e., the vertex of $\chi(w)$ that has index h . \square

4.3.5 Compressing transition pairs

Lemma 15. *Let \mathbf{p} be a graph parameter that has FII. Let also \mathbf{G} be a boundaried graph, $D = (T, \chi, r)$ a binary tree-decomposition of \mathbf{G} of width at most $t - 1$, $t \in \mathbb{N}$, and (a, b) a vertical pair of (T, r) where $4t \cdot \mu_{\mathbf{p}}(t) \leq \text{capacity}_{T,r}(a, b)$. Then there exist a boundaried graph \mathbf{G}' and a binary tree-decomposition $D' = (T', \chi', r)$ of \mathbf{G}' such that $\mathbf{G} \equiv_{\mathbf{p},t} \mathbf{G}'$, $|\mathbf{G}'| < |\mathbf{G}|$, $\mathbf{G}' \leq_m \mathbf{G}$, and D' has width at most $t - 1$. Moreover, if (x, y) is a transition pair where $(x, y) \in \text{outer}_{T,r}(a, b)$, then (x, y) is also a transition pair of (\mathbf{G}', D') and $\text{transp}_{\mathbf{p}}(\mathbf{G}_x, \mathbf{G}_y) = \text{transp}_{\mathbf{p}}(\mathbf{G}'_x, \mathbf{G}'_y)$.*

Proof. From Lemma 13 we know that there exists a binary tree-decomposition $D^* = (T^*, \chi^*, r)$ with the specifications of Lemma 13. Notice that $\text{capacity}_{(T^*,r)}(a, b) \geq \frac{1}{4t} \cdot \text{capacity}_{T,r}(a, b) \geq \mu_{\mathbf{p}}(t)$.

From Lemma 14 we know that there exists a transition pair (a', b') of (\mathbf{G}, D^*) , where $a', b' \in \text{inner}_{(T^*,r)}(a, b)$. Let (\mathbf{G}', D') be the (a', b') -compression of the pair (\mathbf{G}, D^*) . By Lemma 11, $\mathbf{G}' \leq_m \mathbf{G}$, $|\mathbf{G}'| < |\mathbf{G}|$, $\mathbf{G}' \equiv_{\mathbf{p},t} \mathbf{G}$, and $\text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') = \text{transp}_{\mathbf{p}}(\mathbf{G}_{a'}, \mathbf{G}_{b'})$. Notice that $\text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') = \text{transp}_{\mathbf{p}}(\mathbf{G}_{a'}, \mathbf{G}'_{b'}) = \text{transp}_{\mathbf{p}}(\mathbf{G}_{a'}, \mathbf{H}) + \text{transp}_{\mathbf{p}}(\mathbf{H}, \mathbf{G}_{b'})$ where $\mathbf{H} = \text{rep}_{\mathbf{p}}(\mathbf{G}_{a'}) = \text{rep}_{\mathbf{p}}(\mathbf{G}'_{b'})$.

The last statement of the lemma is straightforward in the case where $a \notin \text{desc}_{(T,t)}(y)$ as, in this case, $\mathbf{G}_x = \mathbf{G}'_x$. Suppose now that $a \in \text{desc}_{(T,t)}(y)$. We apply Lemma 10 on \mathbf{G} and D' for $i_1 = x$, $i_2 = y$, $i_3 = a'$ and $\mathbf{G}_{\text{new}} := \mathbf{G}_{b'}$ and we obtain that $\text{transp}_{\mathbf{p}}(\mathbf{G}_x, \mathbf{G}_y) = \text{transp}_{\mathbf{p}}(\mathbf{G}'_x, \mathbf{G}'_y)$ as required. \square

4.4 Boundaried graph compression

Lemma 16. *Let \mathbf{p} be graph parameter that has Fill. Let also $\mathbf{G} = (G, X, \lambda)$ be a t -boundaried graph and $D = (T, \chi, r)$ a binary tree-decomposition of \mathbf{G} of width $t - 1$, $t \in \mathbb{N}$. Then there exists a t -boundaried graph \mathbf{G}' such that (1) $\mathbf{G} \equiv_{\mathbf{p}, t} \mathbf{G}'$, (2) $\mathbf{G}' \leq_m \mathbf{G}$, (3) $\text{tw}(\mathbf{G}') \leq t - 1$, and (4) $|\mathbf{G}'| \leq t \cdot (2^{4t \cdot \mu_{\mathbf{p}}(t)} - 1)$.*

Proof. Let G' be a minimum size t -boundaried graph such that $\mathbf{G}' = (G', X, \lambda)$ satisfies Conditions 1–3 (notice that G' exists as G already satisfies these properties). Let $D' = (T', \chi', r)$. Let also $\mu = 4t \cdot \mu_{\mathbf{p}}(t)$. Let y be a leaf of T' of maximum distance from r . Let d be this distance. Notice that it is enough to prove that $d < \mu - 1$, then we are done because, from Lemma 4, $|\mathbf{G}'| \leq t \cdot (2^\mu - 1)$. Suppose to the contrary that $d \geq \mu - 1$. In this case, the vertical pair (r, y) has capacity at least μ . From Lemma 15, applied on \mathbf{G}' and D' , there exist a boundaried graph \mathbf{G}'' and a binary tree-decomposition D'' of width at most $t - 1$ such that $\mathbf{G}'' \leq_m \mathbf{G}'$, $|\mathbf{G}''| < |\mathbf{G}'|$, and $\mathbf{G}' \equiv_{\mathbf{p}, t} \mathbf{G}''$. Therefore $\mathbf{G}'' \leq_m \mathbf{G}$, $\text{tw}(\mathbf{G}'') \leq t - 1$, and $\mathbf{G} \equiv_{\mathbf{p}, t} \mathbf{G}''$, a contradiction to the choice of G' . \square

