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DOUBLING BIALGEBRAS OF FINITE TOPOLOGIES

MOHAMED AYADI AND DOMINIQUE MANCHON

ABSTRACT. The species of finite topological spaces admits two graded bimonoid structures, recently defined by F. Fauvet, L. Foissy, and the second author. In this article, we define a doubling of this species in two different ways. We build a bimonoid structure on each of these species and describe a cointeraction between them. We also investigate two related associative products obtained by dualisation.

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1. INTRODUCTION AND PRELIMINARIES

The present work considers all finite topological at once, and aims at investigating how they organise themselves into a rich algebraic structure, along the lines opened by L. Foissy, C. Malvenuto and F. Patras in [11, 12]. It is a follow-up of [10] by F. Fauvet, L. Foissy and the second author. The notion of doubling first appears in the work of M. Belhaj Mohamed in the context of Feynman graphs [4], motivated by understanding mathematical aspects of renormalisation. It has been adapted by M. Belhaj Mohamed and the second author to rooted forests [7], and happens to be relevant also in the finite topology framework.

Recall (see e.g. [15, 17, 10]) that a topology on a finite set X is given by the family \mathcal{T} of open subsets of X , subject to the three following axioms:

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- $\emptyset \in \mathcal{T}, X \in \mathcal{T}$,
- The union of (a finite number of) open subsets is an open subset,
- The intersection of a finite number of open subsets is an open subset.

Any topology \mathcal{T} on X defines a quasi-order (i.e. a reflexive transitive relation) denoted by $\leq_{\mathcal{T}}$ on X :

$$(1.1) \quad x \leq_{\mathcal{T}} y \iff \text{any open subset containing } x \text{ also contains } y.$$

Conversely, any quasi-order \leq on X defines a topology \mathcal{T}_{\leq} given by its upper ideals, i.e. subsets $Y \subset X$ such that $(y \in Y \text{ and } y \leq z) \implies z \in Y$. Both operations are inverse to each other:

$$(1.2) \quad \leq_{\mathcal{T}_{\leq}} = \leq, \quad \mathcal{T}_{\leq_{\mathcal{T}}} = \mathcal{T}.$$

Hence there is a natural bijection between topologies and quasi-orders on a finite set X . Any quasi-order (hence any topology \mathcal{T}) on X gives rise to an equivalence relation:

$$(1.3) \quad x \sim_{\mathcal{T}} y \iff (x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} x).$$

More on finite topological spaces can be found in [3, 9, 16, 17].

Let us recall the construction from [9] of two bimonoids [1, 2] in cointeraction on the linear species of finite topological spaces, which originated from a previous Hopf-algebraic approach [11, 12]. Let \mathcal{T} and \mathcal{T}' be two topologies on a finite set X . We say that \mathcal{T}' is finer than \mathcal{T} , and we write $\mathcal{T}' < \mathcal{T}$, when any open subset for \mathcal{T} is an open subset for \mathcal{T}' . This is equivalent to the fact that for any $x, y \in X$, $x \leq_{\mathcal{T}'} y \implies x \leq_{\mathcal{T}} y$.

The *quotient* \mathcal{T}/\mathcal{T}' of two topologies \mathcal{T} and \mathcal{T}' with $\mathcal{T}' < \mathcal{T}$ is defined as follows ([10, Paragraph 2.2]): The associated quasi-order $\leq_{\mathcal{T}/\mathcal{T}'}$ is the transitive closure of the relation \mathcal{R} defined by:

$$(1.4) \quad x \mathcal{R} y \iff (x \leq_{\mathcal{T}} y \text{ or } y \leq_{\mathcal{T}'} x).$$

Recall that a linear species is a contravariant functor from the category of finite sets with bijections into the category of vector spaces (on some field \mathbf{k}). The tensor product of two species \mathbb{E} and \mathbb{F} is given by

$$(1.5) \quad (\mathbb{E} \otimes \mathbb{F})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{E}_Y \otimes \mathbb{F}_Z,$$

where the notation \sqcup stands for disjoint union. The species \mathbb{T} of finite topological spaces is defined as follows: For any finite set X , \mathbb{T}_X is the vector space freely generated by the topologies on X . For any bijection $\varphi : X \longrightarrow X'$, the isomorphism $\mathbb{T}_{\varphi} : \mathbb{T}_{X'} \longrightarrow \mathbb{T}_X$ is defined by the obvious relabelling:

$$\mathbb{T}_{\varphi}(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\}$$

for any topology \mathcal{T} on X' .

For any finite set X , let us recall from [10] the coproduct Γ on \mathbb{T}_X :

$$(1.6) \quad \Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \in \mathcal{T}} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'.$$

The sum runs over topologies \mathcal{T}' which are \mathcal{T} -admissible, i.e

- finer than \mathcal{T} ,
- such that $\mathcal{T}'_{|Y} = \mathcal{T}_{|Y}$ for any subset $Y \subset X$ connected for the topology \mathcal{T}' ,
- such that for any $x, y \in X$,

$$(1.7) \quad x \sim_{\mathcal{T}/\mathcal{T}'} y \iff x \sim_{\mathcal{T}'/\mathcal{T}'} y.$$

A commutative monoid structure ([10, Paragraph 2.3]) on the species of finite topologies is defined as follows: for any pair X_1, X_2 of finite sets we introduce

$$\begin{aligned} m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ \mathcal{T}_1 \otimes \mathcal{T}_2 &\longmapsto \mathcal{T}_1 \mathcal{T}_2, \end{aligned}$$

where $\mathcal{T}_1 \mathcal{T}_2$ is the disjoint union topology characterised by $Y \in \mathcal{T}_1 \mathcal{T}_2$ if and only if $Y \cap X_1 \in \mathcal{T}_1$ and $Y \cap X_2 \in \mathcal{T}_2$. The unit is given by the unique topology on the empty set.

For any topology \mathcal{T} on a finite set X and for any subset $Y \subset X$, we denote by $\mathcal{T}_{|Y}$ the restriction of \mathcal{T} to Y . It is defined by:

$$\mathcal{T}_{|Y} = \{Z \cap Y, Z \in \mathcal{T}\}.$$

The external coproduct Δ on \mathbb{T} is defined as follows:

$$\begin{aligned} \Delta : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|X \setminus Y} \otimes \mathcal{T}_{|Y}. \end{aligned}$$

The internal and external coproducts are compatible, i.e. the following diagram commutes for any finite X .

$$\begin{array}{ccc} \mathbb{T}_X & \xrightarrow{\Gamma} & \mathbb{T}_X \otimes \mathbb{T}_X \\ \Delta \downarrow & & \downarrow I \otimes \Delta \\ (\mathbb{T} \otimes \mathbb{T})_X & & \mathbb{T}_X \otimes (\mathbb{T} \otimes \mathbb{T})_X \\ & \searrow \Gamma \otimes \Gamma & \nearrow m^{1,3} \\ & \bigoplus_{Y \subset X} \mathbb{T}_{X \setminus Y} \otimes \mathbb{T}_{X \setminus Y} \otimes \mathbb{T}_Y \otimes \mathbb{T}_Y & \end{array}$$

Now consider the graded vector space:

$$(1.8) \quad \mathcal{H} = \overline{\mathcal{K}}(\mathbb{T}) = \bigoplus_{n \geq 0} \mathcal{H}_n$$

where $\mathcal{H}_0 = \mathbf{k}.1$, and where \mathcal{H}_n is the linear span of topologies on $\{1, \dots, n\}$ when $n \geq 1$, modulo the action of the symmetric group S_n . The vector space \mathcal{H} can be seen as the quotient of the species \mathbb{T} by the "forget the labels" equivalence relation: $\mathcal{T} \sim \mathcal{T}'$ if \mathcal{T} (resp. \mathcal{T}') is a topology on a finite set X (resp. X'), such that there is a bijection from X onto X' which is a homeomorphism with respect to both topologies. The functor $\overline{\mathcal{K}}$ from linear species to graded vector spaces thus obtained is intensively studied in ([1, chapter 15]) under the name "bosonic Fock functor". This

naturally leads to the following:

- (\mathcal{H}, m, Δ) is a commutative connected Hopf algebra, graded by the number of elements.
- (\mathcal{H}, m, Γ) is a commutative bialgebra, graded by the number of equivalence classes minus the number of connected components.
- (\mathcal{H}, m, Δ) is a comodule-bialgebra on (\mathcal{H}, m, Γ) . In particular the following diagram of unital algebra morphisms commutes:

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{\Gamma} & \mathcal{H} \otimes \mathcal{H} \\
 \Delta \downarrow & & \downarrow I \otimes \Delta \\
 \mathcal{H} \otimes \mathcal{H} & & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \\
 \Gamma \otimes \Gamma \searrow & & \nearrow m^{1,3} \\
 & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} &
 \end{array}$$

On the vector space freely generated by rooted forests, Connes and Kreimer define in [7, 8, 13] a graded bialgebra structure defined using allowable cuts. In [6], D. Calaque, K. Ebrahimi-Fard and the second author introduced bases of a graded Hopf algebra structure defined using contractions of trees. M. Belhaj Mohamed and the second author introduced in [5] the doubling of these two spaces and they built two bialgebra structures on these spaces, which are in interaction. They have also shown that two bialgebra satisfied a commutative diagram similar to the diagram of [6] in the case of rooted trees Hopf algebra, and in the case of directed graphs without cycles [14].

