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On the solution existence and stability of polynomial optimization problems

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Abstract This paper introduces and investigates a regularity condition in the asymptotic sense for the optimization problems whose objective functions are polynomial. We prove two sufficient conditions for the existence of solutions for polynomial optimization problems. Further, when the constraint sets are semi-algebraic, we show results on the stability of the solution map of polynomial optimization problems. At the end of the paper, we discuss the genericity of the regularity condition.

Keywords Polynomial optimization · Regularity condition · Asymptotic cone · Frank–Wolfe type theorem · Eaves type theorem · Upper semicontinuity · Local upper-Hölder stability · Genericity

Mathematics Subject Classification (2000) 90C30 · 14P10

1 Introduction

We consider the following optimization problem

$$\text{minimize } f(x) \text{ subject to } x \in K,$$

where K is a nonempty closed subset of \mathbb{R}^n and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial in n variables of degree $d \geq 2$. The problem and its solution set are denoted by $\text{OP}(K, f)$ and $\text{Sol}(K, f)$, respectively. Let f_d be the homogeneous component of degree d of f , K_∞ be the asymptotic cone of K that will be introduced in Section 2. We say that $\text{OP}(K, f)$ is regular if the solution set of the asymptotic problem $\text{OP}(K_\infty, f_d)$ is bounded, and the problem is non-regular if otherwise.

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The regularity condition has appeared in investigations of the solution existence and stability in quadratic programming (see, e.g., [1, 2] and the references therein).

Asymptotic cones and functions play an important role in optimization and variational inequalities [3]. The normalization argument in asymptotic analysis enables us to see the solution existence and stability not only for quadratic programming, linear complementarity problems, and affine variational inequalities (see, e.g., [1, 4]) but also for polynomial complementarity problems and polynomial variational inequalities having unbounded constraint sets (see, e.g., [5, 6]).

In 1956, Frank and Wolfe [7] proved that if K is polyhedral, f is quadratic and bounded from below over K , then $\text{Sol}(K, f)$ is nonempty. Several extensions of the Frank-Wolfe theorem for quadratic, cubic, and polynomial optimization problems have been shown in [1, 2, 8–12]. The present paper gives another Frank-Wolfe type theorem, which says that if $\text{OP}(K, f)$ is regular and f is bounded from below on K then the problem has a solution. Besides, the Eaves theorem [13] provides us another criterion for the solution existence of quadratic optimization problems. Extensions of this theorem for quadratically constrained quadratic problems has been introduced in [1, 2, 14, 15]. This paper introduces an Eaves type theorem for non-regular pseudoconvex optimization problem $\text{OP}(K, f)$, where K is convex.

Under the assumption that the constraint set K is compact and semi-algebraic, some stability and genericity results for polynomial optimization problems have been shown [16]. If K is compact, then its asymptotic cone is trivial, i.e., $K_\infty = \{0\}$, it follows that $\text{OP}(K, f)$ obviously satisfies the regularity condition. The present paper considers the case that K is unbounded. Under the regularity condition, we prove several local properties of the solution map of polynomial optimization problems such as local boundedness, upper semicontinuity, and local upper-Hölder stability.

We denote by $\mathcal{R} \subset \mathbb{R}[x]$ the set of all polynomials g of degree d such that $\text{OP}(K, g)$ is regular. This set is an open cone in the space $\mathcal{P}_d \subset \mathbb{R}[x]$ of all polynomials of degree at most d . At the end of this work, we prove that \mathcal{R} is generic in \mathcal{P}_d , if K is given by convex polynomials.

The organization of the paper is as follows. Section 2 gives a brief introduction to asymptotic cones, polynomials, and the regular condition. Section 3 proves two criteria of the solution existence. Section 4 investigates properties of the solution map. The last section discusses the genericity of the regularity condition.

2 Preliminaries

Recall that the *asymptotic cone* [3] of a nonempty closed subset S in \mathbb{R}^n is defined and denoted by

$$S_\infty = \left\{ v \in \mathbb{R}^n : \exists t_k \rightarrow +\infty, \exists x_k \in S \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = v \right\}.$$

Clearly, the cone S_∞ is closed and contains 0. The set S is bounded if and only if S_∞ is trivial. Furthermore, if S is convex then S_∞ is a closed convex cone and $S_\infty = 0^+S$, where 0^+S is the *recession* cone of S that is the set of vectors $v \in \mathbb{R}^n$ such that $x + tv \in S$ for any $x \in S$ and $t \geq 0$. Thus, one has $S = S + S_\infty$ when S is convex.

Let $d \geq 2$ be given. The space \mathcal{P}_d of all polynomials of degree at most d has finite dimension; its dimension is denoted by ρ . Let $X(x)$ be the vector consisting of ρ monomials of degree at most d which is listed by lexicographic ordering

$$X(x) := (1, x_1, x_2, \dots, x_n, x_1^2, x_1x_2, \dots, x_1x_n, \dots, x_1^d, x_1^{d-1}x_2, \dots, x_n^d)^T.$$

For every $g \in \mathcal{P}_d$, there exists a unique vector $a = (a_1, \dots, a_\rho) \in \mathbb{R}^\rho$ such that $g(x) = a^T X(x)$. We denote by $\|g\|$ the ℓ_2 -norm of the polynomial g , namely

$$\|g\| := \|a\| = \sqrt{a_1^2 + \dots + a_\rho^2}.$$

The Cauchy–Schwarz inequality claims that $|g(x)| \leq \|X(x)\| \|g\|$. Furthermore, if $\{g^k\}$ is a convergent sequence in \mathcal{P}_d with $g^k \rightarrow g$, then $g_d^k \rightarrow g_d$.

Throughout the paper, we assume that the constraint set $K \subset \mathbb{R}^n$ is nonempty and closed, and the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is polynomial of degree $d \geq 2$.

We say that $\text{OP}(K, f)$ is a *polynomial optimization problem* if K is given by polynomials. With the given set K and the given integer d , the solution map of polynomial optimization problems is defined by

$$\text{Sol}_K(\cdot) : \mathcal{P}_d \rightrightarrows \mathbb{R}^n, \quad g \mapsto \text{Sol}(K, g).$$

Assume that $g \in \mathcal{P}_d$, $\deg g = d$, and $g = g_d + \dots + g_1 + g_0$, where g_l is a homogeneous polynomial of degree l , i.e., $g_l(tx) = t^l g_l(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$, $l \in [d] := \{1, \dots, d\}$, and $g_0 \in \mathbb{R}$. Then, g_d is the *leading term* (or the recession polynomial) of the polynomial g (of degree d). Clearly, one has

$$g_d(x) = \lim_{\lambda \rightarrow +\infty} \frac{g(\lambda x)}{\lambda^d}, \quad \forall x \in \mathbb{R}^n.$$

For the pair (K, f) , we have unique asymptotic pair (K_∞, f_d) . The asymptotic optimization problem $\text{OP}(K_\infty, f_d)$ plays a vital role in the investigation of behavior of $\text{OP}(K, f)$ at infinity. The following remarks point out (without proof) basic properties of the problem.

