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Abstract In this paper, we introduce and investigate a new regularity condition in the asymptotic sense for optimization problems whose objective functions are polynomial. The normalization argument in asymptotic analysis enables us to study the existence as well as the stability of solutions of these problems. We prove a Frank-Wolfe type theorem for regular optimization problems and an Eaves type theorem for non-regular pseudoconvex optimization problems. Moreover, under the regularity condition, we show results on the stability such as upper semicontinuity and local upper-Hölder stability of the solution map of polynomial optimization problems. At the end of the paper, we discuss the genericity of the regularity condition.

Keywords Polynomial optimization · Regularity condition · Asymptotic cone · Frank–Wolfe type theorem · Eaves type theorem · Upper semicontinuity · Local upper-Hölder stability · Genericity

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1 Introduction

We consider the following optimization problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in K,$$

where K is a nonempty, closed subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial in n variables of degree $d \geq 2$. The problem and its solution set are denoted by $\mathrm{OP}(K, f)$ and $\mathrm{Sol}(K, f)$ respectively. Let f_d be the homogeneous component of degree d of f, and let K_∞ be the asymptotic cone of K that will be

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introduced in Section 2. We say that $\mathrm{OP}(K, f)$ is regular if the solution set of the asymptotic problem $\mathrm{OP}(K_\infty, f_d)$ is bounded, and the problem is non-regular otherwise. The regularity condition has appeared in studies about the solution existence and stability in quadratic programming (see, e.g., [1,2] and the references therein).

Asymptotic cones and functions play an important role in optimization and variational inequalities [3]. The normalization argument in asymptotic analysis enables us to study the existence and stability of solutions not only for quadratic programming, linear complementarity problems, and affine variational inequalities (see, e.g., [1,4]), but also for polynomial complementarity problems and polynomial variational inequalities that have unbounded constraint sets (see, e.g., [5,6]). In this paper, the normalization argument is used as the main technique to investigate the existence as well as the stability of solutions to polynomial optimization problems.

In 1956, Frank and Wolfe [7] proved that if K is polyhedral and f is quadratic and bounded from below on K, then $\mathrm{Sol}(K, f)$ is nonempty. Several versions of the Frank-Wolfe theorem for quadratic, cubic, and polynomial optimization problems have been shown in [1,2,8–12]. Belousov and Klatte [9], and Obuchowska [10] have proved Frank-Wolfe type theorems for convex and quasiconvex polynomial optimization problems. Recently, by using a technique from semi-algebraic geometry, Dinh, Ha and Pham [11] have shown a Frank-Wolfe type theorem for nondegenerate problems. The present paper gives another Frank-Wolfe type theorem, which says that if $\mathrm{OP}(K, f)$ is regular and f is bounded from below on K, then the problem has a solution. Besides, the Eaves theorem [13] provides us with another criterion for the existence of solutions to quadratic optimization problems. Extensions of this theorem for quadratically constrained quadratic problems have been investigated in [1,2, 14,15]. This paper introduces an Eaves type theorem for non-regular pseudoconvex optimization problems, where the constraint sets are convex.

Under the assumption that the constraint set K is compact and semi-algebraic, some stability and genericity results for polynomial optimization problems have been shown by Lee and Pham [16]. If K is compact, then its asymptotic cone is trivial, i.e., $K_\infty = \{0\}$; Hence that $\mathrm{OP}(K, f)$ satisfies the regularity condition obviously. In the present paper, K may be unbounded. Under the regularity condition, we prove several local properties of the solution map of polynomial optimization problems such as local boundedness and upper semicontinuity. Furthermore, based on an error bound for a polynomial system in [20], we prove the local upper-Hölder stability of the solution map.

We denote by $\mathbb{R}_d[x]$ the space of all polynomials of degree at most d and by $\mathcal{R}_d$ the set of all polynomials g of degree d such that $\mathrm{OP}(K, g)$ is regular. The set $\mathcal{R}_d$ is an open cone in $\mathbb{R}_d[x]$. At the end of this work, K is defined by convex polynomials, we prove that $\mathcal{R}_d$ is generic in $\mathbb{R}_d[x]$.

The organization of the paper is as follows. Section 2 gives a brief introduction to asymptotic cones, polynomials, and the regularity condition. Section 3 proves two criteria of the solution existence. Section 4 investigates properties

of the solution map. The last section discusses the genericity of the regularity condition.

2 Preliminaries

Recall that the *asymptotic cone* [3] of a nonempty closed subset S in $\mathbb{R}^n$ is defined and denoted by

$$S_\infty = \Big\{v \in \mathbb{R}^n : \exists t_k \to +\infty, \exists x_k \in S \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = v\Big\}.$$

Clearly, the cone S_∞ is closed and contains 0. The set S is bounded if and only if S_∞ is trivial. Furthermore, if S is convex then S_∞ is a closed convex cone and $S_\infty = 0^+S$, where 0^+S is the *recession* cone of S, that consists of all vectors $v \in \mathbb{R}^n$ such that $x + tv \in S$ for any $x \in S$ and $t \geq 0$. Thus, one has $S = S + S_\infty$ when S is convex.

Let $d \geq 2$ be given. The dimension of the space $\mathbb{R}_d[x]$ is finite; its dimension is denoted by ρ. Let $X(x)$ be the vector consisting of ρ monomials of degree at most d which is listed by lexicographic ordering

$$X(x) := (1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, \ldots, x_1^d, x_1^{d-1}x_2, \ldots, x_n^d)^T.$$

For every $g \in \mathbb{R}_d[x]$, there exists a unique vector $a = (a_1, \ldots, a_\rho) \in \mathbb{R}^\rho$ such that $g(x) = a^T X(x)$. We denote by $\|g\|$ the ℓ_2–norm of the polynomial g, namely

$$\|g\| := \|a\| = \sqrt{a_1^2 + \cdots + a_\rho^2}.$$

The Cauchy–Schwarz inequality yields $|g(x)| \leq \|X(x)\|\|g\|$. Furthermore, if $\{g^k\}$ is a convergent sequence in $\mathbb{R}_d[x]$ with $g^k \to g$, then $g_d^k \to g_d$.

Throughout the paper, we assume that the constraint set $K \subset \mathbb{R}^n$ is nonempty and closed, and the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 2$.