In Section 2 of this paper, we define two different doubling species \mathbb{D} and $\widetilde{\mathbb{D}}$ of the species \mathbb{T} . For later use will also consider $\mathcal{D} = \overline{\mathcal{K}}(\mathbb{D})$ and $\widetilde{\mathcal{D}} = \overline{\mathcal{K}}(\widetilde{\mathbb{D}})$. The species \mathbb{D} is defined as follows: For any finite set X , \mathbb{D}_X is the vector space spanned by the pairs (\mathcal{T}, Y) where \mathcal{T} is a topology on X and $Y \in \mathcal{T}$. Similarly, $\widetilde{\mathbb{D}}_X$ is the vector space spanned by the ordered pairs $(\mathcal{T}, \mathcal{T}')$ where \mathcal{T} is a topology on X , and $\mathcal{T}' \otimes \mathcal{T}$. We prove that there exist graded bimonoid structures on \mathbb{D}_X and $\widetilde{\mathbb{D}}_X$, where the external and internal coproducts are defined respectively by

$$(1.9) \quad \Delta(\mathcal{T}, Y) = \sum_{Z \in \mathcal{T}_Y} (\mathcal{T}|_Z, Z) \otimes (\mathcal{T}|_{X \setminus Z}, Y \setminus Z),$$

for all $(\mathcal{T}, Y) \in \mathbb{D}_X$, and

$$(1.10) \quad \Gamma(\mathcal{T}, \mathcal{T}') = \sum_{\mathcal{T}'' \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}'', \mathcal{T}'/\mathcal{T}').$$

for all $(\mathcal{T}, \mathcal{T}') \in \widetilde{\mathbb{D}}_X$. We show the inclusions $\Delta(\mathbb{D}_X) \subset (\mathbb{D} \otimes \mathbb{D})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{D}_Y \otimes \mathbb{D}_Z$ and $\Gamma(\widetilde{\mathbb{D}}_X) \subset \widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X$, and that Δ and Γ are coassociative. It turns out that only the internal coproduct Γ is counital.

In Section 3, after a reminder of the main results of [10], we show an important restriction result, namely the notion of \mathcal{T} -admissibility is stable under restriction to any subset (Proposition

3.2), and we prove that \mathbb{D}_X admits a comodule structure on $\widetilde{\mathbb{D}}_X$ given by the coaction $\Phi : \mathbb{D}_X \longrightarrow \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X$, which is defined for all $(\mathcal{T}, Y) \in \mathbb{D}_X$ by:

$$\Phi(\mathcal{T}, Y) = \sum_{\substack{\mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}' \\ \mathcal{T}'|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}'}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', Y)$$

where, for any topology \mathcal{T} on a finite set X , the finest \mathcal{T} -admissible topology is denoted by $D_{X, \mathcal{T}}$. The connected components of $D_{X, \mathcal{T}}$ are the equivalence classes of \mathcal{T} , and $D_{X, \mathcal{T}}$ restricted to each connected component is the coarse topology. For any $Y \subset X$, we note $D_{Y, \mathcal{T}}$ for $D_{Y, \mathcal{T}|_Y}$.

Remark 1.1. We obviously have $d(D_{X, \mathcal{T}}) = 0$ where d is the grading given by the number of equivalence classes minus the number of connected components [10]. We also clearly have

$$\mathcal{T}/D_{X, \mathcal{T}} = \mathcal{T}.$$

In Section 4, we construct an associative product on \mathbb{D} given by $*$: $\mathbb{D} \otimes \mathbb{D} \longrightarrow \mathbb{D}$, defined for all $(\mathcal{T}_1, Y_1) \in \mathbb{D}_{X_1}$ and $(\mathcal{T}_2, Y_2) \in \mathbb{D}_{X_2}$, (where X_1 and X_2 are two finite sets) by:

$$(\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2) \mapsto \begin{cases} (\mathcal{T}_1, Y_1 \sqcup Y_2) & \text{if } X_2 = X_1 \setminus Y_1 \text{ and } \mathcal{T}_2 = \mathcal{T}_1|_{X_2} \\ 0 & \text{if not.} \end{cases}$$

This product is obtained by dualising the restriction of the coproduct Δ to \mathbb{D}_X , identifying \mathbb{D}_X with its graded dual using the basis $\{(\mathcal{T}, Y), \mathcal{T} \text{ topology and } Y \in \mathcal{T}\}$. We accordingly construct a second associative algebra structure on $\widetilde{\mathbb{D}}_X$ by dualising the restriction of the coproduct Γ to $\widetilde{\mathbb{D}}_X$, yielding the associative product $\ast : \widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X \longrightarrow \widetilde{\mathbb{D}}_X$, defined by:

$$(\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2) \mapsto \begin{cases} (\mathcal{T}_1, \mathcal{U}) & \text{if } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \\ 0 & \text{if not,} \end{cases}$$

where \mathcal{U} is defined by $\mathcal{T}'_2 = \mathcal{U}/\mathcal{T}'_1$.

Finally, we define in Section 5 a new map

$$\xi : \widetilde{\mathbb{D}}_X \otimes \bigoplus_{Y \sqcup Z = X} \mathbb{D}_Y \otimes \mathbb{D}_Z \longrightarrow \widetilde{\mathbb{D}}_X \otimes \bigoplus_{Y \sqcup Z = X} \mathbb{D}_Y \otimes \mathbb{D}_Z$$

by:

$$\xi((\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2)) = (\mathcal{T}|_{Y_1} \mathcal{T}'|_{X \setminus Y_1}, \mathcal{T}'|_{Y_1} \mathcal{T}'|_{X \setminus Y_1}) \otimes (\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2).$$

We prove that the coaction Φ and the map ξ make the following diagram commute:

$$\begin{array}{ccc} \mathbb{D}_X & \xrightarrow{\Phi} & \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \\ \Delta \downarrow & & \downarrow Id \otimes \Delta \\ (\mathbb{D} \otimes \mathbb{D})_X & & \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \\ \Phi \otimes \Phi \downarrow & & \downarrow \xi \\ \bigoplus_{Y \subset X} \widetilde{\mathbb{D}}_Y \otimes \mathbb{D}_Y \otimes \widetilde{\mathbb{D}}_{X \setminus Y} \otimes \mathbb{D}_{X \setminus Y} & \xrightarrow{m^{1,3}} & \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \end{array}$$

Applying the functor $\overline{\mathcal{K}}$ leads to the diagram:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\Phi} & \widetilde{\mathcal{D}} \otimes \mathcal{D} \\
\Delta \downarrow & & \downarrow Id \otimes \Delta \\
\mathcal{D} \otimes \mathcal{D} & & \widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D} \\
\Phi \otimes \Phi \downarrow & & \downarrow \xi \\
\widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \widetilde{\mathcal{D}} \otimes \mathcal{D} & \xrightarrow{m^{1,3}} & \widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D}
\end{array}$$

where we have written Δ for $\overline{\mathcal{K}}(\Delta)$ and so on. All arrows of this diagram are algebra morphisms.

2. DOUBLING BIALGEBRAS OF FINITE TOPOLOGIES

Let X be any finite set, and \mathbb{D}_X be the vector space spanned by the pairs (\mathcal{T}, Y) where \mathcal{T} is a topology, and $Y \in \mathcal{T}$. We define the coproduct Δ by:

$$\begin{aligned}
\Delta : \mathbb{D}_X &\longrightarrow (\mathbb{D} \otimes \mathbb{D})_X = \bigoplus_{Z \subset X} \mathbb{D}_Z \otimes \mathbb{D}_{X \setminus Z} \\
(\mathcal{T}, Y) &\longmapsto \sum_{Z \in \mathcal{T}|_Y} (\mathcal{T}|_Z, Z) \otimes (\mathcal{T}|_{X \setminus Z}, Y \setminus Z).
\end{aligned}$$

Theorem 2.1. \mathbb{D} is a commutative graded connected bimonoid, and $\mathcal{D} = \overline{\mathcal{K}}(\mathbb{D})$ is a commutative graded bialgebra.

