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# On the solution existence and stability of polynomial optimization problems

Vu Trung Hieu

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**Abstract** This paper introduces and investigates a regularity condition in the asymptotic sense for optimization problems whose objective functions are polynomial. Under this regularity condition, the normalization argument in asymptotic analysis enables us to see the solution existence as well as the solution stability of these problems. We prove a Frank-Wolfe type theorem for regular optimization problems and an Eaves type theorem for non-regular pseudoconvex optimization problems. Moreover, we show results on the stability such as upper semicontinuity and local upper-Hölder stability of the solution map of polynomial optimization problems. At the end of the paper, we discuss the genericity of the regularity condition.

**Keywords** Polynomial optimization · Regularity condition · Asymptotic cone · Frank–Wolfe type theorem · Eaves type theorem · Upper semicontinuity · Local upper-Hölder stability · Genericity

**Mathematics Subject Classification (2000)** 90C30 · 14P10

## 1 Introduction

We consider the following optimization problem

$$\text{minimize } f(x) \text{ subject to } x \in K,$$

where  $K$  is a nonempty closed subset of  $\mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a polynomial in  $n$  variables of degree  $d \geq 2$ . The problem and its solution set are denoted by  $\text{OP}(K, f)$  and  $\text{Sol}(K, f)$ , respectively. Let  $f_d$  be the homogeneous component of degree  $d$  of  $f$ ,  $K_\infty$  be the asymptotic cone of  $K$  that will be introduced in

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Section 2. We say that  $\text{OP}(K, f)$  is regular if the solution set of the asymptotic problem  $\text{OP}(K_\infty, f_d)$  is bounded, and the problem is non-regular if otherwise. The regularity condition has appeared in investigations of the solution existence and stability in quadratic programming (see, e.g., [1, 2] and the references therein).

Asymptotic cones and functions play an important role in optimization and variational inequalities [3]. The normalization argument in asymptotic analysis enables us to see the solution existence and stability not only for quadratic programming, linear complementarity problems, and affine variational inequalities (see, e.g., [1, 4]) but also for polynomial complementarity problems and polynomial variational inequalities having unbounded constraint sets (see, e.g., [5, 6]). In this paper, the normalization argument is used as the main technique to investigate the solution existence as well as the stability of polynomial optimization problems.

In 1956, Frank and Wolfe [7] proved that if  $K$  is polyhedral,  $f$  is quadratic and bounded from below on  $K$ , then  $\text{Sol}(K, f)$  is nonempty. Several extensions of the Frank-Wolfe theorem for quadratic, cubic, and polynomial optimization problems have been shown in [1, 2, 8–12]. For convex and quasi-convex polynomial optimization problems, Belousov and Klatte [9], and Obuchowska [10] have proved Frank-Wolfe type theorems. Recently, by using a technique from semi-algebraic geometry, Dinh, Ha and Pham [11] have showed a Frank-Wolfe type theorem for nondegenerate problems. The present paper gives another Frank-Wolfe type theorem, which says that if  $\text{OP}(K, f)$  is regular and  $f$  is bounded from below on  $K$ , then the problem has a solution. Besides, the Eaves theorem [13] provides us another criterion for the solution existence of quadratic optimization problems. Extensions of this theorem for quadratically constrained quadratic problems have been investigated in [1, 2, 14, 15]. This paper introduces an Eaves type theorem for non-regular pseudoconvex optimization problem  $\text{OP}(K, f)$ , where  $K$  is convex.

Under the assumption that the constraint set  $K$  is compact and semi-algebraic, some stability and genericity results for polynomial optimization problems have been shown by Lee and Pham [16]. If  $K$  is compact, then its asymptotic cone is trivial, i.e.,  $K_\infty = \{0\}$ ; obviously, it follows that  $\text{OP}(K, f)$  satisfies the regularity condition. In the present paper,  $K$  may be unbounded. Under the regularity condition, we prove several local properties of the solution map of polynomial optimization problems such as local boundedness, upper semicontinuity; Furthermore, based on an error bound for a polynomial system in [20], we prove the local upper-Hölder stability of the solution map.

We denote by  $\mathcal{R}_d \subset \mathbb{R}[x]$  the set of all polynomials  $g$  of degree  $d$  such that  $\text{OP}(K, g)$  is regular. This set is an open cone in the space  $\mathcal{P}_d \subset \mathbb{R}[x]$  of all polynomials of degree at most  $d$ . At the end of this work, given  $K$  by convex polynomials, we prove that  $\mathcal{R}_d$  is generic in  $\mathcal{P}_d$ .

The organization of the paper is as follows. Section 2 gives a brief introduction to asymptotic cones, polynomials, and the regularity condition. Section 3 proves two criteria of the solution existence. Section 4 investigates properties

of the solution map. The last section discusses the genericity of the regularity condition.

## 2 Preliminaries

Recall that the *asymptotic cone* [3] of a nonempty closed subset  $S$  in  $\mathbb{R}^n$  is defined and denoted by

$$S_\infty = \left\{ v \in \mathbb{R}^n : \exists t_k \rightarrow +\infty, \exists x_k \in S \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = v \right\}.$$

Clearly, the cone  $S_\infty$  is closed and contains 0. The set  $S$  is bounded if and only if  $S_\infty$  is trivial. Furthermore, if  $S$  is convex then  $S_\infty$  is a closed convex cone and  $S_\infty = 0^+S$ , where  $0^+S$  is the *recession cone* of  $S$ , that is the set of vectors  $v \in \mathbb{R}^n$  such that  $x + tv \in S$  for any  $x \in S$  and  $t \geq 0$ . Thus, one has  $S = S + S_\infty$  when  $S$  is convex.

Let  $d \geq 2$  be given. The space  $\mathcal{P}_d$  of all polynomials of degree at most  $d$  has finite dimension; its dimension is denoted by  $\rho$ . Let  $X(x)$  be the vector consisting of  $\rho$  monomials of degree at most  $d$  which is listed by lexicographic ordering

$$X(x) := (1, x_1, x_2, \dots, x_n, x_1^2, x_1x_2, \dots, x_1x_n, \dots, x_1^d, x_1^{d-1}x_2, \dots, x_n^d)^T.$$

For every  $g \in \mathcal{P}_d$ , there exists a unique vector  $a = (a_1, \dots, a_\rho) \in \mathbb{R}^\rho$  such that  $g(x) = a^T X(x)$ . We denote by  $\|g\|$  the  $\ell_2$ -norm of the polynomial  $g$ , namely

$$\|g\| := \|a\| = \sqrt{a_1^2 + \dots + a_\rho^2}.$$

The Cauchy–Schwarz inequality yields  $|g(x)| \leq \|X(x)\| \|g\|$ . Furthermore, if  $\{g^k\}$  is a convergent sequence in  $\mathcal{P}_d$  with  $g^k \rightarrow g$ , then  $g_d^k \rightarrow g_d$ .

Throughout the paper, we assume that the constraint set  $K \subset \mathbb{R}^n$  is nonempty and closed, and the objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a polynomial of degree  $d \geq 2$ .