We say that $\mathrm{OP}(K, f)$ is a *polynomial optimization problem* if K is given by polynomials. With the given set K and the given integer d, the solution map of polynomial optimization problems $\mathrm{OP}(K, g)$, where $g \in \mathbb{R}_d[x]$, is defined by

$$\mathrm{Sol}_K(\cdot) : \mathbb{R}_d[x] \rightrightarrows \mathbb{R}^n, \quad g \mapsto \mathrm{Sol}(K, g).$$

Assume that $g \in \mathbb{R}_d[x]$ with $\deg g = d$ and $g = g_d + \cdots + g_1 + g_0$, where g_l is a homogeneous polynomial of degree l, i.e., $g_l(tx) = t^l g_l(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$, $l \in [d] := \{1, \ldots, d\}$, and $g_0 \in \mathbb{R}$. Then, g_d is the *leading term* (or the recession polynomial) of the polynomial g (of degree d). Clearly, one has

$$g_d(x) = \lim_{\lambda\to+\infty} \frac{g(\lambda x)}{\lambda^d}, \ \forall x \in \mathbb{R}^n.$$

For the pair (K, f), the asymptotic pair (K_∞, f_d) is unique. The asymptotic optimization problem $\mathrm{OP}(K_\infty, f_d)$ plays a vital role in the investigation of behavior of $\mathrm{OP}(K, f)$ at infinity. The following remarks point out (without proof) the basic properties of the asymptotic problem.

Remark 2.1 Since f_d is a homogeneous polynomial and K_∞ is a closed cone, the asymptotic optimization problem $\mathrm{OP}(K_\infty, f_d)$ has a solution if and only if f_d is non-negative on K_∞.

Remark 2.2 Assume that $\mathrm{Sol}(K_\infty, f_d)$ is nonempty. Then, this set is a closed cone with $0 \in \mathrm{Sol}(K_\infty, f_d)$. In addition, $\mathrm{Sol}(K_\infty, f_d)$ coincides with the zero set of f_d in K_∞, i.e.,

$$\mathrm{Sol}(K_\infty, f_d) = \{x \in K_\infty : f_d(x) = 0\}.$$

Now, we introduce the regularity notion concerning the boundedness of the solution set of $\mathrm{OP}(K_\infty, f_d)$.

Definition 2.1 The problem $\mathrm{OP}(K, f)$ is said to be *regular* if $\mathrm{Sol}(K_\infty, f_d)$ is bounded and *non-regular* otherwise.

Denote by $\mathcal{E}_d$ (resp., $\mathcal{O}_d, \mathcal{U}_d$) the set of all polynomials g of degree d such that $\mathrm{Sol}(K_\infty, f_d)$ is the empty set (resp., the trivial cone, an unbounded cone). Clearly, $\mathcal{R}_d = \mathcal{E}_d \cup \mathcal{O}_d$, and one has the following disjoint union:

$$\mathbb{R}_d[x] = \mathbb{R}_{d-1}[x] \cup \mathcal{E}_d \cup \mathcal{O}_d \cup \mathcal{U}_d\,. \tag{2.1}$$

Remark 2.3 The boundedness of $\mathrm{Sol}(K_\infty, f_d)$ implies that of $\mathrm{Sol}(K, f)$. Indeed, assume to the contrary that $\mathrm{Sol}(K, f)$ is unbounded. There exists an unbounded sequence $\{x^k\} \subset \mathrm{Sol}(K, f)$. Without loss of generality, we can assume x^k is nonzero for all k, $\|x^k\| \to +\infty$, and $\|x^k\|^{-1}x^k \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$ with $\|\bar{x}\| = 1$. Note that $f(x^k) = f^*$, where $f^* \in \mathbb{R}$ is the minimum of f over K, for all k. By dividing the last equation by $\|x^k\|^d$ and letting $k \to +\infty$, we obtain $f_d(\bar{x}) = 0$. It follows that $\bar{x} \in \mathrm{Sol}(K_\infty, f_d)$. Since $\bar{x} \neq 0$, the cone $\mathrm{Sol}(K_\infty, f_d)$ is unbounded, which contradicts our assumption. Thus, the claim is proved.

Remark 2.4 We observe that the set $\mathcal{R}_d$ is nonempty. If K is bounded then $\mathcal{R}_d$ coincides with the set of all $g \in \mathbb{R}_d[x]$ such that $\deg g = d$. Hence, we can suppose that K is unbounded. Clearly, the cone K_∞ also is unbounded. Let $\bar{x} \in K_\infty$ be nonzero. There exists $l \in [n]$ such that $\bar{x}_l \neq 0$. Let us define a homogeneous polynomial of degree d as $f(x) := -(\bar{x}_l x_l)^d$. For any $t > 0$, one has $t\bar{x} \in K_\infty$, and $f(t\bar{x}) = -(\bar{x}_l^2)^d t^d \to -\infty$ as $t \to +\infty$. Then, f is not bounded from below on K_∞. This yields $\mathrm{Sol}(K_\infty, f) = \emptyset$ and $f \in \mathcal{R}_d$.

Example 2.1 Consider the case that $n = 1$, $K = \mathbb{R}$ and $d = 2$. One has $\mathbb{R}_2[x] = \{a_2x^2 + a_1x + a_0 : (a_2, a_1, a_0) \in \mathbb{R}^3\}$. Since $K_\infty = \mathbb{R}$, an easy computation shows that $\mathcal{E}_2 = \{a_2x^2 + a_1x + a_0 : a_2 < 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$, $\mathcal{O}_2 = \{a_2x^2 + a_1x + a_0 : a_2 > 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$, and $\mathcal{R}_2 = \{a_2x^2 + a_1x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$.

3 Two criteria for the solution existence

We now introduce two criteria for the solution existence of $\mathrm{OP}(K, f)$. In the proofs, the normalization argument in asymptotic analysis plays a vital role; meanwhile, the semi-algebraicity of K is not required.

3.1 A Frank-Wolfe type theorem for regular problems

The following theorem provides us a criterion for the solution existence of regular optimization problems.

Theorem 3.1 (Frank-Wolfe type theorem) *If* $\mathrm{OP}(K, f)$ *is regular and* f *is bounded from below on* K*, then its solution set is nonempty and compact.*

Proof Suppose that $f \in \mathcal{R}_d$, i.e. $f \in \mathcal{E}_d \cup \mathcal{O}_d$, and there exists $\gamma \in \mathbb{R}$ such that $\gamma \leq f(x)$, for all $x \in K$. For any given $v \in K_\infty$, there are two sequences $\{t_k\} \subset \mathbb{R}_+$ and $\{x^k\} \subset K$ such that $t_k \to +\infty$ and $t_k^{-1} x^k \to v$ as $k \to +\infty$. For any k, one has $\gamma \leq f(x^k)$. Dividing both sides of the last inequality by t_k^d and letting $k \to +\infty$, we obtain $0 \leq f_d(v)$. Thus, f_d is non-negative over K_∞. It follows from Remark 2.1 that f does not belong to $\mathcal{E}_d$; hence, we conclude that f must be in $\mathcal{O}_d$.