Remark 2.1 Assume that $\text{Sol}(K_\infty, f_d)$ is nonempty. Then this set is a closed cone with $0 \in \text{Sol}(K_\infty, f_d)$. Furthermore, $\text{Sol}(K_\infty, f_d)$ equals to the set of zero points of f_d in K_∞ , i.e.,

$$\text{Sol}(K_\infty, f_d) = \{x \in K_\infty : f_d(x) = 0\}.$$

Remark 2.2 Because f_d is a homogeneous and K_∞ is a closed cone, $\text{Sol}(K_\infty, f_d)$ is nonempty if and only if the polynomial f_d is non-negative on K_∞ .

Denote by \mathcal{E} (resp., \mathcal{O} , \mathcal{U}) the set of all polynomials $g \in \mathcal{P}_d$ of degree d such that $\text{Sol}(K_\infty, f_d)$ is the empty set (resp., the trivial cone, an unbounded cone). Clearly, one has the following disjoint unions

$$\mathcal{R} = \mathcal{E} \cup \mathcal{O}, \quad \mathcal{P}_d = \mathcal{P}_{d-1} \cup \mathcal{E} \cup \mathcal{O} \cup \mathcal{U}. \quad (2.1)$$

Now, we introduce the regularity notion concerning to boundedness of the solution set of $\text{OP}(K_\infty, f_d)$.

Definition 2.1 The problem $\text{OP}(K, f)$ is said to be *regular* if $\text{Sol}(K_\infty, f_d)$ is bounded, i.e., $f \in \mathcal{E} \cup \mathcal{O}$, and *non-regular* if otherwise.

Remark 2.3 The boundedness of $\text{Sol}(K_\infty, f_d)$ implies that of $\text{Sol}(K, f)$. Indeed, assume to the contrary that $\text{Sol}(K, f)$ is unbounded. There exists an unbounded sequence $\{x^k\} \subset \text{Sol}(K, f)$. Without loss of generality, we can assume x^k is nonzero for all k , $\|x^k\| \rightarrow +\infty$, and $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$ with $\|\bar{x}\| = 1$. Note that $f(x^k) = f^*$, where $f^* \in \mathbb{R}$ is the minimum of f over K , for all k . By dividing the last equation by $\|x^k\|^d$ and letting $k \rightarrow +\infty$, we obtain $f_d(\bar{x}) = 0$. It follows that $0 \neq \bar{x} \in \text{Sol}(K_\infty, f_d)$. This contradicts to the fact that $\text{Sol}(K_\infty, f_d)$ is bounded. Thus, the claim is proved.

Remark 2.4 We claim that \mathcal{R} is nonempty. Indeed, if K is bounded then \mathcal{R} coincides with the set of all $g \in \mathcal{P}_d$ such that $\deg g = d$. Hence, we can suppose that K is unbounded. Clearly, the cone K_∞ also is unbounded. Let $\bar{x} \in K_\infty$ be nonzero. There exists $l \in [n]$ such that $\bar{x}_l \neq 0$. We define a homogeneous polynomial of degree d as $f(x) := -(\bar{x}_l x_l)^d$. For any $t > 0$, one has $t\bar{x} \in K_\infty$, and $f(t\bar{x}) = -(\bar{x}_l^d)^2 t^d \rightarrow -\infty$ as $t \rightarrow +\infty$. Then f is not bounded from below on K_∞ . This yields $\text{Sol}(K_\infty, f) = \emptyset$ and $f \in \mathcal{R}$.

Example 2.1 Consider the case that $n = 1$, $K = \mathbb{R}$ and $d = 2$. One has $\mathcal{P}_2 = \{a_2 x^2 + a_1 x + a_0 : (a_2, a_1, a_0) \in \mathbb{R}^3\}$. Since $K_\infty = \mathbb{R}$, an easy computation shows that $\mathcal{E} = \{a_2 x^2 + a_1 x + a_0 : a_2 < 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$, $\mathcal{O} = \{a_2 x^2 + a_1 x + a_0 : a_2 > 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$, and $\mathcal{R}_{\mathbb{R}, 2} = \{a_2 x^2 + a_1 x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$.

3 Two criteria for the solution existence

This section introduces two criteria for the solution existence of $\text{OP}(K, f)$. In the proofs, the normalization argument in asymptotic analysis is used; the semi-algebraicity of K is not required.

3.1 A Frank-Wolfe type theorem for regular problems

The following theorem provides a criterion for the solution existence of $\text{OP}(K, f)$, which is a Frank-Wolfe type theorem.

Theorem 3.1 (Frank-Wolfe type theorem) *If $\text{OP}(K, f)$ is regular and f is bounded from below on K , then its solution set is nonempty and compact.*

Proof Suppose that $f \in \mathcal{R}$ and there exists $\gamma \in \mathbb{R}$ such that $\gamma \leq f(x)$, for all $x \in K$. Let \bar{x} be given in K , and $M := \{x \in K : f(x) \leq f(\bar{x})\}$. Clearly, M is closed. It is easy to check that $\text{Sol}(M, f) = \text{Sol}(K, f)$.

We claim that M is bounded. On the contrary, we suppose that there exists an unbounded sequence $\{x^k\} \subset M$ such that x^k is nonzero for all k , $\|x^k\| \rightarrow +\infty$, and $\|x^k\|^{-1}x^k \rightarrow v$ for some $v \in \mathbb{R}^n$ with $\|v\| = 1$. One has

$$\gamma \leq f(\bar{x}) \leq f(x_k) \quad (3.1)$$

for any k . Dividing the values in (3.1) by $\|x^k\|^d$ and letting $k \rightarrow +\infty$, we get $f_d(v) = 0$. Thus, $\text{OP}(K, f)$ is non-regular. This contradicts to our assumption and hence the claim is proved.