We say that  $\text{OP}(K, f)$  is a *polynomial optimization problem* if  $K$  is given by polynomials. With the given set  $K$  and the given integer  $d$ , the solution map of polynomial optimization problems  $\text{OP}(K, g)$ , where  $g \in \mathcal{P}_d$ , is defined by

$$\text{Sol}_K(\cdot) : \mathcal{P}_d \rightrightarrows \mathbb{R}^n, \quad g \mapsto \text{Sol}(K, g).$$

Assume that  $g \in \mathcal{P}_d$  with  $\deg g = d$  and  $g = g_d + \dots + g_1 + g_0$ , where  $g_l$  is a homogeneous polynomial of degree  $l$ , i.e.,  $g_l(tx) = t^l g_l(x)$  for all  $t \geq 0$  and  $x \in \mathbb{R}^n$ ,  $l \in [d] := \{1, \dots, d\}$ , and  $g_0 \in \mathbb{R}$ . Then,  $g_d$  is the *leading term* (or the recession polynomial) of the polynomial  $g$  (of degree  $d$ ). Clearly, one has

$$g_d(x) = \lim_{\lambda \rightarrow +\infty} \frac{g(\lambda x)}{\lambda^d}, \quad \forall x \in \mathbb{R}^n.$$

For the pair  $(K, f)$ , the asymptotic pair  $(K_\infty, f_d)$  is unique. The asymptotic optimization problem  $\text{OP}(K_\infty, f_d)$  plays a vital role in the investigation of

behavior of  $\text{OP}(K, f)$  at infinity. The following remarks point out (without proof) the basic properties of the asymptotic problem.

*Remark 2.1* Since  $f_d$  is a homogeneous polynomial and  $K_\infty$  is a closed cone, the asymptotic optimization problem  $\text{OP}(K_\infty, f_d)$  has a solution if and only if  $f_d$  is non-negative on  $K_\infty$ .

*Remark 2.2* Assume that  $\text{Sol}(K_\infty, f_d)$  is nonempty. Then, this set is a closed cone with  $0 \in \text{Sol}(K_\infty, f_d)$ . In addition,  $\text{Sol}(K_\infty, f_d)$  coincides with the zero set of  $f_d$  in  $K_\infty$ , i.e.,

$$\text{Sol}(K_\infty, f_d) = \{x \in K_\infty : f_d(x) = 0\}.$$

Now, we introduce the regularity notion concerning the boundedness of the solution set of  $\text{OP}(K_\infty, f_d)$ .

**Definition 2.1** The problem  $\text{OP}(K, f)$  is said to be *regular* if  $\text{Sol}(K_\infty, f_d)$  is bounded and *non-regular* if otherwise.

Denote by  $\mathcal{E}_d$  (resp.,  $\mathcal{O}_d, \mathcal{U}_d$ ) the set of all polynomials  $g$  of degree  $d$  such that  $\text{Sol}(K_\infty, f_d)$  is the empty set (resp., the trivial cone, an unbounded cone). Clearly,  $\mathcal{R}_d = \mathcal{E}_d \cup \mathcal{O}_d$ , and one has the following disjoint union:

$$\mathcal{P}_d = \mathcal{P}_{d-1} \cup \mathcal{E}_d \cup \mathcal{O}_d \cup \mathcal{U}_d. \quad (2.1)$$

*Remark 2.3* The boundedness of  $\text{Sol}(K_\infty, f_d)$  implies that of  $\text{Sol}(K, f)$ . Indeed, assume to the contrary that  $\text{Sol}(K, f)$  is unbounded. There exists an unbounded sequence  $\{x^k\} \subset \text{Sol}(K, f)$ . Without loss of generality, we can assume  $x^k$  is nonzero for all  $k$ ,  $\|x^k\| \rightarrow +\infty$ , and  $\|x^k\|^{-1}x^k \rightarrow \bar{x}$  for some  $\bar{x} \in \mathbb{R}^n$  with  $\|\bar{x}\| = 1$ . Note that  $f(x^k) = f^*$ , where  $f^* \in \mathbb{R}$  is the minimum of  $f$  over  $K$ , for all  $k$ . By dividing the last equation by  $\|x^k\|^d$  and letting  $k \rightarrow +\infty$ , we obtain  $f_d(\bar{x}) = 0$ . It follows that  $\bar{x} \in \text{Sol}(K_\infty, f_d)$ . Since  $\bar{x} \neq 0$ , the cone  $\text{Sol}(K_\infty, f_d)$  is unbounded, which contradicts our assumption. Thus, the claim is proved.

*Remark 2.4* We show that  $\mathcal{R}_d$  is nonempty. If  $K$  is bounded then  $\mathcal{R}_d$  coincides with the set of all  $g \in \mathcal{P}_d$  such that  $\deg g = d$ . Hence, we can suppose that  $K$  is unbounded. Clearly, the cone  $K_\infty$  also is unbounded. Let  $\bar{x} \in K_\infty$  be nonzero. There exists  $l \in [n]$  such that  $\bar{x}_l \neq 0$ . Let us define a homogeneous polynomial of degree  $d$  as  $f(x) := -(\bar{x}_l x_l)^d$ . For any  $t > 0$ , one has  $t\bar{x} \in K_\infty$ , and  $f(t\bar{x}) = -(\bar{x}_l^2)^d t^d \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Then,  $f$  is not bounded from below on  $K_\infty$ . This yields  $\text{Sol}(K_\infty, f) = \emptyset$  and  $f \in \mathcal{R}_d$ .

*Example 2.1* Consider the case that  $n = 1$ ,  $K = \mathbb{R}$  and  $d = 2$ . One has  $\mathcal{P}_2 = \{a_2 x^2 + a_1 x + a_0 : (a_2, a_1, a_0) \in \mathbb{R}^3\}$ . Since  $K_\infty = \mathbb{R}$ , an easy computation shows that  $\mathcal{E}_2 = \{a_2 x^2 + a_1 x + a_0 : a_2 < 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$ ,  $\mathcal{O}_2 = \{a_2 x^2 + a_1 x + a_0 : a_2 > 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$ , and  $\mathcal{R}_2 = \{a_2 x^2 + a_1 x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$ .

### 3 Two criteria for the solution existence

This section introduces two criteria for the solution existence of  $\text{OP}(K, f)$ . In the proofs, the normalization argument in asymptotic analysis plays a vital role; meanwhile, the semi-algebraicity of  $K$  is not required.

#### 3.1 A Frank-Wolfe type theorem for regular problems

The following theorem provides us a criterion for the solution existence of regular optimization problems.

**Theorem 3.1 (Frank-Wolfe type theorem)** *If  $\text{OP}(K, f)$  is regular and  $f$  is bounded from below on  $K$ , then its solution set is nonempty and compact.*

*Proof* Suppose that  $f \in \mathcal{R}_d$ , i.e.  $f \in \mathcal{E}_d \cup \mathcal{O}_d$ , and there exists  $\gamma \in \mathbb{R}$  such that  $\gamma \leq f(x)$ , for all  $x \in K$ . For any given  $v \in K_\infty$ , there are two sequences  $\{t_k\} \subset \mathbb{R}_+$  and  $\{x^k\} \subset K$  such that  $t_k \rightarrow +\infty$  and  $t_k^{-1}x^k \rightarrow v$  as  $k \rightarrow +\infty$ . For any  $k$ , one has  $\gamma \leq f(x^k)$ . Dividing both sides of the last inequality by  $t_k^d$  and letting  $k \rightarrow +\infty$ , we obtain  $0 \leq f_d(v)$ . Thus,  $f_d$  is non-negative over  $K_\infty$ . It follows from Remark 2.1 that  $f$  does not belong to  $\mathcal{E}_d$ ; hence, we conclude that  $f$  must be in  $\mathcal{O}_d$ .