Let $\bar{x} \in K$ be given, and $M := \{x \in K : f(x) \leq f(\bar{x})\}$. It is easy to check that $\mathrm{Sol}(M, f) = \mathrm{Sol}(K, f)$. Hence, we need only to prove that $\mathrm{Sol}(M, f)$ is nonempty and compact.

Clearly, M is closed. We claim that M is bounded. On the contrary, we suppose that there exists an unbounded sequence $\{x^k\} \subset M$ such that x^k is nonzero for all k, $\|x^k\| \to +\infty$, and $\|x^k\|^{-1} x^k \to v$ for some $v \in \mathbb{R}^n$ with $\|v\| = 1$. One has

$$\gamma \leq f(x_k) \leq f(\bar{x}), \tag{3.1}$$

for all k. Dividing the values in (3.1) by $\|x^k\|^d$ and letting $k \to +\infty$, we get $f_d(v) = 0$. This yields $v \in \mathrm{Sol}(K_\infty, f_d)$. Because of $v \neq 0$, $\mathrm{Sol}(K_\infty, f_d)$ is unbounded. This contradicts our assumption and, thus, the claim is proved.

The compactness of M and Bolzano-Weierstrass' Theorem allow us to conclude that $\mathrm{Sol}(M, f)$ is nonempty and compact. □

Remark 3.1 From the proof of Theorem 3.1, we see that if f is bounded from below on K then $\mathrm{Sol}(K_\infty, f_d)$ is nonempty, i.e. $f \in \mathcal{O}_d \cup \mathcal{U}_d$. Hence, if $f \in \mathcal{E}_d$ then $\mathrm{OP}(K, f)$ has no solution.

Corollary 3.1 *Assume that* $f = \alpha_1 x_1^d + \cdots + \alpha_n x_n^d + p$ *where* d *is even,* $\alpha_\ell > 0$ *for all* $\ell \in [n]$*,* p *is a polynomial with* $\deg p < d$*. Then,* $\mathrm{Sol}(K, f)$ *is nonempty and compact.*

Proof Clearly, $f_d = \alpha_1 x_1^d + \cdots + \alpha_n x_n^d$ is non-negative over $\mathbb{R}^n$. It follows that f_d is also non-negative over K_∞. From Remarks 2.1 and 2.2, it is clear that $\mathrm{Sol}(K_\infty, f_d)$ is nonempty and

$$\mathrm{Sol}(K_\infty, f_d) = \{x \in K_\infty : \alpha_1 x_1^d + \cdots + \alpha_n x_n^d = 0\} = \{0\}.$$

This means that $f \in \mathcal{O}_d$. Clearly, f is bounded from below on K, and the condition of Theorem 3.1 holds. Therefore, $\mathrm{Sol}(K, f)$ is nonempty and compact. □

The following example illustrates Theorem 3.1, in which the constraint set is neither convex nor semi-algebraic.

Example 3.1 Consider the optimization problem $\mathrm{OP}(K, f)$, where the polynomial f is given by $f(x_1, x_2) = x_2^3 - x_1 x_2$ and the constraint set K is given by

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0, e^{x_1} - x_2 \geq 0\}.$$

Since $f_3(x_1, x_2) = x_2^3$ and $K_\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 - x_1 \geq 0\}$, one has $\mathrm{Sol}(K_\infty, f_3) = \{(0, 0)\}$. According to Theorem 3.1, $\mathrm{Sol}(K, f)$ is nonempty and compact.

3.2 An Eaves type theorem for non-regular problems

In this subsection, we investigate the solution existence of non-regular optimization problems, where the objective functions are pseudoconvex on the constraint sets.

Assume that U is an open subset of $\mathbb{R}^n$. One says the polynomial f is *pseudoconvex* on U if, for any $x, y \in U$ such that $\langle \nabla f(x), y - x\rangle \geq 0$, here ∇f is the gradient of f, we have $f(y) \geq f(x)$. Recall that f is pseudoconvex on U if and only if ∇f is pseudomonotone on U [17, Theorem 3.1], i.e. if, for any $x, y \in U$ such that $\langle \nabla f(x), y - x\rangle \geq 0$, we have $\langle \nabla f(y), y - x\rangle \geq 0$.

Lemma 3.1 *Assume that K is convex and f is pseudoconvex on an open set U containing K. If $x^0 \in \mathrm{Sol}(K, f)$, then $\langle \nabla f(x), x - x^0\rangle \geq 0$ for all $x \in K$.*

Proof Since f is pseudoconvex on the set U, the gradient ∇f is pseudomonotone on U. Suppose that $x^0 \in \mathrm{Sol}(K, f)$, one has $\langle \nabla f(x^0), x - x^0\rangle \geq 0$ for all $x \in K$ (see, e.g., [18, Proposition 5.2]). The pseudomonotonicity of the gradient implies that $\langle \nabla f(x), x - x^0\rangle \geq 0$ for all $x \in K$. The lemma is proved. □

Theorem 3.2 (Eaves type theorem) *Assume that K is convex and f is pseudoconvex on an open set containing K. If $\mathrm{OP}(K, f)$ is non-regular, then the following statements are equivalent:*

(a) *If $v \in \mathrm{Sol}(K_\infty, f_d) \setminus \{0\}$, then there exists $x \in K$ such that $\langle \nabla f(x), v\rangle > 0$;*
(b) *$\mathrm{Sol}(K, f)$ is nonempty and compact.*

Proof Suppose that $\mathrm{OP}(K, f)$ is non-regular. We prove (a) ⇒ (b). Assume that (a) holds. For each $k \in \mathbb{N}$, we denote

$$K_k = \{x \in \mathbb{R}^n : x \in K, \|x\| \leq k\}.$$

Clearly, K_k is compact and convex. Without loss of generality, we can assume that K_k is nonempty. According to Bolzano-Weierstrass' Theorem, $\mathrm{OP}(K_k, f)$ has a solution, denoted by x^k.