By the compactness of M , Weierstrass' Theorem says that the solution set of $\text{Sol}(M, f)$ is nonempty and compact. The theorem is proved. \square

Remark 3.1 The condition of Theorem 3.1 is equivalent to the condition $f \in \mathcal{O}$. Indeed, from the proof of the theorem, we can see that the condition of the theorem implies that $\text{Sol}(K_\infty, f_d)$ is the trivial cone. We now give a sketch of the proof for the inverse conclusion. Assume that $f \in \mathcal{O}$ but, on the contrary, f is unbounded from below on K . By using the normalization argument, we can show that there exists some $v \in K_\infty \setminus \{0\}$ such that $f_d(v) \leq 0$. From Remark 2.2, one has $f_d(v) = 0$. This contracts $f \in \mathcal{O}$.

Remark 3.2 Assume that $f \in \mathcal{E}$, i.e., $\text{Sol}(K_\infty, f_d)$ is empty. We conclude that f is not bounded from below on K , hence $\text{OP}(K, f)$ has no solution. Indeed, assume to the contrary that there is $\gamma \in \mathbb{R}$ such that $f(x) \geq \gamma$ for any $x \in K$. For any given $v \in K_\infty$, there are two sequences $\{t_k\} \subset \mathbb{R}_+$ and $\{x^k\} \subset K$ such that $t_k \rightarrow +\infty$ and $t_k^{-1}x^k \rightarrow v$ as $k \rightarrow +\infty$. One has $f(x^k) \geq \gamma$, for any k . Dividing both sides of the last inequality by t_k^d and letting $k \rightarrow +\infty$, we obtain $f_d(v) \geq 0$. Thus, f_d is non-negative over K_∞ . It follows from Remark 2.2 that $f \in \mathcal{O} \cup \mathcal{U}$; this contradicts to the assumption.

Corollary 3.1 *Assume that $f = \alpha_1 x_1^d + \dots + \alpha_n x_n^d + p$ where d is even, $\alpha_\ell > 0$ for all $\ell \in [n]$, p is a polynomial with $\deg p < d$. Then, $\text{Sol}(K, f)$ is nonempty and compact.*

Proof Clearly, $f_d = \alpha_1 x_1^d + \dots + \alpha_n x_n^d$ is nonnegative over \mathbb{R}^n . It follows that f_d is also nonnegative over K_∞ . From Remarks 2.2 and 2.1, one sees that $\text{Sol}(K_\infty, f_d)$ is nonempty and

$$\text{Sol}(K_\infty, f_d) = \{x \in K_\infty : \alpha_1 x_1^d + \dots + \alpha_n x_n^d = 0\} = \{0\}.$$

This means that $f \in \mathcal{O}$. According to Remark 3.1, the condition of Theorem 3.1 holds. Therefore, $\text{Sol}(K, f)$ is nonempty and compact. \square

The following example illustrates Theorem 3.1, in which the constraint set is neither convex nor semi-algebraic.

Example 3.1 Consider the optimization problem $\text{OP}(K, f)$, where the polynomial f is given by $f(x_1, x_2) = x_2^3 - x_1x_2$ and the constraint set K is given by

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0, e^{x_1} - x_2 \geq 0\}.$$

Since the homogeneous polynomial $f_3(x_1, x_2) = x_2^3$ and the asymptotic cone $K_\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0\}$, one has $\text{Sol}(K_\infty, f_3) = \{(0, 0)\}$. According to Theorem 3.1, $\text{Sol}(K, f)$ is nonempty and compact.

Remark 3.3 The converse of Theorem 3.1 is not true. For example, we consider the polynomial of two variables $f(x_1, x_2) = x_1x_2$ and the constraint set given by $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$. We see that the set $\text{Sol}(K, f) = \{(1, 1)\}$ is nonempty and compact. Meanwhile, $K_\infty = \mathbb{R}_+^2$, $f_2(x_1, x_2) = x_1x_2$, and the solution set $\text{Sol}(K_\infty, f_2) = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1x_2 = 0\}$ is an unbounded cone.

3.2 An Eaves type theorem for non-regular problems

This subsection gives a criterion for the solution existence for non-regular pseudoconvex optimization problems.

Assume that U is an open subset of \mathbb{R}^n . One says the polynomial f is *pseudoconvex* on U if, for any $x, y \in U$ such that $\langle \nabla f(x), y - x \rangle \geq 0$, here ∇f is the gradient of f , we have $f(y) \geq f(x)$. Recall that f is pseudoconvex on U if and only if ∇f is pseudomonotone on U (see, e.g., [17, Theorem 4.4]), i.e. if, for any $x, y \in U$ such that $\langle \nabla f(x), y - x \rangle \geq 0$, we have $\langle \nabla f(y), y - x \rangle \geq 0$.

Lemma 3.1 *Assume that K is convex and f is pseudoconvex on an open set U containing K . If $x^0 \in \text{Sol}(K, f)$, then $\langle \nabla f(x), x - x^0 \rangle \geq 0$ for all $x \in K$.*

Proof Since f is pseudoconvex on the set U , the gradient ∇f is pseudomonotone on U . Suppose that $x^0 \in \text{Sol}(K, f)$, one has $\langle \nabla f(x^0), x - x^0 \rangle \geq 0$ for all $x \in K$ (see, e.g., [17, Proposition 5.2]). The pseudomonotonicity of the gradient implies that $\langle \nabla f(x), x - x^0 \rangle \geq 0$ for all $x \in K$. The lemma is proved. \square

Theorem 3.2 (Eaves type theorem) *Assume that $\text{OP}(K, f)$ is non-regular, K is convex, and f is pseudoconvex on an open set containing K . The two following statements are equivalent:*

- (a) *If $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$, then there exists $x \in K$ such that $\langle \nabla f(x), v \rangle > 0$;*
- (b) *$\text{Sol}(K, f)$ is nonempty and compact.*

Proof (a) \Rightarrow (b) Suppose that (a) holds. For each $k \in \mathbb{N}$, we denote

$$K_k = \{x \in \mathbb{R}^n : x \in K, \|x\| \leq k\}.$$

Clearly, K_k is compact. Without loss of generality, we can assume that K_k is nonempty. According to Weierstrass' Theorem, $\text{OP}(K_k, f)$ has a solution, denoted by x^k .

We assert that the sequence $\{x^k\}$ is bounded. Indeed, suppose on the contrary that $\{x^k\}$ is unbounded, $x^k \neq 0$ for all k , $\|x^k\| \rightarrow +\infty$, and $\|x^k\|^{-1}x^k \rightarrow v$, where $v \in K_\infty$ and $\|v\| = 1$. For each k , one has

$$f(x^k) \leq f(x), \quad \forall x \in K_k. \quad (3.2)$$

Let $y \in K_1$ be given. Hence, $y \in K_k$ and $f(x^k) \leq f(y)$, for any $k \in \mathbb{N}$. By dividing two sides of the last inequality by $\|x^k\|^d$ and letting $k \rightarrow +\infty$, we obtain $f_d(v) \leq 0$. This leads to $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$.