Proof. To show that \mathbb{D} is a bimonoid [1], it is necessary to show that Δ is coassociative, and that the species coproduct Δ and the product defined by:

$$(\mathcal{T}_1, Y_1)(\mathcal{T}_2, Y_2) = (\mathcal{T}_1 \mathcal{T}_2, Y_1 \sqcup Y_2)$$

are compatible. The unit $\mathbf{1}$ is identified to the empty topology, and the grading is given by:

$$(2.1) \quad d(\mathcal{T}, Y) = |Y|.$$

The associativity of the product is given by the direct computation:

$$(\mathcal{T}_1 \mathcal{T}_2, Y_1 \sqcup Y_2)(\mathcal{T}_3, Y_3) = (\mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3, Y_1 \sqcup Y_2 \sqcup Y_3) = (\mathcal{T}_1, Y_1)(\mathcal{T}_2 \mathcal{T}_3, Y_2 \sqcup Y_3).$$

The coassociativity of coproduct Δ is also straightforwardly checked:

$$\begin{aligned}
(\Delta \otimes id)\Delta(\mathcal{T}, Y) &= (\Delta \otimes id) \left(\sum_{Z \in \mathcal{T}|_Y} (\mathcal{T}|_Z, Z) \otimes (\mathcal{T}|_{X \setminus Z}, Y \setminus Z) \right) \\
&= \sum_{W \in \mathcal{T}|_Z, Z \in \mathcal{T}|_Y} (\mathcal{T}|_W, W) \otimes (\mathcal{T}|_{Z \setminus W}, Z \setminus W) \otimes (\mathcal{T}|_{X \setminus Z}, Y \setminus Z),
\end{aligned}$$

and

$$\begin{aligned} (id \otimes \Delta)\Delta(\mathcal{T}, Y) &= (id \otimes \Delta) \left(\sum_{Z \in \mathcal{T}|_Y} (\mathcal{T}|_Z, Z) \otimes (\mathcal{T}|_{X \setminus Z}, Y \setminus Z) \right) \\ &= \sum_{U \in \mathcal{T}|_{Y \setminus Z}, Z \in \mathcal{T}|_Y} (\mathcal{T}|_Z, Z) \otimes (\mathcal{T}|_U, U) \otimes (\mathcal{T}|_{X \setminus (Z \sqcup U)}, Y \setminus (Z \sqcup U)). \end{aligned}$$

Coassociativity then comes from the obvious fact that $(W, Z) \mapsto (W, Z \setminus W)$ is a bijection from the set of pairs (W, Z) with $Z \in \mathcal{T}|_Y$ and $W \in \mathcal{T}|_Z$, onto the set of pairs (W, U) with $W \in \mathcal{T}|_Y$ and $U \in \mathcal{T}|_{Y \setminus W}$. The inverse map is given by $(W, U) \mapsto (W, W \sqcup U)$. Finally, we show immediately that

$$\Delta((\mathcal{T}_1, Y_1)(\mathcal{T}_2, Y_2)) = \Delta(\mathcal{T}_1 \mathcal{T}_2, Y_1 \sqcup Y_2) = \Delta(\mathcal{T}_1, Y_1) \Delta(\mathcal{T}_2, Y_2).$$

□

Remark 2.1. The bimonoid \mathbb{D} is not counitary, because $(\mathcal{T}, Y) \otimes \mathbf{1}$ never occurs in $\Delta(\mathcal{T}, Y)$ unless $Y = X$.

Let $\widetilde{\mathbb{D}}_X$ be the vector space spanned by the ordered pairs $(\mathcal{T}, \mathcal{T}')$ where \mathcal{T} is a topology on X and $\mathcal{T}' \otimes \mathcal{T}$. We define the coproduct Γ for all $(\mathcal{T}, \mathcal{T}') \in \widetilde{\mathbb{D}}_X$ by:

$$\Gamma(\mathcal{T}, \mathcal{T}') = \sum_{\mathcal{T}'' \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}'/\mathcal{T}'', \mathcal{T}'/\mathcal{T}'').$$

Lemma 2.1. ([10, Propostion 2.7]) *Let \mathcal{T} and \mathcal{T}'' be two topologies on X . If $\mathcal{T}'' \otimes \mathcal{T}$, then $\mathcal{T}' \mapsto \mathcal{T}'/\mathcal{T}''$ is a bijection from the set of topologies \mathcal{T}' on X such that $\mathcal{T}'' \otimes \mathcal{T}' \otimes \mathcal{T}$, onto the set of topologies \mathcal{U} on X such that $\mathcal{U} \otimes \mathcal{T}/\mathcal{T}''$.*

Theorem 2.2. $\widetilde{\mathbb{D}}$ is a commutative graded bimonoid, and $\overline{\mathbb{D}} = \overline{\mathcal{K}}(\widetilde{\mathbb{D}})$ is a graded bialgebra.

Proof. To show that $\widetilde{\mathbb{D}}$ is a bimonoid, it is necessary to show that Γ is coassociative and that the species coproduct Γ and the product defined by:

$$m((\mathcal{T}_1, \mathcal{T}'_1)(\mathcal{T}_2, \mathcal{T}'_2)) = (\mathcal{T}_1 \mathcal{T}_2, \mathcal{T}'_1 \mathcal{T}'_2)$$

are compatible. The unit $\mathbf{1}$ is identified to the empty topology, the counit ϵ is given by $\epsilon(\mathcal{T}, \mathcal{T}') = \epsilon(\mathcal{T}')$ and the grading is given by:

$$(2.2) \quad d(\mathcal{T}, \mathcal{T}') = d(\mathcal{T}'),$$

where the grading d on the right-hand side has been defined in the introduction.

We now calculate:

$$\begin{aligned} (\Gamma \otimes id)\Gamma(\mathcal{T}, \mathcal{T}') &= (\Gamma \otimes id) \left(\sum_{\mathcal{T}'' \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}'/\mathcal{T}'', \mathcal{T}'/\mathcal{T}'') \right) \\ &= \sum_{\mathcal{T}''' \otimes \mathcal{T}'' \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}''') \otimes (\mathcal{T}'/\mathcal{T}''', \mathcal{T}''/\mathcal{T}''') \otimes (\mathcal{T}'/\mathcal{T}'', \mathcal{T}'/\mathcal{T}''). \end{aligned}$$

On the other hand;

$$\begin{aligned}
(id \otimes \Gamma)\Gamma(\mathcal{T}, \mathcal{T}') &= (id \otimes \Gamma) \left(\sum_{\mathcal{T}'' \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}'', \mathcal{T}'/\mathcal{T}'') \right) \\
&= \sum_{\mathcal{T}'' \otimes \mathcal{T}', \mathcal{T}_1 \otimes \mathcal{T}'/\mathcal{T}''} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}'', \mathcal{T}_1) \otimes ((\mathcal{T}/\mathcal{T}'')/\mathcal{T}_1, (\mathcal{T}'/\mathcal{T}'')/\mathcal{T}_1) \\
&= \sum_{\mathcal{T}'' \otimes \mathcal{U} \otimes \mathcal{T}'} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}'', \mathcal{U}/\mathcal{T}'') \otimes (\mathcal{T}/\mathcal{U}, \mathcal{T}'/\mathcal{U}).
\end{aligned}$$

The result then comes from Lemma 2.1. Hence, $(\Gamma \otimes id)\Gamma = (id \otimes \Gamma)\Gamma$, and consequently Γ is coassociative. Finally we have directly:

$$\Gamma((\mathcal{T}_1, \mathcal{T}'_1)(\mathcal{T}_2, \mathcal{T}'_2)) = \Gamma(\mathcal{T}_1, \mathcal{T}'_1)\Gamma(\mathcal{T}_2, \mathcal{T}'_2).$$

□

Proposition 2.1. *The second projection*

$$\begin{aligned}
P_2 : \widetilde{\mathbb{D}} &\longrightarrow \mathbb{T} \\
(\mathcal{T}, \mathcal{T}') &\longmapsto \mathcal{T}'
\end{aligned}$$

is a bimonoid morphism with respect to the internal coproducts.