We assert that the sequence $\{x^k\}$ is bounded. Indeed, suppose on the contrary that $\{x^k\}$ is unbounded, $x^k \neq 0$ for all k, $\|x^k\| \to +\infty$, and $\|x^k\|^{-1} x^k \to v$, where $v \in K_\infty$ and $\|v\| = 1$. For each k, one has

$$f(x^k) \leq f(x), \ \forall x \in K_k. \tag{3.2}$$

Let $y \in K$ be given. For k large enough, $y \in K_k$ and $f(x^k) \leq f(y)$. By dividing two sides of the last inequality by $\|x^k\|^d$ and letting $k \to +\infty$, we obtain $f_d(v) \leq 0$. This leads to $v \in \mathrm{Sol}(K_\infty, f_d) \setminus \{0\}$. Furthermore, since f is pseudoconvex on K_k, from Lemma 3.1 we have

$$\langle \nabla f(y), y - x^k \rangle \geq 0. \tag{3.3}$$

Dividing both sides of the inequality in (3.3) by $\|x^k\|$ and letting $k \to +\infty$, we obtain $\langle \nabla f(y), v \rangle \leq 0$. The conclusion holds for any $x \in K$, i.e., $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. This contradicts to our assumption. Hence, $\{x^k\}$ is bounded.

We can assume that $x^k \to \bar{x}$. From (3.2), by the continuity of f, it easy to check that $\bar{x}$ solves $\mathrm{OP}(K, f)$, so $\mathrm{Sol}(K, f)$ is nonempty.

To prove the compactness of the solution set, we can repeat the previous argument by supposing that there is an unbounded solution sequence $\{x^k\}$, and can show that there exists $v \in \mathrm{Sol}(K_\infty, f_d) \setminus \{0\}$ such that $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. This contradicts to (b).

(b) $\Rightarrow$ (a) Since K is convex, one has $K_\infty = 0^+K$ and $K = K + K_\infty$. Suppose that $\mathrm{Sol}(K, f)$ is nonempty and compact, but (b) is wrong, i.e. there exists $v \in \mathrm{Sol}(K_\infty, f_d) \setminus \{0\}$ such that $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. Let $\bar{x}$ be a solution of $\mathrm{OP}(K, f)$. For any $t \geq 0$, one has $\bar{x} + tv \in K$ and $\langle \nabla f(\bar{x} + tv), v \rangle \leq 0$. Thus, we have

$$\langle \nabla f(\bar{x} + tv), \bar{x} - (\bar{x} + tv) \rangle \geq 0.$$

The pseudoconvexity of f yields $f(\bar{x}) \geq f(x^0 + tv)$. Hence, $x^0 + tv$ belongs to $\mathrm{Sol}(K, f)$, for any $t \geq 0$. This shows that $\mathrm{Sol}(K, f)$ is unbounded which contradicts to our assumption. Thus (a) holds, and the proof is complete. □

Corollary 3.2 *Assume that K is convex and f is convex on an open convex set containing K. If $\mathrm{OP}(K, f)$ is non-regular, then the following statements are equivalent:*

(a) *If $v \in \mathrm{Sol}(K_\infty, f_d) \setminus \{0\}$, then there exists $x \in K$ such that $\langle \nabla f(x), v \rangle > 0$;*
(b) *$\mathrm{Sol}(K, f)$ is nonempty and compact.*

Proof Since the convexity implies the pseudoconvexity, by applying Theorem 3.2 for the convex polynomial f, we have the assertion. □

The following example illustrates Corollary 3.2.

Example 3.2 Consider the polynomial optimization problem $\mathrm{OP}(K, f)$ with the objective function $f(x_1, x_2) = \frac{1}{6}x_2^3 + \frac{1}{2}x_1^2 - x_1x_2$ and the constraint set $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 1, x_2 \geq 2\}$. The gradient and the Hessian matrix of f, respectively, are given by

$$\nabla f = \begin{bmatrix} x_1 - x_2 \\ -x_1 + \frac{1}{2}x_2^2 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & -1 \\ -1 & x_2 \end{bmatrix}.$$

It is easy to check that K is convex and H is positive semidefinite on the open set $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 > 0, x_2 > 1\} \supset K$; hence f is convex on K. One has $K_\infty = \mathbb{R}^2_+$ and $f_3(x_1, x_2) = \frac{1}{6}x_2^3$. This yields

$$\mathrm{Sol}(K_\infty, f_3) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}.$$

For every $v = (\alpha, 0)$ in $\mathrm{Sol}(K_\infty, f_3) \setminus \{0\}$, one has $\alpha > 0$. By choosing the point $x = (3, 2)$ in the constraint set, we get $\langle \nabla f(x), v\rangle = \alpha > 0$. Finally, according to Corollary 3.2, the solution set of $\mathrm{OP}(K, f)$ is nonempty and compact.

4 Stability of the solution map

We investigate the local boundedness, the upper semicontinuity, and the local upper-Hölder stability of the solution map under the regularity condition.

4.1 Upper semicontinuity of the solution map

To prove the local boundedness and the upper semicontinuity of the solution map, we need the following lemma.

Lemma 4.1 *The set $\mathcal{R}_d$ is open in $\mathbb{R}_d[x]$.*

Proof To prove the openness of $\mathcal{R}_d$, we only need to show that the complement $\mathbb{R}_d[x] \setminus \mathcal{R}_d$ is closed. Clearly, $\mathbb{R}_d[x] \setminus \mathcal{R}_d = \mathbb{R}_{d-1}[x] \cup \mathcal{U}_d$. Let $\{g^k\}$ be a sequence in $\mathbb{R}_d[x] \setminus \mathcal{R}_d$ such that $g^k \to g$. From the definition of $\mathcal{R}_d$, if $\deg g < d$, i.e., $g \in \mathbb{R}_{d-1}[x]$, then $g \in \mathbb{R}_d[x] \setminus \mathcal{R}_d$. Thus, we can suppose that $\deg g = d$. One has $g_d^k \to g_d$, here g_d^k is the component of degree d of g^k.