From Lemma 3.1, for each $k \in \mathbb{N}$, since f is pseudoconvex on K_k , we have

$$\langle \nabla f(x), x - x^k \rangle \geq 0, \quad \forall x \in K_k. \quad (3.3)$$

Let $x \in K$ be given, then $x \in K_k$ for k large enough. Dividing both sides of the inequality in (3.3) by $\|x^k\|$ and letting $k \rightarrow +\infty$, we obtain $\langle \nabla f(x), v \rangle \leq 0$. The conclusion holds for any $x \in K$, i.e., $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. This contradicts to our assumption. Hence, $\{x^k\}$ is bounded. We now can assume that $x^k \rightarrow \bar{x}$. From (3.2), by the continuity of f , it is easy to check that \bar{x} solves $\text{OP}(K, f)$, so $\text{Sol}(K, f)$ is nonempty.

To prove the compactness of the solution set, we can repeat the previous argument by supposing that there is an unbounded solution sequence $\{x^k\}$, and can show that there exists $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ such that $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. This contradicts to (b).

(b) \Rightarrow (a) Since K is convex, one has $K_\infty = 0^+K$ and $K = K + K_\infty$. Suppose that $\text{Sol}(K, f)$ is nonempty and compact, but (b) is wrong, i.e. there exists $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ such that $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. Let \bar{x} be a solution of $\text{OP}(K, f)$. For any $t \geq 0$, one has $\bar{x} + tv \in K$ and $\langle \nabla f(\bar{x} + tv), v \rangle \leq 0$. Thus, we have

$$\langle \nabla f(\bar{x} + tv), \bar{x} - (\bar{x} + tv) \rangle \geq 0.$$

The pseudoconvexity of f yields $f(\bar{x}) \geq f(\bar{x} + tv)$. Hence, $\bar{x} + tv$ belongs to $\text{Sol}(K, f)$ for any $t \geq 0$. This shows that $\text{Sol}(K, f)$ is unbounded which contradicts to our assumption. Thus (a) holds.

The proof is complete. \square

Corollary 3.2 *Assume that $\text{OP}(K, f)$ is non-regular, K is convex, and f is convex on an open convex set containing K . The two following statements are equivalent:*

- (a) *If $v \in \text{Sol}(K_\infty, f_d) \setminus \{0\}$ then there exists $x \in K$ such that $\langle \nabla f(x), v \rangle > 0$;*
- (b) *$\text{Sol}(K, f)$ is nonempty and compact.*

Proof Since the convexity implies the pseudoconvexity, by applying Theorem 3.2 for the convex polynomial f , we have the assertion. \square

The following example illustrates Corollary 3.2.

Example 3.2 Consider the polynomial optimization problem $\text{OP}(K, f)$ with the objective function $f(x_1, x_2) = \frac{1}{6}x_2^3 + \frac{1}{2}x_1^2 - x_1x_2$ and the constraint set $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 1, x_2 \geq 2\}$. The gradient and the Hessian matrix of f , respectively, given by

$$\nabla f = \begin{bmatrix} x_1 - x_2 \\ -x_1 + \frac{1}{2}x_2^2 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & -1 \\ -1 & x_2 \end{bmatrix}.$$

It is easy to check that K is convex and H is positive semidefinite on the open set $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 > 0, x_2 > 1\} \supset K$; hence f is convex on K . One has $K_\infty = \mathbb{R}_+^2$ and $f_3(x_1, x_2) = \frac{1}{6}x_2^3$. This yields

$$\text{Sol}(K_\infty, f_3) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}.$$

For every $v = (\alpha, 0)$ in $\text{Sol}(K_\infty, f_3) \setminus \{0\}$, one has $\alpha > 0$. Choose the point $x = (3, 2)$ in the constraint set, we have $\langle \nabla f(x), v \rangle = \alpha > 0$. According to Corollary 3.2, the solution set of $\text{OP}(K, f)$ is nonempty and compact.

4 Stability of the solution map

This section investigates properties of the solution map, such as the local boundedness, the upper semicontinuity, and the local upper-Hölder stability.

4.1 Upper semicontinuity of the solution map

Recall that \mathcal{O} is the set of all $g \in \mathcal{P}_d$ of degree d such that $\text{Sol}(K_\infty, g_d) = \{0\}$. From (2.1), one has

$$\mathcal{P}_d \setminus \mathcal{R} = \mathcal{P}_{d-1} \cup \mathcal{U}, \quad \mathcal{P}_d \setminus \mathcal{O} = \mathcal{P}_{d-1} \cup \mathcal{E} \cup \mathcal{U}. \quad (4.1)$$

Lemma 4.1 *The sets \mathcal{R}, \mathcal{O} are open in \mathcal{P}_d .*

Proof To prove \mathcal{R} is open, we only need to show that the complement $\mathcal{P}_d \setminus \mathcal{R}$ is closed. Let $\{g^k\}$ be a sequence in $\mathcal{P}_d \setminus \mathcal{R}$ such that $g^k \rightarrow g$. From the definition of \mathcal{R} , if $\deg g < d$, i.e., $g \in \mathcal{P}_{d-1}$, then $g \in \mathcal{P}_d \setminus \mathcal{R}$. Thus, we can suppose that $\deg g = d$. One has $g_d^k \rightarrow g_d$, here g_d^k is the component of degree d of g^k . For each k , $\text{Sol}(K_\infty, g_d^k)$ is unbounded. There exists an unbounded sequence $\{x^k\}$ such that $x^k \in \text{Sol}(K_\infty, g_d^k)$, $\|x^k\| \rightarrow +\infty$, $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ with $\|\bar{x}\| = 1$. Let $v \in K_\infty$ be given. One has $\|x^k\|v \in K_\infty$ and $g_d^k(\|x^k\|v) \geq g_d^k(x^k)$, for any k . Dividing the last inequality by $\|x^k\|^d$ and letting $k \rightarrow +\infty$, one has $g_d^k(v) \geq g_d^k(\bar{x})$. This conclusion holds for every $v \in K_\infty$. This yields $\bar{x} \in \text{Sol}(K_\infty, g_d)$. As $\|\bar{x}\| = 1$, we have $\bar{x} \neq 0$. It follows that g belongs to \mathcal{U} . The closedness of $\mathcal{P}_d \setminus \mathcal{R}$ is proved.