Proof. The fact that P_2 respects the product is trivial. It suffices to show that P_2 is a coalgebra morphism for any finite set X , analogously to Proposition 1, i.e, P_2 verifies the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{\mathbb{D}}_X & \xrightarrow{P_2} & \mathbb{T}_X \\
\Gamma \downarrow & & \downarrow \Gamma \\
\widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X & \xrightarrow{P_2 \otimes P_2} & \mathbb{T}_X \otimes \mathbb{T}_X
\end{array}$$

which can be seen by direct computation:

$$\begin{aligned}
\Gamma \circ P_2(\mathcal{T}, \mathcal{T}') &= \Gamma(\mathcal{T}') \\
&= \sum_{\mathcal{T}'' \otimes \mathcal{T}'} \mathcal{T}'' \otimes \mathcal{T}'/\mathcal{T}'' \\
&= \sum_{\mathcal{T}'' \otimes \mathcal{T}'} P_2(\mathcal{T}, \mathcal{T}'') \otimes P_2(\mathcal{T}/\mathcal{T}'', \mathcal{T}'/\mathcal{T}'') \\
&= (P_2 \otimes P_2)\Gamma(\mathcal{T}, \mathcal{T}').
\end{aligned}$$

□

3. COMODULE-HOPF ALGEBRA STRUCTURE

3.1. Comodule-Hopf algebra structure on the commutative Hopf algebra of finite topologies [10]. F. Fauvet, L. Foissy and the second author have studied the Hopf algebra (\mathcal{H}, m, Δ) as a comodule-Hopf algebra on the bialgebra (\mathcal{H}, m, Γ) , where $\mathcal{H} = \overline{\mathcal{K}(\mathbb{T})}$. Here the notations m, Δ, Γ

are shorthands for $\overline{\mathcal{K}}(m)$, $\overline{\mathcal{K}}(\Delta)$, $\overline{\mathcal{K}}(\Gamma)$ respectively. The coaction is the map $\overline{\mathcal{K}}(\phi) : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ where ϕ is defined as follows:

$$\phi(\mathcal{T}) = \Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \otimes \mathcal{T}'} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'.$$

Proposition 3.1. [10] *The internal and external coproducts are compatible, i.e. the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\Gamma} & \mathbb{T} \otimes \mathbb{T} \\ \Delta \downarrow & & \downarrow Id \otimes \Delta \\ (\mathbb{T} \otimes \mathbb{T})_X & & \\ \Gamma \otimes \Gamma \downarrow & & \\ \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Y \otimes \mathbb{T}_Z \otimes \mathbb{T}_Z & \xrightarrow{m^{1,3}} & \mathbb{T} \otimes (\mathbb{T} \otimes \mathbb{T})_X \end{array}$$

i.e., the following identity is verified:

$$(Id \otimes \Delta) \circ \Gamma = m^{1,3} \circ (\Gamma \otimes \Gamma) \circ \Delta,$$

where $m^{1,3} : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} \otimes \mathbb{T}_{X_3} \otimes \mathbb{T}_{X_4} \longrightarrow \mathbb{T}_{X_1 \sqcup X_3} \otimes \mathbb{T}_{X_2} \otimes \mathbb{T}_{X_4}$ is defined by

$$m^{1,3}(\mathcal{T}_1 \otimes \mathcal{T}_2 \otimes \mathcal{T}_3 \otimes \mathcal{T}_4) = \mathcal{T}_1 \mathcal{T}_3 \otimes \mathcal{T}_2 \otimes \mathcal{T}_4.$$

Applying the functor $\overline{\mathcal{K}}$ yields the comodule-Hopf algebra structure of (\mathcal{H}, m, Δ) on the bialgebra (\mathcal{H}, m, Γ) . In particular the diagram above yields the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Gamma} & \mathcal{H} \otimes \mathcal{H} \\ \Delta \downarrow & & \downarrow Id \otimes \Delta \\ \mathcal{H} \otimes \mathcal{H} & & \\ \Gamma \otimes \Gamma \downarrow & & \downarrow \\ \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{m^{1,3}} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

3.2. A restriction result. We prove that the admissibility relation \otimes between two topologies on the same finite set X , which is the key to the definition of the internal coproduct Γ , is robust under restriction to any subset.

Proposition 3.2. *Let \mathcal{T} be a topology on a finite set X . For any subset $W \subset X$ and for any $\mathcal{T}' \otimes \mathcal{T}$ we have $\mathcal{T}'|_W \otimes \mathcal{T}|_W$.*

Proof. Let \mathcal{T} be a topology on a finite set X , let W be any subset of X , and let $\mathcal{T}' \otimes \mathcal{T}$. Let R (resp. R') be the relation defined on X by aRb if and only if $a \leq_{\mathcal{T}} b$ or $b \leq_{\mathcal{T}'} a$ (resp. $aR'b$ if and only if $a \leq_{\mathcal{T}'} b$ or $b \leq_{\mathcal{T}} a$). We have $\mathcal{T}' \otimes \mathcal{T}$ hence R' implies R .

- The relation $\mathcal{T}'|_W < \mathcal{T}|_W$ is obvious.
- If $Y \subset W$ connected for the topology \mathcal{T} , and $x \in Y$, we have

$$Y = \{y \in W, \text{ there is a chain } xR't_1 \cdots R't_nR'y, \text{ with } t_1, \dots, t_n \in W\}.$$

The set $\widetilde{Y} := \{y \in W, \text{ there is a chain } xR't_1 \cdots R't_nR'y, \text{ with } t_1, \dots, t_n \in X\}$ is a connected component of X for the topology \mathcal{T}' , so $\mathcal{T}'|_{\widetilde{Y}} = \mathcal{T}|_{\widetilde{Y}}$, hence a fortiori $\mathcal{T}'|_Y = \mathcal{T}|_Y$, because the inclusion $Y \subset \widetilde{Y}$ holds.

- Let $x, y \in W$. If $x \sim_{\mathcal{T}'|_W/\mathcal{T}'|_W} y$ there is $t_1, \dots, t_n \in W$, $j \in [n]$, $y = t_j$ such that $xR't_1 \cdots R't_nR'y$. This implies $xR't_1 \cdots R't_nR'y$, therefore $x \sim_{\mathcal{T}_W/\mathcal{T}'|_W} y$. Conversely, if $x \sim_{\mathcal{T}_W/\mathcal{T}'|_W} y$ there is $t_1, \dots, t_n \in W$ and $j \in [n]$ with $y = t_j$, such that $xR't_1 \cdots R't_nR'y$. For $A = \{x, t_1, \dots, t_n\}$, we have for all a and b in A , $a \sim_{\mathcal{T}/\mathcal{T}'}$ b . Since $\mathcal{T}' \otimes \mathcal{T}$, we have $a \sim_{\mathcal{T}'/\mathcal{T}'}$ b , hence a and b in the same connected component Z for the topology \mathcal{T}' .

We have $\mathcal{T}'|_Z = \mathcal{T}|_Z$ and $A \subset Z$, hence $\mathcal{T}'|_A = \mathcal{T}|_A$. Then for all $a, b \in A$, aRb if and only if $aR'b$, so we have $xR't_1 \cdots R't_nR'y$, therefore $x \sim_{\mathcal{T}'|_W/\mathcal{T}'|_W} y$.

□

3.3. Comodule structure on the doubling bialgebras of finite topologies. For any finite set X , we define $\Phi : \mathbb{D}_X \longrightarrow \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X$, for all $(\mathcal{T}, Y) \in \mathbb{D}_X$ by:

$$\Phi(\mathcal{T}, Y) = \sum_{\substack{\mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}', \\ \mathcal{T}'|_{X \setminus Y} = \mathcal{D}_{X \setminus Y, \mathcal{T}'}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', Y).$$

The map Φ is well defined.

Theorem 3.1. \mathbb{D} admits a comodule structure on $\widetilde{\mathbb{D}}$ given by Φ .