We now prove that g belongs to $\mathcal{U}_d$. For each k, $\mathrm{Sol}(K_\infty, g_d^k)$ is unbounded. There exists an unbounded sequence $\{x^k\}$ such that $x^k \in \mathrm{Sol}(K_\infty, g_d^k)$, $\|x^k\| \to +\infty$, $\|x^k\|^{-1} x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. Let $v \in K_\infty$ be given. One has $\|x^k\| v \in K_\infty$ and $g_d^k(\|x^k\| v) \geq g_d^k(x^k)$, for any k. Dividing the last inequality by $\|x^k\|^d$ and letting $k \to +\infty$, one has $g_d^k(v) \geq g_d^k(\bar{x})$. This conclusion holds for every $v \in K_\infty$. This yields $\bar{x} \in \mathrm{Sol}(K_\infty, g_d)$. As $\|\bar{x}\| = 1$, we have $\bar{x} \neq 0$. It follows that g belongs to $\mathcal{U}_d$. The closedness of $\mathbb{R}_d[x] \setminus \mathcal{R}_d$ is proved. □

Recall that a set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *locally bounded* at $\bar{u}$ if there exists an open neighborhood U of $\bar{u}$ such that $\cup_{u \in U} \Psi(u)$ is bounded [19, Definition 5.14]. The map Ψ is *upper semicontinuous* at $\bar{u} \in T$ if for any open set $V \subset \mathbb{R}^n$ such that $\Psi(\bar{u}) \subset V$ there exists a neighborhood U of $\bar{u}$ such that $\Psi(u) \subset V$ for all $u \in U$. Recall that if Ψ is closed, namely, the graph

$$\mathrm{gph}(\Psi) := \big\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n : v \in \Psi(u)\big\}$$

is closed in $\mathbb{R}^m \times \mathbb{R}^n$, and locally bounded at u, then Ψ is upper semicontinuous at u [19, Theorem 5.19].

Proposition 4.1 *Assume that K is convex. If $\mathrm{OP}(K, f)$ is regular, then the following statements hold:*

(a) *The solution map $\mathrm{Sol}_K(\cdot)$ is locally bounded at f, i.e., there exists $\varepsilon > 0$ such that the set*

$$O_\varepsilon := \bigcup_{g \in \mathcal{B}(\varepsilon, d)} \mathrm{Sol}(K, f + g), \tag{4.1}$$

where $\mathcal{B}(\varepsilon, d)$ is the open ball in $\mathbb{R}_d[x]$ with center 0 and radius ε, is bounded.

(b) *The solution map* $\mathrm{Sol}_K(\cdot)$ *is upper semicontinuous at* f.

Proof (a) According to Lemma 4.1, $\mathcal{R}_d$ is open in $\mathbb{R}_d[x]$. There is a closed ball $\overline{\mathcal{B}}(\varepsilon, d)$ such that

$$f + \overline{\mathcal{B}}(\varepsilon, d) \subset \mathcal{R}_d\,. \tag{4.2}$$

Assume to the contrary that O_ε is unbounded. Then, there exists an unbounded sequence $\{x^k\}$ and a sequence $\{g^k\} \subset \mathcal{B}(\varepsilon, d)$ such that x^k solves $\mathrm{OP}(K, f + g^k)$ with $x^k \neq 0$ for every k, $\|x^k\| \to +\infty$, and $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. By the compactness of $\overline{\mathcal{B}}(\varepsilon, d)$, without loss of generality, we can assume that $g^k \to g$ with $g \in \overline{\mathcal{B}}(\varepsilon, d)$.

From assumptions, for every k, one has

$$(f + g^k)(y) \geq (f + g^k)(x^k), \tag{4.3}$$

for any $y \in K$. Let $y \in K$ be fixed and assume that $v \in K_\infty$. By the convexity of K, one has $y + \|x^k\|v \in K$ for any k. From (4.3), we conclude that

$$(f + g^k)(y + \|x^k\|v) \geq (f + g^k)(x^k).$$

Dividing this inequality by $\|x^k\|^d$ and taking $k \to +\infty$, we obtain

$$(f + g)_d(v) \geq (f + g)_d(\bar{x}).$$

The conclusion hold for any $v \in K_\infty$. It follows that $\bar{x} \in \mathrm{Sol}\,(K_\infty, (f + g)_d)$. Because of (4.2), $\mathrm{Sol}\,(K_\infty, (f + g)_d)$ is contained in $\{0\}$, which contradicts to $\|\bar{x}\| = 1$. Hence, O_ε must be bounded.

(b) It is not difficult to prove that the graph

$$\mathrm{gph}(\mathrm{Sol}) := \big\{(g, x) \in \mathbb{R}_d[x] \times \mathbb{R}^n : x \in \mathrm{Sol}(K, g)\big\}$$

is closed in $\mathbb{R}_d[x] \times \mathbb{R}^n$. Since $\mathrm{Sol}_K(\cdot)$ is locally bounded on $\mathcal{R}_d$, according to [19, Theorem 5.19], $\mathrm{Sol}_K(\cdot)$ is upper semicontinuous at f. □

4.2 Local upper-Hölder stability of the solution map

When the constraint set K is convex and given by polynomials, we can investigate the local upper-Hölder stability of the solution map under the regular condition. To prove the stability, we need the following lemma.

Lemma 4.2 ([20]) *Let U be a semi-algebraic subset in $\mathbb{R}^n$, represented by*

$$U = \{x \in \mathbb{R}^n : u_i(x) = 0, i \in [l], v_j(x) \leq 0, j \in [m]\}\,,$$

where $u_i(x), i \in [l]$, and $v_j(x), j \in [m]$, are polynomials. For any compact set $V \subset \mathbb{R}^n$, there are constants $c > 0$ and $H > 0$ such that

$$d(x, U) \leq c\Big(\sum_{i=1}^{l} |u_i(x)| + \sum_{j=1}^{m} [v_j(x)]_+\Big)^H,$$

for all $x \in V$, here $[r]_+ := \max\{0, r\}$ and $d(x, U)$ the usual distance from x to the set U.

Theorem 4.1 *Assume that* $\mathrm{OP}(K,f)$ *is regular and* K *is a convex set given by*

$$K=\{x\in\mathbb{R}^n : p_i(x)=0, i\in[l], q_j(x)\le 0, j\in[m]\},$$

where all p_i, q_j *are polynomials. If* $\mathrm{Sol}(K,f)$ *is nonempty, then the map* $\mathrm{Sol}_K(\cdot)$ *is locally upper-Hölder stable at* f*, i.e., there exist* $\ell>0, H>0$ *and* $\varepsilon>0$ *such that*

$$\mathrm{Sol}(K,g)\subset \mathrm{Sol}(K,f)+\ell\|g-f\|^H\mathbb{B}, \tag{4.4}$$

for all $g\in\mathbb{R}_d[x]$ *satisfying* $\|g-f\|<\varepsilon$*, where* $\mathbb{B}$ *is the closed unit ball in* $\mathbb{R}^n$*.*