Let  $\bar{x} \in K$  be given, and  $M := \{x \in K : f(x) \leq f(\bar{x})\}$ . It is easy to check that  $\text{Sol}(M, f) = \text{Sol}(K, f)$ . Hence, we need only to prove that  $\text{Sol}(M, f)$  is nonempty and compact.

Clearly,  $M$  is closed. We claim that  $M$  is bounded. On the contrary, we suppose that there exists an unbounded sequence  $\{x^k\} \subset M$  such that  $x^k$  is nonzero for all  $k$ ,  $\|x^k\| \rightarrow +\infty$ , and  $\|x^k\|^{-1}x^k \rightarrow v$  for some  $v \in \mathbb{R}^n$  with  $\|v\| = 1$ . One has

$$\gamma \leq f(x_k) \leq f(\bar{x}), \quad (3.1)$$

for all  $k$ . Dividing the values in (3.1) by  $\|x^k\|^d$  and letting  $k \rightarrow +\infty$ , we get  $f_d(v) = 0$ . This yields  $v \in \text{Sol}(K_\infty, f_d)$ . Because of  $v \neq 0$ ,  $\text{Sol}(K_\infty, f_d)$  is unbounded. This contradicts our assumption and, thus, the claim is proved.

The compactness of  $M$  and Bolzano-Weierstrass' Theorem allow us to conclude that  $\text{Sol}(M, f)$  is nonempty and compact.  $\square$

*Remark 3.1* From the proof of Theorem 3.1, we see that if  $f$  is bounded from below on  $K$  then  $\text{Sol}(K_\infty, f_d)$  is nonempty, i.e.  $f \in \mathcal{O}_d \cup \mathcal{U}_d$ . Hence, if  $f \in \mathcal{E}_d$  then  $\text{OP}(K, f)$  has no solution.

**Corollary 3.1** *Assume that  $f = \alpha_1 x_1^d + \dots + \alpha_n x_n^d + p$  where  $d$  is even,  $\alpha_\ell > 0$  for all  $\ell \in [n]$ ,  $p$  is a polynomial with  $\deg p < d$ . Then,  $\text{Sol}(K, f)$  is nonempty and compact.*

*Proof* Clearly,  $f_d = \alpha_1 x_1^d + \dots + \alpha_n x_n^d$  is nonnegative over  $\mathbb{R}^n$ . It follows that  $f_d$  is also nonnegative over  $K_\infty$ . From Remarks 2.1 and 2.2, it is clear that  $\text{Sol}(K_\infty, f_d)$  is nonempty and

$$\text{Sol}(K_\infty, f_d) = \{x \in K_\infty : \alpha_1 x_1^d + \dots + \alpha_n x_n^d = 0\} = \{0\}.$$

This means that  $f \in \mathcal{O}_d$ . Clearly,  $f$  is bounded from below on  $K$ , and the condition of Theorem 3.1 holds. Therefore,  $\text{Sol}(K, f)$  is nonempty and compact.  $\square$

The following example illustrates Theorem 3.1, in which the constraint set is neither convex nor semi-algebraic.

*Example 3.1* Consider the optimization problem  $\text{OP}(K, f)$ , where the polynomial  $f$  is given by  $f(x_1, x_2) = x_2^3 - x_1x_2$  and the constraint set  $K$  is given by

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0, e^{x_1} - x_2 \geq 0\}.$$

Since  $f_3(x_1, x_2) = x_2^3$  and  $K_\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0\}$ , one has  $\text{Sol}(K_\infty, f_3) = \{(0, 0)\}$ . According to Theorem 3.1,  $\text{Sol}(K, f)$  is nonempty and compact.

### 3.2 An Eaves type theorem for non-regular problems

In this subsection, we investigate the solution existence of non-regular optimization problems, where the objective functions are pseudoconvex on the constraint sets.

Assume that  $U$  is an open subset of  $\mathbb{R}^n$ . One says the polynomial  $f$  is *pseudoconvex* on  $U$  if, for any  $x, y \in U$  such that  $\langle \nabla f(x), y - x \rangle \geq 0$ , here  $\nabla f$  is the gradient of  $f$ , we have  $f(y) \geq f(x)$ . Recall that  $f$  is pseudoconvex on  $U$  if and only if  $\nabla f$  is pseudomonotone on  $U$  [17, Theorem 3.1], i.e. if, for any  $x, y \in U$  such that  $\langle \nabla f(x), y - x \rangle \geq 0$ , we have  $\langle \nabla f(y), y - x \rangle \geq 0$ .

**Lemma 3.1** *Assume that  $K$  is convex and  $f$  is pseudoconvex on an open set  $U$  containing  $K$ . If  $x^0 \in \text{Sol}(K, f)$ , then  $\langle \nabla f(x), x - x^0 \rangle \geq 0$  for all  $x \in K$ .*

*Proof* Since  $f$  is pseudoconvex on the set  $U$ , the gradient  $\nabla f$  is pseudomonotone on  $U$ . Suppose that  $x^0 \in \text{Sol}(K, f)$ , one has  $\langle \nabla f(x^0), x - x^0 \rangle \geq 0$  for all  $x \in K$  (see, e.g., [18, Proposition 5.2]). The pseudomonotonicity of the gradient implies that  $\langle \nabla f(x), x - x^0 \rangle \geq 0$  for all  $x \in K$ . The lemma is proved.  $\square$

**Theorem 3.2 (Eaves type theorem)** *Assume that  $K$  is convex and  $f$  is pseudoconvex on an open set containing  $K$ . If  $\text{OP}(K, f)$  is non-regular, then the following statements are equivalent:*

- (a) *If  $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ , then there exists  $x \in K$  such that  $\langle \nabla f(x), v \rangle > 0$ ;*
- (b)  *$\text{Sol}(K, f)$  is nonempty and compact.*

*Proof* Suppose that  $\text{OP}(K, f)$  is non-regular. We prove (a)  $\Rightarrow$  (b). Assume that (a) holds. For each  $k \in \mathbb{N}$ , we denote

$$K_k = \{x \in \mathbb{R}^n : x \in K, \|x\| \leq k\}.$$

Clearly,  $K_k$  is compact and convex. Without loss of generality, we can assume that  $K_k$  is nonempty. According to Bolzano-Weierstrass' Theorem,  $\text{OP}(K_k, f)$  has a solution, denoted by  $x^k$ .