To prove \mathcal{O} is open, we show that the complement $\mathcal{P}_d \setminus \mathcal{O}$ is closed. Let $\{g^k\} \subset \mathcal{P}_d \setminus \mathcal{O}$ be a sequence with $g^k \rightarrow g$; from the second equation in (4.1), we prove that g belongs to $\mathcal{P}_{d-1} \cup \mathcal{E} \cup \mathcal{U}$. There is a subsequence of $\{g^{k_p}\}$

such that it is contained in \mathcal{P}_{d-1} or $\mathcal{E} \cup \mathcal{U}$. If $\{g^{k_p}\} \subset \mathcal{P}_{d-1}$ then, by the closedness of \mathcal{P}_{d-1} , one has $g^{k_p} \rightarrow g \in \mathcal{P}_{d-1}$. Now we consider the case that $\{g^{k_p}\} \subset \mathcal{E} \cup \mathcal{U}$. By Remark 2.2 (i), g^{k_p} is non-negative over K_∞ , for any p . For any $v \in K_\infty$, one has $g^{k_p}(v) \geq 0$; letting $p \rightarrow +\infty$, we have $g_p(v) \geq 0$. Hence, g_p is non-negative over K_∞ . This means that $g \in \mathcal{E} \cup \mathcal{U}$. The openness of \mathcal{O} is proved. \square

Recall that a set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *locally bounded* at \bar{x} if there exists an open neighborhood U of \bar{x} such that $\cup_{x \in U} \Psi(x)$ is bounded [18, Definition 5.14]. The map Ψ is *upper semicontinuous* at $x \in T$ if for any open set $V \subset \mathbb{R}^n$ such that $\Psi(x) \subset V$ there exists a neighborhood U of x such that $\Psi(x') \subset V$ for all $x' \in U$. If Ψ is upper semicontinuous at every $x \in T \subset \mathbb{R}^m$ then Ψ is said to be upper semicontinuous on T . Recall that if Ψ is closed, namely, the graph

$$\text{gph}(\Psi) := \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n : v \in \Psi(u)\}$$

is closed in $\mathbb{R}^m \times \mathbb{R}^n$, and locally bounded at x , then Ψ is upper semicontinuous at x [18, Theorem 5.19].

Proposition 4.1 *Assume that K is convex. If $\text{OP}(K, f)$ is regular, then the two following statements hold:*

- (a) *The solution map $\text{Sol}_K(\cdot)$ is locally bounded at f , i.e., there exists $\varepsilon > 0$ such that the set*

$$O_\varepsilon := \bigcup_{g \in \mathcal{B}(\varepsilon, d)} \text{Sol}(K, f + g), \quad (4.2)$$

where $\mathcal{B}(\varepsilon, d)$ is the open ball in \mathcal{P}_d with center at 0 and radius ε , is bounded.

- (b) *The solution map $\text{Sol}_K(\cdot)$ is upper semicontinuous on \mathcal{R} .*

Proof (a) According to Lemma 4.1, \mathcal{R} is open in \mathcal{P}_d . There is a closed ball $\overline{\mathcal{B}}(\varepsilon, d)$ such that

$$f + \overline{\mathcal{B}}(\varepsilon, d) \subset \mathcal{R}. \quad (4.3)$$

Assume to the contrary that O_ε is unbounded. Then, there exists an unbounded sequence $\{x^k\}$ and a sequence $\{g^k\} \subset \mathcal{B}(\varepsilon, d)$ such that x^k solves $\text{OP}(K, f + g^k)$ with $x^k \neq 0$ for every k , $\|x^k\| \rightarrow +\infty$, and $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ with $\|\bar{x}\| = 1$. By the compactness of $\overline{\mathcal{B}}(\varepsilon, d)$, without loss of generality, we can assume that $g^k \rightarrow g$ with $g \in \overline{\mathcal{B}}(\varepsilon, d)$.

By assumptions, for every k , one has

$$(f + g^k)(y) \geq (f + g^k)(x^k), \quad \forall y \in K. \quad (4.4)$$

Let $u \in K$ be fixed and $v \in K_\infty$ be arbitrary. By the convexity of K , one has $u + \|x^k\|v \in K$ for any k . From (4.4), we conclude that

$$(f + g^k)(u + \|x^k\|v) \geq (f + g^k)(x^k).$$

Dividing this inequality by $\|x^k\|^d$ and taking $k \rightarrow +\infty$, we obtain

$$(f + g)_d(v) \geq (f + g)_d(\bar{x}).$$

It follows that $\bar{x} \in \text{Sol}(K_\infty, (f + g)_d)$. Because of (4.3), $\text{Sol}(K_\infty, (f + g)_d)$ is contained in $\{0\}$. This contradicts to $\|\bar{x}\| = 1$. Hence, O_ε must be bounded.

(b) It is not difficult to prove that the graph

$$\text{gph}(\text{Sol}) := \{(g, x) \in \mathcal{P}_d \times \mathbb{R}^n : x \in \text{Sol}(K, g)\}$$

is closed in $\mathcal{P}_d \times \mathbb{R}^n$. Since $\text{Sol}_K(\cdot)$ is locally bounded on \mathcal{R} , according to [18, Theorem 5.19], $\text{Sol}_K(\cdot)$ is upper semicontinuous on \mathcal{R} . \square

4.2 Local upper-Hölder stability of the solution map

To prove the local upper-Hölder stability of $\text{Sol}_K(\cdot)$, we need the following lemma.

Lemma 4.2 ([19]) *Let U be a semi-algebraic subset represented by*

$$U = \{x \in \mathbb{R}^n : u_i(x) = 0, i \in [l], v_j(x) \leq 0, j \in [m]\},$$

where $u_i(x), i \in [l]$, and $v_j(x), j \in [m]$, are polynomials. For any compact set $V \subset \mathbb{R}^n$, there are constants $c > 0$ and $H > 0$ such that

$$d(x, U) \leq c \left(\sum_{i=1}^l |u_i(x)| + \sum_{j=1}^m [v_j(x)]_+ \right)^H$$

for all $x \in V$, here $[r]_+ := \max\{r, 0\}$ and $d(x, U)$ the usual distance from x to the set U .