Lemma 17. *Let \mathbf{p} be a graph parameter that has Fill and let $t \in \mathbb{N}$. Let $\mathbf{G}, \mathbf{G}' \in \mathcal{T}^{(\leq t)}$, where $\mathbf{G} \equiv_{\mathbf{p}, t} \mathbf{G}'$. Then $\mathbf{p}(G) = \mathbf{p}(G') + \text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}')$ where G and G' are the underlying graphs of \mathbf{G} and \mathbf{G}' .*

Proof. Let t' be the boundary size of \mathbf{G} and \mathbf{G}' . We denote $\mathbf{G} = (G, X, \lambda)$. The fact that $\mathbf{G} \equiv_{\mathbf{p}, t} \mathbf{G}'$ implies that

$$\forall \mathbf{F} \in \mathcal{B}^{(t')}, \quad \mathbf{p}(\mathbf{G} \oplus \mathbf{F}) - \mathbf{p}(\mathbf{G}' \oplus \mathbf{F}) = \text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') \quad (18)$$

We apply (18) for $\mathbf{F} := \mathbf{B} = (G[X], X, \lambda_X)$. Notice that $\mathbf{G} \oplus \mathbf{B} = G$ and $\mathbf{G}' \oplus \mathbf{B} = G'$. Therefore $\mathbf{p}(G) = \mathbf{p}(G') + \text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}')$. \square

4.5 Rich protrusions

In this section we introduce the concept of a rich protrusion and we prove a series of results on it that will be useful later.

Let \mathbf{p} be a graph parameter and $f : \mathbb{N} \rightarrow \mathbb{N}$ be some function. Let also $\beta \in \mathbb{N}$, a graph G along with a labeling $\lambda : V(G) \rightarrow \mathbb{N}$ of G , and $R \subseteq V(G)$. We say that R is a (β, f) -rich protrusion of G for \mathbf{p} if:

1. $|\partial_G(R)| \leq \beta$ and the boundaried graph $(G[R], \partial_G(R), \lambda|_R)$ has treewidth at most β ,
2. R is tight in G ,
3. the set R is set-maximal with respect to (1) and (2),
4. $|R| > \mathbf{p}(G[R]) \cdot f(\beta)$.

The third condition asks that there is no tight β -protrusion R' of G such that $R \subsetneq R'$ and $\partial_G(R) = \partial_G(R')$ (see Figure 9). Intuitively, this condition demands that all protrusions with the same boundary should be grouped together to a single one. As a consequence of this, the number of rich protrusions with the same boundary B is bounded by $2^{|B|}$. In the proof of Lemma 19, this condition will be combined with conditions 3 and 5 in the definition of a (α, β) -rooted tree-decomposition (given in Subsection 3.4.2).

Let G be a graph and $Y \subseteq V(G)$. We say that a graph A is an *augmented connected component* for (G, Y) if there is some connected component C of $G \setminus Y$ such that $A = G[Y \cup V(C)]$. We denote by $\mathcal{A}(G, Y)$ the set containing the vertex set of the augmented connected components for (G, Y) .

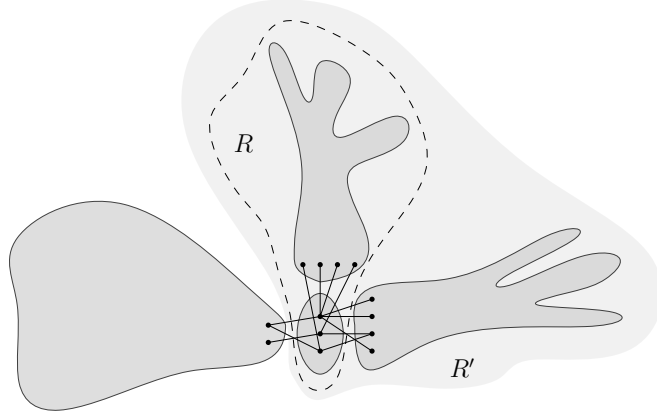


Figure 9: A visualization of the definition of a (β, f) -rich protrusion. Notice that the set R' is a (β, f) -rich protrusion while the set R is not.