We assert that the sequence  $\{x^k\}$  is bounded. Indeed, suppose on the contrary that  $\{x^k\}$  is unbounded,  $x^k \neq 0$  for all  $k$ ,  $\|x^k\| \rightarrow +\infty$ , and  $\|x^k\|^{-1}x^k \rightarrow v$ , where  $v \in K_\infty$  and  $\|v\| = 1$ . For each  $k$ , one has

$$f(x^k) \leq f(x), \quad \forall x \in K_k. \quad (3.2)$$

Let  $y \in K$  be given. For  $k$  large enough,  $y \in K_k$  and  $f(x^k) \leq f(y)$ . By dividing two sides of the last inequality by  $\|x^k\|^d$  and letting  $k \rightarrow +\infty$ , we obtain  $f_d(v) \leq 0$ . This leads to  $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ . Furthermore, since  $f$  is pseudoconvex on  $K_k$ , from Lemma 3.1 we have

$$\langle \nabla f(y), y - x^k \rangle \geq 0. \quad (3.3)$$

Dividing both sides of the inequality in (3.3) by  $\|x^k\|$  and letting  $k \rightarrow +\infty$ , we obtain  $\langle \nabla f(y), v \rangle \leq 0$ . The conclusion holds for any  $x \in K$ , i.e.,  $\langle \nabla f(x), v \rangle \leq 0$  for all  $x \in K$ . This contradicts to our assumption. Hence,  $\{x^k\}$  is bounded.

We can assume that  $x^k \rightarrow \bar{x}$ . From (3.2), by the continuity of  $f$ , it is easy to check that  $\bar{x}$  solves  $\text{OP}(K, f)$ , so  $\text{Sol}(K, f)$  is nonempty.

To prove the compactness of the solution set, we can repeat the previous argument by supposing that there is an unbounded solution sequence  $\{x^k\}$ , and can show that there exists  $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$  such that  $\langle \nabla f(x), v \rangle \leq 0$  for all  $x \in K$ . This contradicts to (b).

(b)  $\Rightarrow$  (a) Since  $K$  is convex, one has  $K_\infty = 0^+K$  and  $K = K + K_\infty$ . Suppose that  $\text{Sol}(K, f)$  is nonempty and compact, but (b) is wrong, i.e. there exists  $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$  such that  $\langle \nabla f(x), v \rangle \leq 0$  for all  $x \in K$ . Let  $\bar{x}$  be a solution of  $\text{OP}(K, f)$ . For any  $t \geq 0$ , one has  $\bar{x} + tv \in K$  and  $\langle \nabla f(\bar{x} + tv), v \rangle \leq 0$ . Thus, we have

$$\langle \nabla f(\bar{x} + tv), \bar{x} - (\bar{x} + tv) \rangle \geq 0.$$

The pseudoconvexity of  $f$  yields  $f(\bar{x}) \geq f(\bar{x} + tv)$ . Hence,  $\bar{x} + tv$  belongs to  $\text{Sol}(K, f)$ , for any  $t \geq 0$ . This shows that  $\text{Sol}(K, f)$  is unbounded which contradicts to our assumption. Thus (a) holds, and the proof is complete.  $\square$

**Corollary 3.2** *Assume that  $K$  is convex and  $f$  is convex on an open convex set containing  $K$ . If  $\text{OP}(K, f)$  is non-regular, then the following statements are equivalent:*

- (a) *If  $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ , then there exists  $x \in K$  such that  $\langle \nabla f(x), v \rangle > 0$ ;*
- (b)  *$\text{Sol}(K, f)$  is nonempty and compact.*

*Proof* Since the convexity implies the pseudoconvexity, by applying Theorem 3.2 for the convex polynomial  $f$ , we have the assertion.  $\square$

The following example illustrates Corollary 3.2.

*Example 3.2* Consider the polynomial optimization problem  $\text{OP}(K, f)$  with the objective function  $f(x_1, x_2) = \frac{1}{6}x_2^3 + \frac{1}{2}x_1^2 - x_1x_2$  and the constraint set



$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1, x_2 \geq 2\}$ . The gradient and the Hessian matrix of  $f$ , respectively, are given by

$$\nabla f = \begin{bmatrix} x_1 - x_2 \\ -x_1 + \frac{1}{2}x_2^2 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & -1 \\ -1 & x_2 \end{bmatrix}.$$

It is easy to check that  $K$  is convex and  $H$  is positive semidefinite on the open set  $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 > 0, x_2 > 1\} \supset K$ ; hence  $f$  is convex on  $K$ . One has  $K_\infty = \mathbb{R}_+^2$  and  $f_3(x_1, x_2) = \frac{1}{6}x_2^3$ . This yields

$$\text{Sol}(K_\infty, f_3) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}.$$

For every  $v = (\alpha, 0)$  in  $\text{Sol}(K_\infty, f_3) \setminus \{0\}$ , one has  $\alpha > 0$ . By choosing the point  $x = (3, 2)$  in the constraint set, we get  $\langle \nabla f(x), v \rangle = \alpha > 0$ . Finally, according to Corollary 3.2, the solution set of  $\text{OP}(K, f)$  is nonempty and compact.

#### 4 Stability of the solution map

This section investigates the local boundedness, the upper semicontinuity, and the local upper-Hölder stability of the solution map under the regularity condition.

##### 4.1 Upper semicontinuity of the solution map

Recall that  $\mathcal{O}_d$  is the set of all  $g \in \mathcal{P}_d$  of degree  $d$  such that  $\text{Sol}(K_\infty, g_d) = \{0\}$ . From (2.1), one has

$$\mathcal{P}_d \setminus \mathcal{R}_d = \mathcal{P}_{d-1} \cup \mathcal{U}_d, \quad \mathcal{P}_d \setminus \mathcal{O}_d = \mathcal{P}_{d-1} \cup \mathcal{E}_d \cup \mathcal{U}_d. \quad (4.1)$$

**Lemma 4.1** *The sets  $\mathcal{R}_d$  and  $\mathcal{O}_d$  are open in  $\mathcal{P}_d$ .*

*Proof* To prove  $\mathcal{R}_d$  is open, we only need to show that the complement  $\mathcal{P}_d \setminus \mathcal{R}_d$  is closed. Let  $\{g^k\}$  be a sequence in  $\mathcal{P}_d \setminus \mathcal{R}_d$  such that  $g^k \rightarrow g$ . From the definition of  $\mathcal{R}_d$ , if  $\deg g < d$ , i.e.,  $g \in \mathcal{P}_{d-1}$ , then  $g \in \mathcal{P}_d \setminus \mathcal{R}_d$ . Thus, we can suppose that  $\deg g = d$ . One has  $g_d^k \rightarrow g_d$ , here  $g_d^k$  is the component of degree  $d$  of  $g^k$ . For each  $k$ ,  $\text{Sol}(K_\infty, g_d^k)$  is unbounded. There exists an unbounded sequence  $\{x^k\}$  such that  $x^k \in \text{Sol}(K_\infty, g_d^k)$ ,  $\|x^k\| \rightarrow +\infty$ ,  $\|x^k\|^{-1}x^k \rightarrow \bar{x}$  with  $\|\bar{x}\| = 1$ . Let  $v \in K_\infty$  be given. One has  $\|x^k\|v \in K_\infty$  and  $g_d^k(\|x^k\|v) \geq g_d^k(x^k)$ , for any  $k$ . Dividing the last inequality by  $\|x^k\|^d$  and letting  $k \rightarrow +\infty$ , one has  $g_d^k(v) \geq g_d^k(\bar{x})$ . This conclusion holds for every  $v \in K_\infty$ . This yields  $\bar{x} \in \text{Sol}(K_\infty, g_d)$ . As  $\|\bar{x}\| = 1$ , we have  $\bar{x} \neq 0$ . It follows that  $g$  belongs to  $\mathcal{U}_d$ . The closedness of  $\mathcal{P}_d \setminus \mathcal{R}_d$  is proved.