Theorem 4.1 *Assume that $\text{OP}(K, f)$ is regular and K is a convex set given by*

$$K = \{x \in \mathbb{R}^n : p_i(x) = 0, i \in [l], q_j(x) \leq 0, j \in [m]\},$$

where all p_i, q_j are polynomials. If $\text{Sol}(K, f)$ is nonempty, then the map $\text{Sol}_K(\cdot)$ is locally upper-Hölder stable at f , i.e., there exist $\ell > 0, H > 0$ and $\varepsilon > 0$ such that

$$\text{Sol}(K, g) \subset \text{Sol}(K, f) + \ell \|g - f\|^H \mathbb{B} \quad (4.5)$$

for all $g \in \mathcal{P}_d$ satisfying $\|g - f\| < \varepsilon$, where \mathbb{B} is the closed unit ball in \mathbb{R}^n .

Proof Suppose $\text{Sol}(K, f)$ is nonempty and its optimal value is f^* . Because $\text{OP}(K, f)$ is regular and K is convex, according to Proposition 4.1, there exists $\varepsilon > 0$ such that $\text{Sol}(K, f) \subset O_\varepsilon$ defined by (4.2) is bounded. Let V be the closure of O_ε . It follows that V is a nonempty compact set. By the assumptions, we see that

$$\text{Sol}(K, f) = \{x \in \mathbb{R}^n : f(x) - f^* = 0, p_i(x) = 0, i \in [l], q_j(x) \leq 0, j \in [m]\}.$$

From this equation, by applying Lemma 4.2 for $U = \text{Sol}(K, f)$ and the compact set V , there are constants $c_0 > 0$ and $H > 0$ such that

$$d(x, \text{Sol}(K, f)) \leq c_0 A(x)^H \quad \forall x \in V, \quad (4.6)$$

where

$$A(x) := |f(x) - f^*| + \sum_{i=1}^l |p_i(x)| + \sum_{j=1}^m [q_j(x)]_+.$$

Let $g \in \mathcal{P}_d$ be arbitrary given such that $\|g - f\| < \varepsilon$. From the definition of V , $\text{Sol}(K, f)$ and $\text{Sol}(K, g)$ are subsets of V . Here, $\text{Sol}(K, g)$ may be empty. By the compactness of V , we define the constant $L := \max\{\|X(x)\| : x \in V\}$. Hence, one has

$$|g(x) - f(x)| = |(g - f)(x)| \leq L\|g - f\| \quad \forall x \in V. \quad (4.7)$$

If $\text{Sol}(K, g)$ is empty, then (4.5) is obvious. Thus, we consider the case that $\text{Sol}(K, g) \neq \emptyset$. Since both $\text{Sol}(K, f)$ and $\text{Sol}(K, g)$ are nonempty and compact, for any $x_g \in \text{Sol}(K, g)$, there is $x_f \in \text{Sol}(K, f)$ such that

$$\|x_g - x_f\| = d(x_g, \text{Sol}(K, f)). \quad (4.8)$$

Because of $p_i(x_g) = 0$ for $i \in [l]$ and $q_j(x_g) \leq 0$ for $j \in [m]$, from the definition of $A(x)$, one has $A(x_g) = |f(x_g) - f^*|$. By (4.8) and (4.6), we see that

$$\|x_g - x_f\| \leq c_0 A(x_g)^H = c_0 |f(x_g) - f^*|^H.$$

Since $x_f \in \text{Sol}(K, f)$, we have $f(x_f) = f^* \leq f(x_g)$. Therefore, we obtain

$$\|x_g - x_f\| \leq c_0 |f(x_g) - f^*|^H = c_0 (f(x_g) - f(x_f))^H. \quad (4.9)$$

It follows from $x_g \in \text{Sol}(K, g)$ that $g(x_g) - g(x_f) \leq 0$. Since $x_g, x_f \in V$, we conclude from (4.7) that

$$\begin{aligned} f(x_g) - f(x_f) &= (f(x_g) - g(x_g)) + (g(x_g) - g(x_f)) + (g(x_f) - f(x_f)) \\ &\leq (f(x_g) - g(x_g)) + (g(x_f) - f(x_f)) \\ &\leq 2L\|g - f\|. \end{aligned}$$

The inequality (4.9) and the last result lead to

$$\|x_g - x_f\| \leq c_0 (2L)^H \|g - f\|^H,$$

consequently,

$$d(x_g, \text{Sol}(K, f)) = \|x_g - x_f\| \leq \ell \|g - f\|^H,$$

where $\ell = c_0 (2L)^H$.

The conclusion holds for any x_g in $\text{Sol}(K, g)$. Hence, the inclusion (4.5) of the theorem is proved. \square

5 Genericity of the regularity condition

In this section, we discuss the genericity of the regularity condition of polynomial optimization problems.

One says that a subset A is generic in \mathbb{R}^m if A contains a countable intersection of dense and open sets in \mathbb{R}^m . If A is generic in \mathbb{R}^m and $A \subset B$ then B also is generic in \mathbb{R}^m . Let \mathcal{T} be a topological space. It is known that if $h : \mathbb{R}^m \rightarrow \mathcal{T}$ is a homeomorphism and A is generic in \mathbb{R}^m then $h(A)$ is generic in \mathcal{T} .

Let $U \subset \mathbb{R}^m$ be a semi-algebraic set. Then there exists a decomposition of U into a disjoint union [20, Theorem 2.3.6], $U = \cup_{i=1}^s U_i$, where each U_i is semi-algebraically homeomorphic to $(0, 1)^{d_i}$. Here, let $(0, 1)^0$ be a point, $(0, 1)^{d_i} \subset \mathbb{R}^{d_i}$ be the set of points $x = (x_1, \dots, x_{d_i})$ such that $x_j \in (0, 1)$ for all $j = 1, \dots, d_i$. The *dimension* of U is defined by $\dim(U) := \max\{d_1, \dots, d_s\}$. The dimension is well-defined and does not depend on the decomposition of S . Recall that if the dimension of a nonempty semi-algebraic set U is zero, then U has finitely many points. Furthermore, if $\dim(\mathbb{R}^m \setminus U) < m$, then U is generic in \mathbb{R}^m (see, e.g. [21, Lemma 2.3]).

The space generated by all monomials of degree d listed by lexicographic ordering $\{x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \dots, x_n^d\}$ is denoted by \mathcal{H}_d . One has the direct sum $\mathcal{P}_d = \mathcal{H}_d \oplus \mathcal{P}_{d-1}$. The dimension of the vector space is denoted by η . For every homogeneous polynomial $h \in \mathcal{H}_d$, one has a unique $b \in \mathbb{R}^\eta$, such that $h(x) = b^T X_d(x)$, where

$$X_d^T(x) = (x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \dots, x_n^d).$$

Here, $\nabla(b^T X_d(x))$ is the gradient vector of $b^T X_d(x)$ and that $D_b[\nabla(b^T X_d(x))]$ is the Jacobian matrix of $b^T X_d(x)$ with respect to b .