Proof. The proof amounts to show that the following diagram is commutative for any finite set X :

$$\begin{array}{ccc} \mathbb{D}_X & \xrightarrow{\Phi} & \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \\ \Phi \downarrow & & \downarrow \Gamma \otimes id \\ \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X & \xrightarrow{id \otimes \Phi} & \widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \end{array}$$

Let $(\mathcal{T}, Y) \in \mathbb{D}_X$:

$$\begin{aligned} (\Gamma \otimes id)\Phi(\mathcal{T}, Y) &= (\Gamma \otimes id) \left(\sum_{\substack{\mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}', \\ \mathcal{T}'|_{X \setminus Y} = \mathcal{D}_{X \setminus Y, \mathcal{T}'}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', Y) \right) \\ &= \sum_{\substack{\mathcal{T}'' \otimes \mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}', \\ \mathcal{T}'|_{X \setminus Y} = \mathcal{D}_{X \setminus Y, \mathcal{T}'}}} (\mathcal{T}, \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}'', \mathcal{T}'/\mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}', Y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(id \otimes \Phi)\Phi(\mathcal{T}, Y) &= (id \otimes \Phi) \left(\sum_{\substack{\mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}', \\ \mathcal{T}'|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}'}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', Y) \right) \\
&= \sum_{\substack{\mathcal{T}' \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}', \\ \mathcal{T}'|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}'}}} \sum_{\substack{\mathcal{T}_1 \otimes \mathcal{T}/\mathcal{T}', Y \in (\mathcal{T}/\mathcal{T}')/\mathcal{T}_1, \\ \mathcal{T}_1|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}_1}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', \mathcal{T}_1) \otimes ((\mathcal{T}/\mathcal{T}')/\mathcal{T}_1, Y) \\
&= \sum_{\substack{\mathcal{T}' \otimes \mathcal{U} \otimes \mathcal{T}, Y \in \mathcal{T}/\mathcal{T}' \\ \mathcal{U}|_{X \setminus Y} = D_{X \setminus Y, \mathcal{U}}}} (\mathcal{T}, \mathcal{T}') \otimes (\mathcal{T}/\mathcal{T}', \mathcal{U}/\mathcal{T}') \otimes (\mathcal{T}/\mathcal{U}, Y).
\end{aligned}$$

Then,

$$(id \otimes \Phi) \circ \Phi = (\Gamma \otimes id) \circ \Phi,$$

and consequently Φ is a coaction. \square

Proposition 3.3. Φ is a monoid morphism, i.e. the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{D}_X & \xrightarrow{\Phi} & \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \\
\uparrow m & & \uparrow Id \otimes m \\
(\mathbb{D} \otimes \mathbb{D})_X & \xrightarrow{\widetilde{\Phi}} & \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \\
\downarrow \Phi \otimes \Phi & & \downarrow \widetilde{m} \otimes Id \otimes Id \\
\bigoplus_{Y \subset X} \widetilde{\mathbb{D}}_Y \otimes \mathbb{D}_Y \otimes \widetilde{\mathbb{D}}_{X \setminus Y} \otimes \mathbb{D}_{X \setminus Y} & \xrightarrow{\tau_{23}} & \bigoplus_{Y \subset X} \widetilde{\mathbb{D}}_Y \otimes \widetilde{\mathbb{D}}_{X \setminus Y} \otimes \mathbb{D}_Y \otimes \mathbb{D}_{X \setminus Y}
\end{array}$$

$\curvearrowright \widetilde{m} \otimes m$

Proof. Let $(\mathcal{T}_1, Y_1) \in \mathbb{D}_{X_1}$ and $(\mathcal{T}_2, Y_2) \in \mathbb{D}_{X_2}$, with $X_1 \sqcup X_2 = X$. Let $\mathcal{T}'_1 \otimes \mathcal{T}_1$ and $\mathcal{T}'_2 \otimes \mathcal{T}_2$. Then $\mathcal{T}'_1 \mathcal{T}'_2 \otimes \mathcal{T}_1 \mathcal{T}_2$. Conversely, any topology \mathcal{U} on X such that $\mathcal{U} \otimes \mathcal{T}_1 \mathcal{T}_2$ can be written $\mathcal{T}'_1 \mathcal{T}'_2$ with $\mathcal{T}'_i = \mathcal{U}|_{X_i}$ for $i = 1, 2$, and we have $\mathcal{T}'_i \otimes \mathcal{T}_i$. We have then:

$$\Phi \circ m((\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2)) = \sum_{\substack{\mathcal{U} \otimes \mathcal{T}, Y_1 \sqcup Y_2 \in \mathcal{T}_1 \mathcal{T}_2 / \mathcal{U} \\ \mathcal{U}|_{X_1 \sqcup X_2 \setminus Y_1 \sqcup Y_2} = D_{X_1 \sqcup X_2 \setminus Y_1 \sqcup Y_2, \mathcal{U}}}} (\mathcal{T}_1 \mathcal{T}_2, \mathcal{U}) \otimes ((\mathcal{T}_1 \mathcal{T}_2) / \mathcal{U}, Y_1 \sqcup Y_2)$$

On the other hand, we have

$$\begin{aligned}
(\Phi \otimes \Phi)((\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2)) &= \\
&= \sum_{\substack{\mathcal{T}'_1 \otimes \mathcal{T}_1, \mathcal{T}'_1|_{X_1 \setminus Y_1} = D_{X_1 \setminus Y_1, \mathcal{T}'_1}, Y_1 \in \mathcal{T}_1 / \mathcal{T}'_1 \\ \mathcal{T}'_2 \otimes \mathcal{T}_2, \mathcal{T}'_2|_{X_2 \setminus Y_2} = D_{X_2 \setminus Y_2, \mathcal{T}'_2}, Y_2 \in \mathcal{T}_2 / \mathcal{T}'_2}} (\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_1 / \mathcal{T}'_1, Y_1) \otimes (\mathcal{T}_2, \mathcal{T}'_2) \otimes (\mathcal{T}_2 / \mathcal{T}'_2, Y_2),
\end{aligned}$$

therefore

$$\begin{aligned}
& (\bar{m} \otimes m) \circ \tau_{23} \circ (\Phi \otimes \Phi)((\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2)) \\
&= \sum_{\substack{\mathcal{T}'_1 \otimes \mathcal{T}_1, \mathcal{T}'_1|_{X_1 \setminus Y_1} = D_{X_1 \setminus Y_1, \mathcal{T}'_1}, Y_1 \in \mathcal{T}'_1 / \mathcal{T}'_1 \\ \mathcal{T}'_2 \otimes \mathcal{T}_2, \mathcal{T}'_2|_{X_2 \setminus Y_2} = D_{X_2 \setminus Y_2, \mathcal{T}'_2}, Y_2 \in \mathcal{T}'_2 / \mathcal{T}'_2}} (\mathcal{T}_1 \mathcal{T}_2, \mathcal{T}'_1 \mathcal{T}'_2) \otimes ((\mathcal{T}_1 / \mathcal{T}'_1)(\mathcal{T}_2 / \mathcal{T}'_2), Y_1 \sqcup Y_2) \\
&= \Phi \circ m((\mathcal{T}_1, Y_1) \otimes (\mathcal{T}_2, Y_2)).
\end{aligned}$$

Hence

$$\Phi \circ m = (\bar{m} \otimes m) \circ \tau_{23} \circ (\Phi \otimes \Phi).$$

□

4. ASSOCIATIVE ALGEBRA STRUCTURES ON THE DOUBLING SPACES

4.1. Associative product on \mathcal{D} . For any finite set X , recall here that an element (\mathcal{T}, Y) belongs to \mathcal{D}_X if \mathcal{T} is a topology on X and $Y \in \mathcal{T}$.

Theorem 4.1. *The product $*$: $\mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D}$ defined for all $(\mathcal{T}_1, Y_1) \in \mathcal{D}_{X_1}$ and $(\mathcal{T}_2, Y_2) \in \mathcal{D}_{X_2}$ by:*

$$(\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2) \longmapsto \begin{cases} (\mathcal{T}_1, Y_1 \sqcup Y_2) & \text{if } X_2 = X_1 \setminus Y_1 \text{ and } \mathcal{T}_2 = \mathcal{T}_1|_{X_2}, \\ 0 & \text{if not} \end{cases}$$

is associative.