We now prove that the complement  $\mathcal{P}_d \setminus \mathcal{O}_d$  is closed. Let  $\{g^k\} \subset \mathcal{P}_d \setminus \mathcal{O}_d$  be a sequence with  $g^k \rightarrow g$ . From the second equation in (4.1), we prove that  $g$  belongs to  $\mathcal{P}_{d-1} \cup \mathcal{E}_d \cup \mathcal{U}_d$ . There is a subsequence of  $\{g^{k_p}\}$  such that it is contained in  $\mathcal{P}_{d-1}$  or  $\mathcal{E}_d \cup \mathcal{U}_d$ . If  $\{g^{k_p}\} \subset \mathcal{P}_{d-1}$  then, by the closedness of  $\mathcal{P}_{d-1}$ , one has  $g^{k_p} \rightarrow g \in \mathcal{P}_{d-1}$ . Now we consider the case that  $\{g^{k_p}\} \subset \mathcal{E}_d \cup \mathcal{U}_d$ , i.e.,

$g^{k_p}$  is non-negative over  $K_\infty$ , for any  $p$ . For any  $v \in K_\infty$ , one has  $g_d^{k_p}(v) \geq 0$ . Let  $p \rightarrow +\infty$ , we have  $g_d(v) \geq 0$ . Hence,  $g_d$  is non-negative over  $K_\infty$ . This means that  $g \in \mathcal{E}_d \cup \mathcal{U}_d$ . The openness of  $\mathcal{O}_d$  is proved.  $\square$

Recall that a set-valued map  $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is *locally bounded* at  $\bar{u}$  if there exists an open neighborhood  $U$  of  $\bar{u}$  such that  $\cup_{u \in U} \Psi(u)$  is bounded [19, Definition 5.14]. The map  $\Psi$  is *upper semicontinuous* at  $\bar{u} \in T$  if for any open set  $V \subset \mathbb{R}^n$  such that  $\Psi(\bar{u}) \subset V$  there exists a neighborhood  $U$  of  $\bar{u}$  such that  $\Psi(u) \subset V$  for all  $u \in U$ . Recall that if  $\Psi$  is closed, namely, the graph

$$\text{gph}(\Psi) := \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n : v \in \Psi(u)\}$$

is closed in  $\mathbb{R}^m \times \mathbb{R}^n$ , and locally bounded at  $u$ , then  $\Psi$  is upper semicontinuous at  $u$  [19, Theorem 5.19].

**Proposition 4.1** *Assume that  $K$  is convex. If  $\text{OP}(K, f)$  is regular, then the following statements hold:*

- (a) *The solution map  $\text{Sol}_K(\cdot)$  is locally bounded at  $f$ , i.e., there exists  $\varepsilon > 0$  such that the set*

$$O_\varepsilon := \bigcup_{g \in \mathcal{B}(\varepsilon, d)} \text{Sol}(K, f + g), \quad (4.2)$$

where  $\mathcal{B}(\varepsilon, d)$  is the open ball in  $\mathcal{P}_d$  with center 0 and radius  $\varepsilon$ , is bounded.

- (b) *The solution map  $\text{Sol}_K(\cdot)$  is upper semicontinuous at  $f$ .*

*Proof* (a) According to Lemma 4.1,  $\mathcal{R}_d$  is open in  $\mathcal{P}_d$ . There is a closed ball  $\bar{\mathcal{B}}(\varepsilon, d)$  such that

$$f + \bar{\mathcal{B}}(\varepsilon, d) \subset \mathcal{R}_d. \quad (4.3)$$

Assume to the contrary that  $O_\varepsilon$  is unbounded. Then, there exists an unbounded sequence  $\{x^k\}$  and a sequence  $\{g^k\} \subset \mathcal{B}(\varepsilon, d)$  such that  $x^k$  solves  $\text{OP}(K, f + g^k)$  with  $x^k \neq 0$  for every  $k$ ,  $\|x^k\| \rightarrow +\infty$ , and  $\|x^k\|^{-1}x^k \rightarrow \bar{x}$  with  $\|\bar{x}\| = 1$ . By the compactness of  $\bar{\mathcal{B}}(\varepsilon, d)$ , without loss of generality, we can assume that  $g^k \rightarrow g$  with  $g \in \bar{\mathcal{B}}(\varepsilon, d)$ .

From assumptions, for every  $k$ , one has

$$(f + g^k)(y) \geq (f + g^k)(x^k), \quad \forall y \in K. \quad (4.4)$$

Let  $y \in K$  be fixed and assume that  $v \in K_\infty$ . By the convexity of  $K$ , one has  $y + \|x^k\|v \in K$  for any  $k$ . From (4.4), we conclude that

$$(f + g^k)(y + \|x^k\|v) \geq (f + g^k)(x^k).$$

Dividing this inequality by  $\|x^k\|^d$  and taking  $k \rightarrow +\infty$ , we obtain

$$(f + g)_d(v) \geq (f + g)_d(\bar{x}).$$

It follows that  $\bar{x} \in \text{Sol}(K_\infty, (f + g)_d)$ . Because of (4.3),  $\text{Sol}(K_\infty, (f + g)_d)$  is contained in  $\{0\}$ , which contradicts to  $\|\bar{x}\| = 1$ . Hence,  $O_\varepsilon$  must be bounded.

- (b) It is not difficult to prove that the graph

$$\text{gph}(\text{Sol}) := \{(g, x) \in \mathcal{P}_d \times \mathbb{R}^n : x \in \text{Sol}(K, g)\}$$

is closed in  $\mathcal{P}_d \times \mathbb{R}^n$ . Since  $\text{Sol}_K(\cdot)$  is locally bounded on  $\mathcal{R}_d$ , according to [19, Theorem 5.19],  $\text{Sol}_K(\cdot)$  is upper semicontinuous at  $f$ .  $\square$

## 4.2 Local upper-Hölder stability of the solution map

When the constraint set  $K$  is convex and given by polynomials, we can investigate the local upper-Hölder stability of the solution map under the regular condition. To prove the stability, we need the following lemma.

**Lemma 4.2** ([20]) *Let  $U$  be a semi-algebraic subset in  $\mathbb{R}^n$ , represented by*

$$U = \{x \in \mathbb{R}^n : u_i(x) = 0, i \in [l], v_j(x) \leq 0, j \in [m]\},$$

where  $u_i(x), i \in [l]$ , and  $v_j(x), j \in [m]$ , are polynomials. For any compact set  $V \subset \mathbb{R}^n$ , there are constants  $c > 0$  and  $H > 0$  such that

$$d(x, U) \leq c \left( \sum_{i=1}^l |u_i(x)| + \sum_{j=1}^m [v_j(x)]_+ \right)^H,$$

for all  $x \in V$ , here  $[r]_+ := \max\{0, r\}$  and  $d(x, U)$  the usual distance from  $x$  to the set  $U$ .