4.5.1 Bounding graphs without rich protrusions

Lemma 18. *Let \mathbf{p} be a graph parameter that has Fll. Let also $\mathbf{G} = (G, X, \lambda)$ be bounded graph and let $D = (T, \chi, r)$ be an (α, β) -tree-decomposition of \mathbf{G} . If $\text{children}_{T,r}(r) = \{r^{(1)}, \dots, r^{(\ell)}\}$, then $\sum_{i \in [\ell]} \mathbf{p}(G_{r^{(i)}}) \leq \mathbf{p}(G) + \alpha \cdot \xi_{\mathbf{p}}(\beta)$.*

Proof. From Lemma 16, for every $i \in [\ell]$, there is a graph $\mathbf{G}'_{r^{(i)}}$ where $\mathbf{G}_{r^{(i)}} \equiv_{\mathbf{p}, \beta} \mathbf{G}'_{r^{(i)}}$, $\mathbf{G}'_{r^{(i)}} \leq_m \mathbf{G}_{r^{(i)}}$, and $|\mathbf{G}'_{r^{(i)}}| \leq m$ where $m = \beta \cdot (2^{4\beta \cdot \mu_{\mathbf{p}}(\beta)} - 1)$. We also set $l_i = \text{transp}_{\mathbf{p}}(\mathbf{G}_{r^{(i)}}, \mathbf{G}'_{r^{(i)}})$, $i \in [\ell]$. From Lemma 17, $\mathbf{p}(\mathbf{G}_{r^{(i)}}) = l_i + \mathbf{p}(\mathbf{G}'_{r^{(i)}}) \leq l_i + \tau_{\mathbf{p}}(|\mathbf{G}'_{r^{(i)}}|) \leq l_i + \tau_{\mathbf{p}}(m)$, therefore:

$$\begin{aligned} \sum_{i \in [\ell]} \mathbf{p}(G_{r^{(i)}}) &\leq \sum_{i \in [\ell]} (l_i + \tau_{\mathbf{p}}(m)) = \left(\sum_{i \in [\ell]} l_i \right) + \ell \cdot \tau_{\mathbf{p}}(m) \\ &\leq \left(\sum_{i \in [\ell]} l_i \right) + \alpha \cdot \tau_{\mathbf{p}}(m) = \left(\sum_{i \in [\ell]} l_i \right) + \alpha \cdot \xi_{\mathbf{p}}(\beta) \end{aligned} \quad (19)$$

It holds that for every $i \in [\ell]$,

$$\forall \mathbf{F} \in \mathcal{B}^{(t, r^{(i)})}, \quad \mathbf{p}(\mathbf{G}_{r^{(i)}} \oplus \mathbf{F}) - \mathbf{p}(\mathbf{G}'_{r^{(i)}} \oplus \mathbf{F}) = l_i \quad (20)$$

Let $G^{(0)} = G$. We set:

$$\begin{aligned} \mathbf{F}_0 &= (G^{(0)}, \chi(r^{(0)}), \lambda|_{G^{(0)}} \setminus \tilde{V}_{r^{(0)}}), & G^{(1)} &= \mathbf{F}_0 \oplus \mathbf{G}'_{r^{(0)}} \\ \mathbf{F}_1 &= (G^{(1)}, \chi(r^{(1)}), \lambda|_{G^{(1)}} \setminus \tilde{V}_{r^{(1)}}), & G^{(2)} &= \mathbf{F}_1 \oplus \mathbf{G}'_{r^{(1)}} \\ &\dots & & \\ \mathbf{F}_{\ell-1} &= (G^{(\ell-1)}, \chi(r^{(\ell-1)}), \lambda|_{G^{(\ell-1)}} \setminus \tilde{V}_{r^{(\ell-1)}}), & G^{(\ell)} &= \mathbf{F}_{\ell-1} \oplus \mathbf{G}'_{r^{(\ell-1)}} \end{aligned}$$

If we repetitively apply (20) and Observation 1 for $i \in [\ell]$ and $\mathbf{F} = \mathbf{F}_i$, we have that $\mathbf{p}(G^{(i)}) = \mathbf{p}(G^{(i-1)}) - l_i$. Combining these equalities altogether we conclude that $\mathbf{p}(G) - \sum_{i \in [\ell]} l_i = \mathbf{p}(G^{(\ell)}) \geq 0$, therefore $\sum_{i \in [\ell]} l_i \leq \mathbf{p}(G)$. This, together with (19), complete the proof of the lemma. \square

The following lemma is an important ingredient of our proof. It provides a constant d such that if $|G| > d \cdot \mathbf{p}(G)$, then G contains a big enough protrusion. In Subection 4.6 we will explain how such a protrusion can be compressed to an equivalent one.

Lemma 19. *Let \mathbf{p} be a graph parameter that has FII and is protrusion decomposable. Let also $c = \text{dec}(\mathbf{p})$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. For every G , if G does not contain any $(2c, f)$ -rich protrusion W for \mathbf{p} , then $|G| \leq d \cdot \mathbf{p}(G)$ where $d = (f(2c) \cdot (c \cdot \xi_{\mathbf{p}}(2c) + 1) + c)$.*