Proof. Let (\mathcal{T}_1, Y_1) , (\mathcal{T}_2, Y_2) and (\mathcal{T}_3, Y_3) be three elements of \mathcal{D}_{X_1} , \mathcal{D}_{X_2} and \mathcal{D}_{X_3} respectively. We suppose first that $X_2 = X_1 \setminus Y_1$ and $\mathcal{T}_2 = \mathcal{T}_1|_{X_2}$, otherwise the result is zero.

$$\begin{aligned}
((\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2)) * (\mathcal{T}_3, Y_3) &= (\mathcal{T}_1, Y_1 \sqcup Y_2) * (\mathcal{T}_3, Y_3) \\
&= (\mathcal{T}_1, Y_1 \sqcup Y_2 \sqcup Y_3),
\end{aligned}$$

whenever $X_3 = X \setminus (Y_1 \sqcup Y_2)$ and $\mathcal{T}_3 = \mathcal{T}_1|_{X_3}$, the left-hand side vanishing otherwise. Hence,

$$((\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2)) * (\mathcal{T}_3, Y_3) = \begin{cases} (\mathcal{T}_1, Y_1 \sqcup Y_2 \sqcup Y_3) & \text{if } X_2 = X_1 \setminus Y_1, X_3 = X_1 \setminus (Y_1 \sqcup Y_2), \mathcal{T}_2 = \mathcal{T}_1|_{X_2} \text{ and } \mathcal{T}_3 = \mathcal{T}_1|_{X_3} \\ 0 & \text{if not.} \end{cases}$$

On the other hand, we have

$$\begin{aligned}
(\mathcal{T}_1, Y_1) * ((\mathcal{T}_2, Y_2) * (\mathcal{T}_3, Y_3)) &= (\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2 \sqcup Y_3) \\
&= (\mathcal{T}_1, Y_1 \sqcup Y_2 \sqcup Y_3),
\end{aligned}$$

whenever $X_3 = X_2 \setminus Y_2$ and $\mathcal{T}_3 = \mathcal{T}_2|_{X_3}$, as well as $X_2 = X_1 \setminus Y_1$ and $\mathcal{T}_2 = \mathcal{T}_1|_{X_2}$. Then

$$(\mathcal{T}_1, Y_1) * ((\mathcal{T}_2, Y_2) * (\mathcal{T}_3, Y_3)) = \begin{cases} (\mathcal{T}_1, Y_1 \sqcup Y_2 \sqcup Y_3) & \text{if } X_2 = X_1 \setminus Y_1, X_3 = X_2 \setminus Y_2, \mathcal{T}_2 = \mathcal{T}_1|_{X_2} \text{ and } \mathcal{T}_3 = \mathcal{T}_2|_{X_3} \\ 0 & \text{if not.} \end{cases}$$

We therefore conclude that for all (\mathcal{T}_1, Y_1) , (\mathcal{T}_2, Y_2) , $(\mathcal{T}_3, Y_3) \in \mathcal{D}_X$, we have

$$((\mathcal{T}_1, Y_1) * (\mathcal{T}_2, Y_2)) * (\mathcal{T}_3, Y_3) = (\mathcal{T}_1, Y_1) * ((\mathcal{T}_2, Y_2) * (\mathcal{T}_3, Y_3)),$$

which proves the associativity of the product $*$.

□

4.2. **Associative product on $\widetilde{\mathbb{D}}$.** Recall here that an element $(\mathcal{T}, \mathcal{T}')$ belongs to $\widetilde{\mathbb{D}}_X$ if \mathcal{T} and \mathcal{T}' are both topologies on X such that $\mathcal{T}' \otimes \mathcal{T}$.

Theorem 4.2. *The product $\ast : \widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X \longrightarrow \widetilde{\mathbb{D}}_X$, defined by:*

$$(\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2) \longmapsto \begin{cases} (\mathcal{T}_1, \mathcal{U}) & \text{if } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1, \\ 0 & \text{if not} \end{cases}$$

is associative, where \mathcal{U} is defined by $\mathcal{T}'_2 = \mathcal{U}/\mathcal{T}'_1$. ([10, Proposition 2.7], see Lemma 2.1).

Proof. Let $(\mathcal{T}_1, \mathcal{T}'_1)$, $(\mathcal{T}_2, \mathcal{T}'_2)$ and $(\mathcal{T}_3, \mathcal{T}'_3)$ be three elements of $\widetilde{\mathbb{D}}_X$, i.e., $\mathcal{T}'_1 \otimes \mathcal{T}_1$, $\mathcal{T}'_2 \otimes \mathcal{T}_2$ and $\mathcal{T}'_3 \otimes \mathcal{T}_3$. We suppose first that $\mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1$, otherwise the result is zero.

$$((\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2)) \ast (\mathcal{T}_3, \mathcal{T}'_3) = (\mathcal{T}_1, \mathcal{U}) \ast (\mathcal{T}_3, \mathcal{T}'_3),$$

where \mathcal{U} is defined by $\mathcal{T}'_2 = \mathcal{U}/\mathcal{T}'_1$, then

$$((\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2)) \ast (\mathcal{T}_3, \mathcal{T}'_3) = (\mathcal{T}_1, \mathcal{V})$$

where \mathcal{V} and \mathcal{U} are defined by $\mathcal{T}'_3 = \mathcal{V}/\mathcal{U}$, and $\mathcal{T}'_2 = \mathcal{U}/\mathcal{T}'_1$. Then,

$$((\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2)) \ast (\mathcal{T}_3, \mathcal{T}'_3) \longmapsto \begin{cases} (\mathcal{T}_1, \mathcal{V}) & \text{if } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \text{ and } \mathcal{T}_3 = \mathcal{T}_1/\mathcal{U} \\ 0 & \text{if not.} \end{cases}$$

Where $\mathcal{T}'_2 = \mathcal{U}/\mathcal{T}'_1$, and $\mathcal{T}'_3 = \mathcal{V}/\mathcal{U}$.

On the other hand, for $\mathcal{T}_3 = \mathcal{T}_2/\mathcal{T}'_2$, we have

$$(\mathcal{T}_1, \mathcal{T}'_1) \ast ((\mathcal{T}_2, \mathcal{T}'_2) \ast (\mathcal{T}_3, \mathcal{T}'_3)) = (\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{W}),$$

where $\mathcal{T}'_3 = \mathcal{W}/\mathcal{T}'_2$, where $\mathcal{T}_3 = \mathcal{T}_2/\mathcal{T}'_2$, and $\mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1$. Then $\mathcal{T}_3 = \mathcal{T}_2/\mathcal{T}'_2 = \mathcal{T}_1/\mathcal{W}$, and $\mathcal{W} = \mathcal{Z}/\mathcal{T}'_1$. Hence

$$(\mathcal{T}_1, \mathcal{T}'_1) \ast ((\mathcal{T}_2, \mathcal{T}'_2) \ast (\mathcal{T}_3, \mathcal{T}'_3)) = \begin{cases} (\mathcal{T}_1, \mathcal{Z}) & \text{if } \mathcal{T}_3 = \mathcal{T}_1/(\mathcal{T}'_2 \sqcup \mathcal{T}'_3) \text{ and } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \\ 0 & \text{if not.} \end{cases}$$

Where \mathcal{Z} is defined by $\mathcal{W} = \mathcal{Z}/\mathcal{T}'_1$, and $\mathcal{T}'_3 = \mathcal{W}/\mathcal{T}'_2$. It remains to show that $\mathcal{V} = \mathcal{Z}$: We have

$$\mathcal{V}/\mathcal{U} = \mathcal{T}'_3 = \mathcal{W}/\mathcal{T}'_2 = (\mathcal{Z}/\mathcal{T}'_1)/(\mathcal{U}/\mathcal{T}'_1) = \mathcal{Z}/\mathcal{U}.$$

Moreover, $\mathcal{T}_3 = \mathcal{T}_2/\mathcal{T}'_2 = (\mathcal{T}_1/\mathcal{T}'_1)/(\mathcal{U}/\mathcal{T}'_1) = \mathcal{T}_1/\mathcal{U}$. We therefore conclude that for all $(\mathcal{T}_1, \mathcal{T}'_1)$, $(\mathcal{T}_2, \mathcal{T}'_2)$ and $(\mathcal{T}_3, \mathcal{T}'_3)$ in $\widetilde{\mathbb{D}}_X$, we have

$$((\mathcal{T}_1, \mathcal{T}'_1) \ast (\mathcal{T}_2, \mathcal{T}'_2)) \ast (\mathcal{T}_3, \mathcal{T}'_3) = (\mathcal{T}_1, \mathcal{T}'_1) \ast ((\mathcal{T}_2, \mathcal{T}'_2) \ast (\mathcal{T}_3, \mathcal{T}'_3)),$$

which proves the associativity of the product \ast . □

5. OTHER RELATIONS BETWEEN BOTH DOUBLING SPECIES \mathbb{D} AND $\widetilde{\mathbb{D}}$

5.1. An action of $\widetilde{\mathbb{D}}$ on \mathbb{D} .

Definition 5.1. *For any finite set X , let $\Psi : \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \longrightarrow \mathbb{D}_X$ be the map defined by:*

$$\Psi((\mathcal{T}, \mathcal{T}') \otimes (\mathcal{U}, Y)) = \begin{cases} (\mathcal{T}, Y) & \text{if } \mathcal{U} = \mathcal{T}/\mathcal{T}' \\ 0 & \text{if not.} \end{cases}$$

Proposition 5.1. Ψ is an action of $\widetilde{\mathbb{D}}$ on \mathbb{D} .