**Theorem 4.1** *Assume that  $\text{OP}(K, f)$  is regular and  $K$  is a convex set given by*

$$K = \{x \in \mathbb{R}^n : p_i(x) = 0, i \in [l], q_j(x) \leq 0, j \in [m]\},$$

where all  $p_i, q_j$  are polynomials. If  $\text{Sol}(K, f)$  is nonempty, then the map  $\text{Sol}_K(\cdot)$  is locally upper-Hölder stable at  $f$ , i.e., there exist  $\ell > 0, H > 0$  and  $\varepsilon > 0$  such that

$$\text{Sol}(K, g) \subset \text{Sol}(K, f) + \ell \|g - f\|^H \mathbb{B}, \quad (4.5)$$

for all  $g \in \mathcal{P}_d$  satisfying  $\|g - f\| < \varepsilon$ , where  $\mathbb{B}$  is the closed unit ball in  $\mathbb{R}^n$ .

*Proof* Suppose  $\text{Sol}(K, f)$  is nonempty and its optimal value is  $f^*$ . Since  $\text{OP}(K, f)$  is regular and  $K$  is convex, according to Proposition 4.1, there exists  $\varepsilon > 0$  such that  $\text{Sol}(K, f) \subset O_\varepsilon$ , defined by (4.2), is bounded. Let  $V$  be the closure of  $O_\varepsilon$ . It follows that  $V$  is a nonempty compact set. By the assumptions, we see that

$$\text{Sol}(K, f) = \{x \in \mathbb{R}^n : f(x) - f^* = 0, p_i(x) = 0, i \in [l], q_j(x) \leq 0, j \in [m]\}.$$

From this equality, by applying Lemma 4.2 for  $U = \text{Sol}(K, f)$  and the compact set  $V$ , there are constants  $c_0 > 0$  and  $H > 0$  such that

$$d(x, \text{Sol}(K, f)) \leq c_0 A(x)^H \quad \forall x \in V, \quad (4.6)$$

where

$$A(x) := |f(x) - f^*| + \sum_{i=1}^l |p_i(x)| + \sum_{j=1}^m [q_j(x)]_+.$$

Let  $g \in \mathcal{P}_d$  be arbitrary given such that  $\|g - f\| < \varepsilon$ . From the definition of  $V$ ,  $\text{Sol}(K, f)$  and  $\text{Sol}(K, g)$  are subsets of  $V$ . Here,  $\text{Sol}(K, g)$  may be empty.

By the compactness of  $V$ , we define the constant  $L := \max\{\|X(x)\| : x \in V\}$ . Hence, one has

$$|g(x) - f(x)| \leq \|X(x)\| \|g - f\| \leq L \|g - f\| \quad \forall x \in V. \quad (4.7)$$

If  $\text{Sol}(K, g)$  is empty, then (4.5) is obvious. Thus, we consider the case that  $\text{Sol}(K, g) \neq \emptyset$ . Since both  $\text{Sol}(K, f)$  and  $\text{Sol}(K, g)$  are nonempty and compact, for any  $x_g \in \text{Sol}(K, g)$ , there is  $x_f \in \text{Sol}(K, f)$  such that

$$\|x_g - x_f\| = d(x_g, \text{Sol}(K, f)). \quad (4.8)$$

Because of  $p_i(x_g) = 0$  for  $i \in [l]$  and  $q_j(x_g) \leq 0$  for  $j \in [m]$ , from the definition of  $A(x)$ , one has  $A(x_g) = |f(x_g) - f^*|$ . By (4.8) and (4.6), we see that

$$\|x_g - x_f\| \leq c_0 A(x_g)^H = c_0 |f(x_g) - f^*|^H.$$

Since  $x_f \in \text{Sol}(K, f)$ , we have  $f(x_f) = f^* \leq f(x_g)$ . Therefore, we obtain

$$\|x_g - x_f\| \leq c_0 |f(x_g) - f^*|^H = c_0 (f(x_g) - f(x_f))^H. \quad (4.9)$$

It follows from  $x_g \in \text{Sol}(K, g)$  that  $g(x_g) - g(x_f) \leq 0$ . Since  $x_g, x_f \in V$ , we conclude from (4.7) that

$$\begin{aligned} f(x_g) - f(x_f) &= (f(x_g) - g(x_g)) + (g(x_g) - g(x_f)) + (g(x_f) - f(x_f)) \\ &\leq (f(x_g) - g(x_g)) + (g(x_f) - f(x_f)) \\ &\leq 2L \|g - f\|. \end{aligned}$$

The inequality (4.9) and the last result lead to

$$\|x_g - x_f\| \leq c_0 (2L)^H \|g - f\|^H,$$

consequently,

$$d(x_g, \text{Sol}(K, f)) = \|x_g - x_f\| \leq \ell \|g - f\|^H,$$

where  $\ell = c_0 (2L)^H$ .

The conclusion holds for any  $x_g$  in  $\text{Sol}(K, g)$ . Hence, the inclusion (4.5) of the theorem is proved.  $\square$

## 5 Genericity of the regularity condition

In this section, we discuss the genericity of the regularity condition of polynomial optimization problems.

A subset  $A$  is called *generic* in  $\mathbb{R}^m$  if  $A$  contains a countable intersection of dense and open sets in  $\mathbb{R}^m$ . If  $A$  is generic in  $\mathbb{R}^m$  and  $A \subset B$  then  $B$  also is generic in  $\mathbb{R}^m$ . Let  $\mathcal{T}$  be a topological space. It is known that if  $h : \mathbb{R}^m \rightarrow \mathcal{T}$  is a homeomorphism and  $A$  is generic in  $\mathbb{R}^m$ , then  $h(A)$  is generic in  $\mathcal{T}$ .

Let  $U \subset \mathbb{R}^m$  be a semi-algebraic set. Then, there exists a decomposition of  $U$  into a disjoint union [21, Theorem 2.3.6],  $U = \cup_{i=1}^s U_i$ , where each  $U_i$

is semi-algebraically homeomorphic to  $(0, 1)^{d_i}$ . Here, let  $(0, 1)^0$  be a point,  $(0, 1)^k \subset \mathbb{R}^k$  be the set of points  $x = (x_1, \dots, x_k)$  such that  $x_j \in (0, 1)$  for all  $j \in [k]$ . The *dimension* of  $U$  is defined by  $\dim(U) := \max\{d_1, \dots, d_s\}$ . The dimension is well-defined and does not depend on the decomposition of  $U$ . Recall that if  $U$  is nonempty and  $\dim(U)$  is zero, then  $U$  has finitely many points. Furthermore, if  $\dim(\mathbb{R}^m \setminus U) < m$ , then  $U$  is generic in  $\mathbb{R}^m$  (see, e.g. [22, Lemma 2.3]).