Proof. According to the definition of protrusion decomposability in (2), G has a $(c \cdot \mathbf{p}(G), c)$ -protrusion decomposition. Let λ be a labeling of G . From Lemma 9 G is the underlying graph of some boundaried graph $\mathbf{G} = (G, X, \lambda)$ that has a $(c \cdot \mathbf{p}(G), 2c)$ -tree-decomposition $D = (T, \chi, r)$. We consider such a $(c \cdot \mathbf{p}(G), 2c)$ -tree-decomposition D so that $\chi(r)$ is minimized. Let $\{r^{(1)}, \dots, r^{(\ell)}\}$ be the set of children of r in (T, r) . A consequence of the minimality of $\chi(r)$ and Conditions (4) and (5) of the definition of an (a, b) -tree-decomposition, is that property (3) in the definition of a $(2c, f)$ -rich protrusion is satisfied for all $V_{r^{(i)}}, i \in [\ell]$. As none of $V_{r^{(i)}}$ can be a $(2c, f)$ -rich protrusion of G for \mathbf{p} , we deduce that:

$$\forall i \in [\ell], \quad |\mathbf{G}_{r^{(i)}}| \leq \mathbf{p}(G_{r^{(i)}}) \cdot f(t_{r^{(i)}}) \quad (21)$$

From Lemma 18 it holds that:

$$\sum_{i \in [\ell]} \mathbf{p}(G_{r^{(i)}}) \leq \mathbf{p}(G) + c \cdot \mathbf{p}(G) \cdot \xi_{\mathbf{p}}(2c) = \mathbf{p}(G) \cdot (1 + c \cdot \xi_{\mathbf{p}}(2c)) \quad (22)$$

Using (21) and (22) we deduce that $|G| \leq |X| + \sum_{i \in [\ell]} |\mathbf{G}_{r^{(i)}}| \leq c \cdot \mathbf{p}(G) + f(2c) \cdot \sum_{i \in [\ell]} \mathbf{p}(G_{r^{(i)}}) \leq c \cdot \mathbf{p}(G) + f(2c) \cdot \mathbf{p}(G) \cdot (1 + c \cdot \xi_{\mathbf{p}}(2c)) \leq (f(2c) \cdot (c \cdot \xi_{\mathbf{p}}(2c) + 1) + c) \cdot \mathbf{p}(G)$ which gives the claimed upper bound. \square

4.6 Compressing a null-transition pair

In this subsection we prove a series of lemmata that permit us to compress a graph with a rich protrusion to an equivalent one. This is achieved in Lemma 23 and is prepared in Lemmata 20, 21, and 22.

Let \mathbf{G} be a boundaried graph and $D = (T, \chi, r)$ a (α, β) -tree-decomposition of it. The *potential* of a transition pair (a, b) of (\mathbf{G}, D) is defined as $\text{potential}_{(\mathbf{G}, D)}(a, b) = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b)$. If the potential of a transition pair (a, b) is 0, then we say that (a, b) is a *null* transition pair. A *transition collection* of (\mathbf{G}, D) is a pair collection \mathcal{P} of (T, r) consisting of transition pairs of (\mathbf{G}, D) . The *potential* of \mathcal{P} , denoted by $\text{potential}_{(\mathbf{G}, D)}(\mathcal{P})$ is the sum of the potentials of the pairs of \mathcal{P} .

4.6.1 Bounding potentials

Lemma 20. *Let \mathbf{p} be a graph parameter that has FII and let $t \in \mathbb{N}$. Let also \mathbf{G} be a boundaried graph and $D = (T, \chi, r)$ a binary tree-decomposition of \mathbf{G} of width at most t . For every transition collection \mathcal{P} of (\mathbf{G}, D) , it holds that $\text{potential}_{(\mathbf{G}, D)}(\mathcal{P}) \leq \mathbf{p}(G)$.*

Proof. Notice that the lemma is obvious when $\text{potential}_{(\mathbf{G}, D)}(\mathcal{P}) = 0$. Assume now that the lemma holds for every choice of (\mathbf{G}, D) and \mathcal{P} with $\text{potential}_{(\mathbf{G}, D)}(\mathcal{P}) < k$, where $k \in \mathbb{N}^+$. Let \mathcal{P} be a transition collection of (\mathbf{G}, D) where $\text{potential}_{(\mathbf{G}, D)}(\mathcal{P}) = k > 0$. Clearly, \mathcal{P} contains some non-null-transition pair (a, b) . We set $t := t_a = t_b$. Assume that $\ell = \text{potential}_{(\mathbf{G}, D)}(a, b) = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b)$ and keep in mind that $\ell > 0$. We set $\mathcal{P}' = \mathcal{P} \setminus \{(a, b)\}$.

Let (\mathbf{G}', D') be the (a, b) -compression of (\mathbf{G}, D) .

Claim: $\text{potential}_{(\mathbf{G}', D')}(\mathcal{P}') = k - \ell$.