Proof. We have to verify the commutativity of this diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{D}}_X \otimes \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X & \xrightarrow{id \otimes \Psi} & \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \\ \downarrow \ast \otimes id & & \downarrow \Psi \\ \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X & \xrightarrow{\Psi} & \mathbb{D}_X \end{array}$$

Let $(\mathcal{T}_1, \mathcal{T}'_1)$ and $(\mathcal{T}_2, \mathcal{T}'_2)$ be two elements of $\widetilde{\mathbb{D}}_X$, and $(\mathcal{U}, Y) \in \mathbb{D}_X$. We have

$$(id \otimes \Psi)((\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_2, \mathcal{T}'_2) \otimes (\mathcal{U}, Y)) = \begin{cases} (\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_2, Y) & \text{if } \mathcal{U} = \mathcal{T}_2/\mathcal{T}'_2 \\ 0 & \text{if not.} \end{cases}$$

Then,

$$\Psi \circ (id \otimes \Psi)((\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_2, \mathcal{T}'_2) \otimes (\mathcal{U}, Y)) = \begin{cases} (\mathcal{T}_1, Y) & \text{if } \mathcal{U} = \mathcal{T}_2/\mathcal{T}'_2 \text{ and } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \\ 0 & \text{if not.} \end{cases}$$

On the other hand, we have

$$(\ast \otimes id)((\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_2, \mathcal{T}'_2) \otimes (\mathcal{U}, Y)) = \begin{cases} (\mathcal{T}_1, \mathcal{V}) \otimes (\mathcal{U}, Y) & \text{if } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \\ 0 & \text{if not.} \end{cases}$$

where $\mathcal{T}'_2 = \mathcal{V}/\mathcal{T}'_1$. Then,

$$\Psi \circ (\ast \otimes id)((\mathcal{T}_1, \mathcal{T}'_1) \otimes (\mathcal{T}_2, \mathcal{T}'_2) \otimes (\mathcal{U}, Y)) = \begin{cases} (\mathcal{T}_1, Y) & \text{if } \mathcal{T}_2 = \mathcal{T}_1/\mathcal{T}'_1 \text{ and } \mathcal{U} = \mathcal{T}_1/\mathcal{V} \\ 0 & \text{if not.} \end{cases}$$

Moreover, $\mathcal{U} = \mathcal{T}_2/\mathcal{T}'_2 = (\mathcal{T}_1/\mathcal{T}'_1)/(\mathcal{V}/\mathcal{T}'_1) = \mathcal{T}_1/\mathcal{V}$. We conclude then

$$\Psi \circ (\ast \otimes id) = \Psi \circ (id \otimes \Psi).$$

which proves that Ψ is an action of $\widetilde{\mathbb{D}}$ on \mathbb{D} . □

5.2. The cointeraction.

Theorem 5.1. For any finite set X , let $\xi : \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \longrightarrow \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X$ be the map defined by:

$$(5.1) \quad \xi((\mathcal{T}, \widetilde{\mathcal{T}}) \otimes (\mathcal{T}_1, Z) \otimes (\mathcal{T}_2, W)) = (\mathcal{T}|_Z \mathcal{T}|_{X \setminus Z}, \widetilde{\mathcal{T}}|_Z \widetilde{\mathcal{T}}|_{X \setminus Z}) \otimes (\mathcal{T}_1, Z) \otimes (\mathcal{T}_2, W).$$

The following diagram is commutative:

$$\begin{array}{ccc} \mathbb{D}_X & \xrightarrow{\Phi} & \widetilde{\mathbb{D}}_X \otimes \mathbb{D}_X \\ \Delta \downarrow & & \downarrow Id \otimes \Delta \\ (\mathbb{D} \otimes \mathbb{D})_X & & \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \\ \Phi \otimes \Phi \downarrow & & \downarrow \xi \\ \bigoplus_{Y \subset X} \widetilde{\mathbb{D}}_Y \otimes \mathbb{D}_Y \otimes \widetilde{\mathbb{D}}_{X \setminus Y} \otimes \mathbb{D}_{X \setminus Y} & \xrightarrow{m^{1,3}} & \widetilde{\mathbb{D}}_X \otimes (\mathbb{D} \otimes \mathbb{D})_X \end{array}$$

i.e.,

$$\xi \circ (id \otimes \Delta) \circ \Phi = (m^{1,3}) \circ (\Phi \otimes \Phi) \circ \Delta.$$

Proof. In (5.1), the finite set X is partitioned into two subsets X_1 and X_2 , and Z (resp. W) is an open subset of X_1 (resp. X_2) for \mathcal{T}_1 (resp. \mathcal{T}_2). According to Proposition 3.2, the relation $\mathcal{T}|_Z \mathcal{T}|_{X \setminus Z} \otimes \mathcal{T}'|_Z \mathcal{T}'|_{X \setminus Z}$ holds, hence the map ξ is well defined. For any $(\mathcal{T}, Y) \in \mathbb{D}_X$, we have

$$\begin{aligned} (id \otimes \Delta) \circ \Phi(\mathcal{T}, Y) &= (id \otimes \Delta) \left(\sum_{\substack{\tilde{\mathcal{T}} \otimes \mathcal{T}, Y \in \mathcal{T}/\tilde{\mathcal{T}} \\ \tilde{\mathcal{T}}|_{X \setminus Y} = D_{X \setminus Y, \tilde{\mathcal{T}}} }} (\mathcal{T}, \tilde{\mathcal{T}}) \otimes (\mathcal{T}/\tilde{\mathcal{T}}, Y) \right) \\ &= \sum_{\substack{\tilde{\mathcal{T}} \otimes \mathcal{T}, Y \in \mathcal{T}/\tilde{\mathcal{T}} \\ \tilde{\mathcal{T}}|_{X \setminus Y} = D_{X \setminus Y, \tilde{\mathcal{T}}} }} \sum_{Z \in (\mathcal{T}/\tilde{\mathcal{T}})|_Y} (\mathcal{T}, \tilde{\mathcal{T}}) \otimes ((\mathcal{T}/\tilde{\mathcal{T}})|_Z, Z) \otimes ((\mathcal{T}/\tilde{\mathcal{T}})|_{X \setminus Z}, Y \setminus Z) \end{aligned}$$

therefore

$$\xi \circ (id \otimes \Delta) \circ \Phi(\mathcal{T}, Y) = \sum_{\substack{\tilde{\mathcal{T}} \otimes \mathcal{T}, Y \in \mathcal{T}/\tilde{\mathcal{T}} \\ \tilde{\mathcal{T}}|_{X \setminus Y} = D_{X \setminus Y, \tilde{\mathcal{T}}} }} \sum_{Z \in (\mathcal{T}/\tilde{\mathcal{T}})|_Y} (\mathcal{T}|_Z \mathcal{T}|_{X \setminus Z}, \tilde{\mathcal{T}}|_Z \tilde{\mathcal{T}}|_{X \setminus Z}) \otimes ((\mathcal{T}/\tilde{\mathcal{T}})|_Z, Z) \otimes ((\mathcal{T}/\tilde{\mathcal{T}})|_{X \setminus Z}, Y \setminus Z).$$

On the other hand, we have

$$\begin{aligned} (\Phi \otimes \Phi) \circ \Delta(\mathcal{T}, Y) &= \sum_{Z \in \mathcal{T}|_Y} \sum_{\mathcal{T}' \otimes \mathcal{T}|_Z} \sum_{\substack{\mathcal{T}'' \otimes \mathcal{T}|_{X \setminus Z}, Y \setminus Z \in \mathcal{T}|_{X \setminus Z}/\mathcal{T}'' \\ \mathcal{T}''|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}''}}} (\mathcal{T}|_Z, \mathcal{T}') \otimes (\mathcal{T}|_Z/\mathcal{T}', Z) \otimes (\mathcal{T}|_{X \setminus Z}, \mathcal{T}'') \otimes (\mathcal{T}|_{X \setminus Z}/\mathcal{T}'', Y \setminus Z), \end{aligned}$$

therefore

$$\begin{aligned} m^{1,3} \circ (\Phi \otimes \Phi) \circ \Delta(\mathcal{T}, Y) &= \sum_{Z \in \mathcal{T}|_Y} \sum_{\mathcal{T}' \otimes \mathcal{T}|_Z} \sum_{\substack{\mathcal{T}'' \otimes \mathcal{T}|_{X \setminus Z}, Y \setminus Z \in \mathcal{T}|_{X \setminus Z}/\mathcal{T}'' \\ \mathcal{T}''|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}''}}} (\mathcal{T}|_Z \mathcal{T}|_{X \setminus Z}, \mathcal{T}' \mathcal{T}'') \otimes (\mathcal{T}/\mathcal{T}', Z) \otimes (\mathcal{T}|_{X \setminus Z}/\mathcal{T}'', Y \setminus Z). \end{aligned}$$

To show that the both expressions above coincide, we use the two following lemmas.