The space generated by all monomials of degree  $d$  listed by lexicographic ordering  $\{x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \dots, x_n^d\}$  is denoted by  $\mathcal{H}_d$ . One has the direct sum  $\mathcal{P}_d = \mathcal{H}_d \oplus \mathcal{P}_{d-1}$ . The dimension of  $\mathcal{H}_d$  is denoted by  $\eta$ . For every homogeneous polynomial  $h \in \mathcal{H}_d$ , one has a unique  $b \in \mathbb{R}^\eta$ , such that  $h(x) = b^T X_d(x)$ , where

$$X_d^T(x) = (x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \dots, x_n^d).$$

Here,  $\nabla(b^T X_d(x))$  is the gradient vector of  $b^T X_d(x)$  and  $D_b[\nabla(b^T X_d(x))]$  is the Jacobian matrix of  $b^T X_d(x)$  with respect to  $b$ .

**Lemma 5.1** *One has  $\text{rank}(D_b[\nabla(b^T X_d(x))]) = n$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ .*

*Proof* In the proof, we are only interested in the monomials  $x_i^{d-1}x_j$ , where  $i, j \in [n]$ . Hence, for convenience, we rewrite  $X_d^T(x)$  and  $b^T$  respectively as follows:

$$(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_n; \dots; x_n^{d-1}x_1, x_n^{d-1}x_2, \dots, x_n^d; \dots)$$

and

$$(b_{11}, b_{12}, \dots, b_{1n}; \dots; b_{n1}, b_{n2}, \dots, b_{nn}; \dots).$$

Then, we have

$$b^T X_d(x) = \sum_{j \in [n]} b_{1j} x_1^{d-1} x_j + \dots + \sum_{j \in [n]} b_{nj} x_n^{d-1} x_j + Q, \quad (5.1)$$

where  $Q$  is a homogeneous polynomial of degree  $d$ .

From (5.1), an easy computation shows that

$$\frac{\partial(b^T X_d(x))}{\partial x_i} = db_{ii} x_i^{d-1} + (d-1) \sum_{j \neq i} b_{ij} x_i^{d-2} x_j + \sum_{j \neq i} b_{ji} x_j^{d-1} + \frac{\partial Q}{\partial x_i},$$

and the  $n \times \eta$ -matrix  $D_b[\nabla(b^T X_d(x))]$  can be described as follows

$$D_b[\nabla(b^T X_d(x))] = \left[ M_1, M_2, \dots, M_n, \dots \right],$$

where the submatrix  $M_i$ , for  $i \in [n]$ , is defined by

$$M_i = \begin{bmatrix} x_i^{d-1} I_{i-1} & O_{i \times 1} & O_{i \times (n-i)} \\ L_{1 \times (i-1)} & dx_i^{d-1} & R_{1 \times (n-i)} \\ O_{i \times (n-i)} & O_{(n-i) \times 1} & x_i^{d-1} I_{n-i} \end{bmatrix}$$

with  $I_k$  being the unit  $k \times k$ -matrix,  $O_{k \times s}$  being the zero  $k \times s$ -matrix,

$$L_{1 \times (i-1)} = \left( (d-1)x_i^{d-2}x_1, \dots, (d-1)x_i^{d-2}x_{i-1} \right),$$

and

$$R_{1 \times (n-i)} = \left( (d-1)x_i^{d-2}x_{i+1}, \dots, (d-1)x_i^{d-2}x_n \right).$$

We observe that  $\det(M_i) = dx_i^{d(d-1)}$ , for all  $i \in [n]$ . Since  $x \neq 0$ , there exists  $l \in [n]$  such that  $x_l \neq 0$ . This implies that  $\text{rank}(M_l) = n$ . Hence, the rank of  $D_b[\nabla(b^T X_d(x))]$  is  $n$ , for any  $x \neq 0$ .  $\square$

Suppose that  $C$  is a polyhedral cone given by

$$C = \{x \in \mathbb{R}^n : Ax \geq 0\}, \quad (5.2)$$

where  $A = (a_{ij}) \in \mathbb{R}^{p \times n}$ . Let  $\text{KKT}(C, g)$ , where  $g \in \mathcal{P}_d$ , be the set of the Karush-Kuhn-Tucker points of  $\text{OP}(C, g)$ , i.e.,  $x \in \text{KKT}(C, g)$  if and only if there exists  $\lambda \in \mathbb{R}^p$  such that

$$\begin{cases} \nabla g(x) - A^T \lambda = 0, \\ \lambda^T (Ax) = 0, \lambda \geq 0, Ax \geq 0. \end{cases} \quad (5.3)$$

From the Karush-Kuhn-Tucker conditions, we see that  $\text{Sol}(C, g) \subset \text{KKT}(C, g)$  for all  $g \in \mathcal{P}_d$ .

For each index set  $\alpha \subset [p]$ , we associate the *pseudo-face*  $C_\alpha$  of  $C$ , which is denoted and defined by

$$C_\alpha := \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{ij}x_j = 0 \ \forall i \in \alpha, \sum_{j=1}^n a_{ij}x_j > 0 \ \forall i \in [p] \setminus \alpha \right\},$$

where  $a_{ij}$  is the element in the  $i$ -th row and the  $j$ -th column of  $A$ . The number of pseudo-faces of  $C$  is finite. These pseudo-faces establish a disjoint decomposition of  $C$ . So, we obtain

$$\text{KKT}(C, g) = \bigcup_{\alpha \subset [p]} (\text{KKT}(C, g) \cap C_\alpha), \quad (5.4)$$

The following proposition shows that the Karush-Kuhn-Tucker set-valued map of homogeneous polynomial optimization problems

$$\text{KKT}_C : \mathbb{R}^p \rightrightarrows \mathbb{R}^n, \ b \mapsto \text{KKT}_C(b) = \text{KKT}(C, b^T X_d(x)),$$

is finite-valued, i.e., the cardinal  $\#\text{KKT}_C(b)$  is finite, on a generic semi-algebraic set of  $\mathbb{R}^p$ .

**Proposition 5.1** *Assume that  $C$  is a polyhedral cone given by (5.2) and the matrix  $A$  is full rank. Then, there exists a generic semi-algebraic set  $\mathbb{S} \subset \mathbb{R}^p$  such that  $\#\text{KKT}_C(b) < \infty$  for any  $b \in \mathbb{S}$ .*

*Proof* Let  $C_\alpha$  be a nonempty pseudo-face of  $C$  and  $0 \notin C_\alpha$ . This implies that  $X_d(x)$  is nonzero on this pseudo-face. We consider the function

$$\Phi_\alpha : \mathbb{R}^\eta \times C_\alpha \times \mathbb{R}_+^{|\alpha|} \rightarrow \mathbb{R}^{n+|\alpha|},$$

which is defined by

$$\Phi_\alpha(b, x, \lambda_\alpha) = \left( \nabla(b^T X_d(x)) + \sum_{i \in \alpha} \lambda_i A_i, A_\alpha x \right),$$

where  $A_\alpha x = (A_{i_1} x, \dots, A_{i_{|\alpha|}} x)$ ,  $i_j \in \alpha$ . Clearly,  $C_\alpha$  is smooth and  $\Phi_\alpha$  is a semi-algebraic function of class  $C^\infty$ . The Jacobian matrix of  $\Phi_\alpha$  is determined as follows

$$D\Phi_\alpha = \left[ \begin{array}{c|c|c} D_b[\nabla(b^T X_d(x))] & * & A_\alpha^T \\ \hline O_{|\alpha| \times \eta} & A_\alpha & O_{|\alpha| \times |\alpha|} \end{array} \right].$$