Proof of claim: Let $\mathcal{P}_{\text{down}}$ contain all the pairs $(i, j) \in \mathcal{P}$ such that $i \in \text{children}_{T, r}(b)$, let $\mathcal{P}_{\text{side}}$ contains all the pairs $(i, j) \in \mathcal{P}$ such that $i \neq_{T, r} a$ and let \mathcal{P}_{up} contains all the pairs $(i, j) \in \mathcal{P}$ such that $i \geq_{T, r} a$. The fact that all pairs \mathcal{P} are pairwise non-interfering pairs of (T, r) ,

implies that $\{\mathcal{P}_{\text{down}}, \mathcal{P}_{\text{up}}, \mathcal{P}_{\text{side}}\}$ is a partition of \mathcal{P}' . Notice that if $(x, y) \in \mathcal{P}_{\text{down}} \cup \mathcal{P}_{\text{side}}$, then $\mathbf{G}'_x = \mathbf{G}_x$, $\mathbf{G}'_y = \mathbf{G}_y$, therefore $\text{potential}_{(\mathbf{G}', D')}(x, y) = \text{potential}_{(\mathbf{G}, D)}(x, y)$. Suppose that $(x, y) \in \mathcal{P}_{\text{up}}$. We apply Lemma 10 on \mathbf{G} by setting $i_1 := x$, $i_2 = y$, $i_3 = a$, and $\mathbf{G}_{\text{new}} := \mathbf{G}_b$ and deduce that $\mathbf{G}'_x \equiv_{\mathbf{p}, t} \mathbf{G}'_y$ and $\text{transp}_{\mathbf{p}}(\mathbf{G}'_x, \mathbf{G}'_y) = \text{transp}_{\mathbf{p}}(\mathbf{G}_x, \mathbf{G}_y)$. This implies that $\text{potential}_{(\mathbf{G}', D')}(x, y) = \text{potential}_{(\mathbf{G}, D)}(x, y)$. As the potentials of the pairs in \mathcal{P}' do not change, we conclude that $\text{potential}_{(\mathbf{G}', D')}(\mathcal{P}') = \text{potential}_{(\mathbf{G}, D)}(\mathcal{P}) - \text{potential}_{(\mathbf{G}, D)}(a, b) = k - \ell$. The induction hypothesis along with the above claim yield that $\mathbf{p}(G') \geq k - \ell$. Recall now that $\mathbf{G}_a \equiv_{\mathbf{p}, t} \mathbf{G}_b$ and $\mathbf{G}'_a = \mathbf{G}_b$. This implies $\mathbf{G}_a \equiv_{\mathbf{p}, t} \mathbf{G}'_a$ and that $\text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}'_a) = \ell$. We have

$$\forall \mathbf{F} \in \mathcal{B}^{(t)}, \quad \mathbf{p}(\mathbf{G}_a \oplus \mathbf{F}) - \mathbf{p}(\mathbf{G}'_a \oplus \mathbf{F}) = \ell \quad (23)$$

By applying (23) for $\mathbf{F} = \overline{\mathbf{G}}_a$ and taking into account that $\mathbf{G}_a \oplus \overline{\mathbf{G}}_a = G$ and $\mathbf{G}'_a \oplus \overline{\mathbf{G}}_a = G'$, we conclude that $\mathbf{p}(G') = \mathbf{p}(G) - \ell$. This, together with the fact that $\mathbf{p}(G') \geq k - \ell$, imply that $k \leq \mathbf{p}(G)$ as required. \square

The following lemma is an important ingredient of our proof. Notice that it is the first time that we ask \mathbf{p} to be minor-closed.

Lemma 21. *Let \mathbf{p} be a graph parameter that has Fill and is minor-closed and let $t \in \mathbb{N}$. If \mathbf{G} is a bounded graph, D is a binary rooted tree-decomposition of \mathbf{G} , and (a, b) is a transition pair of (\mathbf{G}, D) , then $\text{potential}_{(\mathbf{G}, D)}(a, b) \geq 0$.*

Proof. Let $\mathbf{Q} = \text{rep}_{\mathbf{p}}(\mathbf{G}_a) = \text{rep}_{\mathbf{p}}(\mathbf{G}_b)$. Recall that $\text{potential}_{(\mathbf{G}, D)}(a, b) = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b)$. It remains to prove that $\text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b) \geq 0$. Let $t' = t_a = t_b$ and $\mathbf{H} = (H_a, \chi(a), \lambda_a)$. Notice that $\mathbf{G}_a \oplus \mathbf{H} = G_a$ and $\mathbf{G}_b \oplus \mathbf{H} = G_b$. Recall that $\mathbf{G}_a \equiv_{\mathbf{p}, t} \mathbf{G}_b$ implies that

$$\forall \mathbf{F} \in \mathcal{B}^{(t')}, \quad \mathbf{p}(\mathbf{G}_a \oplus \mathbf{F}) - \mathbf{p}(\mathbf{G}_b \oplus \mathbf{F}) = \text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b) \quad (24)$$

By applying (24) for $\mathbf{F} := \mathbf{H}$, we have that $\text{transp}_{\mathbf{p}}(\mathbf{G}_a, \mathbf{G}_b) = \mathbf{p}(G_a) - \mathbf{p}(G_b)$. Recall that $G_b \leq_m G_a$. As \mathbf{p} is minor closed, it holds that $\mathbf{p}(G_b) \leq \mathbf{p}(G_a)$ and the lemma follows. \square

4.6.2 A combinatorial lemma on pair collections

Lemma 22. *Let (T, r) be a rooted binary tree and $z \geq 2$ an integer. There exists a pair collection of minimum capacity at least z that has more than $\frac{|T|+1}{(z-1)^2+z}$ elements.*

Proof. In our proof we “arrange” T so that for every two leafs of T the one in the left is the one of the biggest depth. Let f be a function that takes as input a rooted binary tree (T, r) and finds a vertex u such that $\text{capacity}_{(T, r)}(r, u) \geq z$, rTu is the leftmost such path in T , and $|V(rTu)|$ is the minimum possible, or outputs null when there is no such vertex in T . Notice that if $f(T, r) = \text{null}$ then $|T| \leq z - 1$.