Lemma 5.1. *Let \mathcal{T} and $\tilde{\mathcal{T}}$ be two topologies on X , such that $\tilde{\mathcal{T}} \otimes \mathcal{T}$ and let $Y \in \mathcal{T}$. Then $Y \in \mathcal{T}/\tilde{\mathcal{T}}$ if and only if both Y and $X \setminus Y$ are open subsets of X for $\tilde{\mathcal{T}}$.*

Proof. From $Y \in \mathcal{T}$ and $\tilde{\mathcal{T}} \otimes \mathcal{T}$ we immediately get $Y \in \tilde{\mathcal{T}}$. Now let $x \in X \setminus Y$ and $y \in X$ with $x \leq_{\tilde{\mathcal{T}}} y$. From $x \leq_{\tilde{\mathcal{T}}} y$ we get $y \leq_{\mathcal{T}/\tilde{\mathcal{T}}} x$. Suppose that $y \in Y$ then $x \in Y$ (because $Y \in \mathcal{T}/\tilde{\mathcal{T}}$), which is absurd. Hence $X \setminus Y \in \tilde{\mathcal{T}}$.

Conversely, suppose that both Y and $X \setminus Y$ are open subsets of X for $\tilde{\mathcal{T}}$, let $y \in Y$ and let $z \in X$ with $y \leq_{\mathcal{T}/\tilde{\mathcal{T}}} z$. There is a chain

$$y R t_1 \cdots R t_k R z$$

with $t_1, \dots, t_k \in X$, where $a R b$ means $a \leq_{\mathcal{T}} b$ or $a \geq_{\tilde{\mathcal{T}}} b$. Supposing $a \in Y$ we have either $a \leq_{\mathcal{T}} b$ which yields $b \in Y$, or $b \leq_{\tilde{\mathcal{T}}} a$, which would yield the contradiction $a \in X \setminus Y$ if b were to belong

to $X \setminus Y$. Hence we necessarily have $b \in Y$. Progressing along the chain, from $y \in Y$ we therefore infer $z \in Y$. \square

Lemma 5.2. *Let \mathcal{T} be a topology on a finite set X , and let Z be an open subset of X for T . Let \mathcal{U} be the topology $\mathcal{T}|_Z \mathcal{T}|_{X \setminus Z}$. Then we have*

$$\mathcal{U} \otimes \mathcal{T}.$$

Proof. The relation $\mathcal{U} < \mathcal{T}$ is obvious. Any connected component W for \mathcal{U} is contained either in Z or in $X \setminus Z$, hence $\mathcal{T}|_W = \mathcal{U}|_W$. Finally, consider $x, y \in X$ such that $x \sim_{\mathcal{T}/\mathcal{U}} y$. There is a chain $xRt_1R \cdots Rt_kRx$ with some $j \in \{1, \dots, k\}$ such that $t_j = y$. By the argument which was used in the end of the proof of Proposition 3.2, the whole chain belongs to the same \mathcal{U} -connected component, hence we have $x\tilde{R}t_1\tilde{R} \cdots \tilde{R}t_k\tilde{R}x$, where $a\tilde{R}b$ stands for $a \leq_{\mathcal{U}} b$ or $b \leq_{\mathcal{U}} a$. Hence $x \sim_{\mathcal{U}/\mathcal{U}} y$. \square

Proof of theorem 5.1 (continued). Let \mathcal{E} be the set of triples $(Z, \mathcal{T}', \mathcal{T}'')$ which occur in the expression of $m^{1,3} \circ (\Phi \otimes \Phi) \circ \Delta(\mathcal{T}, Y)$ above, i.e. subject to the conditions

$$Z \in \mathcal{T}|_Y, \quad \mathcal{T}'' \otimes \mathcal{T}|_{X \setminus Z}, \quad \mathcal{T}' \otimes \mathcal{T}|_Z, \quad \mathcal{T}''|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}''}, \quad Y \setminus Z \in \mathcal{T}|_{X \setminus Z} / \mathcal{T}'',$$

and let \mathcal{F} be the set of pairs $(Z, \tilde{\mathcal{T}})$ which occur in the expression of $\xi \circ (id \otimes \Delta) \circ \Phi(\mathcal{T}, Y)$ above, i.e. subject to the conditions

$$\tilde{\mathcal{T}} \otimes \mathcal{T}, \quad Y \in \mathcal{T} / \tilde{\mathcal{T}}, \quad \tilde{\mathcal{T}}|_{X \setminus Y} = D_{X \setminus Y, \tilde{\mathcal{T}}}, \quad Z \in (\mathcal{T} / \tilde{\mathcal{T}})|_Y.$$

To prove Theorem 5.1, it suffices to show that $(Z, \mathcal{T}', \mathcal{T}'') \mapsto (Z, \mathcal{T}'\mathcal{T}'')$ is a bijection from \mathcal{E} onto \mathcal{F} . For any $(Z, \mathcal{T}'\mathcal{T}'') \in \mathcal{E}$, it is clear from Lemma 5.2 that $\tilde{\mathcal{T}} \otimes \mathcal{T}$ holds, with $\tilde{\mathcal{T}} := \mathcal{T}'\mathcal{T}''$. From $Y \in \mathcal{T}$ we get $Y \setminus Z \in \mathcal{T}|_{X \setminus Z}$. Together with $Y \setminus Z \in \mathcal{T}|_{X \setminus Z} / \mathcal{T}''$ one deduces from Lemma 5.1 that both $Y \setminus Z$ and $X \setminus Y$ are open subsets of $X \setminus Z$ for \mathcal{T}'' . Hence we have a partition

$$(5.2) \quad X = Z \sqcup Y \setminus Z \sqcup X \setminus Y$$

of X into three open subsets for the topology $\tilde{\mathcal{T}}$. From $Y \in \tilde{\mathcal{T}}$ and $X \setminus Y \in \tilde{\mathcal{T}}$ we get $Y \in \mathcal{T} / \tilde{\mathcal{T}}$ from Lemma 5.1. We also have

$$\tilde{\mathcal{T}}|_{X \setminus Y} = \mathcal{T}''|_{X \setminus Y} = D_{X \setminus Y, \mathcal{T}''} = D_{X \setminus Y, \tilde{\mathcal{T}}}.$$

Finally, from the fact that both Z and $X \setminus Z$ are open for $\tilde{\mathcal{T}}$ and $Z \in T$, we get $Z \in \mathcal{T} / \tilde{\mathcal{T}}$ from Lemma 5.1, hence $Z \in (\mathcal{T} / \tilde{\mathcal{T}})|_Y$. This proves $(Z, \tilde{\mathcal{T}}) \in \mathcal{F}$.

Conversely, for any $(Z, \tilde{\mathcal{T}}) \in \mathcal{F}$, from $Y \in \mathcal{T} / \tilde{\mathcal{T}}$ and Lemma 5.1 we get that both Y and $X \setminus Y$ are open subsets of X for $\tilde{\mathcal{T}}$, and from $Z \in \mathcal{T} / \tilde{\mathcal{T}}$ and the same lemma we get that both Z and $X \setminus Z$ are open subsets of X for $\tilde{\mathcal{T}}$. Hence the splitting (5.2) into three open subsets holds, and we have $\tilde{\mathcal{T}} = \mathcal{T}'\mathcal{T}''$, with $\mathcal{T}' := \tilde{\mathcal{T}}|_Z$ and $\mathcal{T}'' := \tilde{\mathcal{T}}|_{X \setminus Z}$. The triple $(Y, \mathcal{T}', \mathcal{T}'')$ verifies the five required conditions to belong to \mathcal{E} . Both correspondences from \mathcal{E} to \mathcal{F} are obviously inverse to each other, which ends up the proof of Theorem 5.1. \square

Remark 5.1. If we apply the functor $\overline{\mathcal{K}}$, we notice here that this diagram yields the commutative diagram:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\Phi} & \widetilde{\mathcal{D}} \otimes \mathcal{D} \\
\Delta \downarrow & & \downarrow Id \otimes \Delta \\
\mathcal{D} \otimes \mathcal{D} & & \widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D} \\
\Phi \otimes \Phi \downarrow & & \downarrow \xi \\
\widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \widetilde{\mathcal{D}} \otimes \mathcal{D} & \xrightarrow{m^{1,3}} & \widetilde{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D}
\end{array}$$

where Φ is a shorthand for $\overline{\mathcal{K}(\Phi)}$ and so on.

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