Proof Suppose $\mathrm{Sol}(K,f)$ is nonempty and its optimal value is f^*. Since $\mathrm{OP}(K,f)$ is regular and K is convex, according to Proposition 4.1, there exists $\varepsilon>0$ such that $\mathrm{Sol}(K,f)\subset O_\varepsilon$, defined by (4.1), is bounded. Let V be the closure of O_ε. It follows that V is a nonempty compact set. By the assumptions, we see that

$$\mathrm{Sol}(K,f)=\{x\in\mathbb{R}^n : f(x)-f^*=0, p_i(x)=0, i\in[l], q_j(x)\le 0, j\in[m]\}.$$

From this equality, by applying Lemma 4.2 for $U=\mathrm{Sol}(K,f)$ and the compact set V, there are constants $c_0>0$ and $H>0$ such that

$$d(x,\mathrm{Sol}(K,f))\le c_0 A(x)^H \quad \forall x\in V, \tag{4.5}$$

where

$$A(x):=|f(x)-f^*|+\sum_{i=1}^{l}|p_i(x)|+\sum_{j=1}^{m}[q_j(x)]_+.$$

Let $g\in\mathbb{R}_d[x]$ be arbitrary given such that $\|g-f\|<\varepsilon$. From the definition of V, $\mathrm{Sol}(K,f)$ and $\mathrm{Sol}(K,g)$ are subsets of V. Here, $\mathrm{Sol}(K,g)$ may be empty. By the compactness of V, we define the constant $L:=\max\{\|X(x)\| : x\in V\}$. Hence, one has

$$|g(x)-f(x)|\le\|X(x)\|\|g-f\|\le L\|g-f\| \quad \forall x\in V. \tag{4.6}$$

If $\mathrm{Sol}(K,g)$ is empty, then (4.4) is obvious. Thus, we consider the case that $\mathrm{Sol}(K,g)\neq\emptyset$. Since both $\mathrm{Sol}(K,f)$ and $\mathrm{Sol}(K,g)$ are nonempty and compact, for any $x_g\in\mathrm{Sol}(K,g)$, there is $x_f\in\mathrm{Sol}(K,f)$ such that

$$\|x_g-x_f\|=d(x_g,\mathrm{Sol}(K,f)). \tag{4.7}$$

Because of $p_i(x_g)=0$ for $i\in[l]$ and $q_j(x_g)\le 0$ for $j\in[m]$, from the definition of $A(x)$, one has $A(x_g)=|f(x_g)-f^*|$. By (4.7) and (4.5), we see that

$$\|x_g-x_f\|\le c_0A(x_g)^H=c_0|f(x_g)-f^*|^H.$$

Since $x_f\in\mathrm{Sol}(K,f)$, we have $f(x_f)=f^*\le f(x_g)$. Therefore, we obtain

$$\|x_g-x_f\|\le c_0|f(x_g)-f^*|^H=c_0(f(x_g)-f(x_f))^H. \tag{4.8}$$

It follows from $x_g \in \mathrm{Sol}(K, g)$ that $g(x_g) - g(x_f) \leq 0$. Since $x_g, x_f \in V$, we conclude from (4.6) that

$$\begin{aligned} f(x_g) - f(x_f) = & \ (f(x_g) - g(x_g)) + (g(x_g) - g(x_f)) + (g(x_f) - f(x_f)) \\ \leq & \ (f(x_g) - g(x_g)) + (g(x_f) - f(x_f)) \\ \leq & \ 2L\|g - f\|. \end{aligned}$$

The inequality (4.8) and the last result lead to

$$\|x_g - x_f\| \leq c_0(2L)^H \|g - f\|^H,$$

consequently,

$$d(x_g, \mathrm{Sol}(K, f)) = \|x_g - x_f\| \leq \ell \|g - f\|^H,$$

where $\ell = c_0(2L)^H$.

The conclusion holds for any x_g in $\mathrm{Sol}(K, g)$. Hence, the inclusion (4.4) of the theorem is proved. □

5 Genericity of the regularity condition

In this section, we discuss the genericity of the regularity condition of polynomial optimization problems.

A subset A is called *generic* in $\mathbb{R}^m$ if A contains a countable intersection of dense and open sets in $\mathbb{R}^m$. If A is generic in $\mathbb{R}^m$ and $A \subset B$ then B also is generic in $\mathbb{R}^m$. Let $\mathcal{T}$ be a topological space. It is known that if $h : \mathbb{R}^m \to \mathcal{T}$ is a homeomorphism and A is generic in $\mathbb{R}^m$, then $h(A)$ is generic in $\mathcal{T}$.

Let $U \subset \mathbb{R}^m$ be a semi-algebraic set. Then, there exists a decomposition of U into a disjoint union [21, Theorem 2.3.6], $U = \cup_{i=1}^s U_i$, where each U_i is semi-algebraically homeomorphic to $(0,1)^{d_i}$. Here, let $(0,1)^0$ be a point, $(0,1)^k \subset \mathbb{R}^k$ be the set of points $x = (x_1, \ldots, x_k)$ such that $x_j \in (0,1)$ for all $j \in [k]$. The *dimension* of U is defined by $\dim(U) := \max\{d_1, \ldots, d_s\}$. The dimension is well-defined and does not depends on the decomposition of U. Recall that if U is nonempty and $\dim(U)$ is zero, then U has finitely many points. Furthermore, if $\dim(\mathbb{R}^m \setminus U) < m$, then U is generic in $\mathbb{R}^m$ (see, e.g. [22, Lemma 2.3]).

The space generated by all monomials of degree d listed by lexicographic ordering $\{x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \ldots, x_n^d\}$ is denoted by $\mathcal{H}_d$. One has the direct sum $\mathbb{R}_d[x] = \mathcal{H}_d \oplus \mathbb{R}_{d-1}[x]$. The dimension of $\mathcal{H}_d$ is denoted by η. For every homogeneous polynomial $h \in \mathcal{H}_d$, one has a unique $b \in \mathbb{R}^\eta$, such that $h(x) = b^T X_d(x)$, where

$$X_d^T(x) = (x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \ldots, x_n^d).$$

Here, $\nabla(b^T X_d(x))$ is the gradient vector of $b^T X_d(x)$ and $D_b[\nabla(b^T X_d(x))]$ is the Jacobian matrix of $b^T X_d(x)$ with respect to b.

Lemma 5.1 *One has* $\mathrm{rank}(D_b[\nabla(b^T X_d(x))]) = n$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$.