We denote the left child of a vertex u by u^L and the right by u^R . We also define the set $\mathcal{V}_u = \mathbf{U}_{v \in V(rTu)} \text{children}_{T, r}(v) \setminus (V(rTu) \cup \text{children}_{T, r}(u))$ (see Figure 10). Notice that, if u is the output of f , then $|\mathcal{V}_u| \leq z - 1$.

We define a pair collection $C(T, r)$ using the following recursive formula:

$$C(T, r) = \begin{cases} C(T_{f(T, r)^L}, f(T, r)^L) \cup C(T_{f(T, r)^R}, f(T, r)^R) & , \text{ if } |T| > z \text{ and} \\ \cup \{(r, f(T, r))\} & \bigwedge_{v \in \mathcal{V}_{f(T, r)}} |T_v| \leq z - 1 \\ C(T_{r^L}, r^L) \cup C(T_{r^R}, r^R) & , \text{ if } |T| > z \text{ and} \\ \bigvee_{v \in \mathcal{V}_{f(T, r)}} |T_v| \geq z \\ \{(r, f(T, r))\} & , \text{ if } |T| = z \\ \emptyset & , \text{ if } |T| < z \end{cases}$$

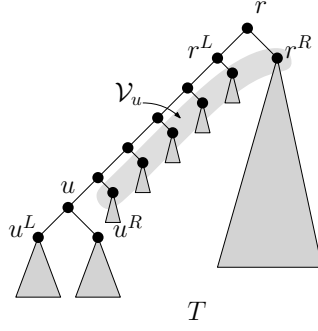


Figure 10: The trees and vertices defined in the proof of Lemma 22, where $u = f(T, r)$.

It is straightforward to show that the pairs in $C(T, r)$ are non-interfering and have capacity at least z . Let $a(T, r)$ be the number of elements of $C(T, r)$. This number can be computed from the following recurrence relation:

$$a(T, r) = \begin{cases} a(T_{f(T,r)^L}, f(T, r)^L) + a(T_{f(T,r)^R}, f(T, r)^R) + 1 & , \text{ if } |T| > z \text{ and} \\ & \bigwedge_{v \in \mathcal{V}_{f(T,r)}} |T_v| \leq z - 1 \\ a(T_{r^L}, r^L) + a(T_{r^R}, r^R) & , \text{ if } |T| > z \text{ and} \\ & \bigvee_{v \in \mathcal{V}_{f(T,r)}} |T_v| \geq z \\ 1 & , \text{ if } |T| = z \\ 0 & , \text{ if } |T| < z \end{cases}$$

We will prove by induction that $a(T, r) \geq \frac{|T|+1}{(z-1)^2+z}$: Clearly, if $|T| = z$, $C(T, r)$ contains only one pair, namely $(r, f(T, r))$, thus $a(T, r) = 1$. As $z + 1/((z-1)^2 + z)$ for $z \geq 2$ is at most 1 the claim holds.

Let $|T| > z$ and let $u = f(T, r)$. We assume that the claim holds for every tree with at most $|T| - 1$ vertices. We distinguish two cases:

Case 1: if $\bigwedge_{v \in \mathcal{V}_u} |T_v| \leq z - 1$ then:

$$\begin{aligned} a(T, r) &= a(T_{u^L}, u^L) + a(T_{u^R}, u^R) + 1 \geq \frac{|T_{u^L}| + 1}{(z-1)^2 + z} + \frac{|T_{u^R}| + 1}{(z-1)^2 + z} + 1 \\ &\geq \frac{|T_{u^L}| + |T_{u^R}| + 2}{(z-1)^2 + z} + 1 \end{aligned}$$

Notice that $|T_{u^L}| + |T_{u^R}| \geq |T| - (|\mathcal{V}_u| \cdot (z-1) + |rTu|) \geq |T| - ((z-1) \cdot (z-1) + z)$ (see Figure 10). Therefore:

$$a(T, r) \geq \frac{|T| - (z-1)^2 - z + 2}{(z-1)^2 + z} + 1 = \frac{|T| + 2}{(z-1)^2 + z} > \frac{|T| + 1}{(z-1)^2 + z}$$

Case 2: if $\bigvee_{v \in \mathcal{V}_{f(T,r)}} |T_v| \geq z$ then:

$$\begin{aligned} a(T, r) &= a(T_{r^L}, r^L) + a(T_{r^R}, r^R) \geq \frac{|T_{r^L}| + 1}{(z-1)^2 + z} + \frac{|T_{r^R}| + 1}{(z-1)^2 + z} \\ &= \frac{|T| - 1 + 2}{(z-1)^2 + z} = \frac{|T| + 1}{(z-1)^2 + z} \end{aligned}$$