From Lemma 5.1, for all  $x \in C_\alpha$ , the rank of  $D_b[\nabla(b^T X_d(x))]$  is  $n$ . Since the rank of  $A_\alpha$  is  $|\alpha|$ , we conclude that the rank of the matrix  $D\Phi_\alpha$  is  $n + |\alpha|$  for all  $x \in C_\alpha$ . Therefore,  $0 \in \mathbb{R}^{n+|\alpha|+|J|}$  is a regular value of  $\Phi_\alpha$ . According to the Sard Theorem with parameter [22, Theorem 2.4], there exists a generic semi-algebraic set  $\mathbb{S}_\alpha \subset \mathbb{R}^\eta$  such that if  $b \in \mathbb{S}_\alpha$  then  $0$  is a regular value of the map

$$\Phi_{\alpha,b} : C_\alpha \times \mathbb{R}^{|\alpha|} \rightarrow \mathbb{R}^{n+|\alpha|}, \quad \Phi_{\alpha,b}(x, \lambda_\alpha) = \Phi_\alpha(b, x, \lambda_\alpha).$$

We see that  $\Omega(\alpha, b) := \Phi_{\alpha,b}^{-1}(0)$  is a semi-algebraic set. From the Regular Level Set Theorem [23, Theorem 9.9], we can claim that if  $\Omega(\alpha, b)$  is nonempty then the set is a  $0$ -dimensional semi-algebraic set. It follows that  $\Omega(\alpha, b)$  is a finite set. Moreover, from (5.3), one has  $\text{KKT}_C(b) \cap C_\alpha = \pi(\Omega(\alpha, b))$ , where  $\pi$  is the projection  $\mathbb{R}^{n+|\alpha|} \rightarrow \mathbb{R}^n$  which is defined by  $\pi(x, \lambda_\alpha) = x$ . Hence,  $\text{KKT}_C(b) \cap C_\alpha$  is a finite set.

We consider the case that  $0 \in C_\alpha$  and define  $U := C_\alpha \setminus \{0\}$ . As is clear,  $U$  is semi-algebraic. From (5.3), we see that  $0 \in \text{KKT}_C(b)$ . Hence,

$$\text{KKT}_C(b) \cap C_\alpha = \{0\} \cup (\text{KKT}_C(b) \cap U).$$

From the previous argument,  $\text{KKT}_C(b) \cap U$  is a finite set. By the decomposition (5.4),  $\text{KKT}_C(b)$  is a finite set.

Take  $\mathbb{S} = \bigcap_{\alpha \subset [p]} \mathbb{S}_\alpha$ , it follows that  $\mathbb{S}$  is generic in  $\mathbb{R}^\eta$  and  $\text{KKT}_C(b)$  has finite points for any  $b \in \mathbb{S}$ . Hence,  $\#\text{KKT}_C(b) < \infty$  for all  $b \in \mathbb{S}$ . The proof is complete.  $\square$

**Corollary 5.1** *Assume that  $C$  is a polyhedral cone given by (5.2) and the matrix  $A$  is full rank. Then there exists a generic set  $\mathcal{G}_d$  in  $\mathcal{H}_d$  such that  $\#\text{Sol}(C, g) < \infty$  for any  $g \in \mathcal{G}_d$ .*

*Proof* Since  $\mathbb{R}^\eta$  and  $\mathcal{H}_d$  are homeomorphic, with the isomorphism  $\Pi : \mathbb{R}^\eta \rightarrow \mathcal{H}_d$  defined by  $\Pi(b) = b^T X_d(x)$ . According to Proposition 5.1, there exists a generic set  $\mathbb{S} \subset \mathbb{R}^\eta$  such that the Karush-Kuhn-Tucker set  $\text{KKT}(C, b)$  is finite, for any  $b \in \mathbb{S}$ . Clearly,  $\mathcal{G}_d := \Pi(\mathbb{S})$  is generic in  $\mathcal{H}_d$ . Since  $\text{Sol}(C, b^T X_d(x)) \subset \text{KKT}(C, b)$ , one has  $\#\text{Sol}(C, g) < \infty$ , for any  $g \in \mathcal{G}_d$ .  $\square$

*Remark 5.1* If the constraint  $K$  is represented by

$$K = \{x \in \mathbb{R}^n : q_1(x) \leq 0, \dots, q_m(x) \leq 0\}, \quad (5.5)$$

where  $q_1, \dots, q_m$  are convex polynomials, then the recession cone of  $K$  is a nonempty polyhedral cone. We denote

$$K^j = \{x \in \mathbb{R}^n : q_j(x) \leq 0\}, \quad j \in [m].$$

For each  $j \in [m]$ ,  $K^j$  is closed convex set, and  $K_\infty^j$  is polyhedral (see [9, p.39]). Since  $K = K^1 \cap \dots \cap K^m$ , according to [3, Proposition 2.1.9], one has

$$K_\infty = \bigcap_{j \in [m]} K_\infty^j.$$

It follows that  $K_\infty$  is a nonempty polyhedral cone. Hence, there exists a matrix  $A \in \mathbb{R}^{p \times n}$  such that

$$K_\infty = \{x \in \mathbb{R}^n : Ax \geq 0\}. \quad (5.6)$$

**Theorem 5.1** *Assume that  $K$  be represented by (5.5) and the cone  $K_\infty$  represented by (5.6), where  $A$  is full rank. Then, the set  $\mathcal{R}_d$  is generic in  $\mathcal{P}_d$ .*

*Proof* From Remark 5.1, the recession cone  $K_\infty$  is a nonempty polyhedral cone, where  $K_\infty = \{x \in \mathbb{R}^n : Ax \geq 0\}$ . According to Corollary 5.1, there exists a generic set  $\mathcal{G}_d$  in  $\mathcal{H}_d$  such that  $\#\text{Sol}(K_\infty, g) < \infty$  for any  $g \in \mathcal{G}_d$ . Because of the direct sum  $\mathcal{P}_d = \mathcal{H}_d \oplus \mathcal{P}_{d-1}$ , the set  $\mathcal{G}_d \oplus \mathcal{P}_{d-1}$  is generic in  $\mathcal{P}_d$ . It is easy to check that  $\mathcal{G}_d \oplus \mathcal{P}_{d-1} \subset \mathcal{R}_d$ . Hence,  $\mathcal{R}_d$  is generic in  $\mathcal{P}_d$ .  $\square$

*Example 5.1* Consider the problem  $\text{OP}(K, f)$  given in Example 2.1, we see that

$$\mathcal{R}_2 = \{a_2x^2 + a_1x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$$

is open and dense in  $\mathcal{P}_2$ .

## Perspectives

The regularity condition enables us to investigate the stability of the optimal value function of polynomial optimization problems. Furthermore, the regularity condition is useful to study the connectedness of the solution sets of convex polynomial vector optimization problems.

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