Proof In the proof, we are only interested in the monomials $x_i^{d-1}x_j$, where $i, j \in [n]$. Hence, for convenience, we rewrite $X_d^T(x)$ and b^T respectively as follows:

$$(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_n;\ \ldots;\ x_n^{d-1}x_1, x_n^{d-1}x_2, \ldots, x_n^d;\ \ldots)$$

and

$$(b_{11}, b_{12}, \ldots, b_{1n};\ \ldots;\ b_{n1}, b_{n2}, \ldots, b_{nn};\ \ldots).$$

Then, we have

$$b^T X_d(x) = \sum_{j\in[n]} b_{1j}x_1^{d-1}x_j + \cdots + \sum_{j\in[n]} b_{nj}x_n^{d-1}x_j + Q, \tag{5.1}$$

where Q is a homogeneous polynomial of degree d.

From (5.1), an easy computation shows that

$$\frac{\partial(b^T X_d(x))}{\partial x_i} = db_{ii}x_i^{d-1} + (d-1)\sum_{j\neq i} b_{ij}x_i^{d-2}x_j + \sum_{j\neq i} b_{ji}x_j^{d-1} + \frac{\partial Q}{\partial x_i},$$

and the $n \times \eta$-matrix $D_b[\nabla(b^T X_d(x))]$ can be described as follows

$$D_b[\nabla(b^T X_d(x))] = \Big[M_1, M_2, \cdots, M_n, \cdots\ \Big],$$

where the submatrix M_i, for $i \in [n]$, is defined by

$$M_i = \begin{bmatrix} x_i^{d-1}I_{i-1} & O_{i\times 1} & O_{i\times(n-i)} \\ L_{1\times(i-1)} & dx_i^{d-1} & R_{1\times(n-i)} \\ O_{i\times(n-i)} & O_{(n-i)\times 1} & x_i^{d-1}I_{n-i} \end{bmatrix}$$

with I_k being the unit $k \times k$-matrix, $O_{k\times s}$ being the zero $k \times s$-matrix,

$$L_{1\times(i-1)} = \Big((d-1)x_i^{d-2}x_1, \ldots, (d-1)x_i^{d-2}x_{i-1}\Big),$$

and

$$R_{1\times(n-i)} = \Big((d-1)x_i^{d-2}x_{i+1}, \ldots, (d-1)x_i^{d-2}x_n\Big).$$

We observe that $\det(M_i) = dx_i^{d(d-1)}$, for all $i \in [n]$. Since $x \neq 0$, there exists $l \in [n]$ such that $x_l \neq 0$. This implies that $\operatorname{rank}(M_l) = n$. Hence, the rank of $D_b[\nabla(b^T X_d(x))]$ is n, for any $x \neq 0$. □

Suppose that C is a polyhedral cone given by

$$C = \{x \in \mathbb{R}^n : Ax \geq 0\}, \tag{5.2}$$

where $A = (a_{ij}) \in \mathbb{R}^{p\times n}$. Let $\mathrm{KKT}(C, g)$, where $g \in \mathbb{R}_d[x]$, be the set of the Karush-Kuhn-Tucker points of $\mathrm{OP}(C, g)$, i.e., $x \in \mathrm{KKT}(C, g)$ if and only if there exists $\lambda \in \mathbb{R}^p$ such that

$$\begin{cases} \nabla g(x) - A^T\lambda = 0, \\ \lambda^T(Ax) = 0,\ \lambda \geq 0,\ Ax \geq 0. \end{cases} \tag{5.3}$$

From the Karush-Kuhn-Tucker conditions, we see that $\mathrm{Sol}(C,g) \subset \mathrm{KKT}(C,g)$ for all $g \in \mathbb{R}_d[x]$.

For each index set $\alpha \subset [p]$, we associate the *pseudo-face* C_α of C, which is denoted and defined by

$$C_\alpha := \Big\{ x \in \mathbb{R}^n \,:\, \sum_{j=1}^n a_{ij}x_j = 0 \;\; \forall i \in \alpha, \;\; \sum_{j=1}^n a_{ij}x_j > 0 \;\; \forall i \in [p] \setminus \alpha \Big\},$$

where a_{ij} is the element in the i-th row and the j-th column of A. The number of pseudo-faces of C is finite. These pseudo-faces establish a disjoint decomposition of C. So, we obtain

$$\mathrm{KKT}(C,g) = \bigcup_{\alpha \subset [p]} \left(\mathrm{KKT}(C,g) \cap C_\alpha \right), \tag{5.4}$$

The following proposition shows that the Karush-Kuhn-Tucker set-valued map of homogeneous polynomial optimization problems

$$\mathrm{KKT}_C : \mathbb{R}^\eta \rightrightarrows \mathbb{R}^n, \;\; b \mapsto \mathrm{KKT}_C(b) = \mathrm{KKT}(C, b^T X_d(x)),$$

is finite-valued, i.e., the cardinal $\#\,\mathrm{KKT}_C(b)$ is finite, on a generic semi-algebraic set of $\mathbb{R}^\eta$.

Proposition 5.1 *Assume that C is a polyhedral cone given by* (5.2) *and the matrix A is full rank. Then, there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^\eta$ such that $\#\,\mathrm{KKT}_C(b) < \infty$ for any $b \in \mathbb{S}$.*

Proof Let C_α be a nonempty pseudo-face of C and $0 \notin C_\alpha$. This implies that $X_d(x)$ is nonzero on this pseudo-face. We consider the function

$$\Phi_\alpha : \mathbb{R}^\eta \times C_\alpha \times \mathbb{R}_+^{|\alpha|} \to \mathbb{R}^{n+|\alpha|},$$

which is defined by

$$\Phi_\alpha(b,x,\lambda_\alpha) = \Big(\nabla(b^T X_d(x)) + \sum_{i \in \alpha} \lambda_i A_i, A_\alpha x \Big),$$

where $A_\alpha x = (A_{i_1}x, \ldots, A_{i_{|\alpha|}}x), i_j \in \alpha$. Clearly, C_α is smooth and Φ_α is a semi-algebraic function of class C^∞. The Jacobian matrix of Φ_α is determined as follows

$$D\Phi_\alpha = \left[\begin{array}{c|c|c} D_b[\nabla(b^T X_d(x))] & * & A_\alpha^T \\ \hline O_{|\alpha| \times \eta} & A_\alpha & O_{|\alpha| \times |\alpha|} \end{array} \right].$$