Hence, the claim holds for every case. \square

4.6.3 Protrusion compression

Lemma 23. *Let \mathbf{p} be a graph parameter that has Fill, and is minor-closed. Let also G^* be a graph and R a $(t, \delta_{\mathbf{p}})$ -rich protrusion of G^* for \mathbf{p} , $t \in \mathbb{N}$. Then there exists a graph G^{**} such that $|G^{**}| < |G^*|$, $G^{**} \leq_m G^*$, and $\mathbf{p}(G^{**}) = \mathbf{p}(G^*)$.*

Proof. Recall that $\delta_{\mathbf{p}}(x) = x((4x \cdot \mu_{\mathbf{p}}(x) - 1)^2 + 4x \cdot \mu_{\mathbf{p}}(x))$. We set $Z = \partial_{G^*}(R)$, $G = G^*[R]$, and $\mathbf{G} = (G, Z, \lambda|_R)$ where λ is a some labelling of G . From Lemma 6 there exists a binary tree-decomposition of \mathbf{G} , say $D = (T, \chi, r)$, of width $t - 1$.

Notice that $|T| \geq |R|/t$, because of Lemma 4. From Lemma 22 there exists a pair collection \mathcal{Q} of (T, r) of minimum capacity $z = 4t \cdot \mu_{\mathbf{p}}(t)$ such that:

$$\begin{aligned} |\mathcal{Q}| &\geq \frac{|T| + 1}{(z - 1)^2 + z} > \frac{|T|}{(z - 1)^2 + z} \geq \frac{|R|}{t((z - 1)^2 + z)} \\ &= \frac{|R|}{t((4t \cdot \mu_{\mathbf{p}}(t) - 1)^2 + 4t \cdot \mu_{\mathbf{p}}(t))} = \frac{|R|}{\delta_{\mathbf{p}}(t)} \geq \mathbf{p}(G) \end{aligned}$$

We assume that $\mathcal{Q} = \{(a_1, b_1), \dots, (a_s, b_s)\}$ where we know that $s \geq \mathbf{p}(G) + 1$.

Using Lemma 13 for every $(a_i, b_i) \in \mathcal{Q}$, $i \in [s]$, we conclude that there exists a binary tree-decomposition $D' = (T', \chi', r)$ of \mathbf{G} of width at most $t - 1$ such that, for every $i \in [s]$, D' is (a_i, b_i) -linked and $\text{capacity}_{(T', r)}(a_i, b_i) \geq \frac{1}{4t} \cdot z \geq \mu_{\mathbf{p}}(t)$.

From Lemma 14, we know that each $\text{inner}_{T', r}(a_i, b_i)$, $i \in [s]$ contains a vertical pair (a'_i, b'_i) that is a transition pair for (\mathbf{G}, D') . Let $\mathcal{Q}' = \{(a'_1, b'_1), \dots, (a'_s, b'_s)\}$ be such a collection of transition pairs for (\mathbf{G}, D') . From Lemma 20, $|\mathcal{Q}'| = s > \mathbf{p}(G) \geq \text{potential}_{(\mathbf{G}, D')}(\mathcal{Q}')$, therefore $|\mathcal{Q}'| > \text{potential}_{(\mathbf{G}, D')}(\mathcal{Q}')$. From Lemma 21, all transition pairs in \mathcal{Q}' are of non-negative potential, therefore \mathcal{Q}' contains at least one null-transition pair, say (a, b) . Clearly (a, b) is a vertical pair of T' where a, b belong to the same $\text{inner}_{T', r}(a_i, b_i)$ for some $i \in [s]$.

Let (\mathbf{G}', D'') be the (a, b) -compression of (\mathbf{G}, D') . From Lemma 11 we deduce that $\mathbf{G}' \leq_m \mathbf{G}$, $|\mathbf{G}'| < |\mathbf{G}|$, $\mathbf{G}' \equiv_{\mathbf{p}, t} \mathbf{G}$ and $\text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}') = \text{transp}_{\mathbf{p}}(\mathbf{G}_x, \mathbf{G}_y) = 0$. Let $F = G^* \setminus \tilde{V}_r$, i.e., F contains all vertices of G^* except from the non-boundary vertices of \mathbf{G} . Set $\mathbf{F} = (G^*[F], \chi'(r), \lambda|_F)$ and observe that $G^* = \mathbf{F} \oplus \mathbf{G}$. Let $G^{**} = \mathbf{F} \oplus \mathbf{G}'$. From $|\mathbf{G}'| < |\mathbf{G}|$ and $\mathbf{G}' \leq_m \mathbf{G}$, we obtain $|G^{**}| < |G^*|$, $G^{**} \leq_m G^*$. Moreover, $\mathbf{G}' \equiv_{\mathbf{p}, t} \mathbf{G}$ implies that

$$\forall \mathbf{B} \in \mathcal{B}^{(t_r)}, \quad \mathbf{p}(\mathbf{G} \oplus \mathbf{B}) - \mathbf{p}(\mathbf{G}' \oplus \mathbf{B}) = \text{transp}_{\mathbf{p}}(\mathbf{G}, \mathbf{G}'). \quad (25)$$

By setting $\mathbf{B} := \mathbf{F}$ in (25), we obtain that $\mathbf{p}(G^*) - \mathbf{p}(G^{**}) = 0$ and the lemma follows. \square

4.6.4 The proof of Theorem 2

Now we have all the tools we need to prove Theorem 2.