Lemma 5.1 *One has $\text{rank}(D_b[\nabla(b^T X_d(x))]) = n$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

Proof In the proof, we are only interested in the monomials $x_i^{d-1}x_j$, where $i, j \in [n]$. Hence, for convenience, we rewrite $X_d^T(x)$ and b^T respectively as follows:

$$(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_n; x_2^{d-1}x_1, x_2^d, \dots, x_2^{d-1}x_n; \dots; x_n^{d-1}x_1, x_n^{d-1}x_2, \dots, x_n^d; \dots)$$

and

$$(b_{11}, b_{12}, \dots, b_{1n}; b_{21}, b_{22}, \dots, b_{2n}; \dots; b_{n1}, b_{n2}, \dots, b_{nn}; \dots).$$

Then, we have

$$b^T X_d(x) = \sum_{j \in [n]} b_{1j} x_1^{d-1} x_j + \sum_{j \in [n]} b_{2j} x_2^{d-1} x_j + \dots + \sum_{j \in [n]} b_{nj} x_n^{d-1} x_j + Q, \quad (5.1)$$

where Q is a homogeneous polynomial of degree d .

From (5.1), an easy computation shows that

$$\frac{\partial(b^T X_d(x))}{\partial x_i} = db_{ii} x_i^{d-1} + (d-1) \sum_{j \neq i} b_{ij} x_i^{d-2} x_j + \sum_{j \neq i} b_{ji} x_j^{d-1} + \frac{\partial Q}{\partial x_i},$$

and the $n \times \eta$ -matrix $D_b[\nabla(b^T X_d(x))]$ can be described as follows

$$D_b[\nabla(b^T X_d(x))] = [M_1, M_2, \dots, M_n, \dots],$$

where the submatrix M_i , for $i \in [n]$, is defined by

$$M_i = \begin{bmatrix} x_i^{d-1} E_{i-1} & O_{i \times 1} & O_{i \times (n-i)} \\ L_{1 \times (i-1)} & dx_i^{d-1} & R_{1 \times (n-i)} \\ O_{i \times (n-i)} & O_{(n-i) \times 1} & x_i^{d-1} E_{n-i} \end{bmatrix}$$

with E_k being the unit $k \times k$ -matrix, $O_{k \times s}$ being the zero $k \times s$ -matrix,

$$L_{1 \times (i-1)} = \left((d-1)x_i^{d-2}x_1, \dots, (d-1)x_i^{d-2}x_{i-1} \right),$$

and

$$R_{1 \times (n-i)} = \left((d-1)x_i^{d-2}x_{i+1}, \dots, (d-1)x_i^{d-2}x_n \right).$$

We observe that $\det(M_i) = dx_i^{d(d-1)}$, for all $i \in [n]$. Since $x \neq 0$, there exists $l \in [n]$ such that $x_l \neq 0$. This implies that $\text{rank}(M_l) = n$. Hence, the rank of $D_b[\nabla(b^T X_d(x))]$ is n , for any $x \neq 0$. \square

Suppose that C is a polyhedral cone given by

$$C = \{x \in \mathbb{R}^n : Ax \geq 0\}, \quad (5.2)$$

where $A = (a_{ij}) \in \mathbb{R}^{p \times n}$. Let $\text{KKT}(C, g)$, where $g \in \mathcal{P}_d$, be the set of the Karush-Kuhn-Tucker points of $\text{OP}(C, g)$, i.e. $x \in \text{KKT}(C, g)$ if and only if there exists $\lambda \in \mathbb{R}^p$ such that

$$\begin{cases} \nabla g(x) - A^T \lambda = 0, \\ \lambda^T (Ax) = 0, \lambda \geq 0, Ax \geq 0. \end{cases} \quad (5.3)$$

From the Karush-Kuhn-Tucker conditions, we see that $\text{Sol}(C, g) \subset \text{KKT}(C, g)$ for all $g \in \mathcal{P}_d$.

For each index set $\alpha \subset [p]$, we associate the *pseudo-face* C_α of C , which is denoted and defined by

$$C_\alpha := \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{ij} x_j = 0 \quad \forall i \in \alpha, \sum_{j=1}^n a_{ij} x_j > 0 \quad \forall i \in [p] \setminus \alpha \right\},$$

where a_{ij} is the element in the i -th row and the j -th column of A . The number of pseudo-faces of C is finite. These pseudo-faces establish a disjoint decomposition of C . So, we obtain

$$\text{KKT}(C, g) = \bigcup_{\alpha \subset [p]} (\text{KKT}(C, g) \cap C_\alpha), \quad (5.4)$$

The following proposition shows that the Karush-Kuhn-Tucker set-valued map of homogeneous polynomial optimization problems

$$\text{KKT}_C : \mathbb{R}^\eta \rightrightarrows \mathbb{R}^n, \quad b \mapsto \text{KKT}_C(b) = \text{KKT}(C, b^T X_d(x))$$

is finite-valued, i.e., the cardinal $\#\text{KKT}_C(b)$ is finite, on a generic semi-algebraic set of \mathbb{R}^η .

Proposition 5.1 *Assume that the cone C is polyhedral given by (5.2) and the matrix A is full rank. Then, there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^n$ such that $\#\text{KKT}_C(b) < \infty$ for any $b \in \mathbb{S}$.*

Proof Let C_α be a nonempty pseudo-face of C and $0 \notin C_\alpha$. This implies that $X_d(x)$ is nonzero on this pseudo-face. We consider the function

$$\Phi_\alpha : \mathbb{R}^n \times C_\alpha \times \mathbb{R}_+^{|\alpha|} \rightarrow \mathbb{R}^{n+|\alpha|},$$

which is defined by

$$\Phi_\alpha(b, x, \lambda_\alpha) = \left(\nabla(b^T X_d(x)) + \sum_{i \in \alpha} \lambda_i A_i, A_\alpha x \right),$$

where $A_\alpha x = (A_{i_1} x, \dots, A_{i_{|\alpha|}} x)$, $i_j \in \alpha$. Clearly, C_α is smooth and Φ_α is a semi-algebraic function of class C^∞ . The Jacobian matrix of Φ_α is determined as follows

$$D\Phi_\alpha = \left[\begin{array}{c|c|c} D_b[\nabla(b^T X_d(x))] & * & A_\alpha^T \\ \hline O_{|\alpha| \times n} & A_\alpha & O_{|\alpha| \times |\alpha|} \end{array} \right].$$