From Lemma 5.1, for all $x \in C_\alpha$, the rank of $D_b[\nabla(b^T X_d(x))]$ is n. Since the rank of A_α is $|\alpha|$, we conclude that the rank of the matrix $D\Phi_\alpha$ is $n + |\alpha|$ for all $x \in C_\alpha$. Therefore, $0 \in \mathbb{R}^{n+|\alpha|+|J|}$ is a regular value of Φ_α. According to the Sard Theorem with parameter [22, Theorem 2.4], there exists a generic semi-algebraic set $\mathbb{S}_\alpha \subset \mathbb{R}^\eta$ such that if $b \in \mathbb{S}_\alpha$ then 0 is a regular value of the map

$$\Phi_{\alpha,b} : C_\alpha \times \mathbb{R}^{|\alpha|} \to \mathbb{R}^{n+|\alpha|}, \;\; \Phi_{\alpha,b}(x,\lambda_\alpha) = \Phi_\alpha(b,x,\lambda_\alpha).$$

We see that $\Omega(\alpha, b) := \Phi_{\alpha,b}^{-1}(0)$ is a semi-algebraic set. From the Regular Level Set Theorem [23, Theorem 9.9], we can claim that if $\Omega(\alpha, b)$ is nonempty then the set is a $0-$dimensional semi-algebraic set. It follows that $\Omega(\alpha, b)$ is a finite set. Moreover, from (5.3), one has $\mathrm{KKT}_C(b) \cap C_\alpha = \pi(\Omega(\alpha, b))$, where π is the projection $\mathbb{R}^{n+|\alpha|} \to \mathbb{R}^n$ which is defined by $\pi(x, \lambda_\alpha) = x$. Hence, $\mathrm{KKT}_C(b) \cap C_\alpha$ is a finite set.

We consider the case that $0 \in C_\alpha$ and define $U := C_\alpha \setminus \{0\}$. As is clear, U is semi-algebraic. From (5.3), we see that $0 \in \mathrm{KKT}_C(b)$. Hence,

$$\mathrm{KKT}_C(b) \cap C_\alpha = \{0\} \cup (\mathrm{KKT}_C(b) \cap U).$$

From the previous argument, $\mathrm{KKT}_C(b) \cap U$ is a finite set. By the decomposition (5.4), $\mathrm{KKT}_C(b)$ is a finite set.

Take $\mathbb{S} = \cap_{\alpha \subset [p]} \mathbb{S}_\alpha$, it follows that $\mathbb{S}$ is generic in $\mathbb{R}^\eta$ and $\mathrm{KKT}_C(b)$ has finite points for any $b \in \mathbb{S}$. Hence, $\#\,\mathrm{KKT}_C(b) < \infty$ for all b in $\mathbb{S}$. The proof is complete. □

Corollary 5.1 *Assume that C is a polyhedral cone given by* (5.2) *and the matrix A is full rank. Then there exists a generic set $\mathcal{G}_d$ in $\mathcal{H}_d$ such that $\#\,\mathrm{Sol}(C, g) < \infty$ for any $g \in \mathcal{G}_d$.*

Proof Since $\mathbb{R}^\eta$ and $\mathcal{H}_d$ are homeomorphic, with the isomorphism $\Pi : \mathbb{R}^\eta \to \mathcal{H}_d$ defined by $\Pi(b) = b^T X_d(x)$. According to Proposition 5.1, there exists a generic set $\mathbb{S} \subset \mathbb{R}^\eta$ such that the Karush-Kuhn-Tucker set $\mathrm{KKT}(C, b)$ is finite, for any $b \in \mathbb{S}$. Clearly, $\mathcal{G}_d := \Pi(\mathbb{S})$ is generic in $\mathcal{H}_d$. Since $\mathrm{Sol}(C, b^T X_d(x)) \subset \mathrm{KKT}(C, b)$, one has $\#\,\mathrm{Sol}(C, g) < \infty$, for any $g \in \mathcal{G}_d$. □

Remark 5.1 If the constraint K is represented by

$$K = \{x \in \mathbb{R}^n : q_1(x) \le 0, \ldots, q_m(x) \le 0\}, \tag{5.5}$$

where $q_1, \ldots, q_m$ are convex polynomials, then the recession cone of K is a nonempty polyhedral cone. We denote

$$K^j = \{x \in \mathbb{R}^n : q_j(x) \le 0\}, \ j \in [m].$$

For each $j \in [m]$, K^j is closed convex set, and K_∞^j is polyhedral (see [9, p.39]). Since $K = K^1 \cap \cdots \cap K^m$, according to [3, Proposition 2.1.9], one has

$$K_\infty = \bigcap_{j \in [m]} K_\infty^j.$$

If follows that K_∞ is a nonempty polyhedral cone. Hence, there exists a matrix $A \in \mathbb{R}^{p \times n}$ such that

$$K_\infty = \{x \in \mathbb{R}^n : Ax \ge 0\}. \tag{5.6}$$

Theorem 5.1 *Assume that K be represented by* (5.5) *and the cone K_∞ represented by* (5.6)*, where A is full rank. Then, the set $\mathcal{R}_d$ is generic in $\mathbb{R}_d[x]$.*

Proof From Remark 5.1, the recession cone K_∞ is a nonempty polyhedral cone, where $K_\infty = \{x \in \mathbb{R}^n : Ax \geq 0\}$. According to Corollary 5.1, there exists a generic set $\mathcal{G}_d$ in $\mathcal{H}_d$ such that $\#\,\mathrm{Sol}(K_\infty, g) < \infty$ for any $g \in \mathcal{G}_d$. Because of the direct sum $\mathbb{R}_d[x] = \mathcal{H}_d \oplus \mathbb{R}_{d-1}[x]$, the set $\mathcal{G}_d \oplus \mathbb{R}_{d-1}[x]$ is generic in $\mathbb{R}_d[x]$. It is easy to check that $\mathcal{G}_d \oplus \mathbb{R}_{d-1}[x] \subset \mathcal{R}_d$. Hence, $\mathcal{R}_d$ is generic in $\mathbb{R}_d[x]$. □

Example 5.1 Consider the problem $\mathrm{OP}(K, f)$ given in Example 2.1, we see that

$$\mathcal{R}_2 = \{a_2x^2 + a_1x + a_0 : a_2 \neq 0, a_1 \in \mathbb{R}, a_0 \in \mathbb{R}\}$$

is open and dense in $\mathbb{R}_2[x]$.

Perspectives

The regularity condition enables us to investigate the stability of the optimal value function of polynomial optimization problems. Furthermore, the regularity condition is useful to study the connectedness of the solution sets of convex polynomial vector optimization problems.

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