Proof of Theorem 2. Let $\beta = 2 \cdot \text{dec}(\mathbf{p})$ and $f = \delta_{\mathbf{p}}$. Let G' be a minimum size graph satisfying conditions 1 and 2 of Theorem 2, i.e., $G' \leq_m G$ and $\mathbf{p}(G') = \mathbf{p}(G)$. It remains to prove that $|G'| \leq c_{\mathbf{p}} \cdot \mathbf{p}(G)$. Suppose, to the contrary that $|G'| > c_{\mathbf{p}} \cdot \mathbf{p}(G)$. By the minimality of the choice of G' and Lemma 23 G' does not contain any (β, f) -rich protrusion for \mathbf{p} . Therefore, from Lemma 19 we conclude that $|G'| \leq c_{\mathbf{p}} \cdot \mathbf{p}(G')$ (recall that $c_{\mathbf{p}} = (\delta_{\mathbf{p}}(2 \cdot \text{dec}(\mathbf{p})) \cdot (2 \cdot \text{dec}(\mathbf{p}) \cdot \xi_{\mathbf{p}}(2 \cdot \text{dec}(\mathbf{p})) + 1) + \text{dec}(\mathbf{p}))$), a contradiction. \square

5 Conclusions and open problems

In this paper we study the obstructions of the graph class $\mathcal{A}_k(\mathcal{H})$, i.e., the k -extensions of \mathcal{H} -minor free graphs when \mathcal{H} consists of connected graphs and at least one of the graphs in

\mathcal{H} is planar. We prove that such obstructions have size linear in k when they satisfy some certain sparsity properties. The proof is based on a more abstract result whose statement and proof use algorithm-driven notions, such as Fill and protrusion decomposability. A first improvement of our result would be to waive the connectivity requirement on the graphs of \mathcal{H} .

It is worth to comment here some results from parameterized complexity and, in particular, from the theory of kernelization (see [17, 26, 48] for textbooks on parameterized algorithms and the corresponding parameterized complexity classes – see also [28] for the relation between kernelization and obstructions).

Consider the problem, known as PLANAR \mathcal{H} -DELETION, that, given a graph G and a non-negative integer k , asks whether G is an k -apex extension of some $\text{exc}(\mathcal{H})$ (for some planar collection \mathcal{H}). It was proven in [47] that, when $\mathcal{H} = \{K_2\}$ (i.e., we consider VERTEX COVER), this problem admits a kernel of size $2k$. This means the existence of a preprocessing procedure that, given G and k , can produce an equivalent instance G', k' of at most $2k$ vertices where, moreover, G' is a subgraph of G . This last property implies a bound of size $2k + 1$ to the size of all graphs in $\text{obs}(\mathcal{A}_k(\{K_2\}))$. In the much more general result in [30], a kernel of size $k^{O(1)}$ was given for PLANAR \mathcal{H} -DELETION. Moreover this kernel produces an equivalent instance whose graph is a *minor* of the original one. Again, this last property implies that each graph in $\text{obs}(\mathcal{A}_k(\mathcal{H}))$ has at most $k^{c_{\mathcal{H}}}$ vertices. In [34], it was proved that this problem does not admit a uniformly polynomial kernel, i.e., a kernel of size at most $c_{\mathcal{H}} \cdot k^c$, for all \mathcal{H} , unless $\text{NP} \subseteq \text{coNP/poly}$.

Given the above complexity bounds, one may be tempted to conjecture that the obstruction bound of [30] *cannot* be improved to a uniform one, i.e., one of the type $c_{\mathcal{H}} \cdot k^c$ where c is a universal constant and $c_{\mathcal{H}}$ a constant depending only on \mathcal{H} . Actually we would like to argue to the opposite direction. Empirical evidence on obstruction detection indicates that the *sparser an obstruction is, the bigger becomes the number of its vertices*. For apex extensions this evidence is supported by the graphs in $\text{obs}(\mathcal{A}_k(\{K_2\}))$ in [12, 24, 25] or $\text{obs}(\mathcal{A}_k(\{K_3\}))$ in [22] or in $\text{obs}(\mathcal{A}_k(\{K_4, K_{2,3}\}))$ in [20]. Furthermore, the same empiric observation can also be done by inspecting other known obstruction sets (see [5, 7, 9, 27, 36, 40, 50, 55, 56]). Based on this, we conjecture the following:

Conjecture 1. *For every collection \mathcal{H} of graphs containing at least one planar graph and for every non-negative integer k , every graph in $\text{obs}(\mathcal{A}_k(\mathcal{H}))$ has at most $c_{\mathcal{H}} \cdot k^{c'}$ vertices, for some universal c' and some $c_{\mathcal{H}}$ depending only on \mathcal{H} .*

An other interesting question is whether polynomial bounds can also be proved when \mathcal{H} does not contain any planar graph. We would avoid to make a conjecture on this.

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