From Lemma 5.1, for all $x \in C_\alpha$, the rank of $D_b[\nabla(b^T X_d(x))]$ is n . By assumptions, we conclude that the rank of the matrix $D\Phi_\alpha$ is $n + |\alpha|$ for all $x \in C_\alpha$. Therefore, $0 \in \mathbb{R}^{n+|\alpha|+|\mathcal{J}|}$ is a regular value of Φ_α . According to the Sard Theorem with parameter [21, Theorem 2.4], there exists a generic semi-algebraic set $\mathbb{S}_\alpha \subset \mathbb{R}^n$ such that if $b \in \mathbb{S}_\alpha$ then 0 is a regular value of the map

$$\Phi_{\alpha,b} : C_\alpha \times \mathbb{R}^{|\alpha|} \rightarrow \mathbb{R}^{n+|\alpha|}, \quad \Phi_{\alpha,b}(x, \lambda_\alpha) = \Phi_\alpha(b, x, \lambda_\alpha).$$

We see that $\Omega(\alpha, b) := \Phi_{\alpha,b}^{-1}(0)$ is a semi-algebraic set. Furthermore, as the Regular Level Set Theorem [22, Theorem 9.9], we can claim that if the set is nonempty then it is a 0 -dimensional semi-algebraic set. It follows that $\Omega(\alpha, b)$ is a finite set. Moreover, from (5.3), one has $\text{KKT}_C(b) \cap C_\alpha = \pi(\Omega(\alpha, b))$, where π is the projection $\mathbb{R}^{n+|\alpha|} \rightarrow \mathbb{R}^n$ which is defined by $\pi(x, \lambda_\alpha) = x$. Hence, $\text{KKT}_C(b) \cap C_\alpha$ is a finite set.

We consider the case that $0 \in C_\alpha$ and define $U := C_\alpha \setminus \{0\}$. Clearly, U is semi-algebraic since C_α and $\{0\}$ are semi-algebraic. From (5.3), we see that $0 \in \text{KKT}_C(b)$. Hence,

$$\text{KKT}_C(b) \cap C_\alpha = \{0\} \cup (\text{KKT}_C(b) \cap U).$$

From the previous argument, $\text{KKT}_C(b) \cap U$ is a finite set. By the decomposition (5.4), $\text{KKT}_C(b)$ is a finite set.

Take $\mathbb{S} = \bigcap_{\alpha \subset [p]} \mathbb{S}_\alpha$, it follows that \mathbb{S} is generic in \mathbb{R}^n and $\text{KKT}_C(b)$ has finite points for any $b \in \mathbb{S}$. Hence, $\#\text{KKT}_C(b) < \infty$ for all b in \mathbb{S} . The proof is complete. \square

Corollary 5.1 *Assume that the cone C is polyhedral given by (5.2) and the matrix A is full rank. Then there exists a generic set \mathcal{G}_d in \mathcal{H}_d such that $\#\text{Sol}(C, g) < \infty$ for any $g \in \mathcal{G}_d$.*

Proof Since \mathbb{R}^η and \mathcal{H}_d are homeomorphic, with the isomorphism $\Pi : \mathbb{R}^\eta \rightarrow \mathcal{H}_d$ defined by $\Pi(b) = b^T X_d(x)$. According to Proposition 5.1, there exists a generic set $\mathbb{S} \subset \mathbb{R}^\eta$ such that the Karush-Kuhn-Tucker set $\text{KKT}(C, b)$ is finite, for any $b \in \mathbb{S}$. Clearly, $\mathcal{G}_d := \Pi(\mathbb{S})$ is generic in \mathcal{H}_d . Since $\text{Sol}(C, b^T X_d(x)) \subset \text{KKT}(C, b)$, one has $\#\text{Sol}(C, g) < \infty$, for any $g \in \mathcal{G}_d$. \square

Remark 5.1 If the constraint K is represented by

$$K = \{x \in \mathbb{R}^n : q_1(x) \leq 0, \dots, q_m(x) \leq 0\}, \quad (5.5)$$

where q_1, \dots, q_m are convex polynomials, then the recession cone of K is a nonempty polyhedral cone. We denote

$$K^j = \{x \in \mathbb{R}^n : q_j(x) \leq 0\}, \quad j \in [m].$$

For each $j \in [m]$, K^j is closed convex set, and K_∞^j is polyhedral (see [9, p.39]). Since $K = K^1 \cap \dots \cap K^m$, according to [3, Proposition 2.1.9], one has

$$K_\infty = \bigcap_{j \in [m]} K_\infty^j.$$

It follows that K_∞ is a nonempty polyhedral cone. Hence, there exists a matrix $A \in \mathbb{R}^{p \times n}$ such that

$$K_\infty = \{x \in \mathbb{R}^n : Ax \geq 0\}. \quad (5.6)$$

Theorem 5.1 *Assume that K be represented by (5.5) and the cone K_∞ represented by (5.6), where A is full rank. Then, the set \mathcal{R} is generic in \mathcal{P}_d .*

Proof From Remark 5.1, the recession cone K_∞ is a nonempty polyhedral cone, where $K_\infty = \{x \in \mathbb{R}^n : Ax \geq 0\}$. According to Corollary 5.1, there exists a generic set \mathcal{G}_d in \mathcal{H}_d such that $\#\text{Sol}(K_\infty, g) < \infty$ for any $g \in \mathcal{G}_d$. Because of the direct sum $\mathcal{P}_d = \mathcal{H}_d \oplus \mathcal{P}_{d-1}$, the set $\mathcal{G}_d \oplus \mathcal{P}_{d-1}$ is generic in \mathcal{P}_d . It is easy to check that $\mathcal{G}_d \oplus \mathcal{P}_{d-1} \subset \mathcal{R}$. Hence, \mathcal{R} is generic in \mathcal{P}_d . \square

Example 5.1 Consider the problem $\text{OP}(K, f)$ given in Example 2.1, we see that

$$\mathcal{R}_{\mathbb{R}, 2} = \{a_2 x^2 + a_1 x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$$

is open and dense in \mathcal{P}_2 .

Conclusions and future work

We introduced a regularity condition in the asymptotic sense of polynomial optimization problems. Two criteria for the solution existence, namely, Frank-Wolfe type theorem and Eaves type theorem have been obtained. Under the regularity condition, we established several local properties of the solution map of polynomial optimization problems. When the constraint set is convex and semi-algebraic, we proved the genericity of the regular property.

Based on the regularity condition, the stability of the optimal value function of polynomial optimization problems will be studied in future work. Furthermore, the regularity condition enables us to investigate the connectedness of the solution sets of convex polynomial vector optimization problems.

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